Average-Case Analysis and Randomization

Textbook Reading
Chapter 7 & Sections 8.4, 9.2
Overview

Design principle
• Do the easy thing and hope it works for most inputs
• Make random choices and hope they’re good

Problems
• Sorting (Quick Sort)
• Permuting
• Selection
• Game tree evaluation
Quick Sort Revisited

The problem with deterministic Quick Sort:

The running time is in $O(n \lg n)$, but the algorithm for finding the pivot is non-trivial (and slow).
Quick Sort Revisited

The problem with deterministic Quick Sort:

The running time is in $O(n \log n)$, but the algorithm for finding the pivot is non-trivial (and slow).

Remedy:

Blindly use the last element as pivot.

SimpleQuickSort($A, \ell, r$)

1. if $r \leq \ell$
2. then return
3. $m =$ Partition($A, \ell, r$)
4. SimpleQuickSort($A, \ell, m - 1$)
5. SimpleQuickSort($A, m + 1, r$)

Partition($A, \ell, r$)

1. $i = \ell - 1$
2. for $j = \ell$ to $r - 1$
3. do if $A[j] \leq A[r]$
4. then $i = i + 1$
5. swap $A[i]$ and $A[j]$
6. swap $A[i + 1]$ and $A[r]$
7. return $i + 1$
Average-Case Analysis of Simple Quick Sort

**Lemma:** The average-case running time of SimpleQuickSort is in $O(n \lg n)$. 
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We defined the *average-case running time* of an algorithm as the average of its running time over all possible inputs of size $n$. 
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Observation: Simple Quick Sort behaves the same on all inputs whose elements have the same relative order.
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⇒ The input to SimpleQuickSort is a permutation $\pi$ of the sorted output sequence $\langle x_1, x_2, \ldots, x_n \rangle$ we expect as the output.
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**Problem:** There are infinitely many different inputs of size $n$!

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$\Rightarrow$ The input to SimpleQuickSort is a permutation $\pi$ of the sorted output sequence $\langle x_1, x_2, \ldots, x_n \rangle$ we expect as the output.

$\Rightarrow$ The average-case running time of SimpleQuickSort is the same as its expected running time on a uniformly random input permutation.
Lemma: If $A[\ell \ldots r]$ is a uniform random permutation of the elements in $A[\ell \ldots r]$, then the two subarrays $A[\ell \ldots m - 1]$ and $A[m + 1 \ldots r]$ produced by $\text{Partition}(A, \ell, r)$ are also uniform random permutations of the elements they contain.
Partitioning Maintains Uniformity

**Lemma:** If $A[\ell \ldots r]$ is a uniform random permutation of the elements in $A[\ell \ldots r]$, then the two subarrays $A[\ell \ldots m - 1]$ and $A[m + 1 \ldots r]$ produced by $\text{Partition}(A, \ell, r)$ are also uniform random permutations of the elements they contain.
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The behaviour of $\text{Partition}$ depends only on the sequence of $-$s and $+$s!

The $-$s are exactly the elements that end up in $A[\ell \ldots m - 1]$, the $+$s end up in $A[m + 1 \ldots r]$.
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The behaviour of \( \text{Partition} \) depends only on the sequence of \(-s\) and \(+s\)!

The \(-s\) are exactly the elements that end up in \( A[\ell \ldots m - 1] \), the \(+s\) end up in \( A[m + 1 \ldots r] \).

In a uniformly random permutation, any permutation of the \(-s\) or \(+s\) is equally likely.
**Partitioning Maintains Uniformity**

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Each such permutation produces a different permutation of $A[\ell \ldots m-1]$ or $A[m+1 \ldots r]$. 

\[
\begin{array}{ccccccccc}
1 & 7 & 3 & 4 & 8 & 6 & 2 & 5 \\
\hline
- & + & - & - & + & + & - & p
\end{array}
\]
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The $-s$ are exactly the elements that end up in $A[\ell \ldots m - 1]$, the $+s$ end up in $A[m + 1 \ldots r]$.

In a uniformly random permutation, any permutation of the $-s$ or $+s$ is equally likely.

Each such permutation produces a different permutation of $A[\ell \ldots m - 1]$ or $A[m + 1 \ldots r]$.

$\Rightarrow A[\ell \ldots m - 1]$ and $A[m + 1 \ldots r]$ are uniform random permutations.
Observation: The running time of SimpleQuickSort is in $O(n + C)$, where $C$ is the number of comparisons it performs between input elements.

**SimpleQuickSort**$(A, \ell, r)$

1. if $r \leq \ell$
2. then return
3. $m = \text{Partition}(A, \ell, r)$
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**Partition**$(A, \ell, r)$

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- There are $O(n)$ recursive calls in total.
- The cost of each recursive call, excluding the call to Partition is constant.
- The cost of Partition is $O(1 + \# \, \text{comparisons it performs})$. 
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**Partition(A, ℓ, r)**

1. $i = ℓ – 1$
2. for $j = ℓ$ to $r – 1$
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- There are $O(n)$ recursive calls in total.
- The cost of each recursive call, excluding the call to Partition is constant.
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⇒ It suffices to prove that $E[C] ∈ O(n \lg n)$. 
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Observation: Two elements $x_i$ and $x_j$ are compared at most once.
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- The pivot is not part of the recursive calls `SimpleQuickSort(A, ℓ, m - 1)` and `SimpleQuickSort(A, m + 1, r)`.

```
SimpleQuickSort(A, ℓ, r)
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2 then return
3 m = Partition(A, ℓ, r)
4 SimpleQuickSort(A, ℓ, m - 1)
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Partition(A, ℓ, r)
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5 swap A[i] and A[j]
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7 return i + 1
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Average-Case Analysis of Simple Quick Sort

Observation: Two elements $x_i$ and $x_j$ are compared at most once.

- Each call $\text{SimpleQuickSort}(A, \ell, r)$ compares every element in $A[\ell \ldots r - 1]$ to the pivot $p$ stored in $A[r]$.
- The pivot is not part of the recursive calls $\text{SimpleQuickSort}(A, \ell, m - 1)$ and $\text{SimpleQuickSort}(A, m + 1, r)$.

Let $C_{ij} = \begin{cases} 1 & \text{if } x_i \text{ and } x_j \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$

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$\Rightarrow C = \sum_{i,j} C_{ij}$

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$E[C] = E \left[ \sum_{i,j} C_{ij} \right] = \sum_{i,j} E[C_{ij}]$

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Corollary: $E[C_{ij}] = \frac{2}{j - i + 1}$. 
Average-Case Analysis of Simple Quick Sort

\[ E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}] \]
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\[ E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k + 1} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k} \]
Average-Case Analysis of Simple Quick Sort

\[
E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}]
\]

\[
= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}
\]

\[
= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k + 1}
\]

\[
< 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}
\]

\[
= 2(n - 1)H_n.
\]

\[
H_n = \sum_{i=1}^{n} \frac{1}{i} = n\text{th Harmonic Number}
\]
Average-Case Analysis of Simple Quick Sort

\[ \sum_{i=1}^{n} \frac{1}{i} \]
Average-Case Analysis of Simple Quick Sort

\[
\int_{1}^{n+1} \frac{d}{x} < \sum_{i=1}^{n} \frac{1}{i}
\]
Average-Case Analysis of Simple Quick Sort

\[ \ln(n + 1) = \int_{1}^{n+1} \frac{d}{x} < \sum_{i=1}^{n} \frac{1}{i} \]
Average-Case Analysis of Simple Quick Sort

\[
\ln(n + 1) = \int_{1}^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_{1}^{n} \frac{dx}{x}
\]
Average-Case Analysis of Simple Quick Sort

\[
\ln(n + 1) = \int_{1}^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_{1}^{n} \frac{dx}{x} = 1 + \ln n
\]
Average-Case Analysis of Simple Quick Sort

\[ \ln(n+1) = \int_1^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_1^{n} \frac{dx}{x} = 1 + \ln(n) \]

\[ \Rightarrow E[C] \leq 2(n - 1)H_n \in \mathcal{O}(n \lg n) \]
Interpretation of Average-Case Analysis

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Algorithms that are fast **on average** are often simpler and on average faster than worst-case efficient algorithms.
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Worst-case efficiency is desirable if we need performance guarantees every single time we run the algorithm.

Algorithms that are fast on average are often simpler and on average faster than worst-case efficient algorithms.

They are a good choice when we want good performance most of the time and possibly averaged over running the algorithm many times.
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What exactly is the meaning of the following statement?

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Example:

SimpleQuickSort takes $\Theta(n^2)$ time on almost sorted inputs.

There are applications where the inputs to be sorted are all almost sorted.

SimpleQuickSort is a poor choice of a sorting algorithm in such applications.
Randomization

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Since a randomized algorithm behaves differently every time it runs, there is no way to force it to exhibit its worst-case running time!
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The expected running time of a randomized algorithm is an expectation over the random choices the algorithm makes.
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A randomized algorithm makes no assumptions about the input and ensures randomness by making random choices.

Since a randomized algorithm behaves differently every time it runs, there is no way to force it to exhibit its worst-case running time!

The expected running time of a randomized algorithm is an expectation over the random choices the algorithm makes.

⇒ No more assumptions about the probability distribution. We know the distribution of the choices the algorithm makes.
Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in $O(n \lg n)$. 
Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in $O(n \lg n)$.

So why don’t we just ensure the input is a uniform random permutation?

RandomPermutationQuickSort(A)

1. RandomPermute(A)
2. SimpleQuickSort(A, 1, n)
Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in $O(n \lg n)$.

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We can compute a uniform random permutation in $O(n)$ time in the worst case.
Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in \( O(n \lg n) \).

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**RandomPermutationQuickSort(A)**

1. RandomPermute(A)
2. SimpleQuickSort(A, 1, n)

We can compute a uniform random permutation in \( O(n) \) time in the worst case.

**Corollary:** The expected running time of RandomPermutationQuickSort is in \( O(n \lg n) \).
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.
Randomized Quick Sort, Take 2

The key to the analysis of **SimpleQuickSort**: If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.

So why don’t we make sure we choose a uniform random pivot, no matter the input permutation?

**RandomPivotQuickSort(A, ℓ, r)**

1. if $r \leq ℓ$
2. then return
3. $p = \text{RandomNumber}(ℓ, r)$
4. swap $A[p]$ and $A[r]$
5. $m = \text{Partition}(A, ℓ, r)$
6. RandomPivotQuickSort(A, ℓ, m – 1)
7. RandomPivotQuickSort(A, m + 1, r)
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.

So why don’t we make sure we choose a uniform random pivot, no matter the input permutation?

RandomPivotQuickSort(A, ℓ, r)

1. if r ≤ ℓ
2. then return
3. p = RandomNumber(ℓ, r)
4. swap A[p] and A[r]
5. m = Partition(A, ℓ, r)
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7. RandomPivotQuickSort(A, m + 1, r)

Lemma: The expected running time of RandomPivotQuickSort is in O(n lg n).

The analysis is 100% identical to that of SimpleQuickSort!
Uniform Random Permutation In Linear Time

RandomPermute(A)

1: \( n = |A| \)
2: for \( j = n \) downto 2
3: do  \( i = \text{RandomNumber}(1, j) \)
4: swap \( A[i] \) and \( A[j] \)
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Uniform Random Permutation In Linear Time

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1  \text{n} = \vert A \vert
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1. \( n = |A| \)
2. \( \text{for} \ j = n \ \text{downto} \ 2 \)
3. \( \text{do} \ i = \text{RandomNumber}(1, j) \)
4. \( \text{swap} \ A[i] \ \text{and} \ A[j] \)

**Observation:** RandomPermute takes \( \mathcal{O}(n) \) time.
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
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Observation: RandomPermute takes \( O(n) \) time.

Lemma: RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).
Uniform Random Permutation In Linear Time

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Induction on \( n \).
**Uniform Random Permutation In Linear Time**

**RandomPermute(A)**

1. \( n = |A| \)
2. \( \text{for } j = n \text{ downto } 2 \)
3. \( \text{do } i = \text{RandomNumber}(1, j) \)
4. \( \text{swap } A[i] \text{ and } A[j] \)

**Observation:** RandomPermute takes \( O(n) \) time.

**Lemma:** RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).

**Induction on** \( n \).

If \( n = 1 \), then it produces the only possible permutation with probability \( 1 = \frac{1}{1!} \).
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
2. for \( j = n \) downto 2
3. do \( i = \text{RandomNumber}(1, j) \)
4. swap \( A[i] \) and \( A[j] \)

Observation: RandomPermute takes \( O(n) \) time.

Lemma: RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).

If \( n > 1 \), then to produce the permutation \( \langle x_1, x_2, \ldots, x_n \rangle \) (event \( E \)), we need to
- Place \( x_n \) into \( A[n] \) (event \( E_1 \)) and
- Place \( x_1, x_2, \ldots, x_{n-1} \) into \( A[1 \ldots n-1] \) (event \( E_2 \)).
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
2. for \( j = n \) downto 2
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Lemma: RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).

If \( n > 1 \), then to produce the permutation \( \langle x_1, x_2, \ldots, x_n \rangle \) (event \( E \)), we need to

- Place \( x_n \) into \( A[n] \) (event \( E_1 \)) and
- Place \( x_1, x_2, \ldots, x_{n-1} \) into \( A[1 \ldots n-1] \) (event \( E_2 \)).

So \( P[E] = P[E_1 \cap E_2] = P[E_1] \cdot P[E_2|E_1] = \frac{1}{n} \cdot \frac{1}{(n-1)!} = \frac{1}{n!} \).
Randomized Selection

RandomizedSelection(A, ℓ, r, k)

1. if $r \leq ℓ$
   2. then return $A[ℓ]$
3. $p = \text{RandomNumber}(ℓ, r)$
4. swap $A[p]$ and $A[r]$
5. $m = \text{Partition}(A, ℓ, r)$
6. if $m - ℓ = k - 1$
   7. then return $A[m]$
8. else if $m - ℓ \geq k$
   9. then RandomizedSelection(A, ℓ, $m - 1, k$)
10. else RandomizedSelection(A, $m + 1, r, k - (m + 1 - ℓ)$)
Randomized Selection

RandomizedSelection(A, L, R, K)

1. if R ≤ L
2. then return A[L]
3. p = RandomNumber(L, R)
4. swap A[p] and A[R]
5. m = Partition(A, L, R)
6. if m – L = K – 1
7. then return A[m]
8. else if m – L ≥ K
9. then RandomizedSelection(A, L, m – 1, K)
10. else RandomizedSelection(A, m + 1, R, K – (m + 1 – L))

Lemma: The expected running time of RandomizedSelection is in O(n).
Randomized Selection

Observation: If we choose the $i$th smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(\text{max}(n - i, i - 1))].$$
Randomized Selection

**Observation:** If we choose the $i$th smallest element as pivot, then
$$E[T(n)] \leq O(n) + E[T(\max(n - i, i - 1))]$$.

**Corollary:**
$$E[T(n)] \leq O(n) + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(n - i, i - 1))]$$. 
Randomized Selection

Observation: If we choose the $i$th smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(\max(n - i, i - 1))].$$

Corollary: $E[T(n)] \leq O(n) + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(n - i, i - 1))].$

Claim: $E[T(n)] \leq cn$, for some $c > 0$. 
Randomized Selection

Observation: If we choose the ith smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(\max(n - i, i - 1))]$$.

Corollary: $$E[T(n)] \leq O(n) + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(n - i, i - 1))]$$.

Claim: $$E[T(n)] \leq cn$$, for some $$c > 0$$.

Base case: $$1 \leq n < 4$$.

$$T(n) \leq c \leq cn$$.
Randomized Selection

Inductive step: $n \geq 4$.

$$E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))]$$
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq a_n + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))] \\
\leq a_n + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} E[T(i)]
\]
Randomized Selection

Inductive step: $n \geq 4$.

\[
E[T(n)] \leq a_n + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]
\]

\[
\leq a_n + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} E[T(i)]
\]

\[
\leq a_n + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} c_i
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\]

\[
\leq an + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} ci
\]

\[
= an + \frac{2c}{n} \left( \sum_{i=1}^{n-1} i - \sum_{i=1}^{\lfloor n/2 \rfloor - 1} i \right)
\]
Randomized Selection

Inductive step: \( n \geq 4. \)

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]
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= an + \frac{2c}{n} \left( \frac{n(n-1)}{2} - \frac{\lfloor n/2 \rfloor (\lfloor n/2 \rfloor - 1)}{2} \right)
\]
Randomized Selection

Inductive step: \( n \geq 4. \)

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))]
\]

\[
\leq an + \frac{2}{n} \sum_{i=\lceil n/2 \rceil}^{n-1} E[T(i)]
\]

\[
\leq an + \frac{2}{n} \sum_{i=\lceil n/2 \rceil}^{n-1} ci
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\[
= an + \frac{2c}{n} \left( \sum_{i=1}^{\lceil n/2 \rceil - 1} i - \sum_{i=1}^{\lceil n/2 \rceil - 1} i \right)
\]

\[
= an + \frac{2c}{n} \left( \frac{n(n - 1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right)
\]

\[
\leq an + \frac{c}{n} \left[ n(n - 1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]
\]
Randomized Selection

Inductive step: $n \geq 4$.

$$E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(max(i - 1, n - i))]$$

$$\leq an + \frac{c}{n} \left[ n(n - 1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]$$
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\]

\[
= an + \frac{c}{n} \left( \frac{3n^2}{4} + \frac{n}{2} \right)
\]
Randomized Selection

Inductive step: $n \geq 4.$

$$E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))]$$

$$\leq an + \frac{c}{n} \left[ n(n - 1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]$$

$$= an + \frac{c}{n} \left( \frac{3n^2}{4} + \frac{n}{2} \right)$$

$$= \left( a + \frac{3c}{4} + \frac{c}{2n} \right) n$$
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]
\]

\[
\leq an + \frac{c}{n} \left[ n(n-1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]
\]

\[
= an + \frac{c}{n} \left( \frac{3n^2}{4} + \frac{n}{2} \right)
\]

\[
= \left( a + \frac{3c}{4} + \frac{c}{2n} \right) n
\]

\[
\leq cn \quad \forall c \geq 8a.
\]
Sorting in Linear Time?

Using comparisons only, as Insertion Sort, Merge Sort, Quick Sort do, it is impossible to sort faster than in $\Omega(n \lg n)$ time.
Sorting in Linear Time?

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**Counting sort:** Sorts $n$ integers between 1 and $n$ in $O(n)$ time.
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**Radix sort**: Sorts $n$ integers between 1 and $n^c$ in $O(cn)$ time. This is $O(n)$ if $c$ is a constant.
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**Radix sort:** Sorts $n$ integers between 1 and $n^c$ in $O(cn)$ time. This is $O(n)$ if $c$ is a constant.

**Bucket sort:** Sorts $n$ real numbers drawn uniformly at random from an interval $[a, b)$ in expected linear time.
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval $[a, b)$.
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval 
\([a, b)\).

We can normalize this to the interval \([0, 1)\).
**Bucket Sort**

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Divide \([0, 1)\) into subintervals of length \(\frac{1}{n}\).
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Divide \([0, 1)\) into subintervals of length \(\frac{1}{n}\).

How many elements do we expect to end up in each subinterval?
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval \([a, b)\).

We can normalize this to the interval \([0, 1)\).

Divide \([0, 1)\) into subintervals of length \(\frac{1}{n}\).

How many elements do we expect to end up in each subinterval? \(\frac{1}{n}\)!
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval $[a, b)$.

We can normalize this to the interval $[0, 1)$.

Divide $[0, 1)$ into subintervals of length $\frac{1}{n}$.

How many elements do we expect to end up in each subinterval? 1!

⇒ Strategy:
- Bucket items according to the subinterval they belong to.
- Sort each bucket, hopefully in constant time.
- Concatenate the sorted buckets.
Bucket Sort

BucketSort(A)

1. \( n = |A| \)
2. \( B = \) an array of \( n \) empty singly-linked lists
3. for \( i = 1 \) to \( n \)
4. do prepend \( A[i] \) to list \( B[1 + \lfloor n \cdot A[i] \rfloor] \)
5. for \( i = 1 \) to \( n \)
6. do InsertionSort(B[i])
7. \( j = 0 \)
8. for \( i = 1 \) to \( n \)
9. do for every element \( x \in B[i] \)
10. do \( A[j] = x \)
11. \( j = j + 1 \)
Bucket Sort

BucketSort(A)

1. $n = |A|$
2. $B$ = an array of $n$ empty singly-linked lists
3. for $i = 1$ to $n$
   4. do prepend $A[i]$ to list $B[1 + \lfloor n \cdot A[i] \rfloor]$  
5. for $i = 1$ to $n$
   6. do InsertionSort($B[i]$)
7. $j = 0$
8. for $i = 1$ to $n$
   9. do for every element $x \in B[i]$
      10. do $A[j] = x$
        11. $j = j + 1$

This is where we depart from using comparisons only!
Bucket Sort

BucketSort(A)

1. \( n = |A| \)
2. \( B = \) an array of \( n \) empty singly-linked lists
3. \( \text{for } i = 1 \text{ to } n \) do prepend \( A[i] \) to list \( B[l + \lfloor n \cdot A[i] \rfloor] \)
4. \( \text{for } i = 1 \text{ to } n \) do InsertionSort(B[i])
5. \( j = 0 \)
6. \( \text{for } i = 1 \text{ to } n \) do for every element \( x \in B[i] \) do \( A[j] = x \)
7. \( j = j + 1 \)

Worst-case running time: \( O(n^2) \)

This is where we depart from using comparisons only!
Bucket Sort

BucketSort(A)

1. n = |A|
2. B = an array of n empty singly-linked lists
3. for i = 1 to n
do prepend A[i] to list B[1 + ⌊n · A[i]⌋]
4. for i = 1 to n
do InsertionSort(B[i])
5. j = 0
6. for i = 1 to n
do for every element x ∈ B[i]
do A[j] = x
7. j = j + 1

Worst-case running time: $O(n^2)$

This is where we depart from using comparisons only!

Why not Merge Sort?
Bucket Sort

BucketSort(A)

1. \( n = |A| \)
2. \( B = \) an array of \( n \) empty singly-linked lists
3. for \( i = 1 \) to \( n \)
   4. do prepend \( A[i] \) to list \( B[1 + \lfloor n \cdot A[i] \rfloor] \)
5. for \( i = 1 \) to \( n \)
   6. do InsertionSort(\( B[i] \) )
7. \( j = 0 \)
8. for \( i = 1 \) to \( n \)
   9. do for every element \( x \in B[i] \) 
      10. do \( A[j] = x \)
      11. \( j = j + 1 \)

Worst-case running time: \( O(n^2) \)

This is where we depart from using comparisons only!

Why not Merge Sort?

- It only helps in the worst case.
- It's more complicated.
- It actually hurts when buckets are small, which is what we expect.
Bucket Sort

Running time: \( T(n) \in O \left( n + \sum_{i=1}^{n} n_i^2 \right) \)

\( n_i \) = the number of elements in \( B[i] \)
Bucket Sort

Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right)$

$n_i = \text{the number of elements in } B[i]$

$E[T(n)] \in O\left(n + \sum_{i=1}^{n} E[n_i]^2\right)$
Bucket Sort

Running time: \( T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right) \)

\( n_i = \text{the number of elements in } B[i] \)

Lemma: \( E[n_i^2] < 2. \)
Bucket Sort

Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right)$

$n_i = \text{the number of elements in } B[i]$

$E[T(n)] \in O\left(n + \sum_{i=1}^{n} E[n_i]^2\right)$

Lemma: $E[n_i^2] < 2$.

Corollary: $E[T(n)] \in O(n)$. 
Bucket Sort

Lemma: $E[n_i^2] < 2.$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

$$X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in B}[i] \\
0 & \text{otherwise}
\end{cases}$$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

$X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in } \text{B}[i] \\
0 & \text{otherwise}
\end{cases}$

$n_i = \sum_{j=1}^{n} X_j$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

$$X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in B}[i] \\
0 & \text{otherwise}
\end{cases}$$

$$n_i = \sum_{j=1}^{n} X_j$$

$$E[n_i^2] = E\left[\left(\sum_{j=1}^{n} X_j\right)^2\right]$$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

$$X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in } B[i] \\
0 & \text{otherwise}
\end{cases}$$

$$n_i = \sum_{j=1}^{n} X_j$$

$$E[n_i^2] = E\left[\left(\sum_{j=1}^{n} X_j\right)^2\right] = E\left[\sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k\right]$$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

Let $X_j = \begin{cases} 1 & \text{A[j] ends up in } B[i] \\ 0 & \text{otherwise} \end{cases}$

Then:

$$X_j = \begin{cases} 1 & \text{A[j] ends up in } B[i] \\ 0 & \text{otherwise} \end{cases}$$

$$n_i = \sum_{j=1}^{n} X_j$$

$$E[n_i^2] = \mathbb{E} \left[ \left( \sum_{j=1}^{n} X_j \right)^2 \right] = \mathbb{E} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E}[X_j X_k]$$
Bucket Sort

**Lemma:** \( E[n_i^2] < 2. \)

\[
X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in } B[i] \\
0 & \text{otherwise}
\end{cases}
\]

\[
n_i = \sum_{j=1}^{n} X_j
\]

\[
E[n_i^2] = E \left[ \left( \sum_{j=1}^{n} X_j \right)^2 \right] = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[X_j X_k]
\]

\[
= \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n} \sum_{k=1}^{n} E[X_j] E[X_k]
\]

\[
= \sum_{j=1}^{n} E[X_j^2]
\]
Bucket Sort

Lemma: $E[n_i^2] < 2.$

\[
X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in } B[i] \\
0 & \text{otherwise}
\end{cases}
\]

\[
n_i = \sum_{j=1}^{n} X_j
\]

\[
E[n_i^2] = E \left[ \left( \sum_{j=1}^{n} X_j \right)^2 \right] = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[X_j X_k]
\]

\[
= \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n} \sum_{\substack{k=1 \atop k \neq j}}^{n} E[X_j]E[X_k]
\]

$X_j$ and $X_j$ are clearly not independent. $X_j$ and $X_k$ are independent.
Bucket Sort

\[ E[X_j] = \frac{1}{n} \cdot 1 + \left(1 - \frac{1}{n}\right) \cdot 0 = \frac{1}{n} \]
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\[ E[n_i^2] = \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n} \sum_{k=1 \atop k \neq j}^{n} E[X_j]E[X_k] = n \cdot \frac{1}{n} + \frac{n(n-1)}{n^2} < 2 \]
Randomized Bucket Sort?

For Quick Sort, we were able to eliminate assumptions about the input distribution using randomization.
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Does that work for Bucket Sort?
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Randomly permuting the input to guarantee this does not affect the final result of the algorithm.
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Bucket Sort relies on the random distribution of the input values. We can’t simply change them without changing the algorithm’s output.
Consider a game where two players, Max and Minnie, take turns. Assume the game cannot end in a draw.
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We can model all possible games as a **game tree**:
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Max-node: \[ \text{label}(v) = \bigvee_{\text{child } w} \text{label}(w) \]

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Game Tree Evaluation

Restrict ourselves to binary game trees of height $2k \Rightarrow n = 2^{2k}$ leaves

Max

Minnie

Max

Minnie

Max
Restrict ourselves to binary game trees of height $2^k$ \(\Rightarrow n = 2^{2k}\) leaves

\[(a \land b) \lor (c \land d) = (a \land b) \land (c \land d)\]
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Game Tree Evaluation: A Deterministic Algorithm

GameValue(v)
1  if v is a leaf
2  then return its value
3  else return not (GameValue(v.leftChild) and GameValue(v.rightChild))
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- One recursive call per node
- 2n – 1 nodes
⇒ Running time O(n)
Game Tree Evaluation: A Deterministic Algorithm

GameValue(v)
1 if v is a leaf
2 then return its value
3 if not GameValue(v.leftChild)
4 then return 1
5 else return not GameValue(v.rightChild)

• One recursive call per node
• 2n – 1 nodes
⇒ Running time O(n)
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Observation: Any deterministic algorithm has to inspect every leaf in the worst case and thus takes $\Omega(n)$ time in the worst case.
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Observation: Any deterministic algorithm has to inspect every leaf in the worst case and thus takes $\Omega(n)$ time in the worst case.

Adversary argument:

Can be used to construct a worst-case input for any deterministic algorithm, based on how the algorithm behaves.

- Fix every input element the first time the algorithm inspects it.
- Choose this to ensure the algorithm runs as long as possible.

When a leaf is the last unknown leaf in a subtree, we cannot prevent the algorithm from learning the value of the root of the subtree.

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Choose the leaf value so the algorithm doesn't learn more than that.
Game Tree Evaluation: Randomized Algorithm

RandomizedGameValue(v)

1. if v is a leaf
2. then return its value
3. coinFlip = RandomNumber(0, 1)
4. if coinFlip = 1
5. then first = v.leftChild
6. second = v.rightChild
7. else first = v.rightChild
8. second = v.leftChild
9. if not f = GameValue(first)
10. then return 1
11. else return not GameValue(second)
Game Tree Evaluation: Randomized Algorithm

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Lemma: The expected running time of RandomizedGameValue on any input is in \( O(n^{0.754}) \).
Game Tree Evaluation: Randomized Algorithm

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$$E_i[T(n)] = \text{expected running time on } n \text{ leaves if the result is } i \quad (i \in \{0, 1\})$$
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$$E_i[T(n)] = \text{expected running time on } n \text{ leaves if the result is } i \quad (i \in \{0, 1\})$$

$$E_0[T(n)] = 2 \cdot E_1\left[T\left(\frac{n}{2}\right)\right] + O(1)$$
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= 2 \cdot E_1 \left[ T \left( \frac{n}{4} \right) \right] + \frac{1}{2} \cdot E_1 \left[ T \left( \frac{n}{2} \right) \right] + O(1)
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= 2 \cdot E_1\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_1\left[T\left(\frac{n}{2}\right)\right] + O(1)
\]

\[
E[T(n)] \leq \max\left(2 \cdot E_1\left[T\left(\frac{n}{2}\right)\right], E_1[T(n)]\right)
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$E_i[T(n)] = \text{expected running time on } n \text{ leaves if the result is } i \quad (i \in \{0, 1\})$

$E_0[T(n)] = 2 \cdot E_1 \left( T \left( \frac{n}{2} \right) \right) + O(1)$

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$E_1[T(n)] \in O(n^{0.754}) \Rightarrow E[T(n)] \in O(n^{0.754})$
Claim: $E[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

$\alpha = \log \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754$. 

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Claim: \( E[T(n)] \leq cn^\alpha - d \) for some \( c > d > 0 \) and all \( n \geq 1 \), where
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\]

**Base case:** \( 1 \leq n < 2 \).

\( T(n) \in O(1) \Rightarrow E[T(n)] \leq cn^\alpha - d \) for any \( d \) and \( c \) sufficiently larger than \( d \).
Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where $\alpha = \log \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754$.

Inductive step: $n \geq 2$.

$$E_1[T(n)] \leq 2 \cdot E_1 \left[ T \left( \frac{n}{4} \right) \right] + \frac{1}{2} \cdot E_1 \left[ T \left( \frac{n}{2} \right) \right] + a$$
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Inductive step: \( n \geq 2 \).

\[
E_1[T(n)] \leq 2 \cdot E_1[T \left( \frac{n}{4} \right)] + \frac{1}{2} \cdot E_1[T \left( \frac{n}{2} \right)] + a
\leq 2 \cdot \left[ c \left( \frac{n}{4} \right)^\alpha - d \right] + \frac{1}{2} \cdot \left[ c \left( \frac{n}{2} \right)^\alpha - d \right] + a
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Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where $\alpha = \log \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754$.

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**Inductive step:** \( n \geq 2 \).

\[
E_1[T(n)] \leq 2 \cdot E_1[T\left(\frac{n}{4}\right)] + \frac{1}{2} \cdot E_1[T\left(\frac{n}{2}\right)] + a
\]

\[
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\]

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= cn^\alpha \left( 2 \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha} \right) + a - \frac{5d}{2} \right)
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\[
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\]

\[
= cn^\alpha \left( \frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2 + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}}} \right) - d
\]
Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where $\alpha = \lg \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754$.

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\]

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Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

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Inductive step: $n \geq 2$.

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$$= cn^\alpha \left(\frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2}\right) - d$$

$$= cn^\alpha \left(\frac{34 + 2 \cdot \sqrt{33}}{34 + 2 \cdot \sqrt{33}}\right) - d$$
Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

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\[
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\]
Summary

Algorithms that are fast on average are often easier to design and faster in practice than worst-case efficient algorithms.

In some applications, worst-case guarantees matter!

Average-case analysis provides a valid performance prediction only if the inputs are uniformly distributed.

Randomized algorithms remove this dependence on the input distribution (but rely on a good random number generator).

There are problems where randomized algorithms are provably faster than the best possible deterministic algorithm.