Average-Case Analysis and Randomization

Textbook Reading

Chapter 7 & Sections 8.4, 9.2

Overview

Design principle

- Do the easy thing and hope it works for most inputs
- Make random choices and hope they're good

Problems

- Sorting (Quick Sort)
- Permuting
- Selection
- Game tree evaluation

Quick Sort Revisited

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Remedy:

Blindly use the last element as pivot.

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$\Rightarrow$ The average-case running time of SimpleQuickSort is the same as its expected running time on a uniformly random input permutation.

Partitioning Maintains Uniformity

Lemma: If $A[\ell \mathinner{..} r]$ is a uniform random permutation of the elements in $A[\ell \mathinner{..} r]$, then the two subarrays $A[\ell \mathinner{..} m-1]$ and $A[m+1 \mathinner{..} r]$ produced by Partition(A, ℓ, r) are also uniform random permutations of the elements they contain.

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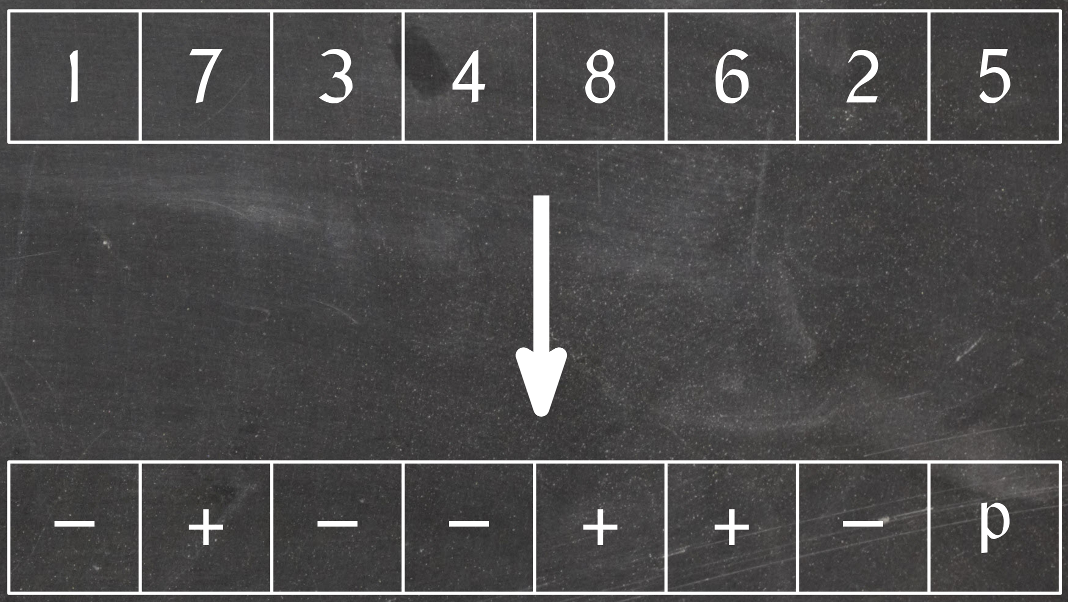

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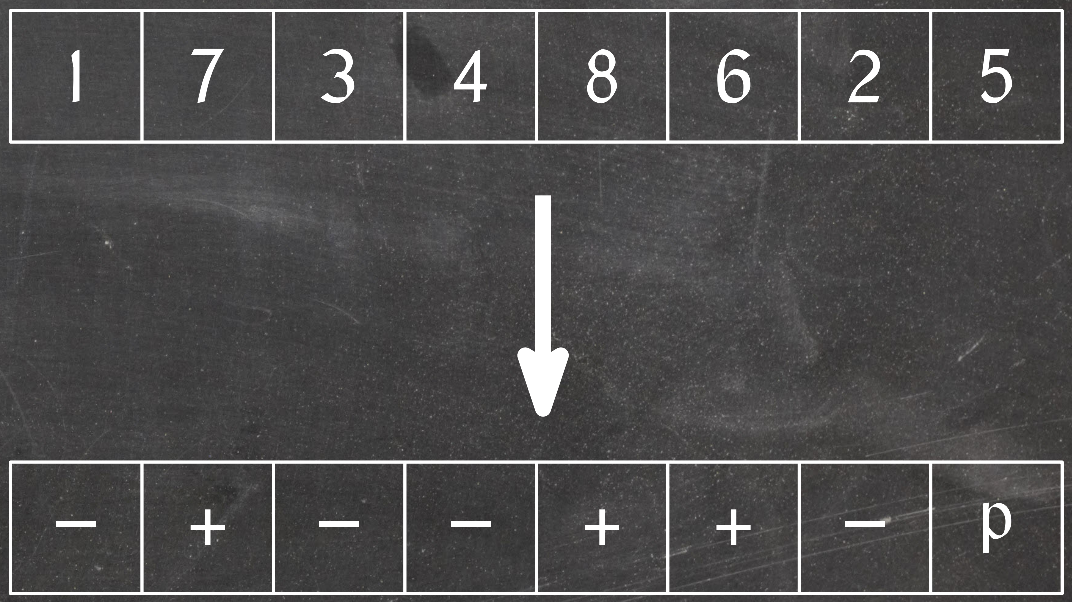

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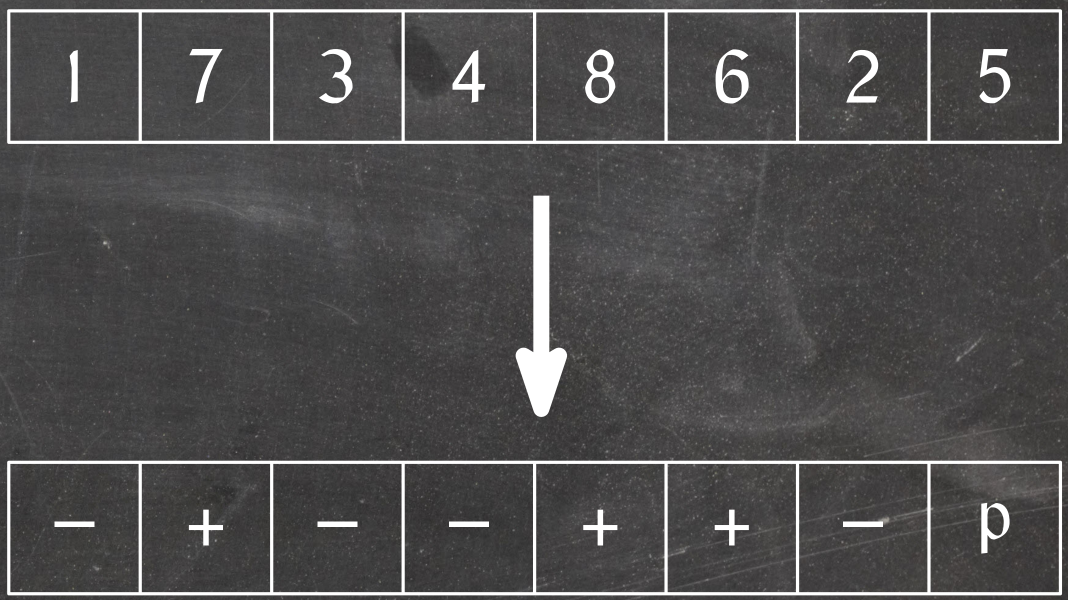

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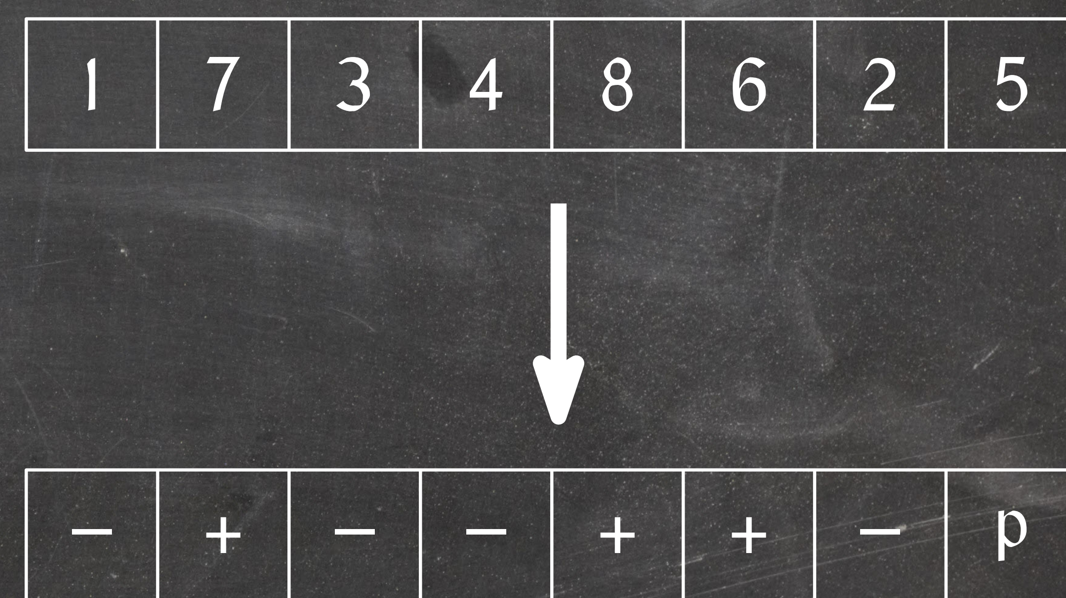

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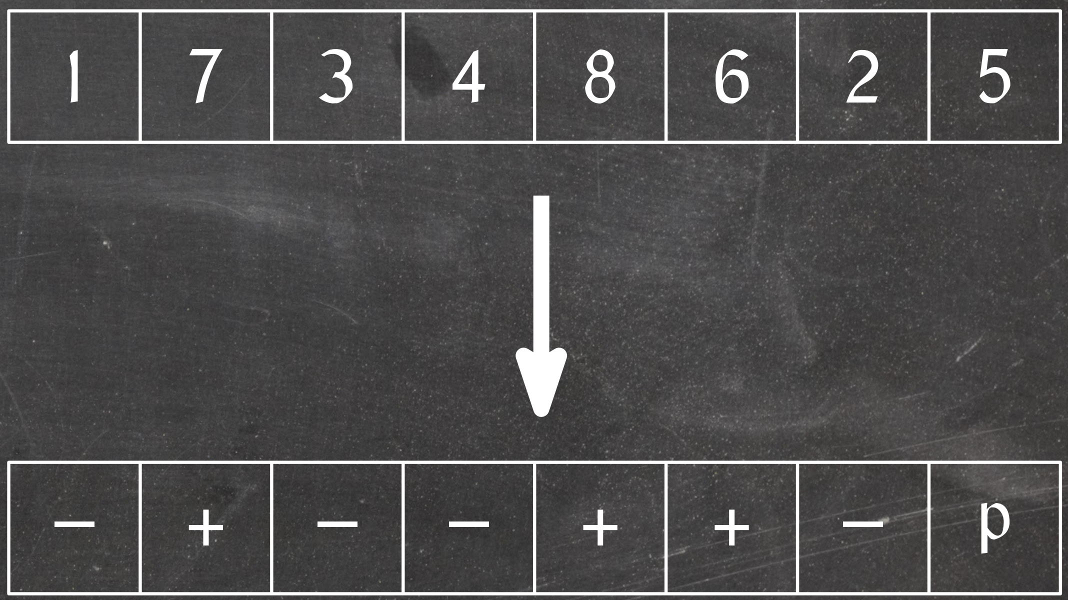

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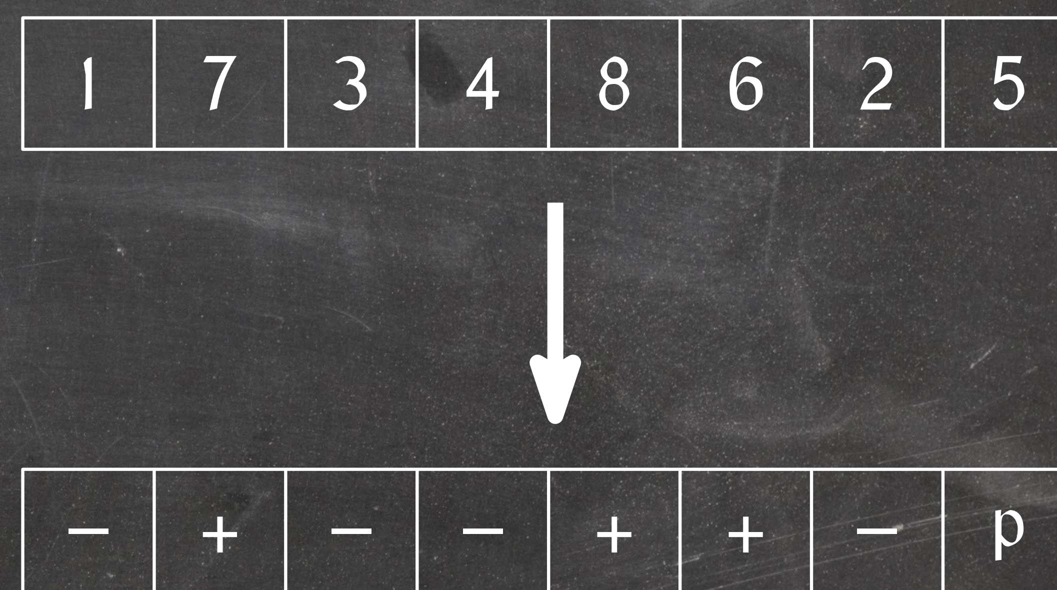

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$\Rightarrow$ It suffices to prove that $E[C] \in O(n \lg n)$.

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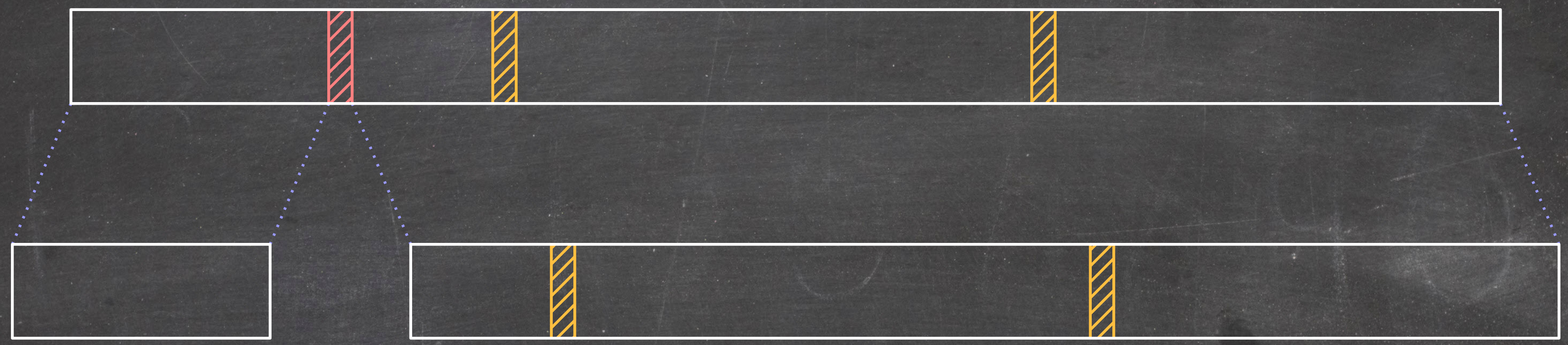

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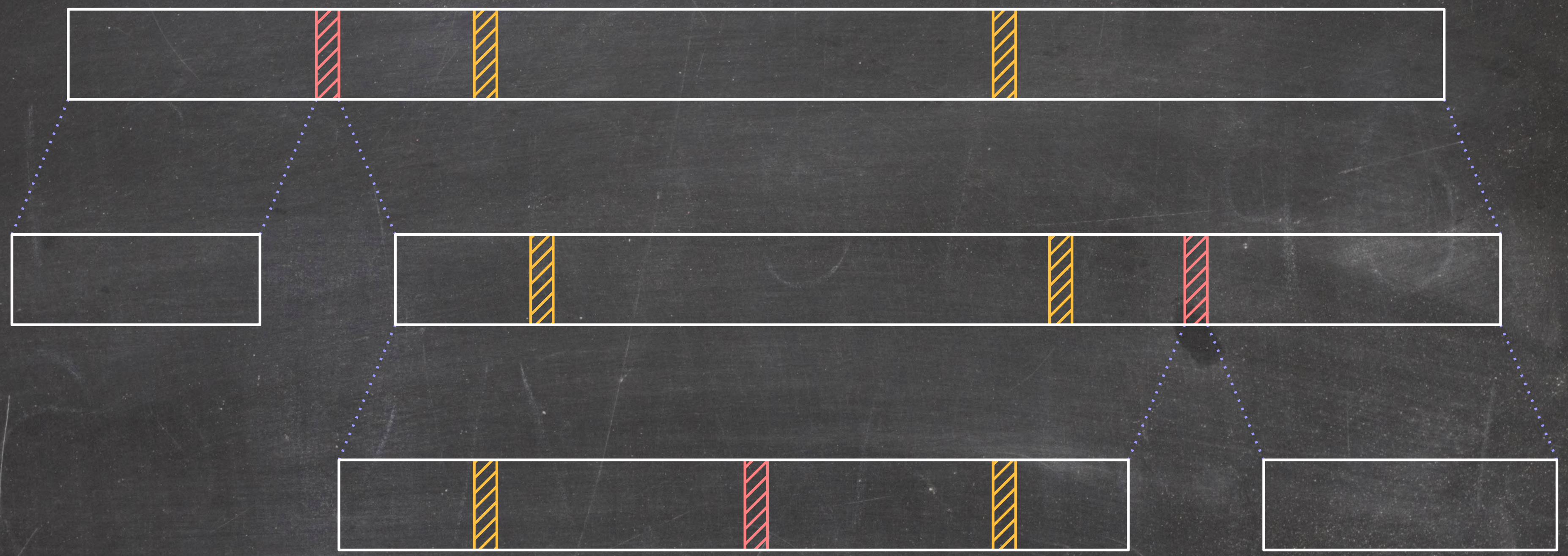

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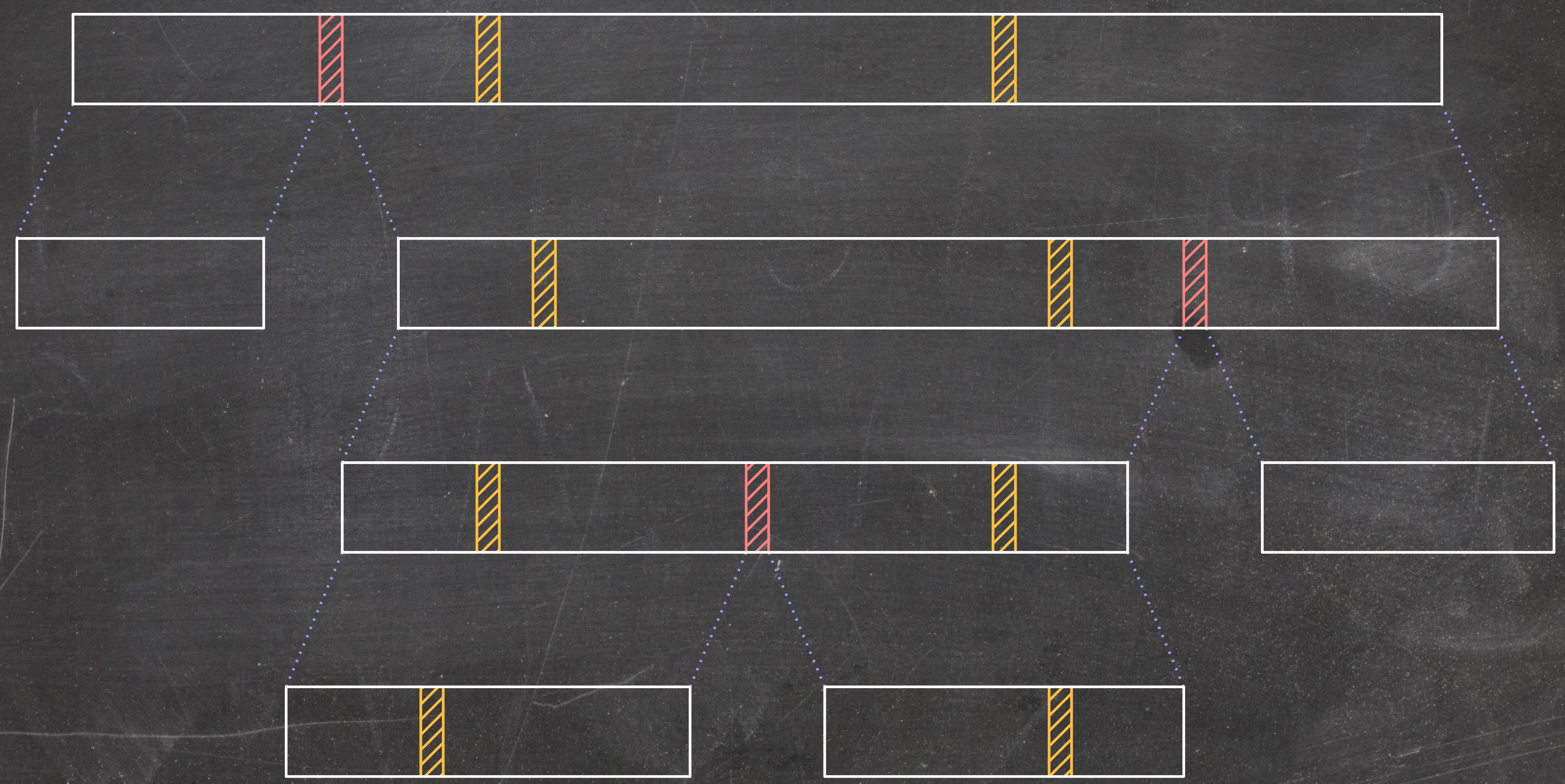

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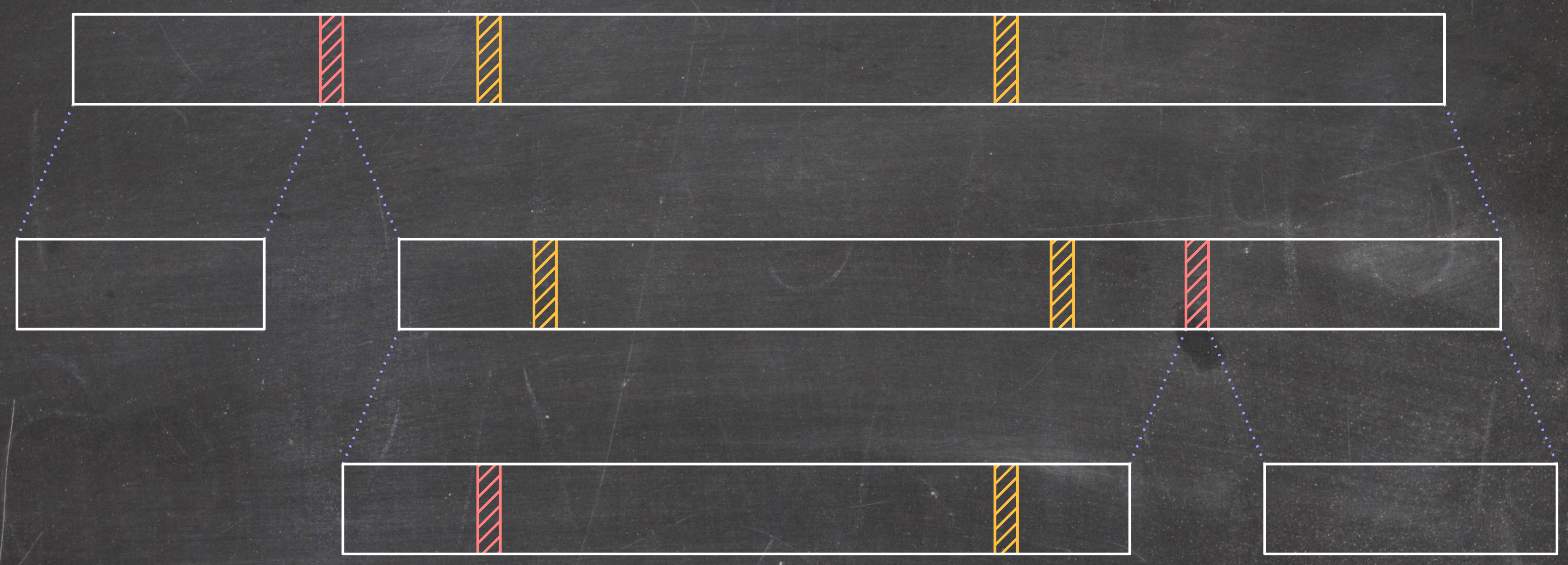

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Corollary: $E[C_{ij}] = \frac{2}{j - i + 1}$.

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$$= 2(n-1)H_n.$$

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \text{nth Harmonic Number}$$

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$$\sum_{i=1}^{n} \frac{1}{i}$$

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$$\int_1^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i}$$

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$$\Rightarrow E[C] \le 2(n-1)H_n \in O(n \lg n)$$

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They are a good choice when we want good performance most of the time and possibly averaged over running the algorithm many times.

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Example:

SimpleQuickSort takes $\Theta(n^2)$ time on almost sorted inputs.

There are applications where the inputs to be sorted are all almost sorted.

SimpleQuickSort is a poor choice of a sorting algorithm in such applications.

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$\Rightarrow$ No more assumptions about the probability distribution. We know the distribution of the choices the algorithm makes.

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The expected running time of SimpleQuickSort on a uniform random permutation is in O(n lg n).

So why don't we just ensure the input is a uniform random permutation?

RandomPermutationQuickSort(A)

```
RandomPermute(A)
SimpleQuickSort(A, 1, n)
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Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in O(n lg n).

So why don't we just ensure the input is a uniform random permutation?

RandomPermutationQuickSort(A)

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RandomPermute(A)
SimpleQuickSort(A, 1, n)
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We can compute a uniform random permutation in O(n) time in the worst case.

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Corollary: The expected running time of RandomPermutationQuickSort is in O(n lg n).

Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

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So why don't we make sure we choose a uniform random pivot, no matter the input permutation?

RandomPivotQuickSort(A, ℓ, r)

```
if r ≤ ℓ
   then return
p = RandomNumber(ℓ, r)
swap A[p] and A[r]
m = Partition(A, ℓ, r)
RandomPivotQuickSort(A, ℓ, m − 1)
RandomPivotQuickSort(A, m + 1, r)
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Lemma: The expected running time of RandomPivotQuickSort is in $O(n \lg n)$.

The analysis is 100% identical to that of SimpleQuickSort!

Uniform Random Permutation In Linear Time

RandomPermute(A)

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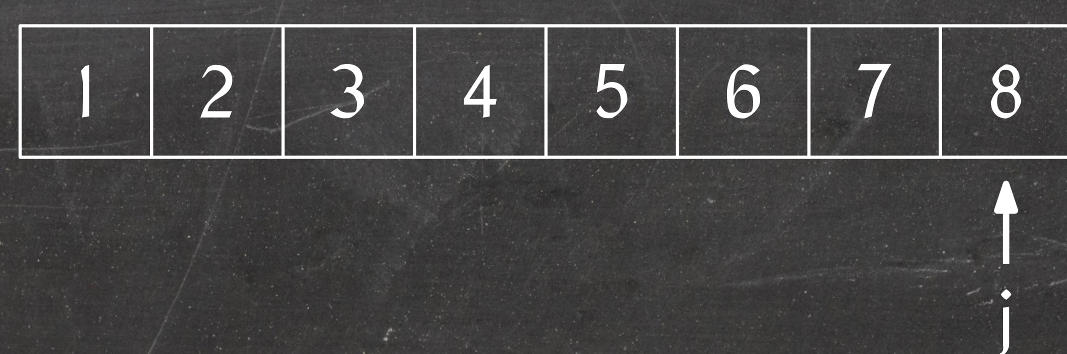

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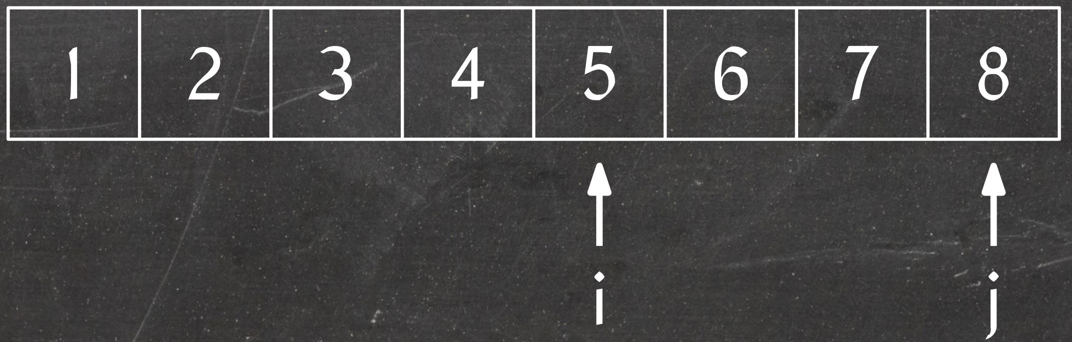

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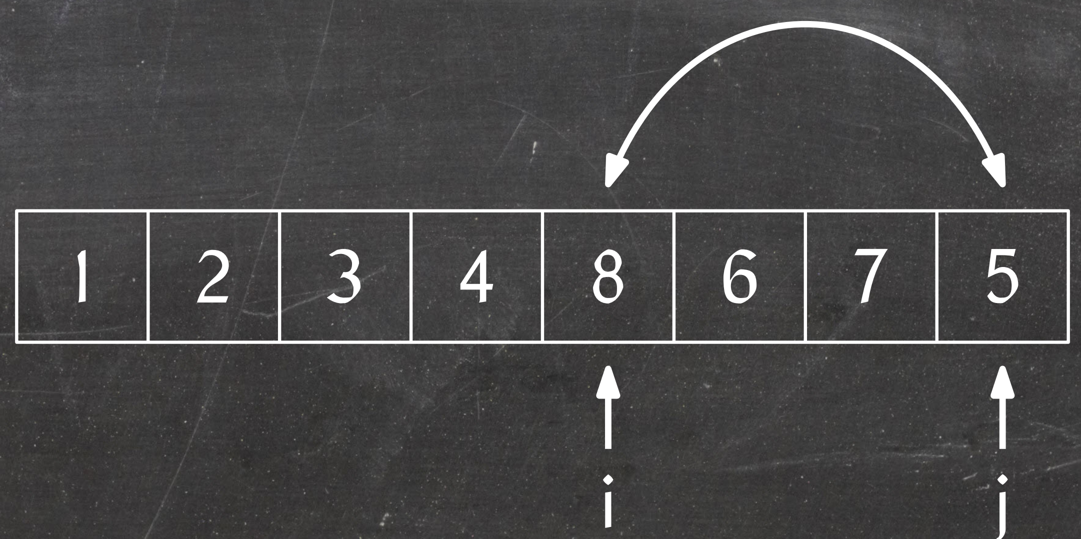

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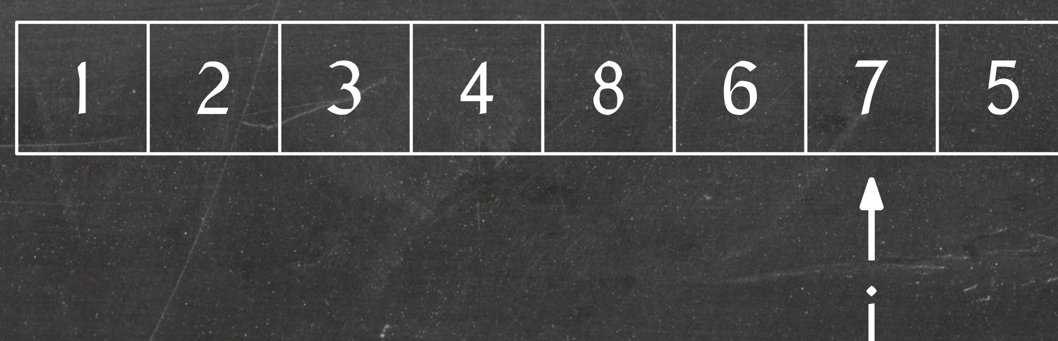

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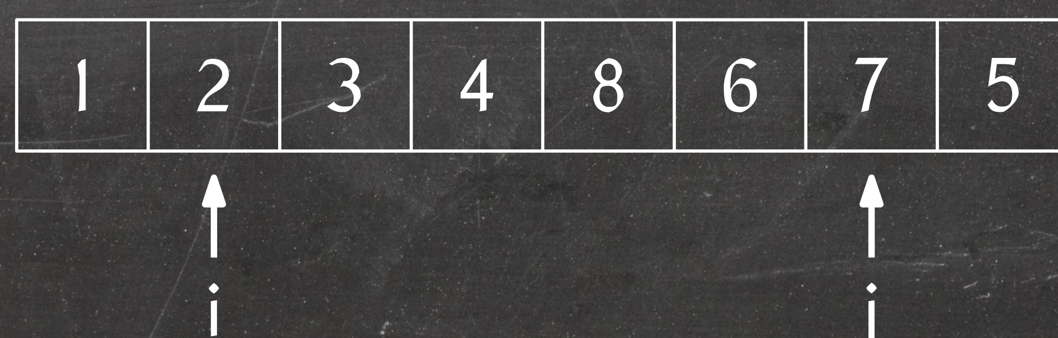

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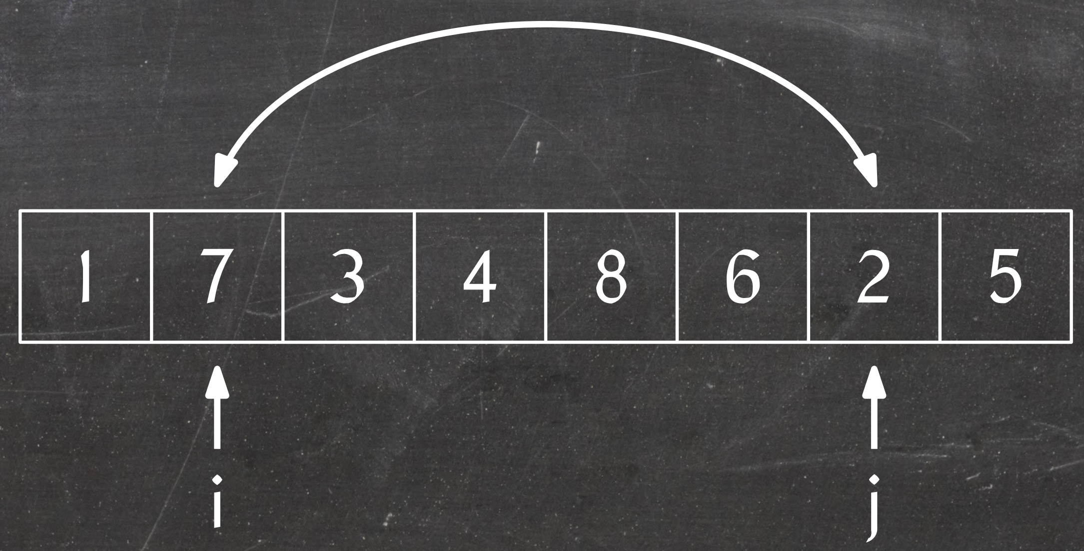

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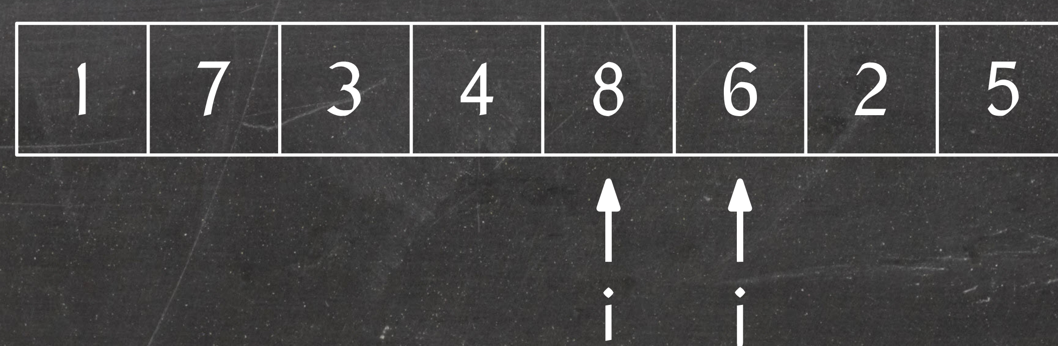

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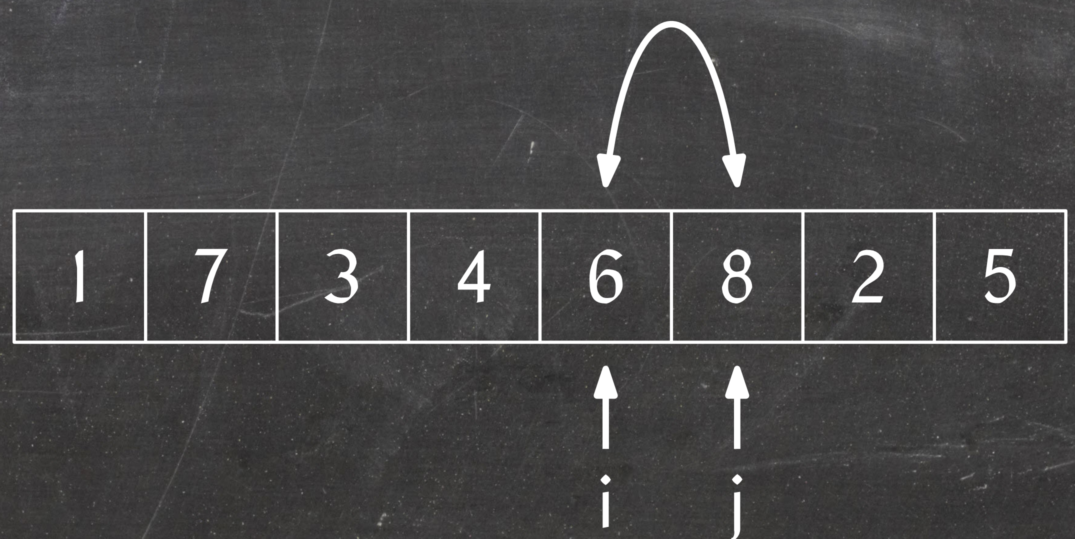

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If $n = 1$, then it produces the only possible permutation with probability $1 = \frac{1}{1!}$.

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If $n > 1$, then to produce the permutation $\langle x_1, x_2, \ldots, x_n \rangle$ (event E), we need to

- Place x_n into $A[n]$ (event E_1) and
- Place $x_1, x_2, \ldots, x_{n-1}$ into $A[1 \,..\, n-1]$ (event E_2).

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- Place x_n into $A[n]$ (event E_1) and
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So $P[E] = P[E_1 \cap E_2] = P[E_1] \cdot P[E_2 | E_1] = \frac{1}{n} \cdot \frac{1}{(n-1)!} = \frac{1}{n!}$.

Randomized Selection

RandomizedSelection(A, ℓ, r, k)

```
if r ≤ ℓ
    then return A[ℓ]
p = RandomNumber(ℓ, r)
swap A[p] and A[r]
m = Partition(A, ℓ, r)
if m − ℓ = k − 1
    then return A[m]
    else if m − ℓ ≥ k
            then RandomizedSelection(A, ℓ, m − 1, k)
            else RandomizedSelection(A, m + 1, r, k − (m + 1 − ℓ))
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```

Lemma: The expected running time of RandomizedSelection is in $O(n)$.

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Observation: If we choose the ith smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(\max(n - i, i - 1))].$$

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Base case: $1 \leq n < 4$.

$T(n) \leq c \leq cn.$

Randomized Selection

Inductive step: $n \geq 4$.

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Inductive step: $n \geq 4$.

$$E[T(n)] \leq an + \frac{1}{n}\sum_{i=1}^{n} E[T(\max(i-1, n-i))]$$

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Inductive step: $n \geq 4$.

$$\begin{aligned}
E[T(n)] &\leq an + \frac{1}{n}\sum_{i=1}^{n} E[T(\max(i-1, n-i))] \\
&\leq an + \frac{2}{n}\sum_{i=\lfloor n/2 \rfloor}^{n-1} E[T(i)] \\
&\leq an + \frac{2}{n}\sum_{i=\lfloor n/2 \rfloor}^{n-1} ci \\
&= an + \frac{2c}{n}\left(\sum_{i=1}^{n-1} i - \sum_{i=1}^{\lfloor n/2 \rfloor - 1} i\right) \\
&= an + \frac{2c}{n}\left(\frac{n(n-1)}{2} - \frac{\lfloor n/2 \rfloor(\lfloor n/2 \rfloor - 1)}{2}\right) \\
&\leq an + \frac{c}{n}\left[n(n-1) - \left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)\right]
\end{aligned}$$

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&= \left(a + \frac{3c}{4} + \frac{c}{2n}\right) n \\
&\leq cn \quad \forall c \geq 8a.
\end{aligned}
$$

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Using comparisons only, as Insertion Sort, Merge Sort, Quick Sort do, it is impossible to sort faster than in $\Omega(n \lg n)$ time.

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Bucket sort: Sorts n real numbers drawn uniformly at random from an interval $[a, b)$ in expected linear time.

Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval [a, b).

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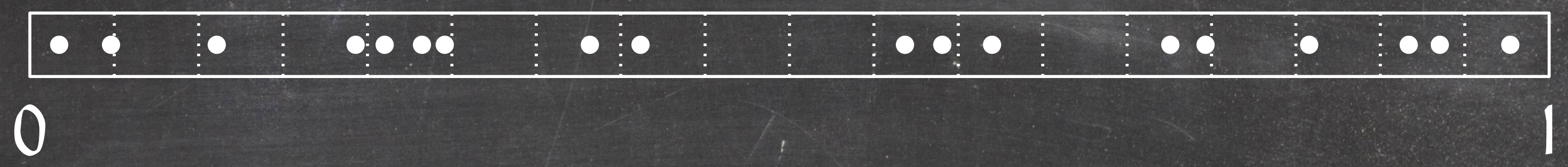

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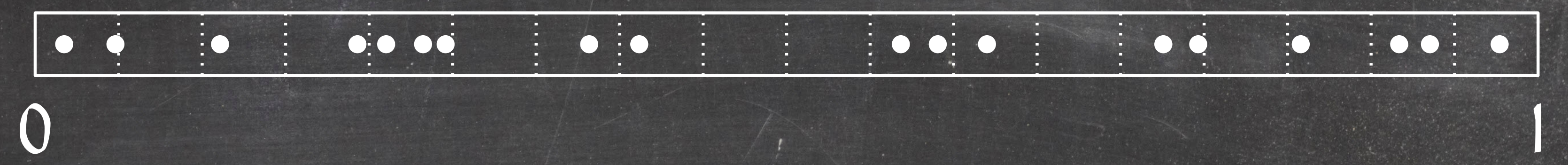

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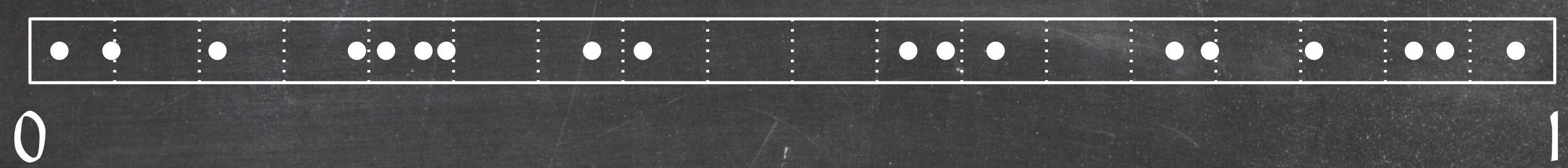

How many elements do we expect to end up in each subinterval? 1!

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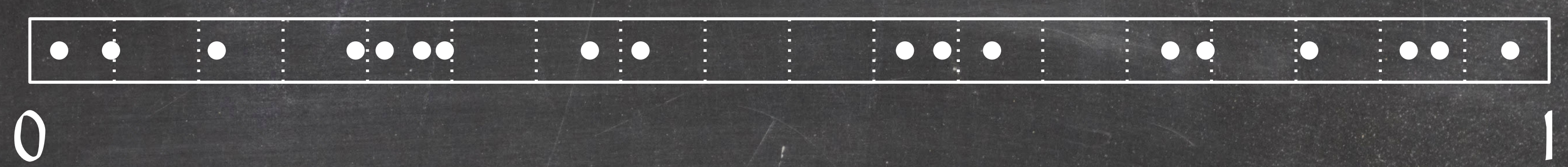

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$\Rightarrow$ Strategy:

- Bucket items according to the subinterval they belong to.
- Sort each bucket, hopefully in constant time.
- Concatenate the sorted buckets.

Bucket Sort

BucketSort(A)

```
n = |A|
B = an array of n empty singly-linked lists
for i = 1 to n
    do prepend A[i] to list B[1 + ⌊n · A[i]⌋]
for i = 1 to n
    do InsertionSort(B[i])
j = 0
for i = 1 to n
    do for every element x ∈ B[i]
           do A[j] = x
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This is where we depart from using comparisons only! (line 4)

Why not Merge Sort? (line 6)

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Why not Merge Sort?

It only helps in the worst case.

It's more complicated.

It actually hurts when buckets are small, which is what we expect.

Worst-case running time: $O(n^2)$

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Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right)$

n_i = the number of elements in B[i]

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Lemma: $E[n_i^2] < 2$.

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Corollary: $E[T(n)] \in O(n)$.

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X_j and X_j are clearly not independent.

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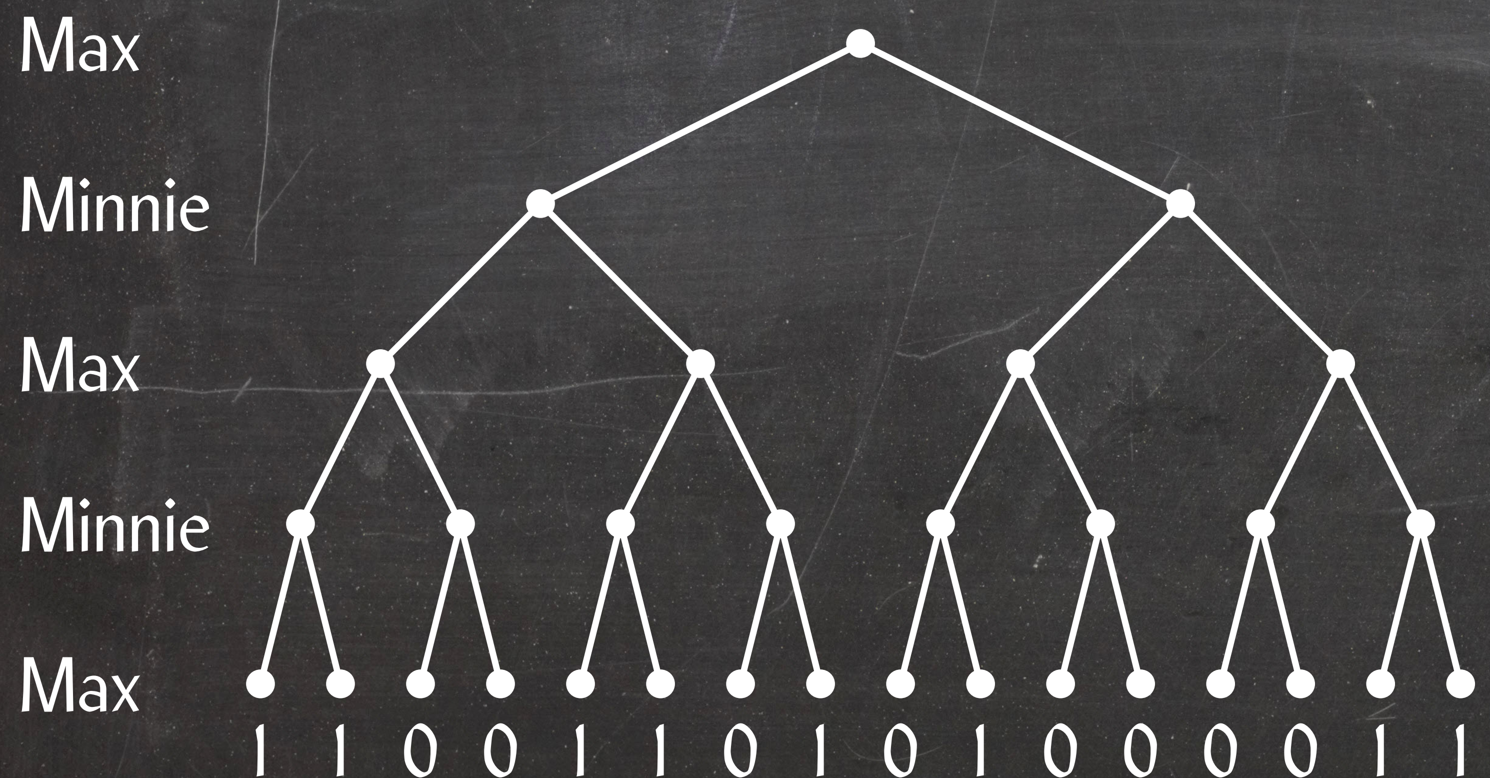

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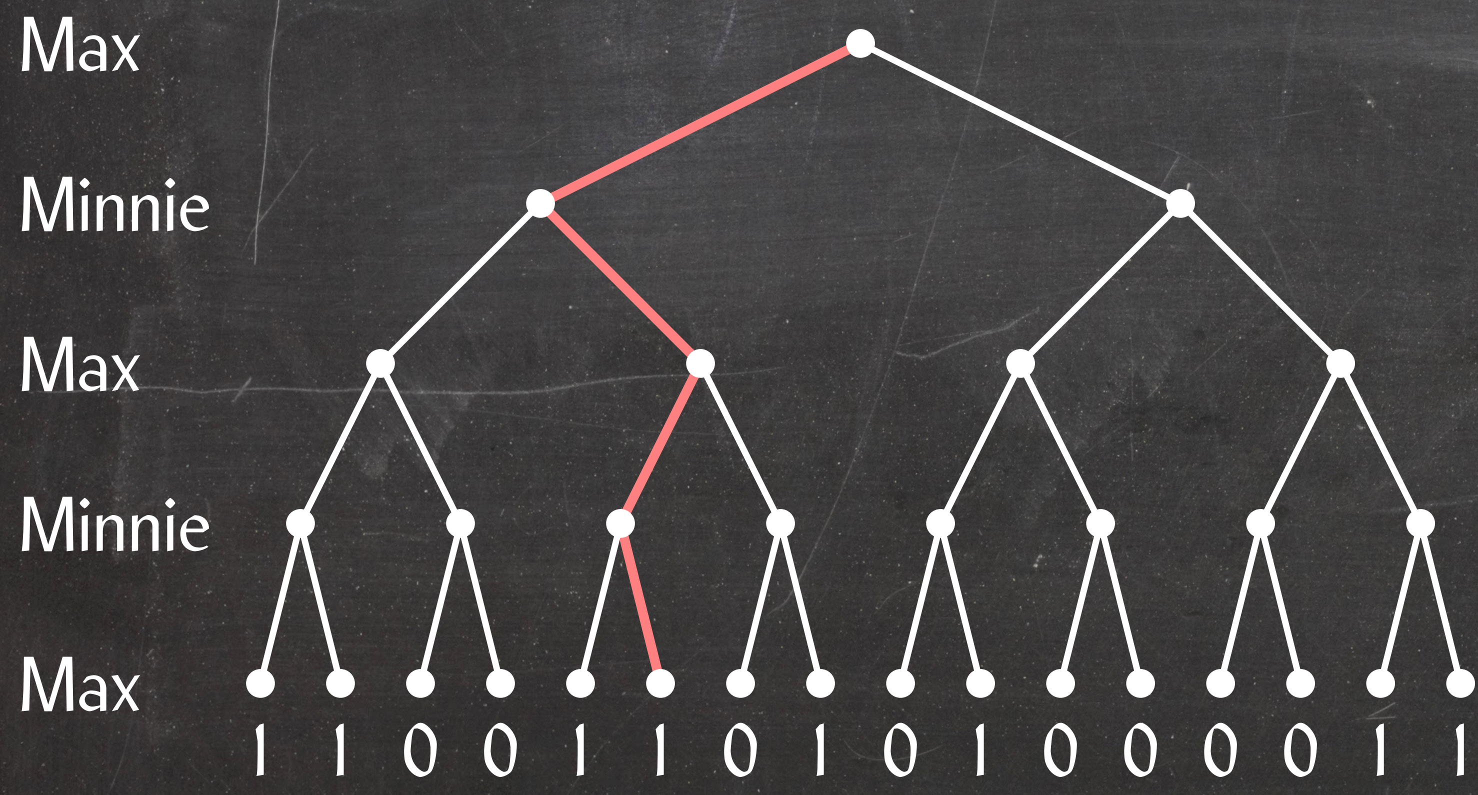

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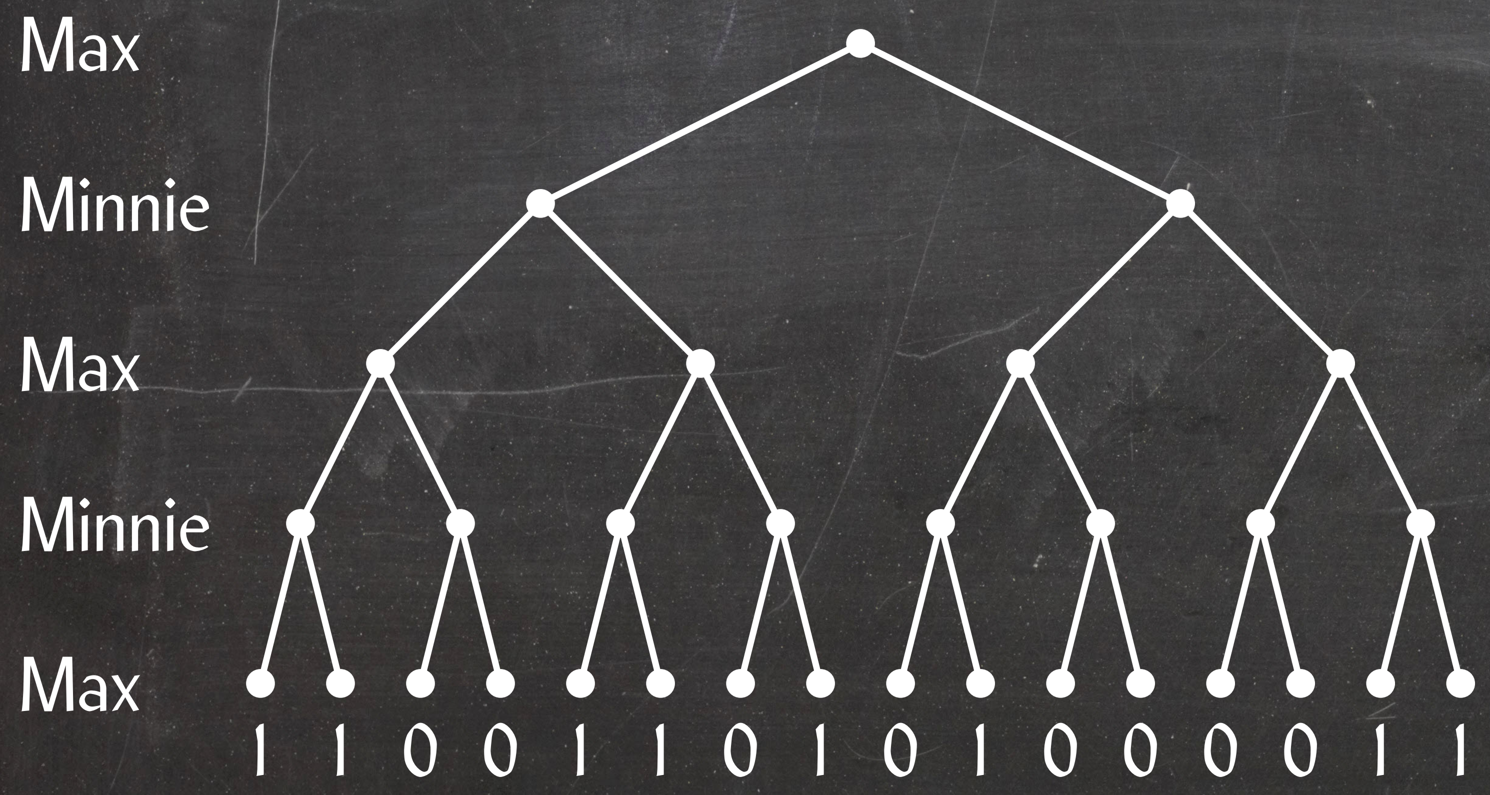

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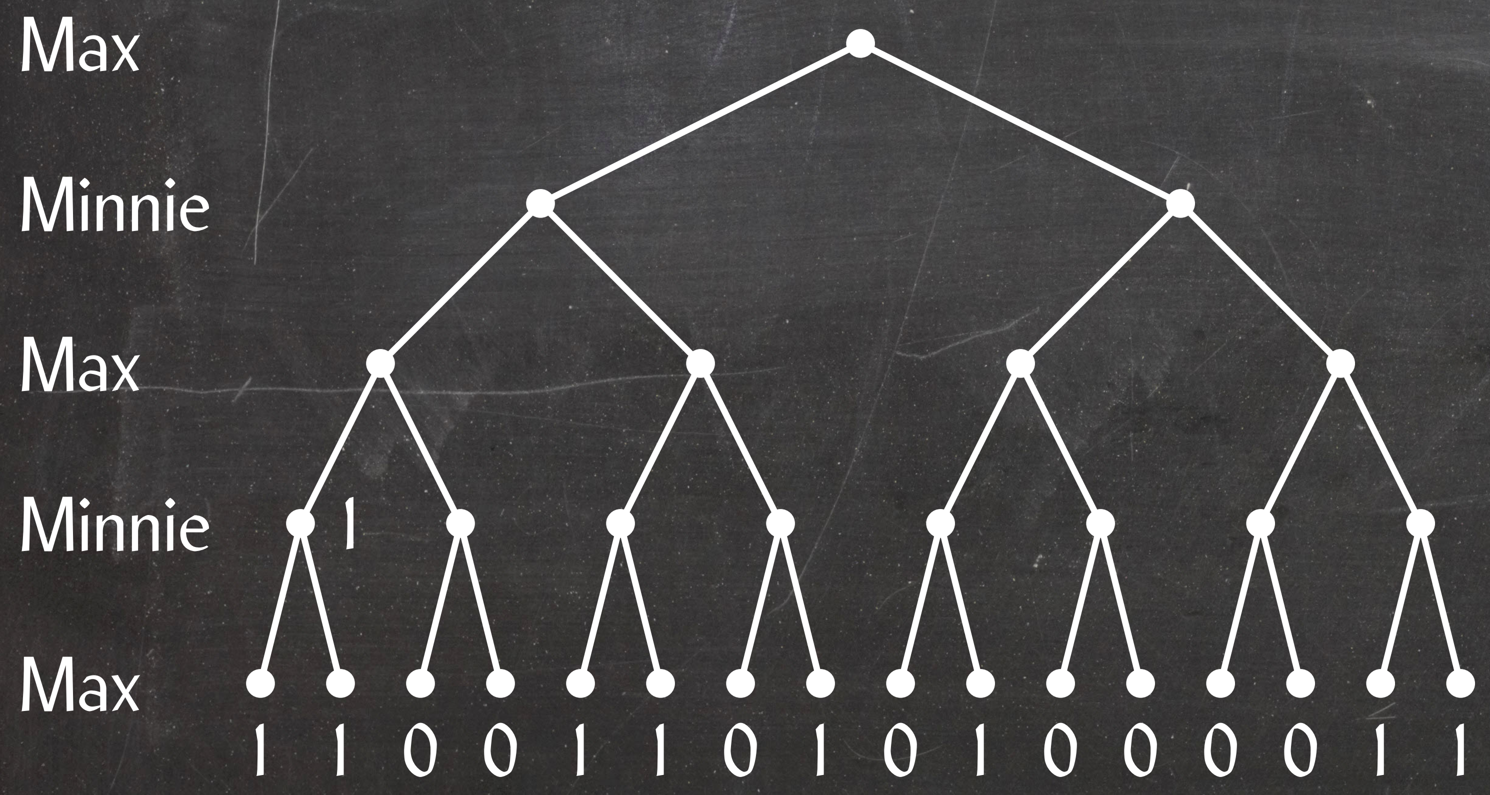

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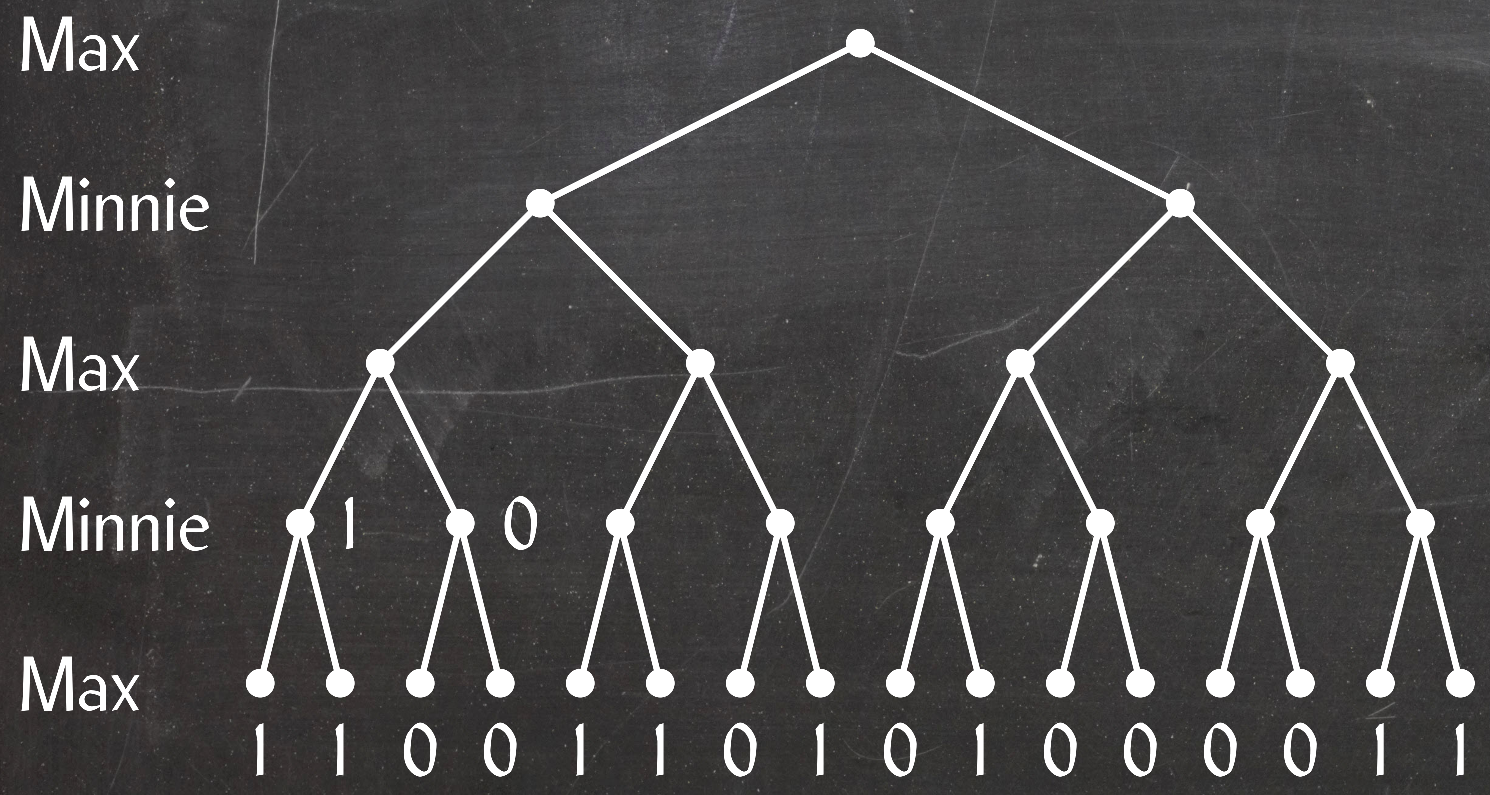

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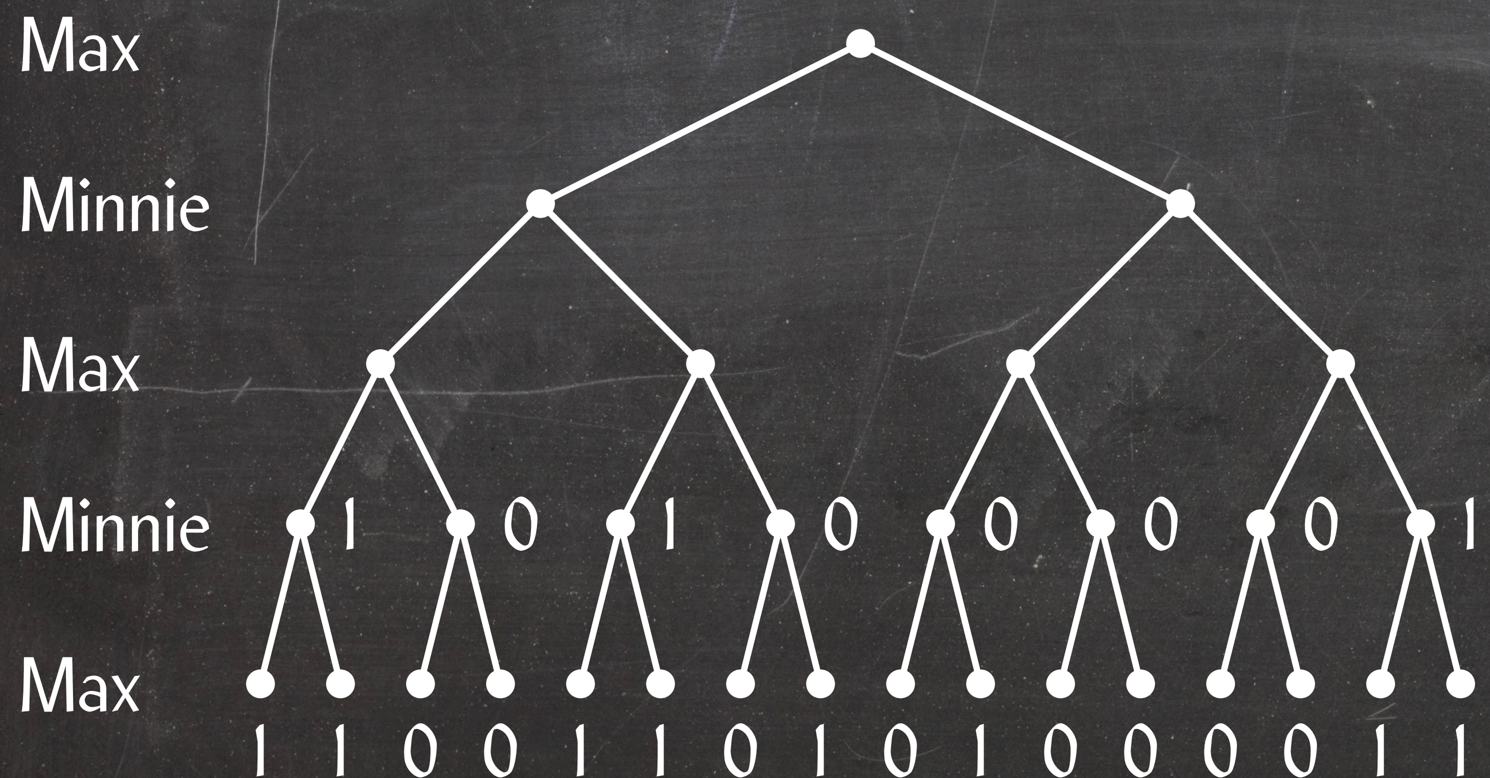

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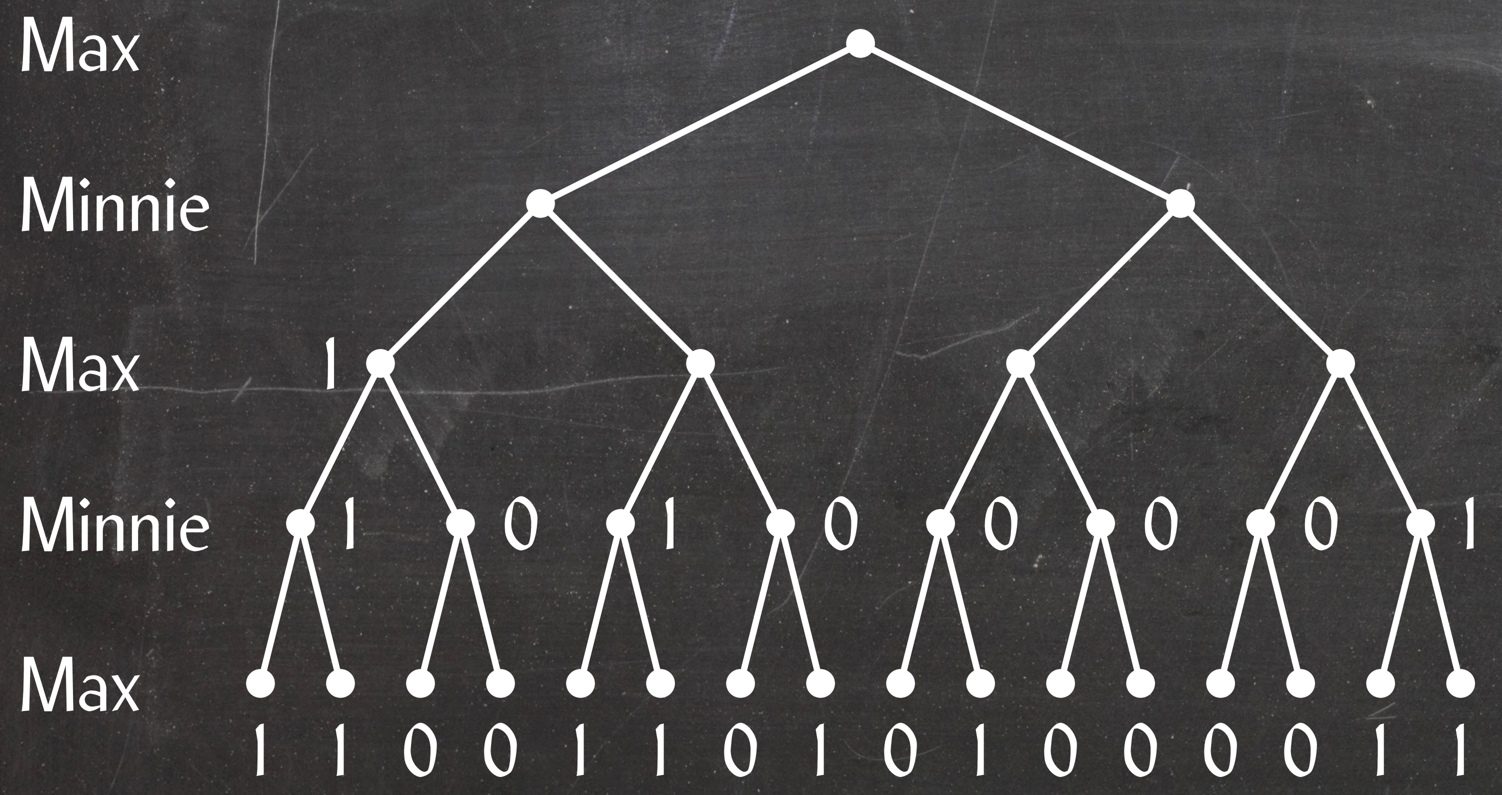

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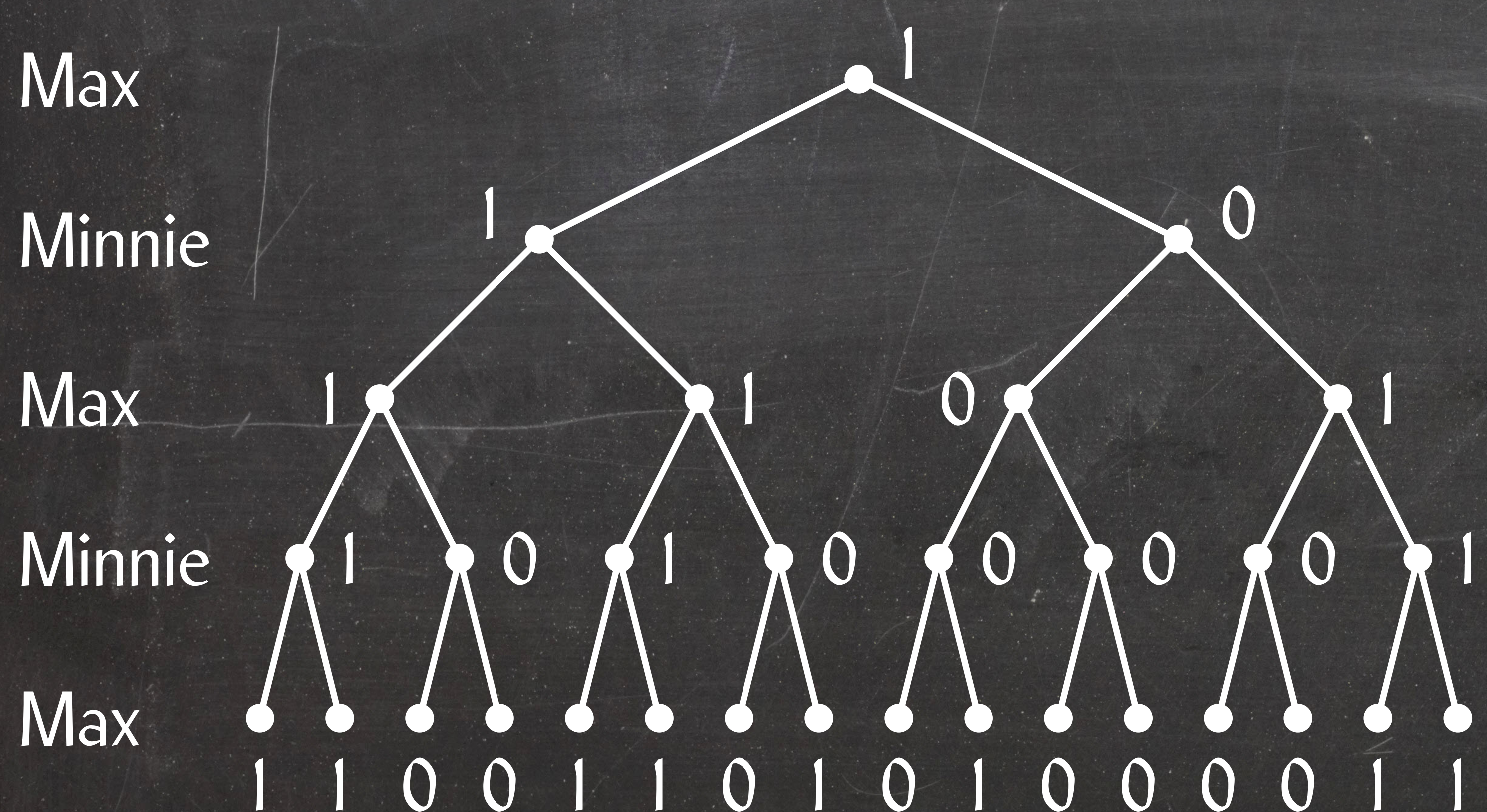

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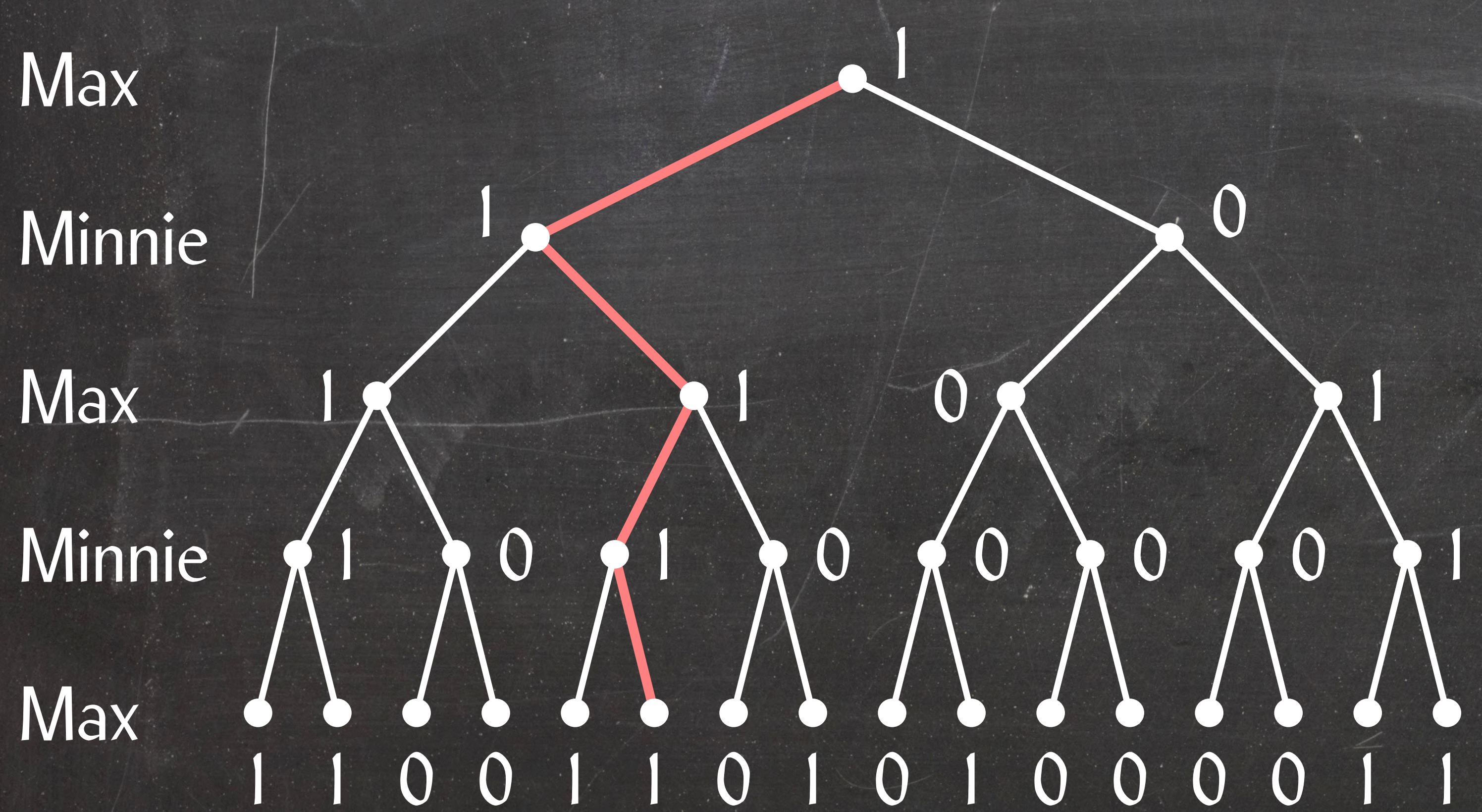

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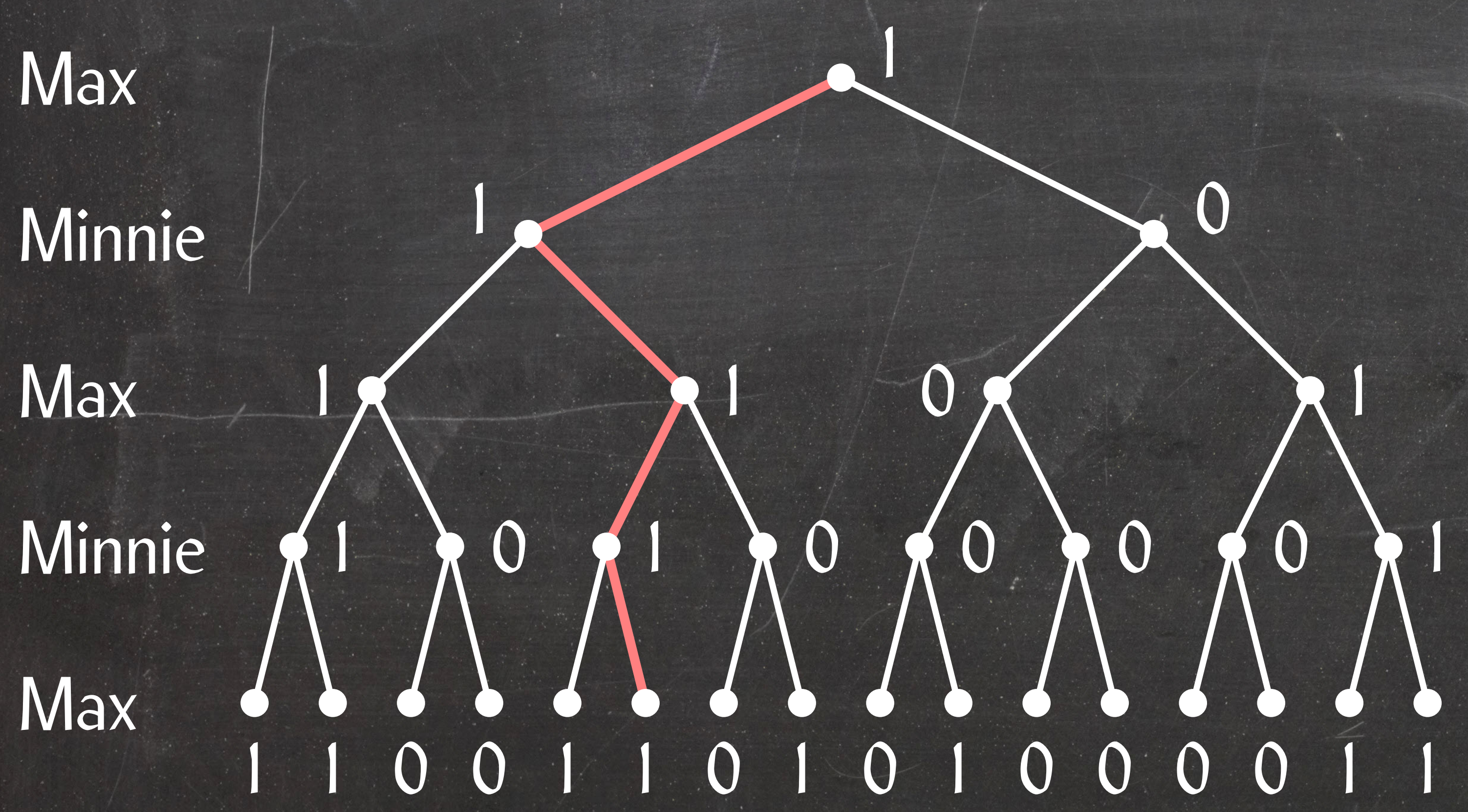

Max-node:

$$\text{label}(v) = \max_{\text{child } w} \text{label}(w)$$

Minnie-node:

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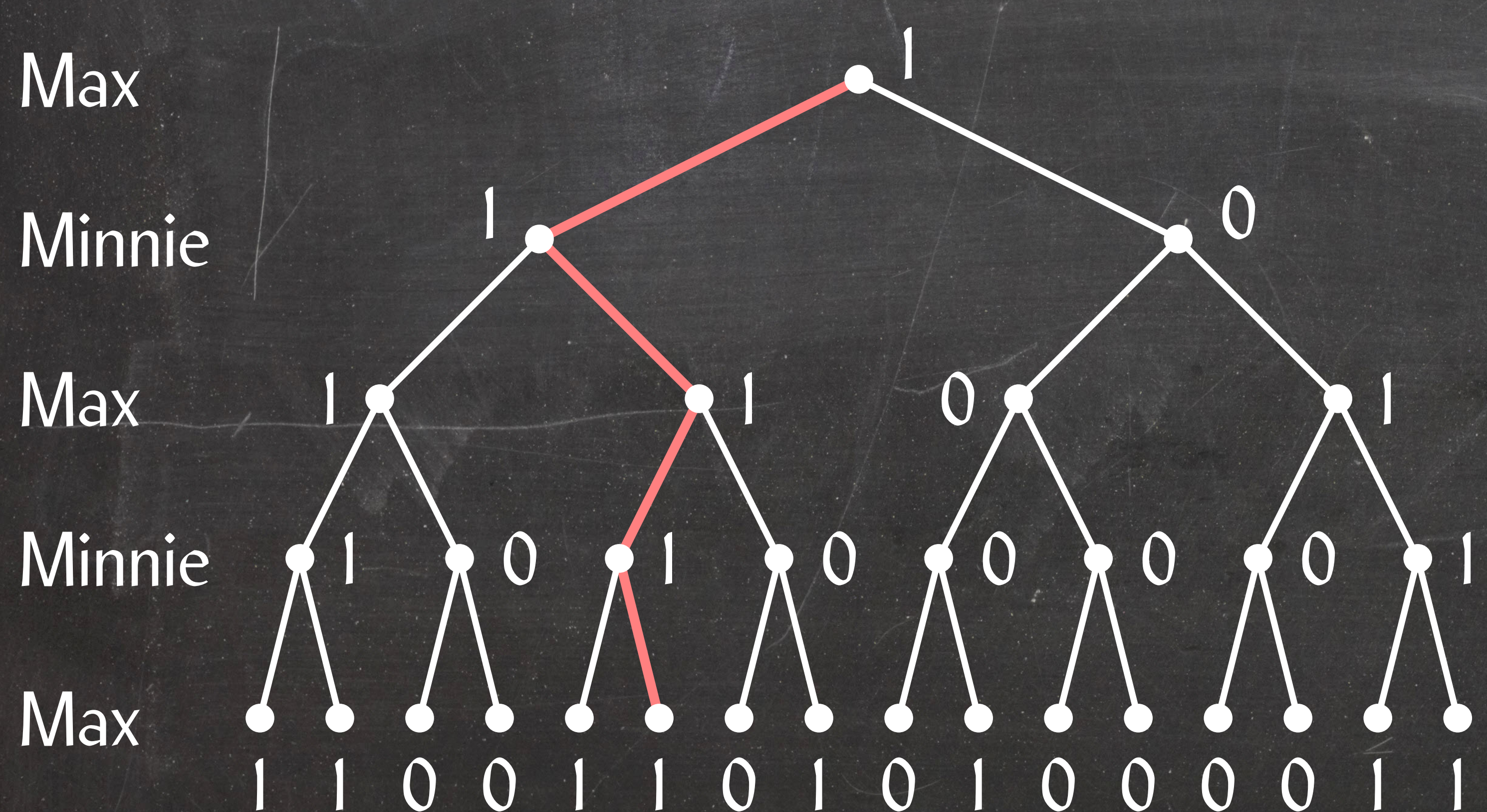

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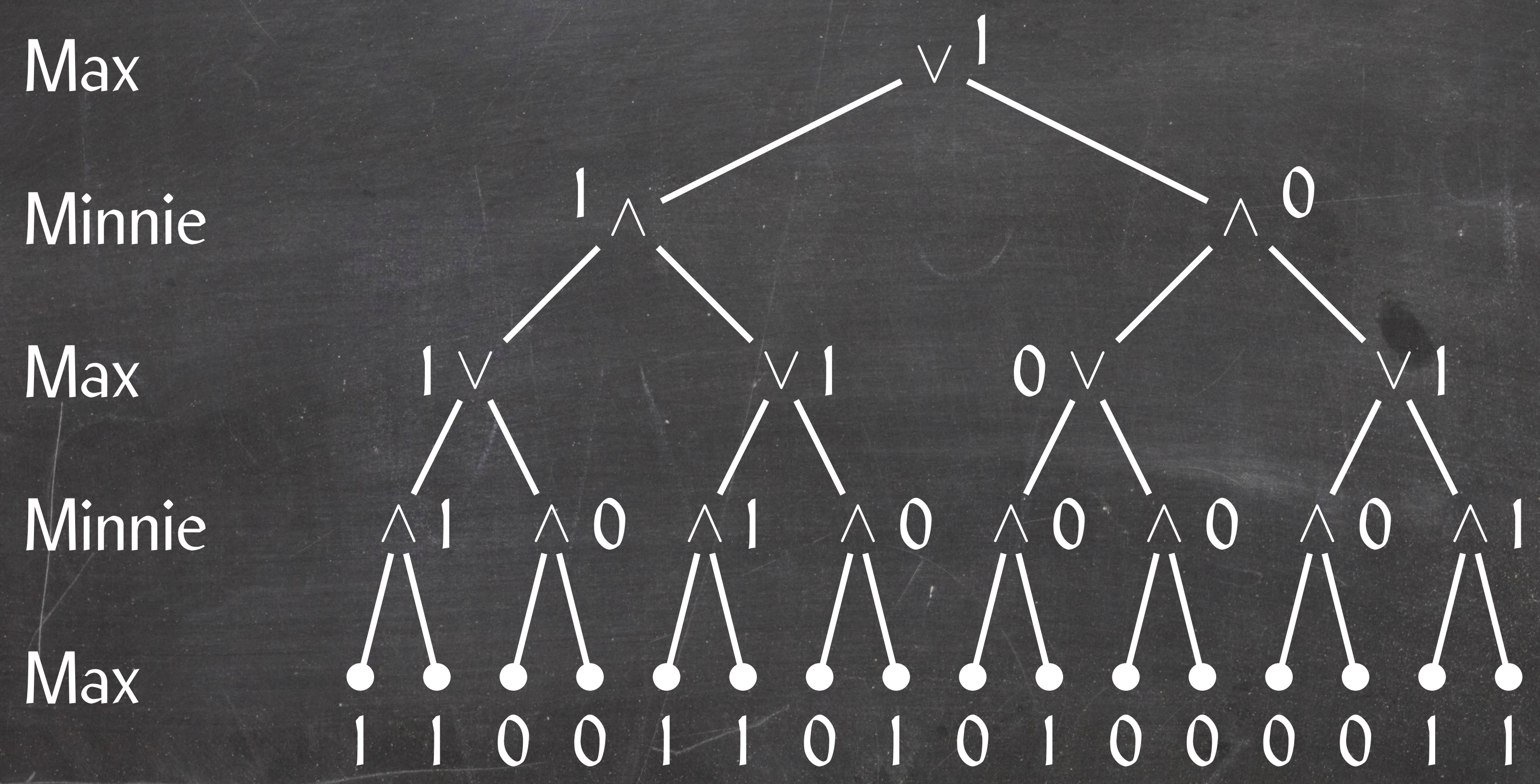
Restrict ourselves to binary game trees of height 2k $\Rightarrow n = 2^{2k}$ leaves



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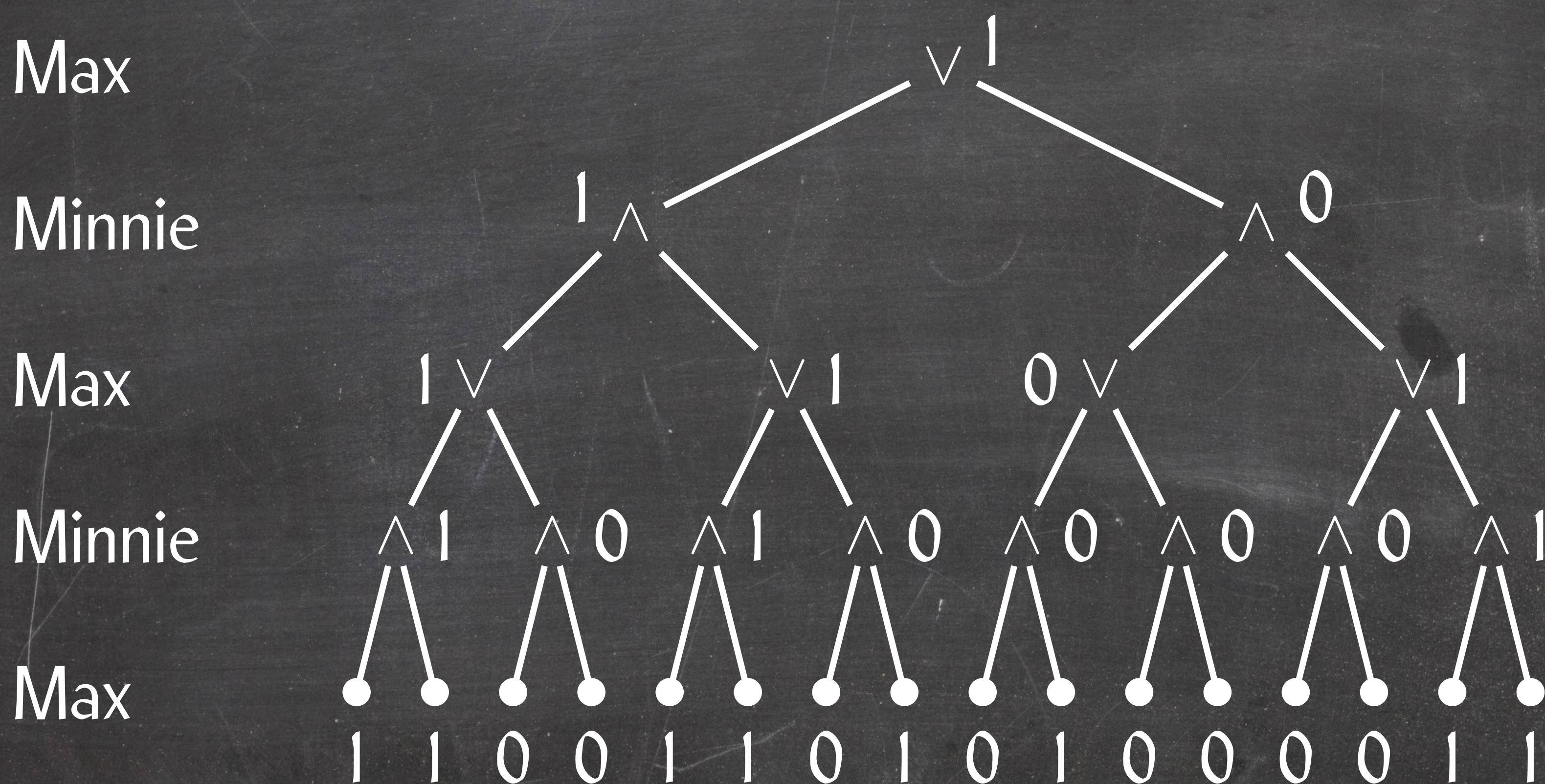

$$(a \wedge b) \vee (c \wedge d) = \overline{\overline{(a \wedge b)} \wedge \overline{(c \wedge d)}}$$

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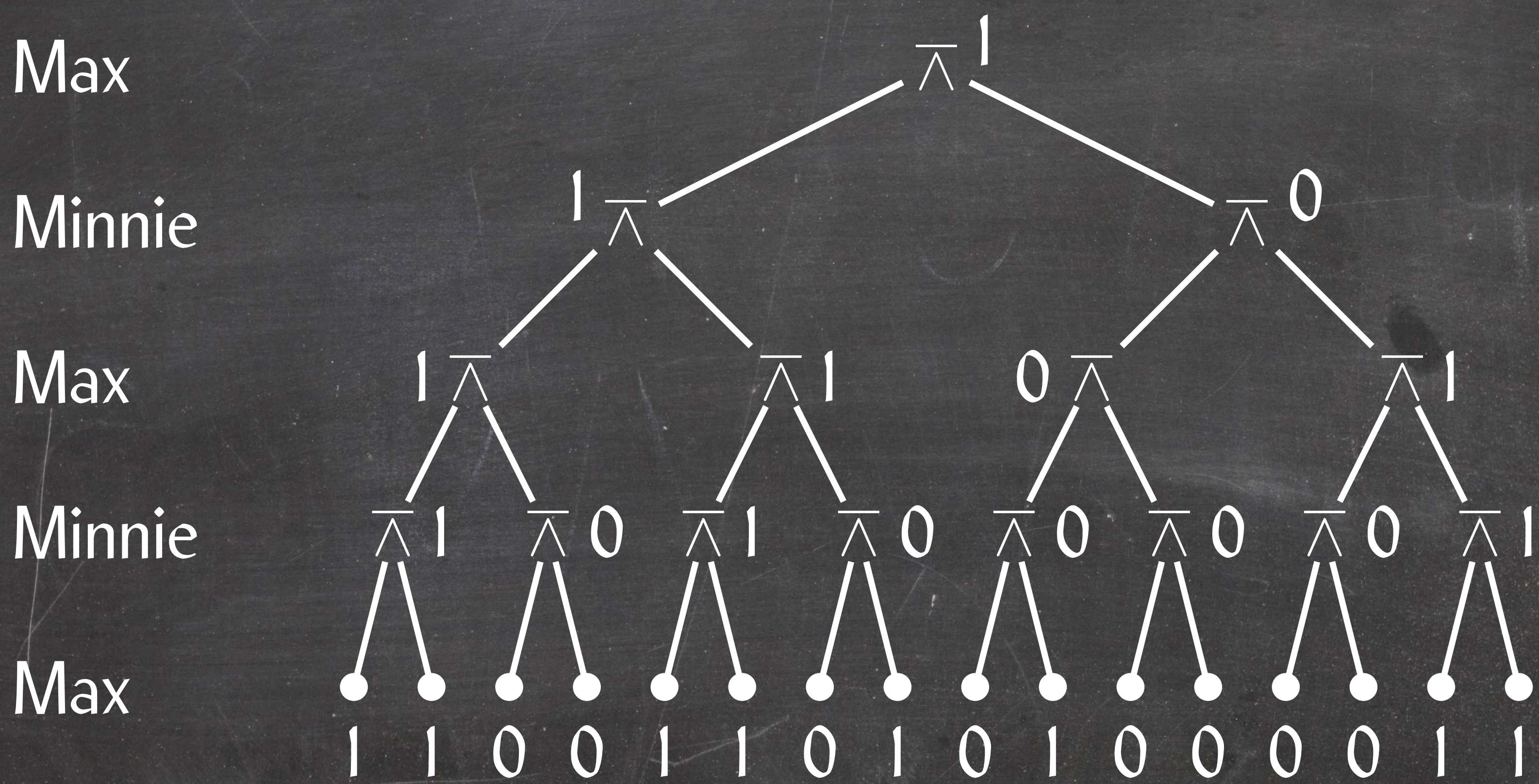

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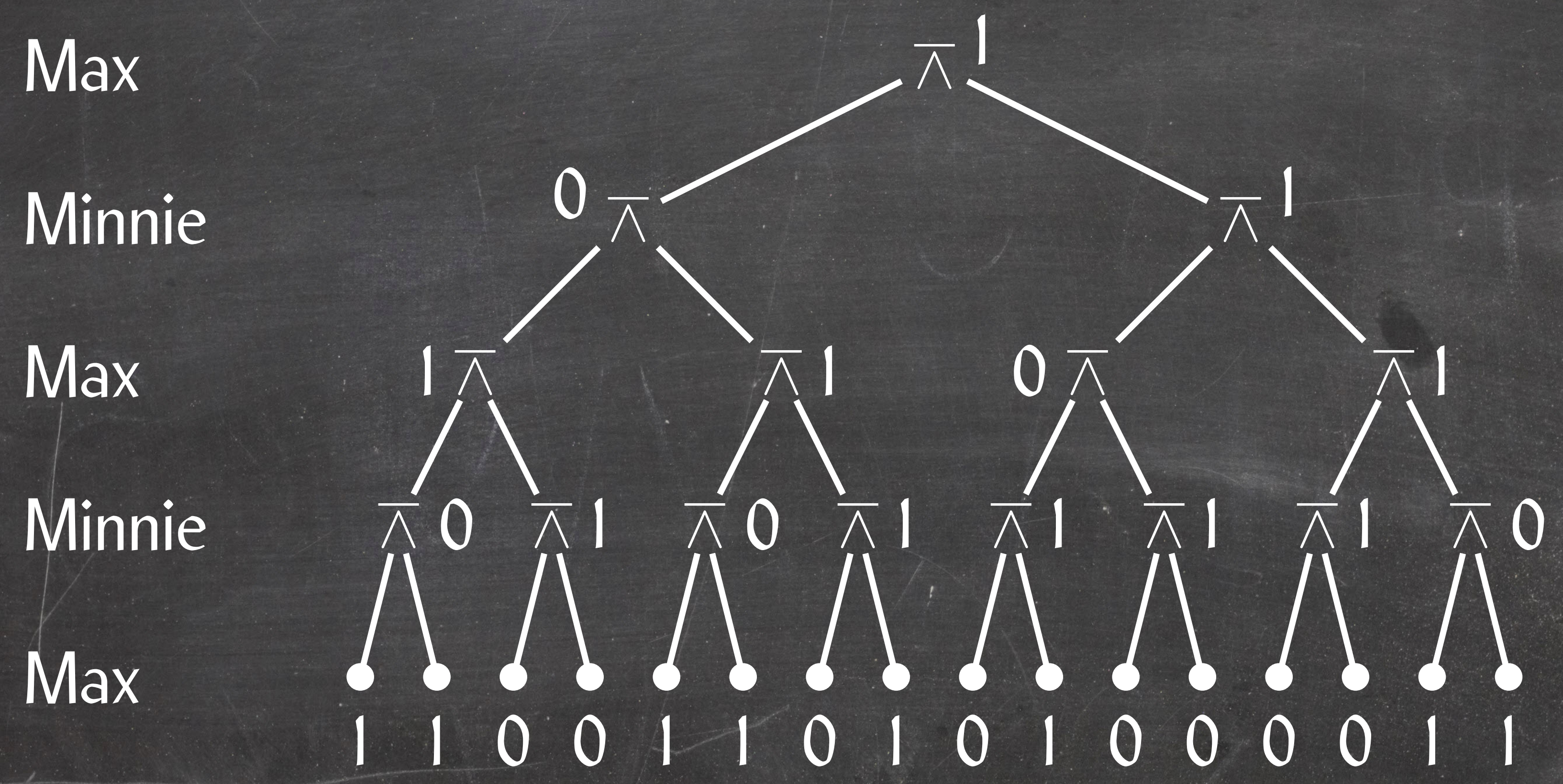

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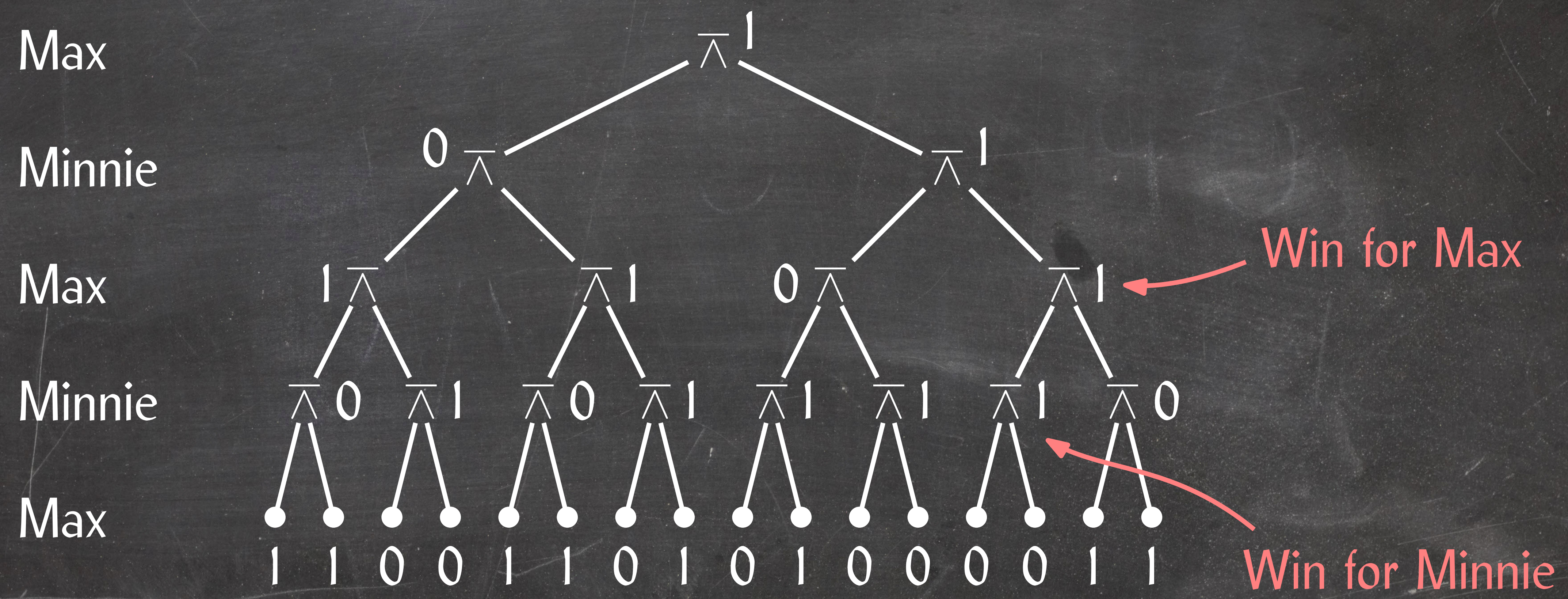

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```
if v is a leaf
    then return its value
    else  return not (GameValue(v.leftChild) and GameValue(v.rightChild))
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- One recursive call per node
- $2n - 1$ nodes

$\Rightarrow$ Running time $O(n)$

Game Tree Evaluation: A Deterministic Algorithm

GameValue(v)

```
if v is a leaf
    then return its value
if not GameValue(v.leftChild)
    then return 1
    else return not GameValue(v.rightChild)
```

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Observation: Any deterministic algorithm has to inspect every leaf in the worst case and thus takes $\Omega(n)$ time in the worst case.

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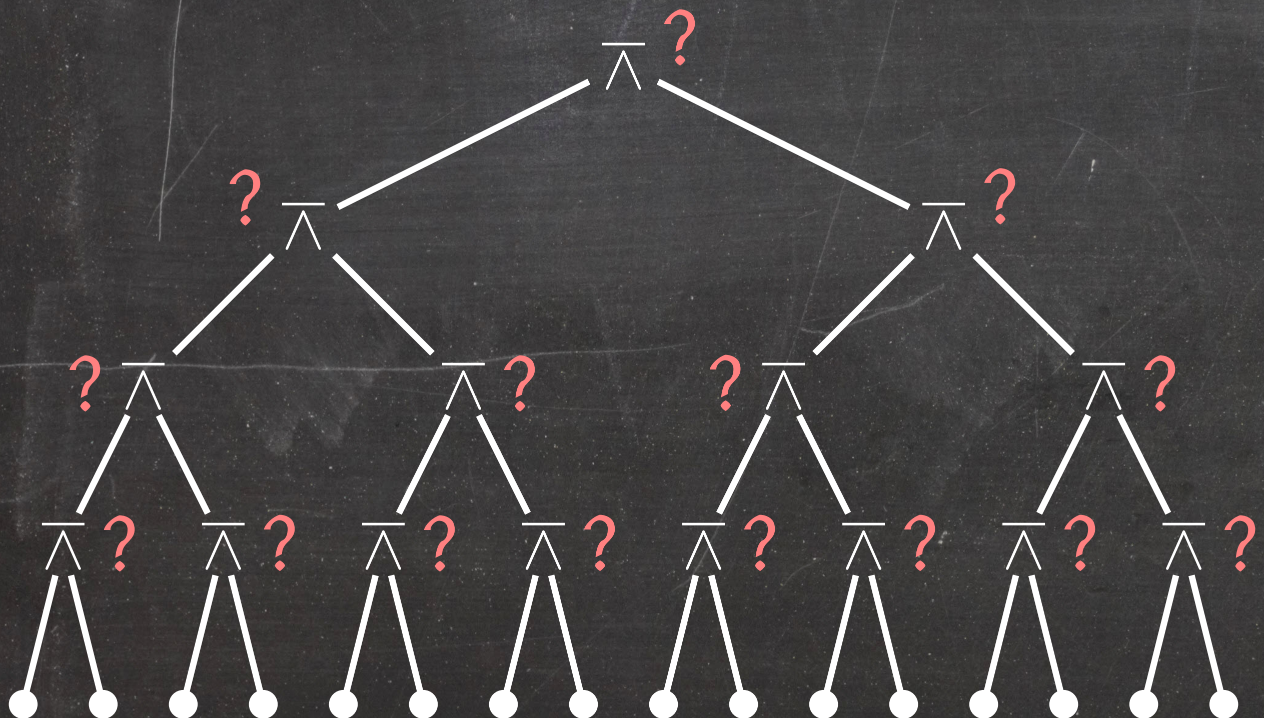

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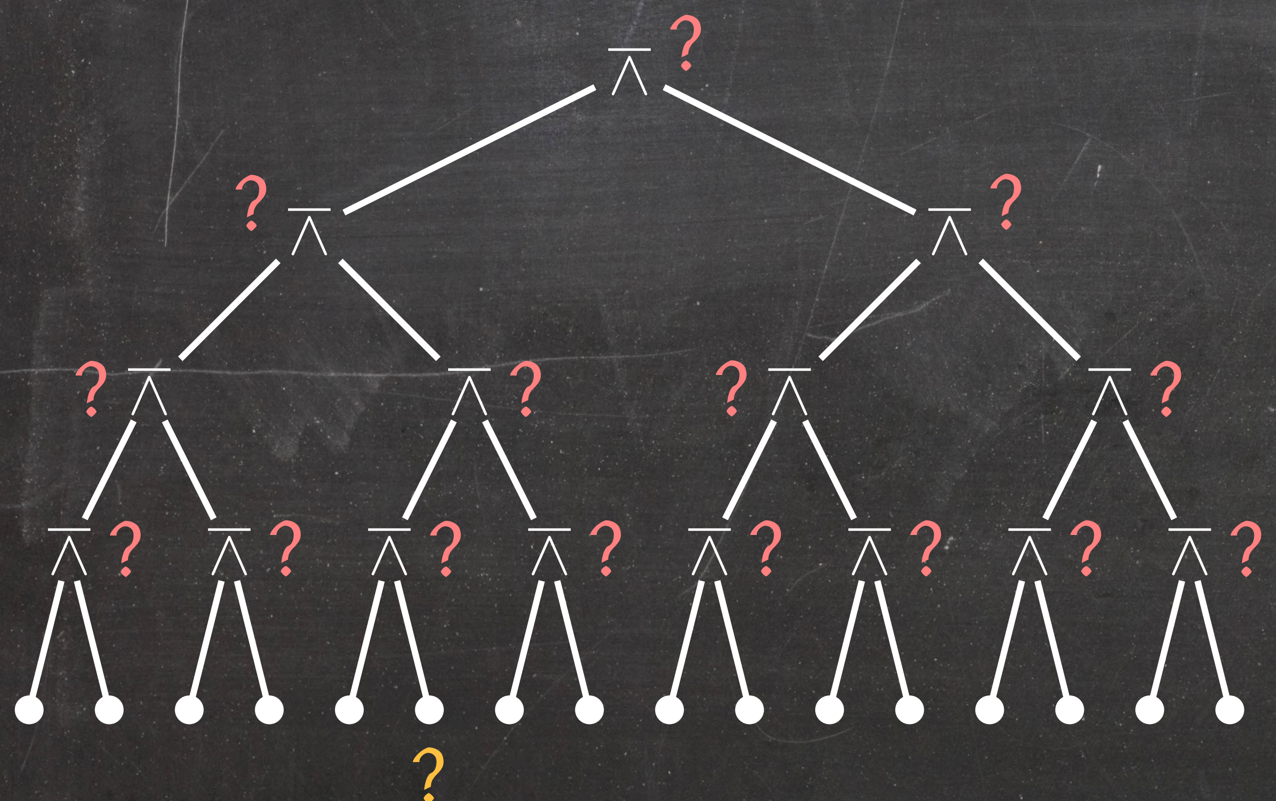

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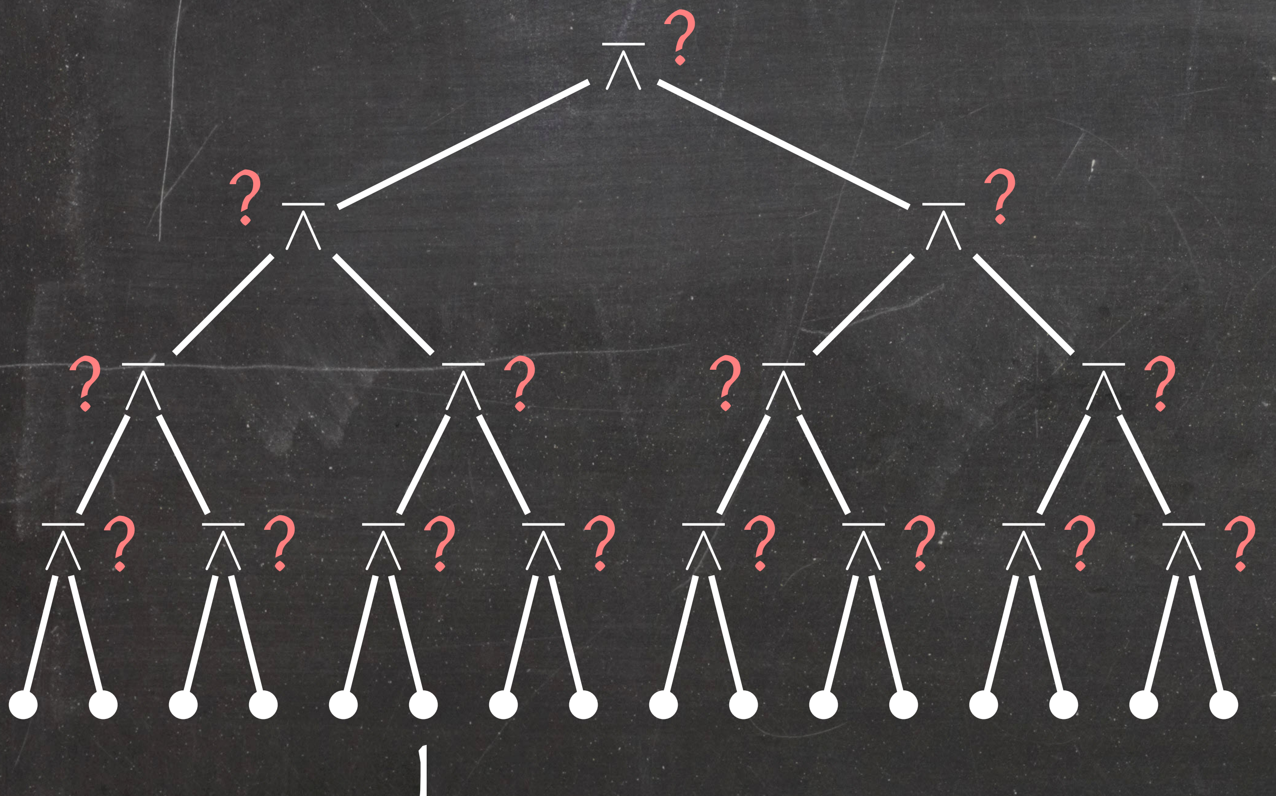

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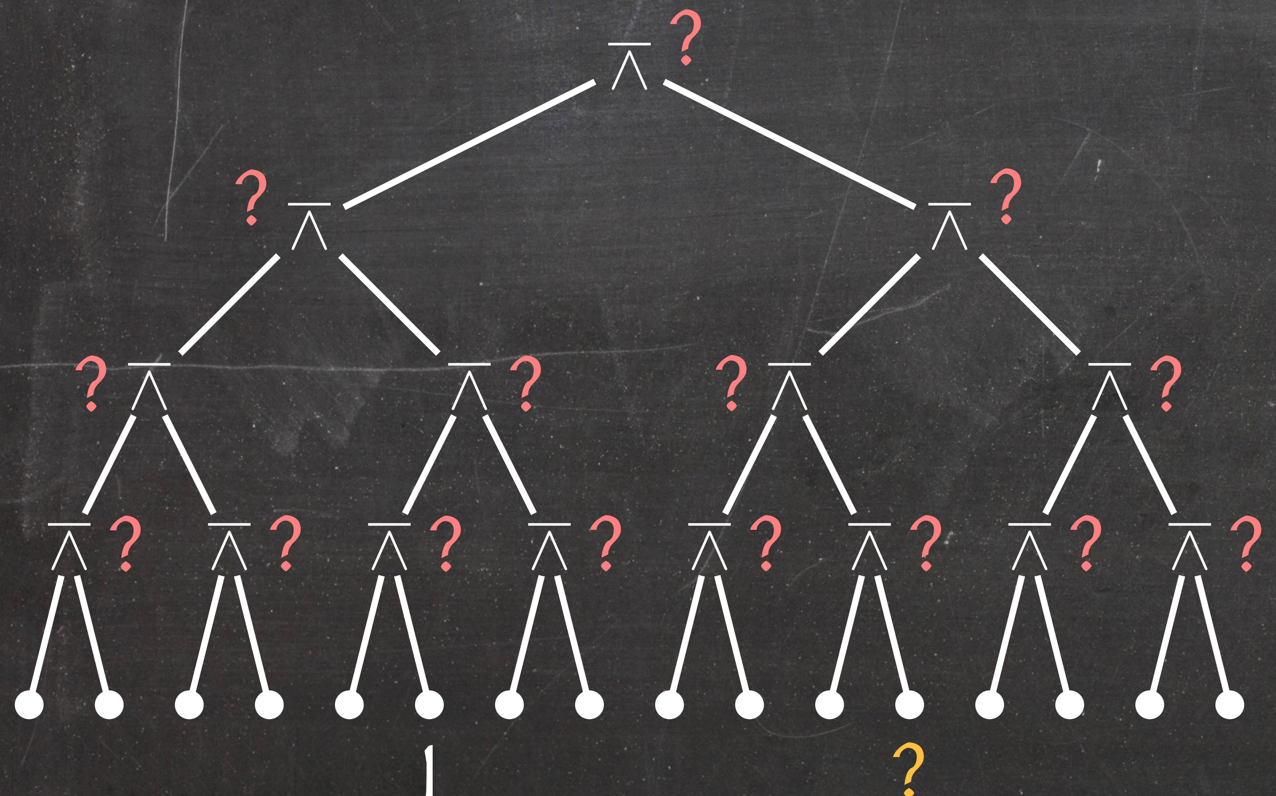

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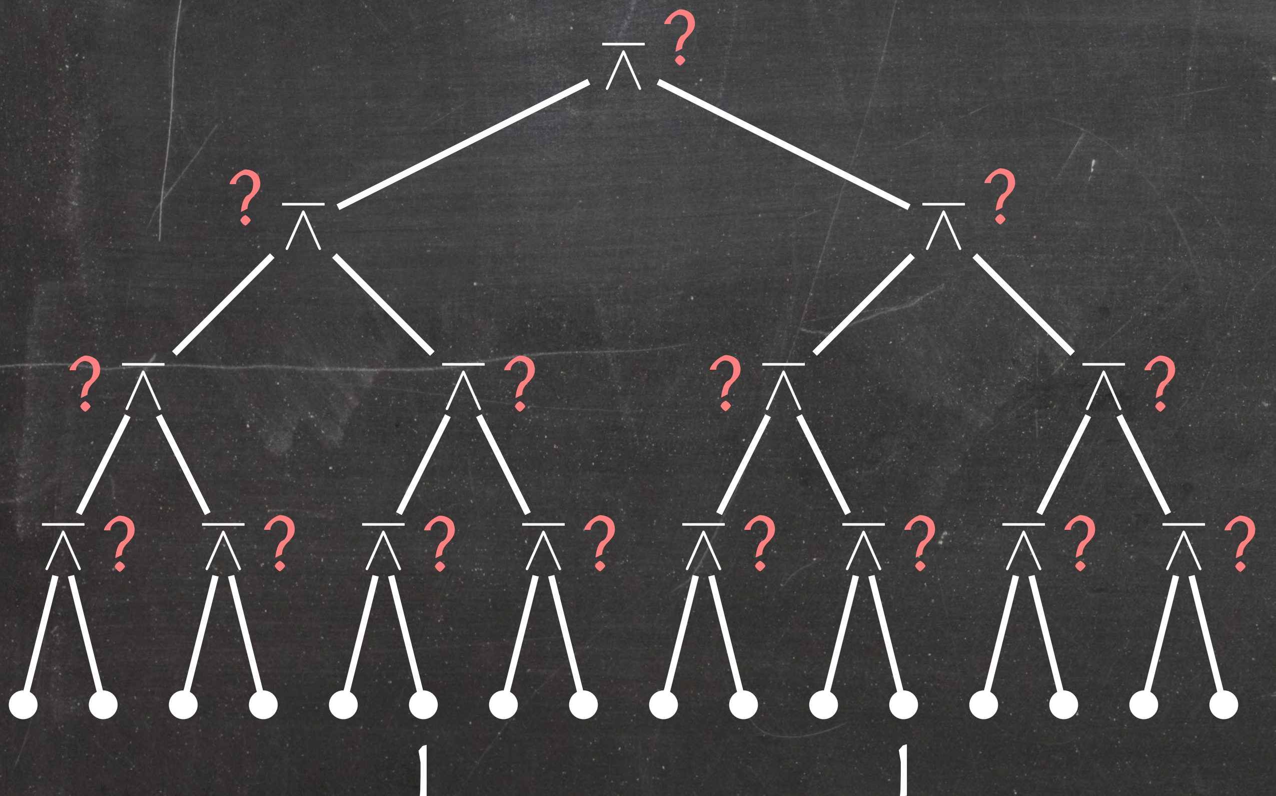

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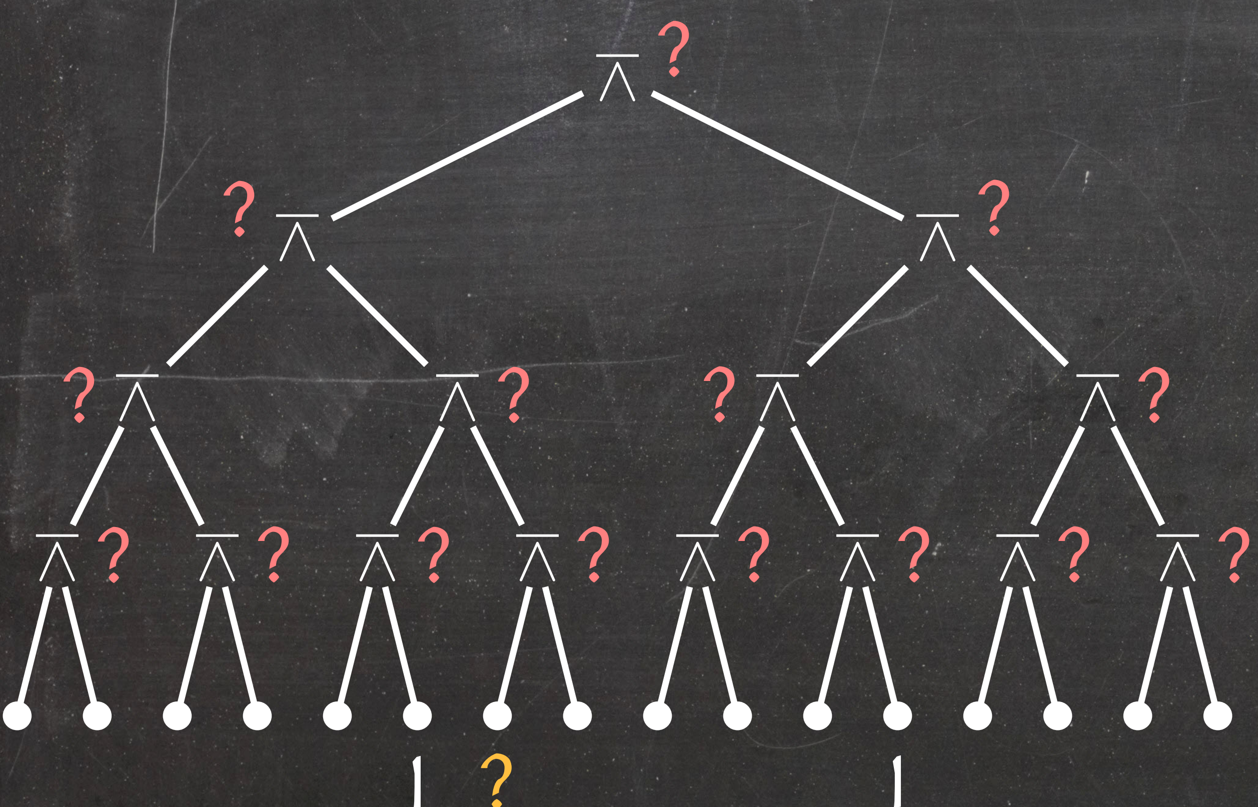

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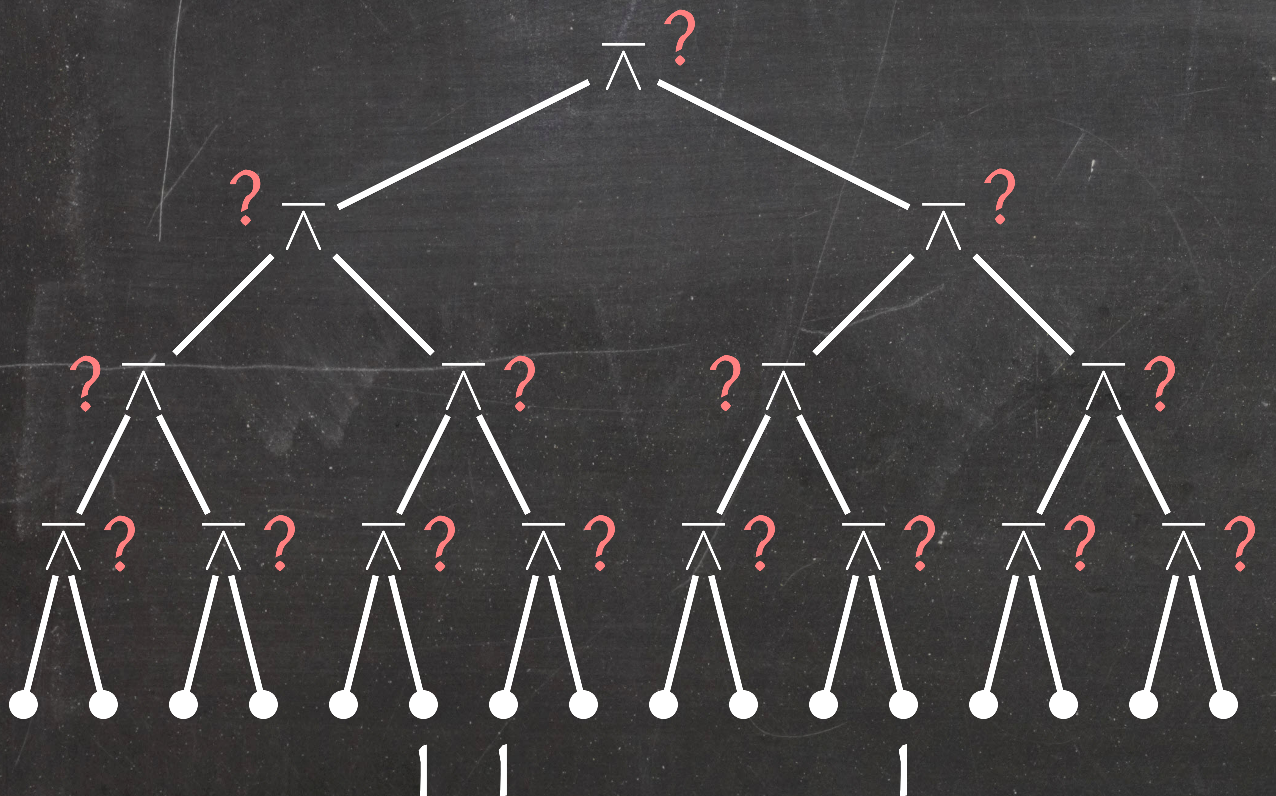

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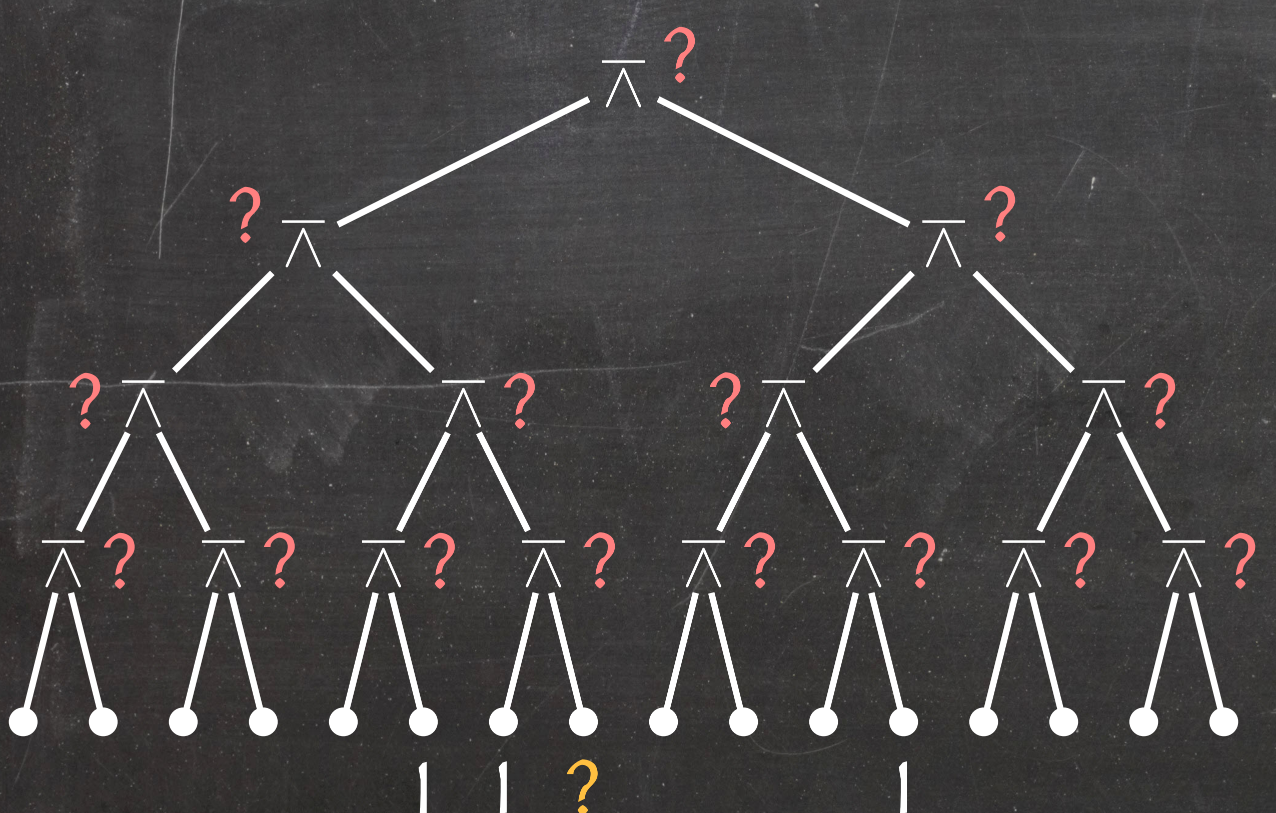

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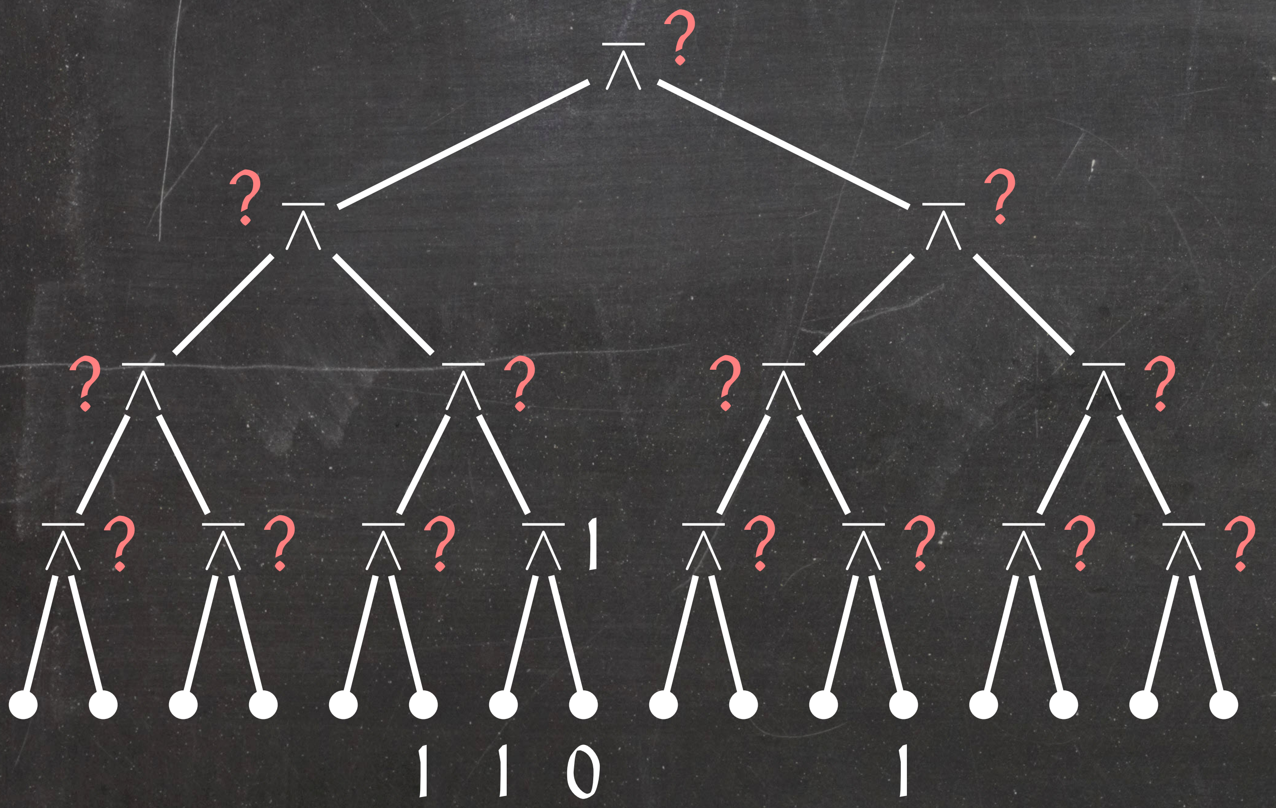

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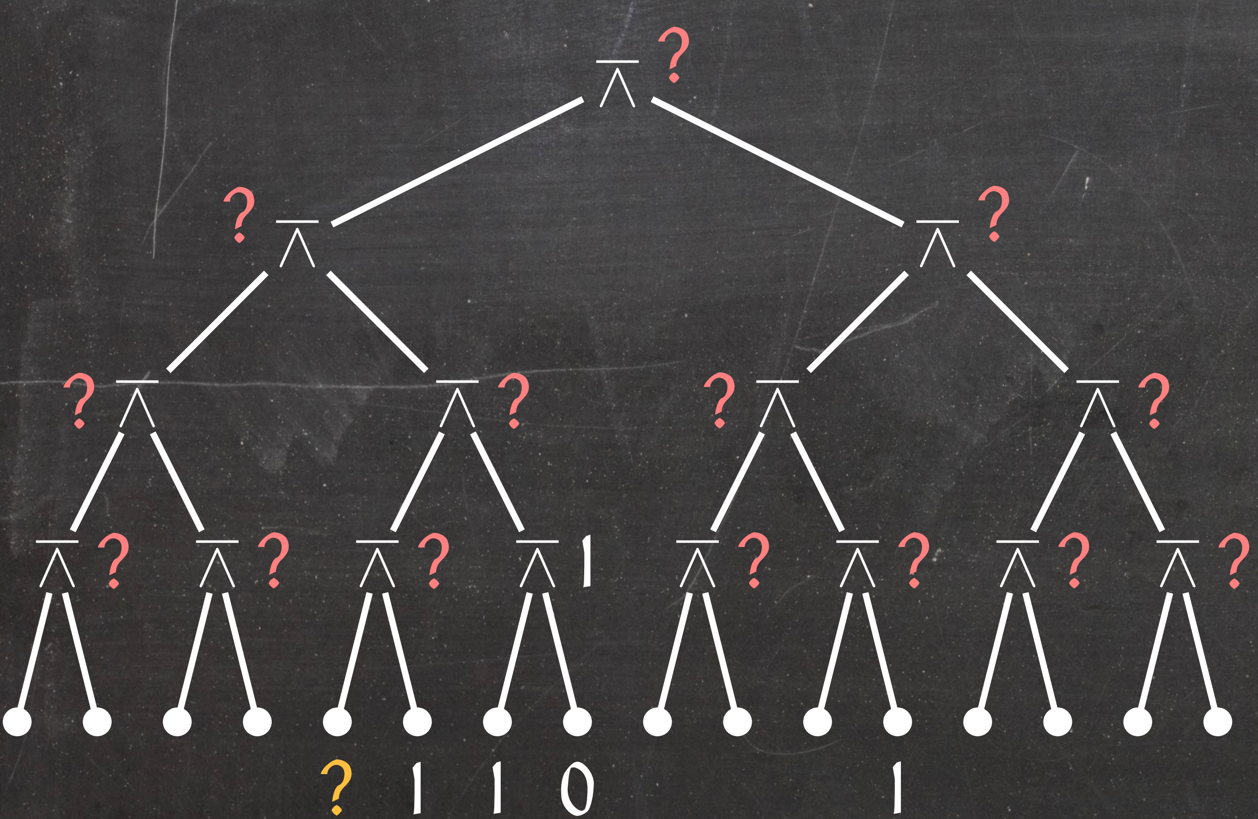

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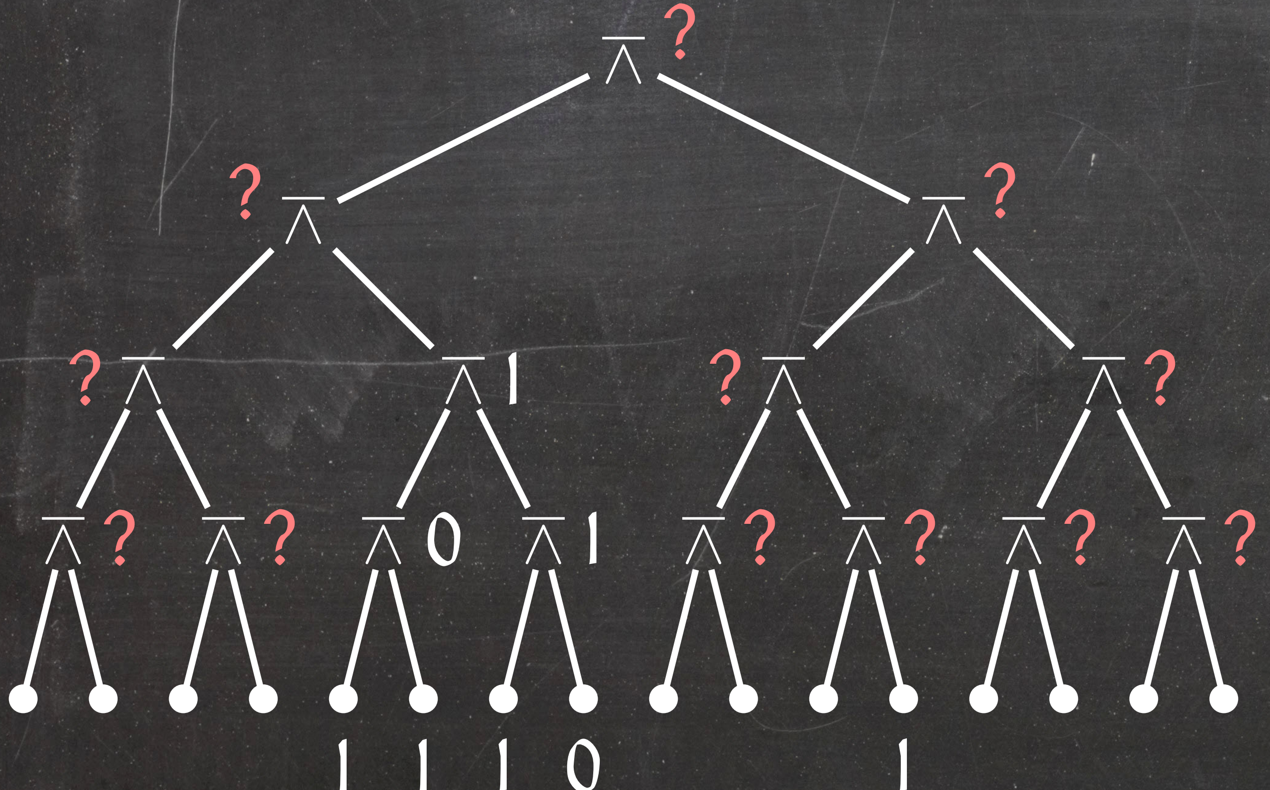

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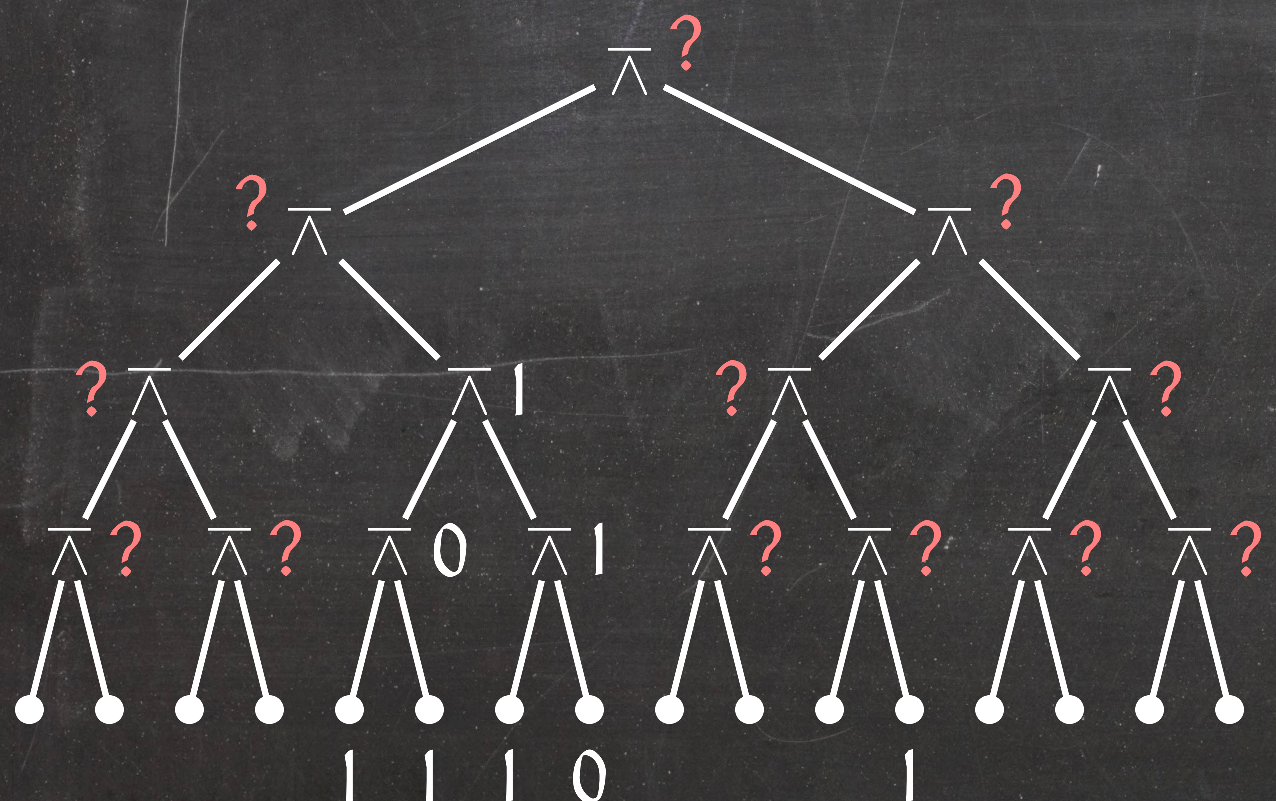

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Game Tree Evaluation: A Lower Bound

Observation: Any deterministic algorithm has to inspect every leaf in the worst case and thus takes $\Omega(n)$ time in the worst case.

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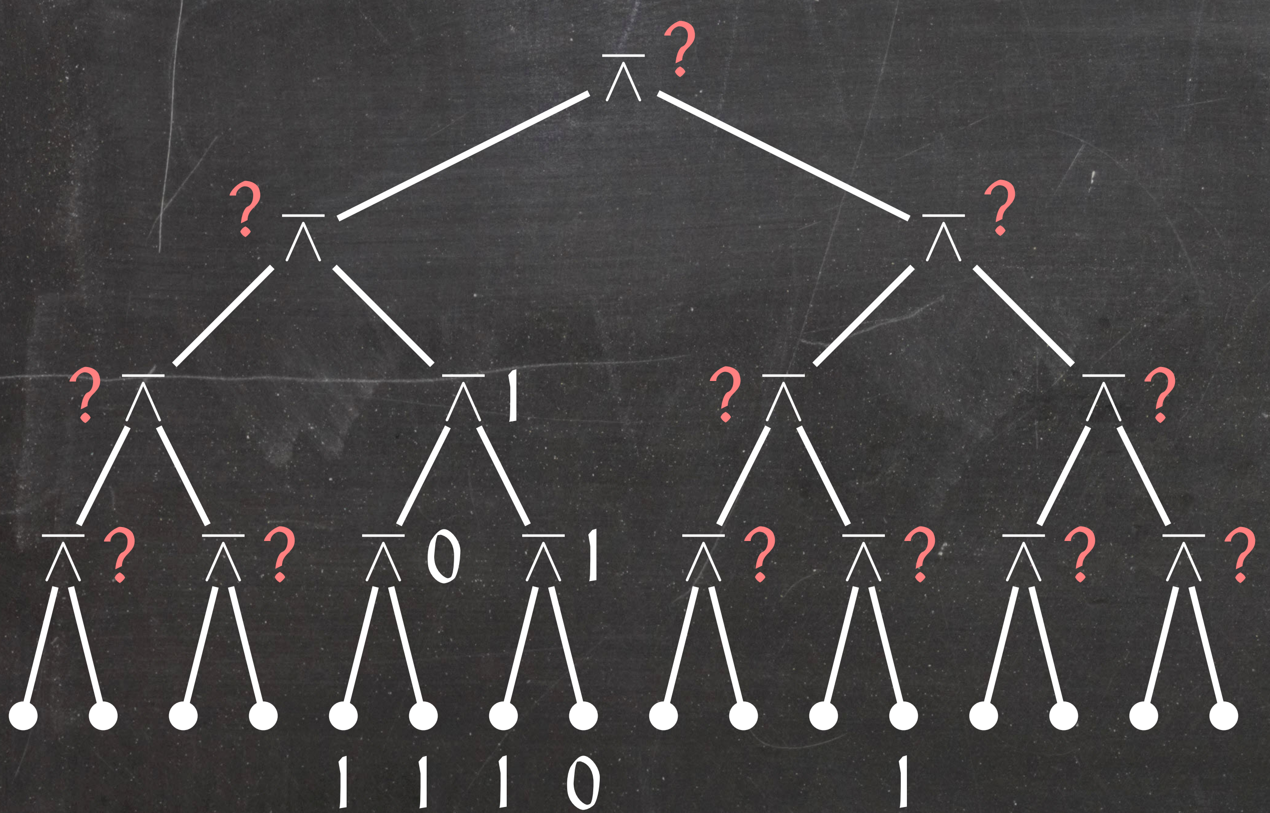

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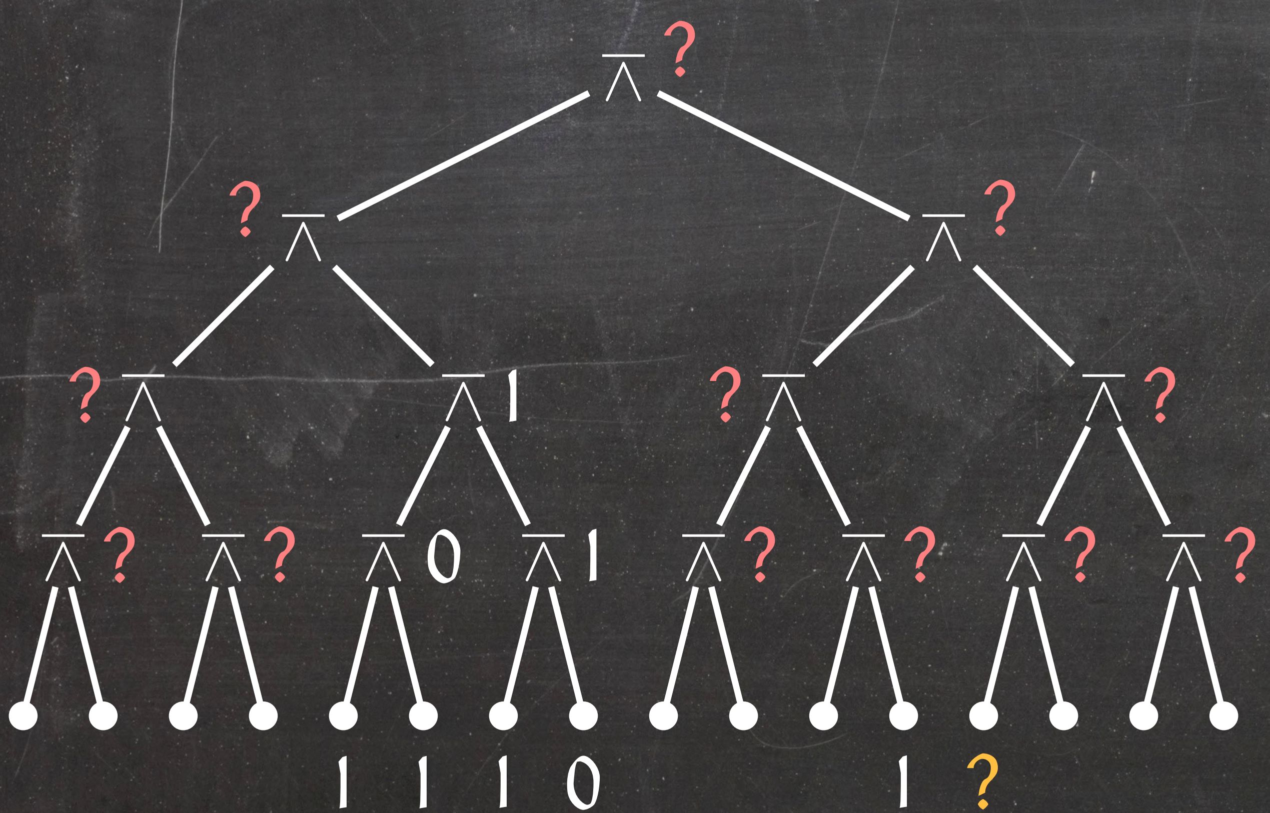

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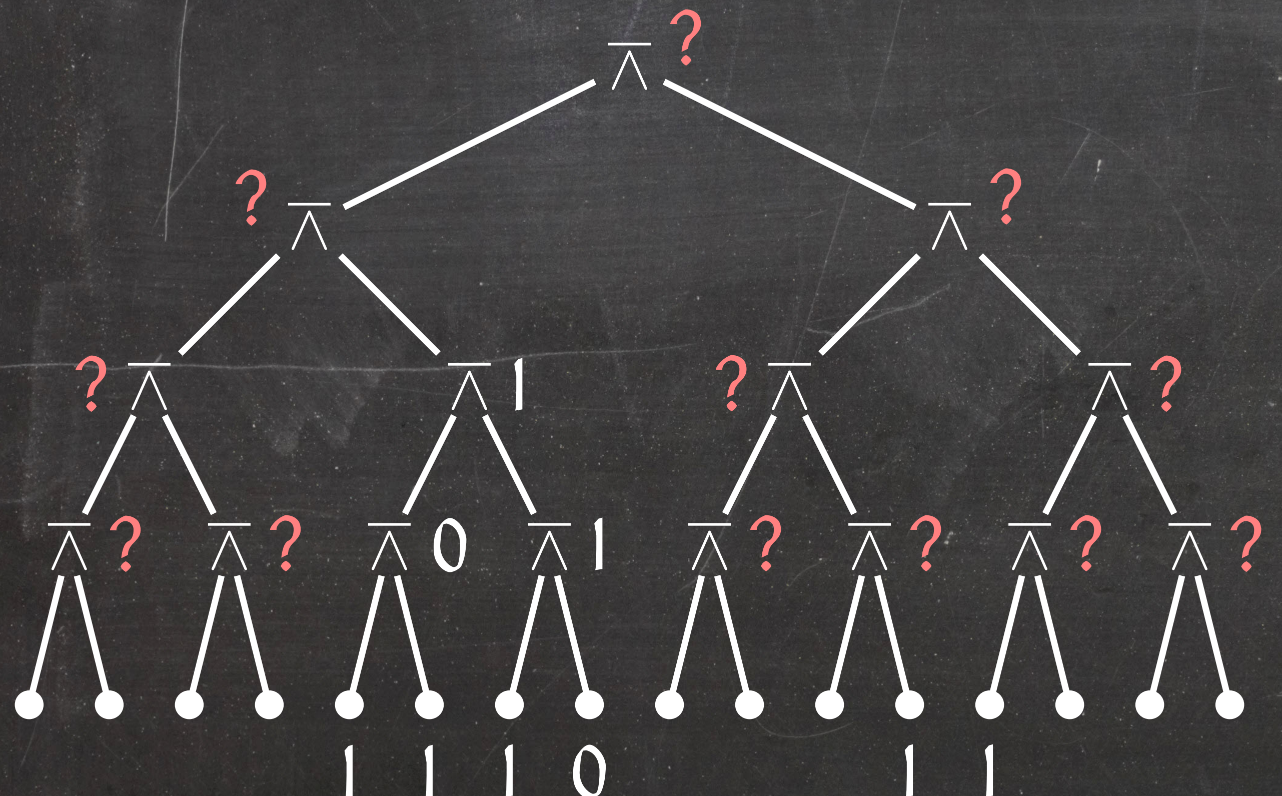

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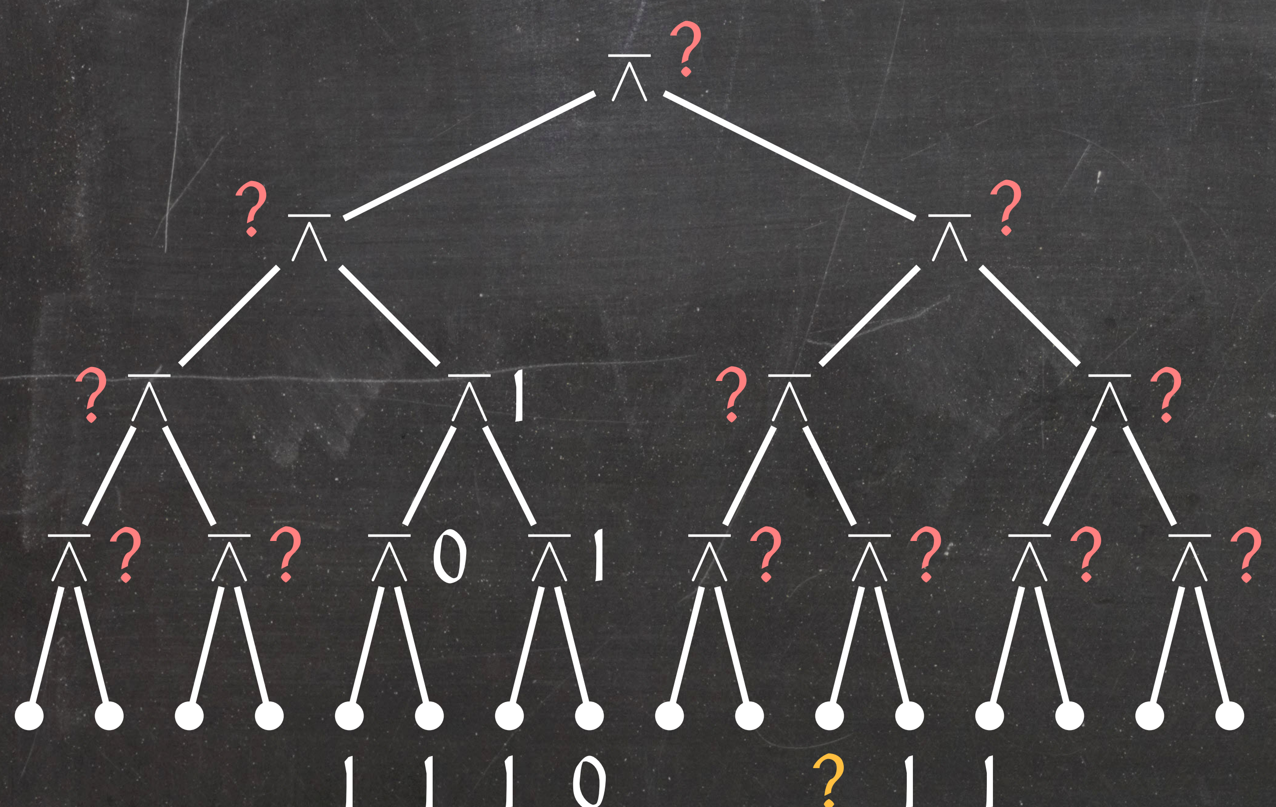

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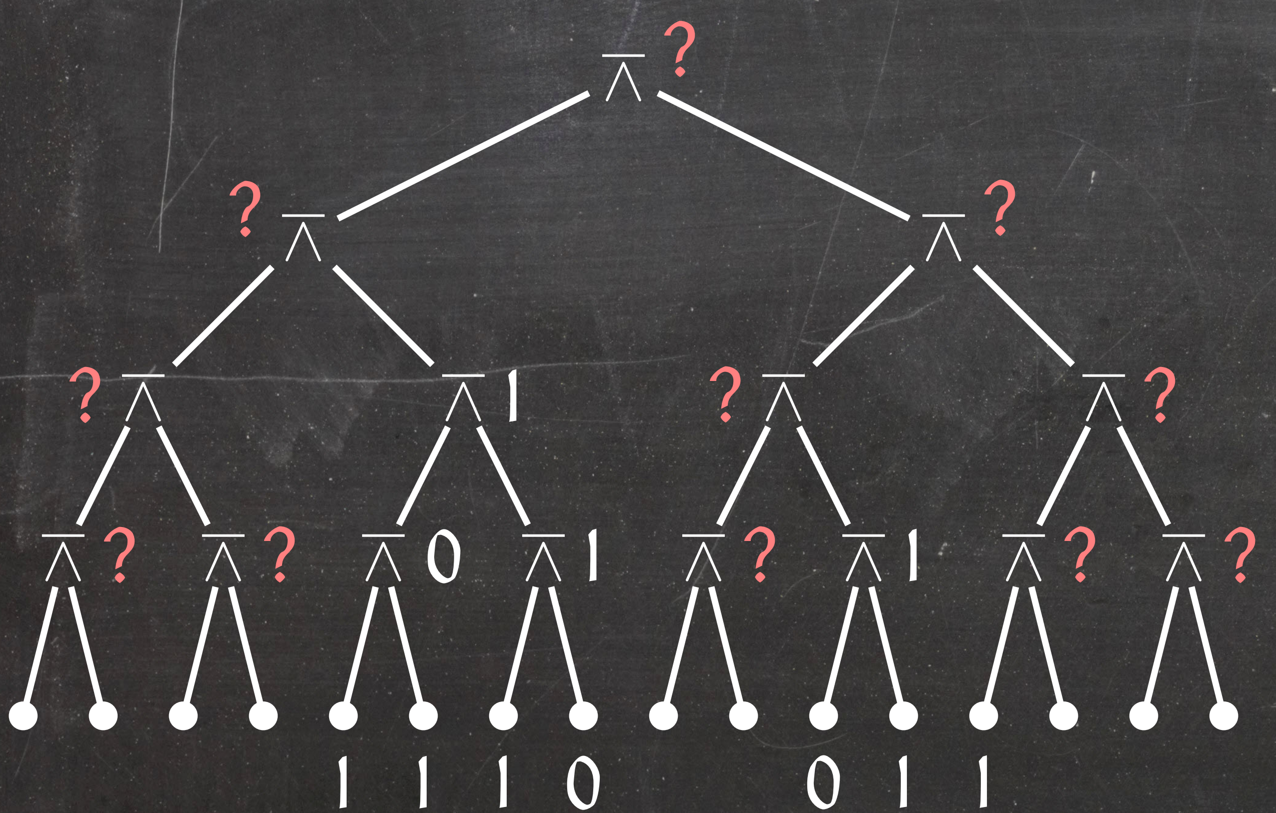

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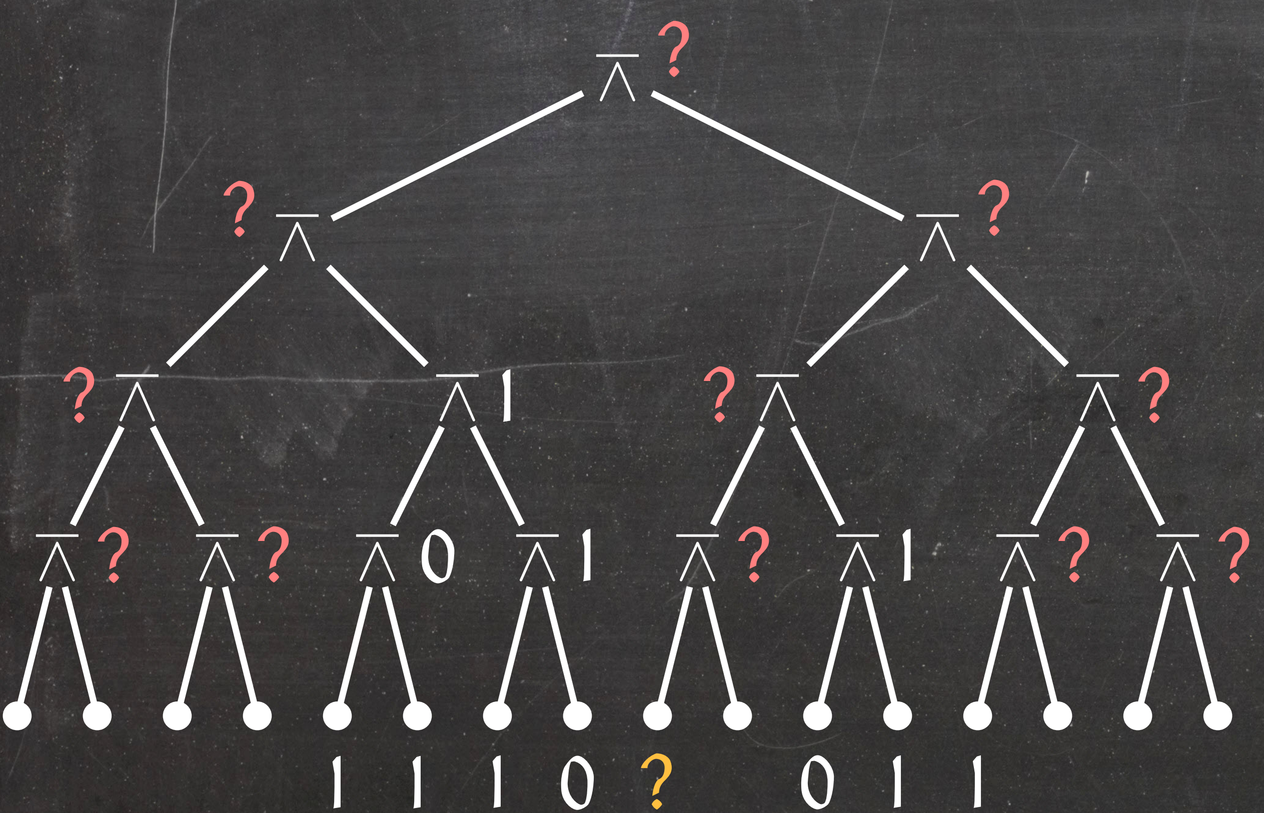

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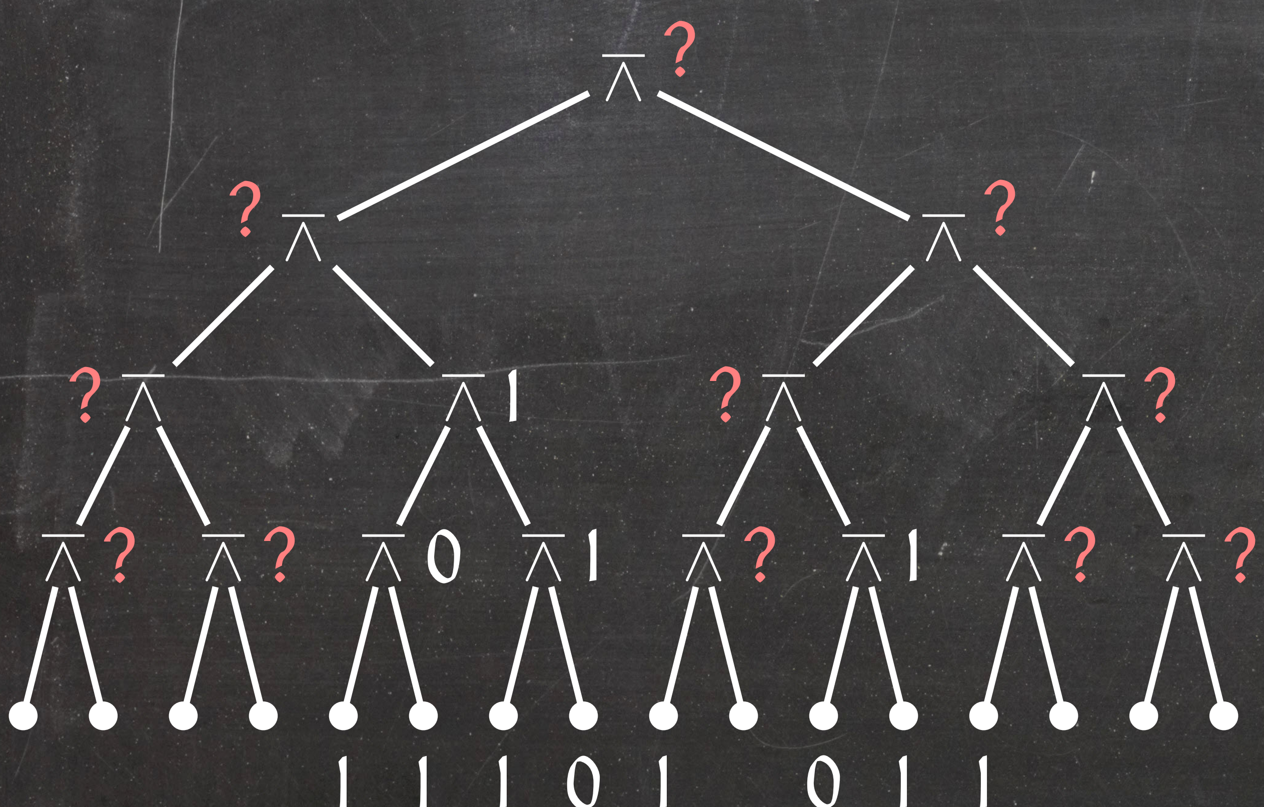

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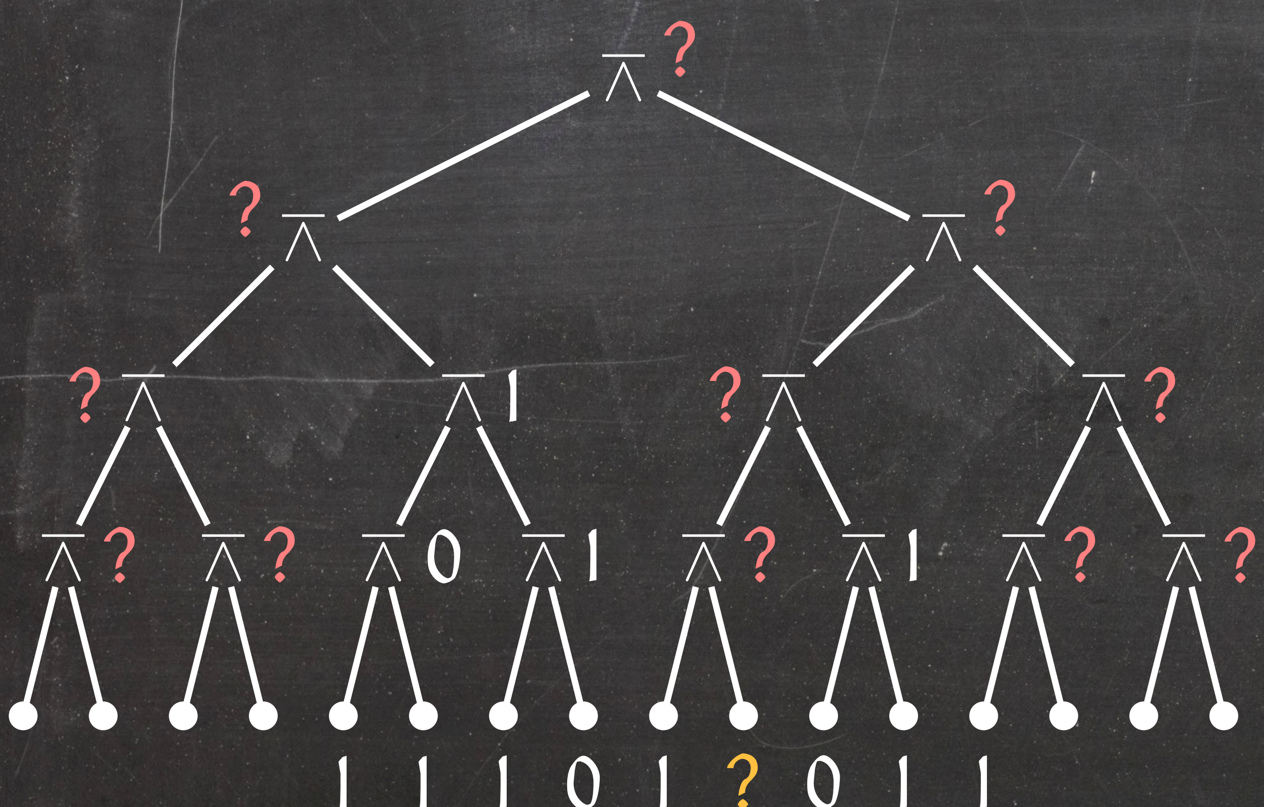

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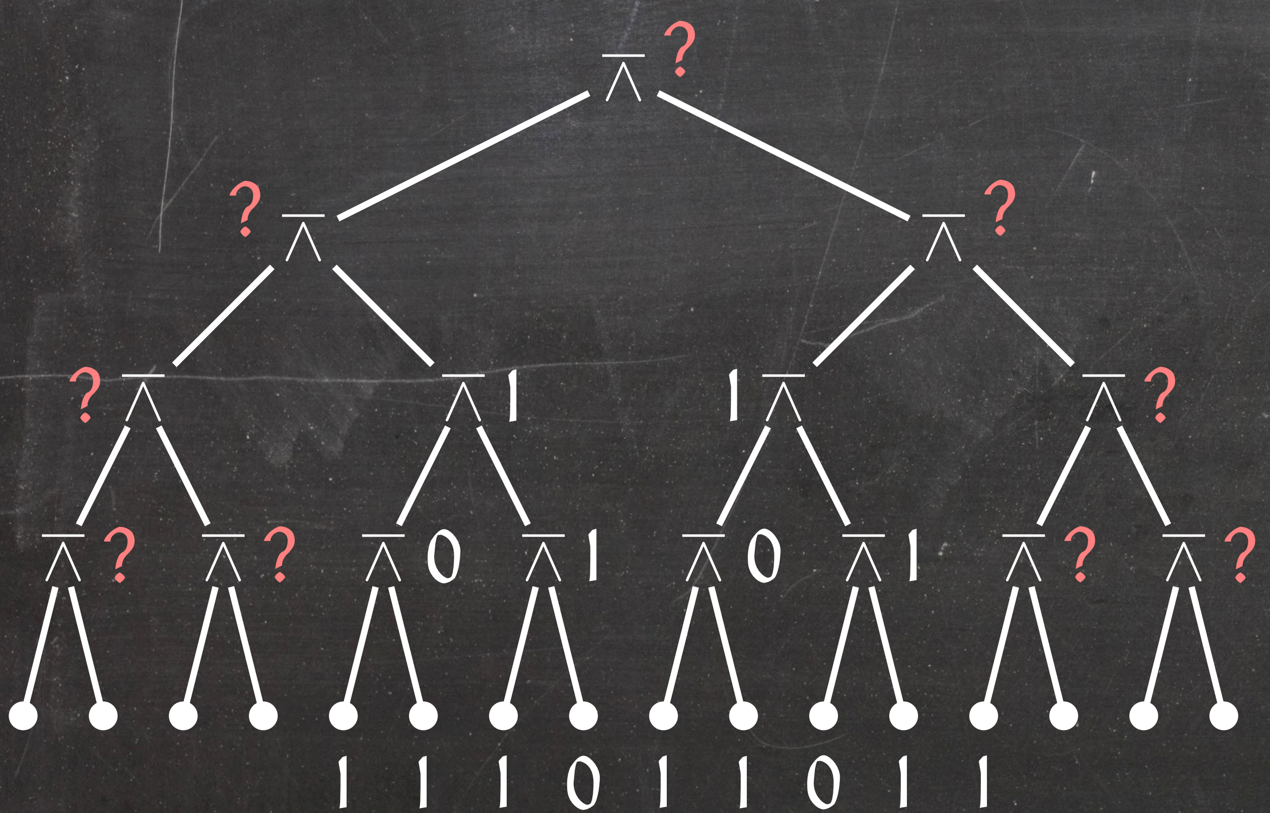

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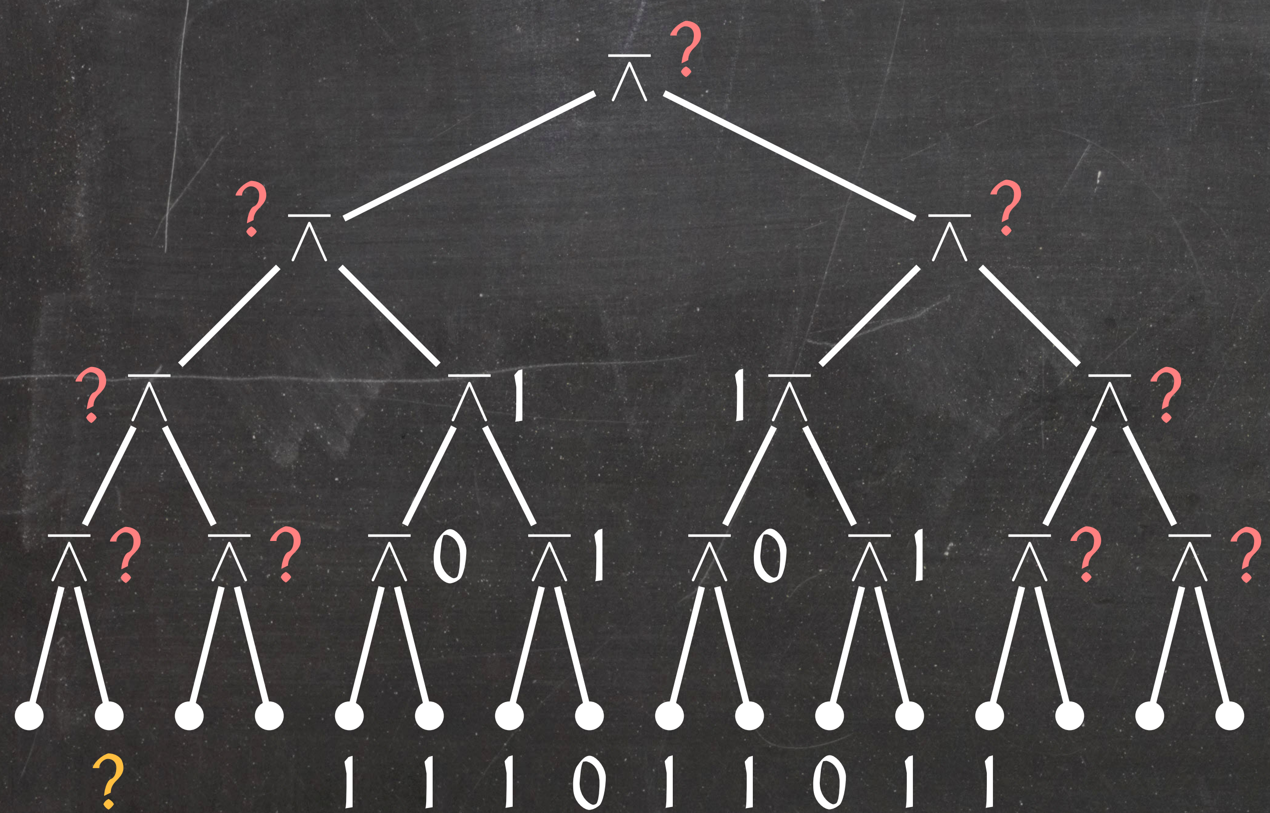

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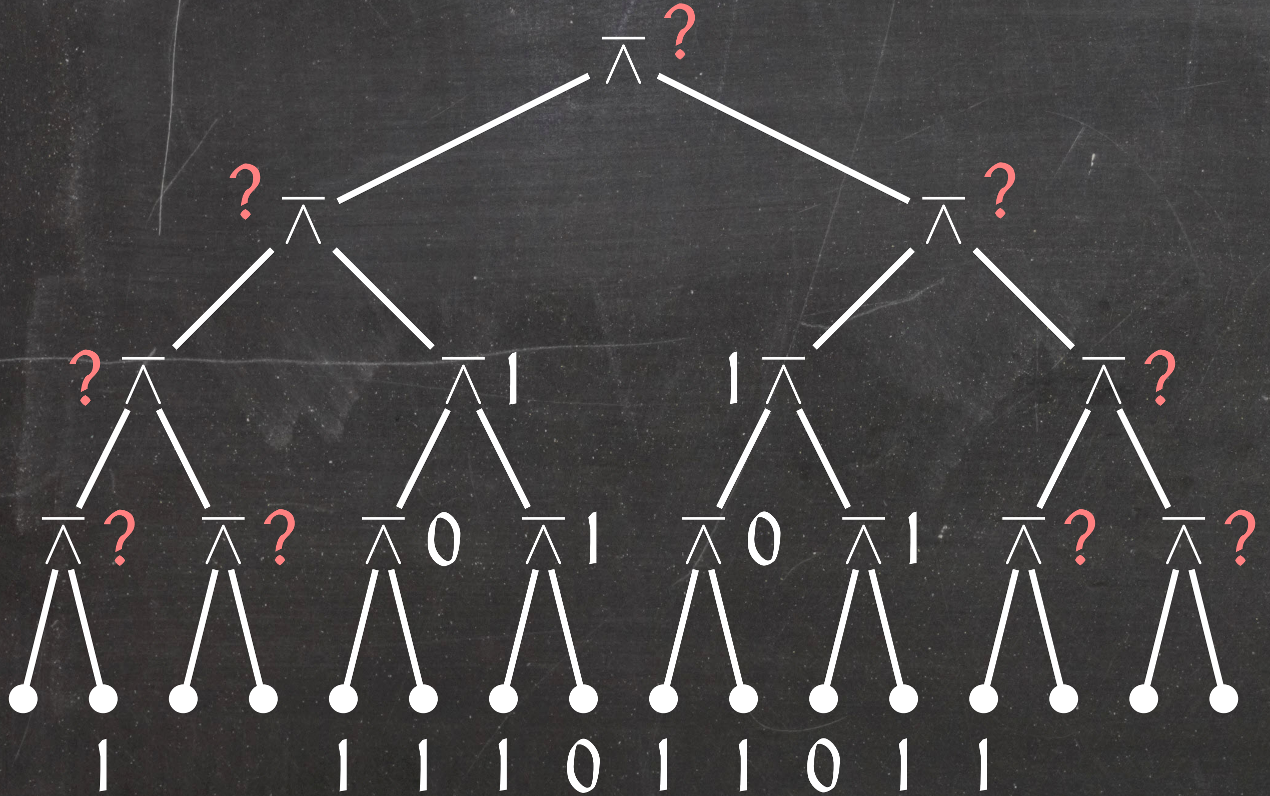

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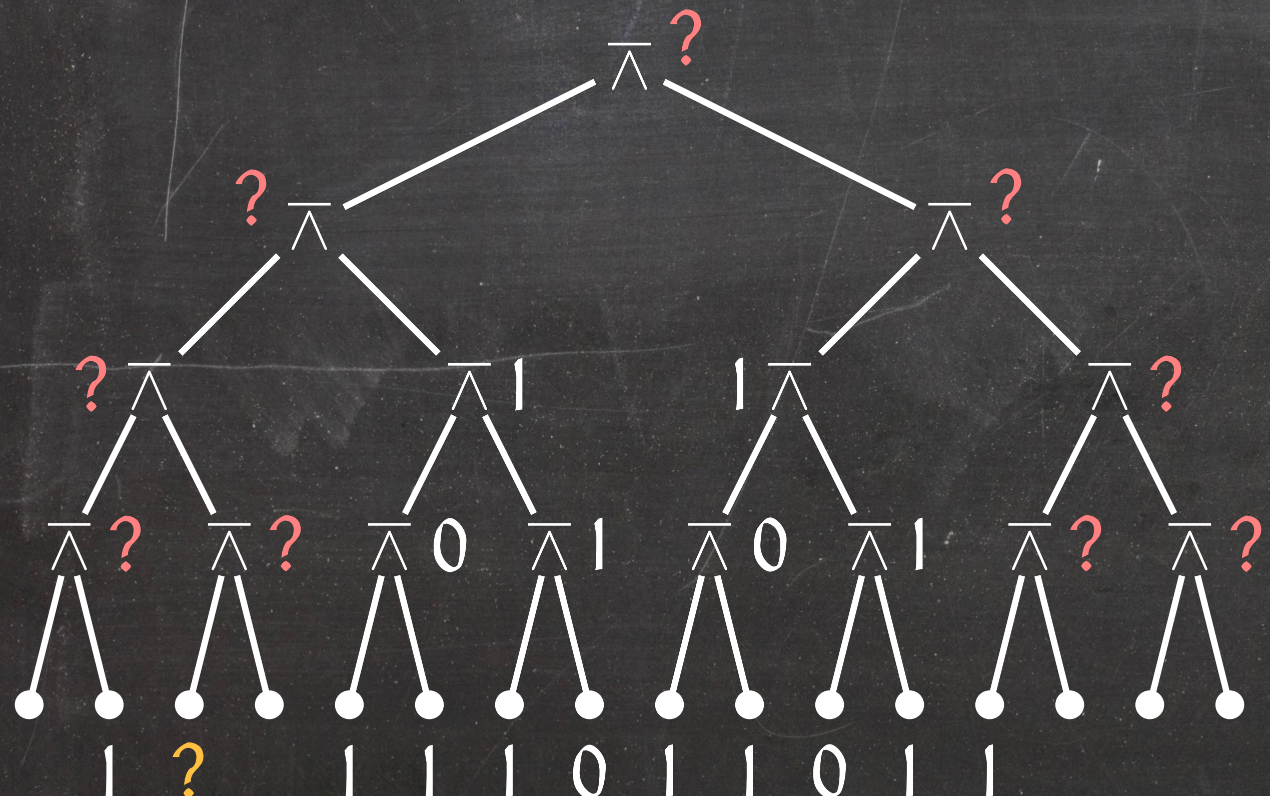

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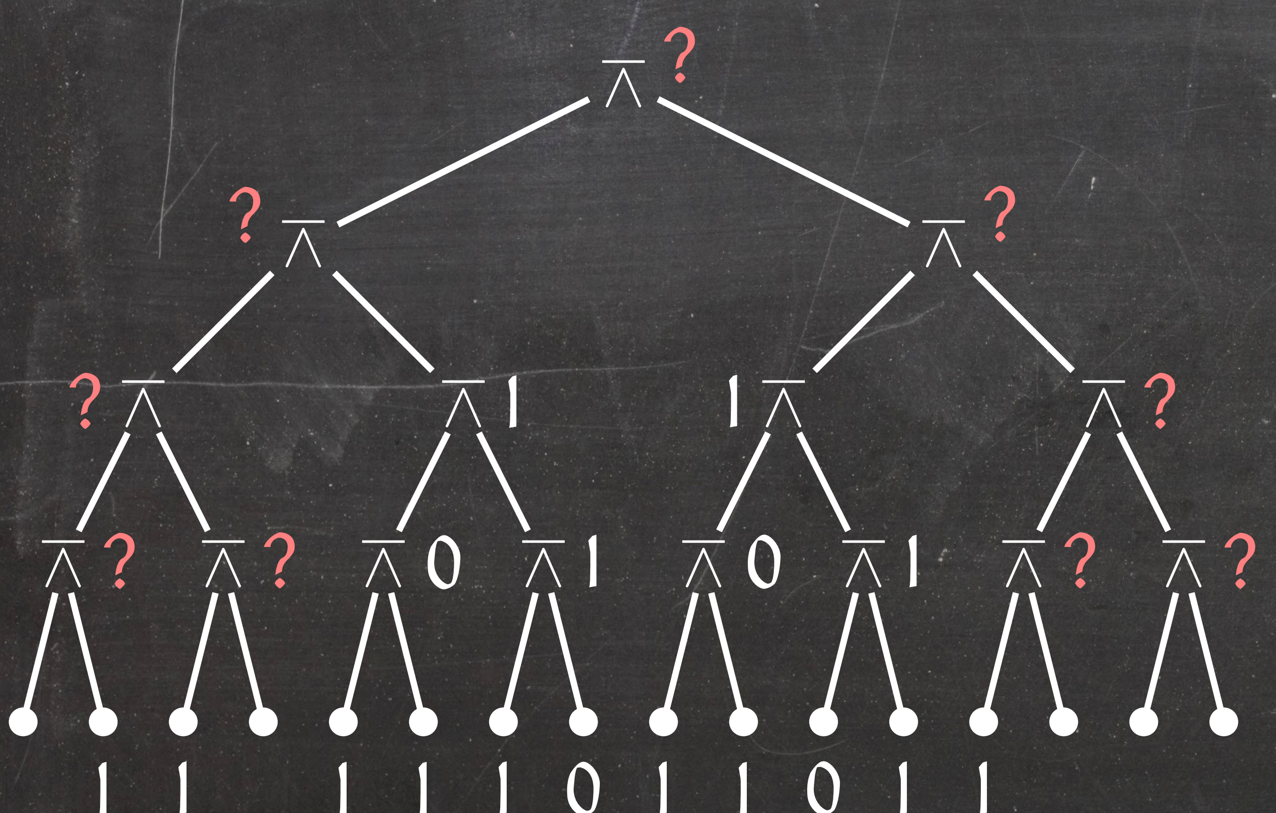

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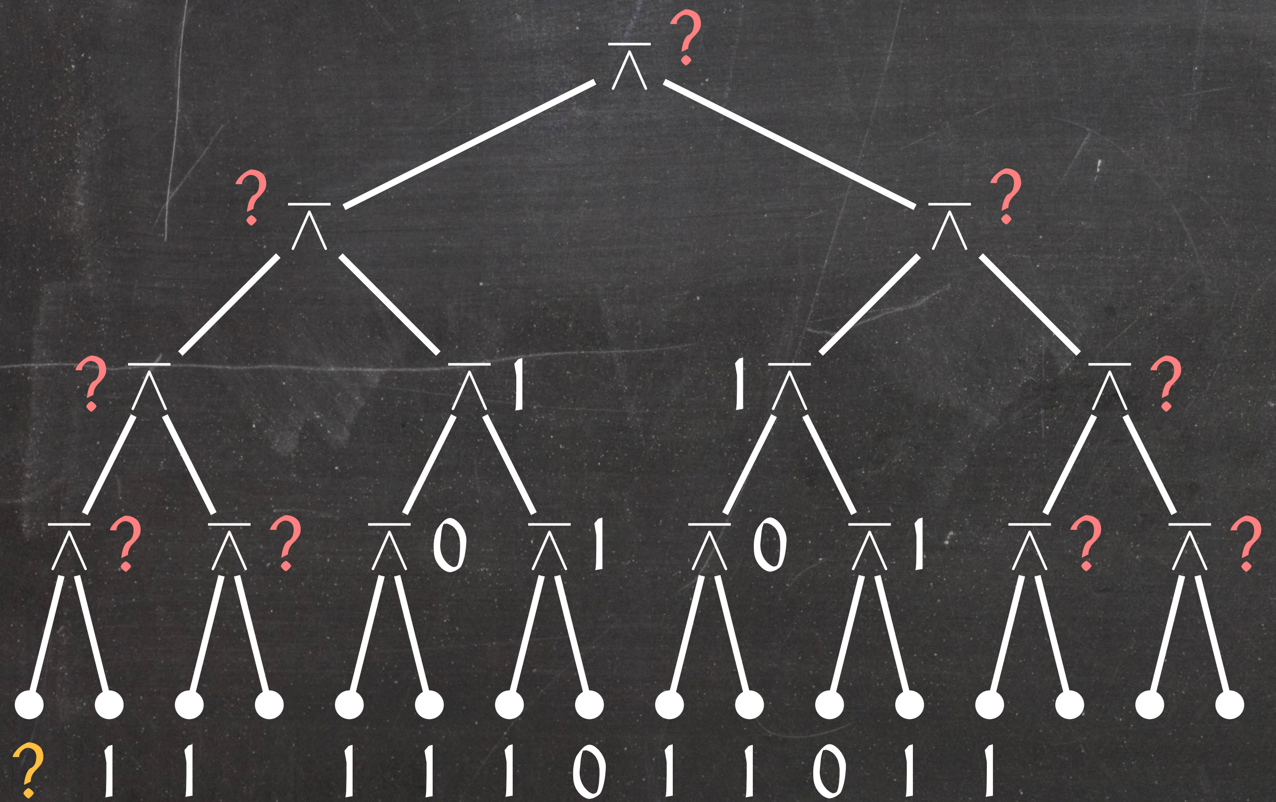

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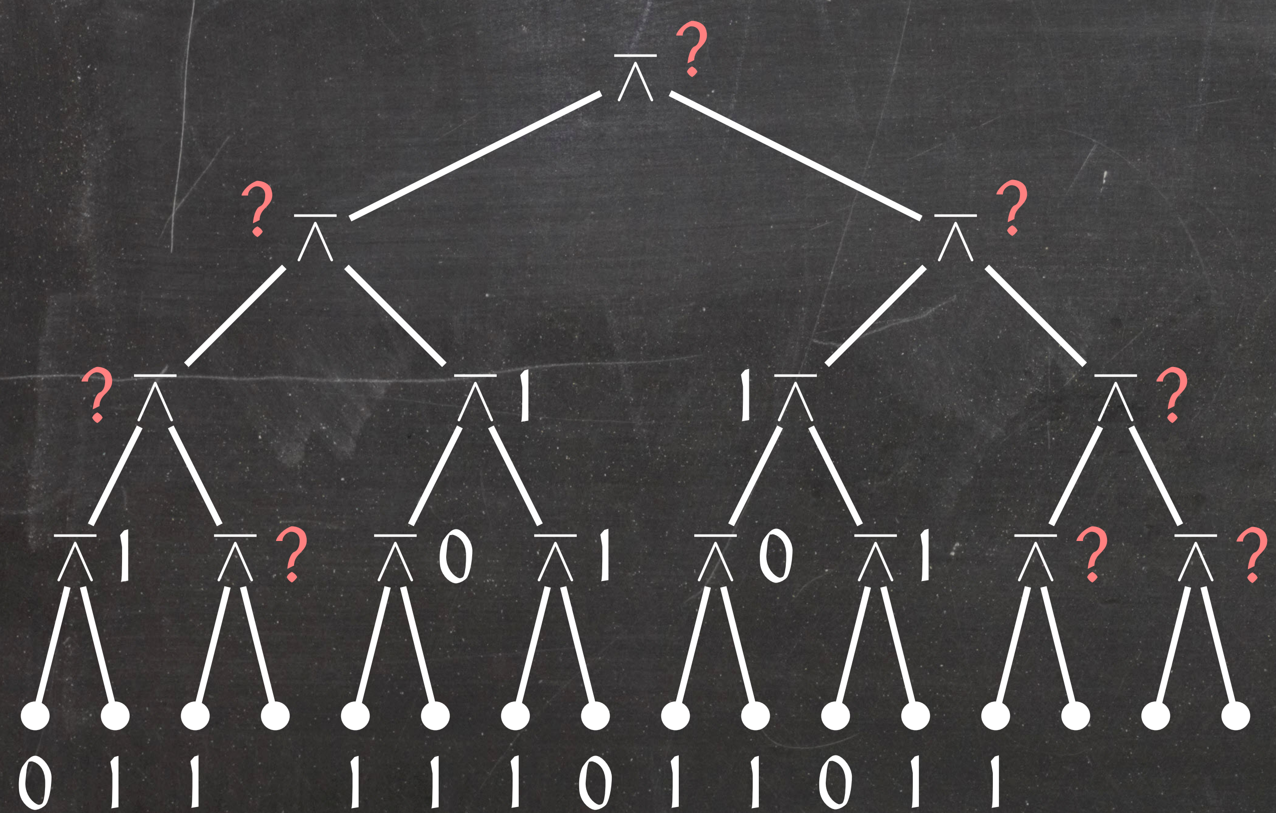

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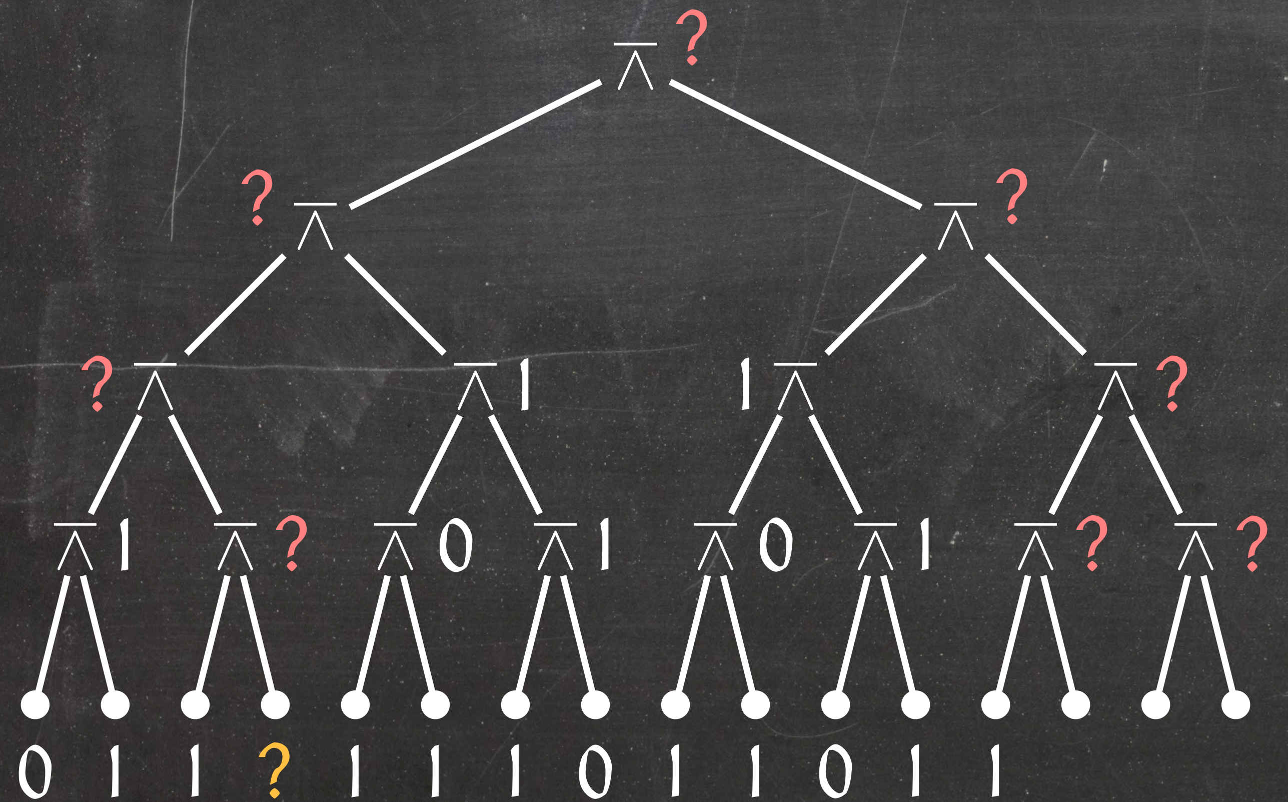

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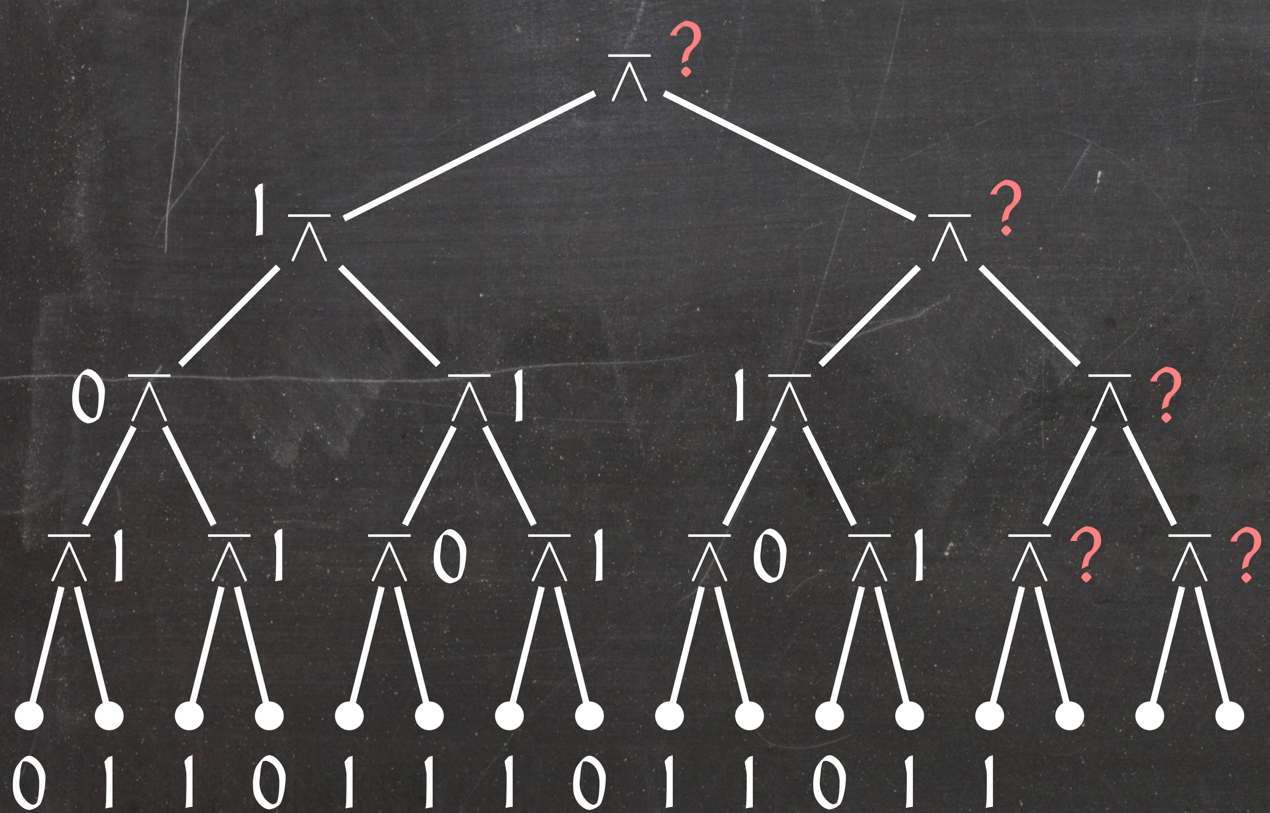

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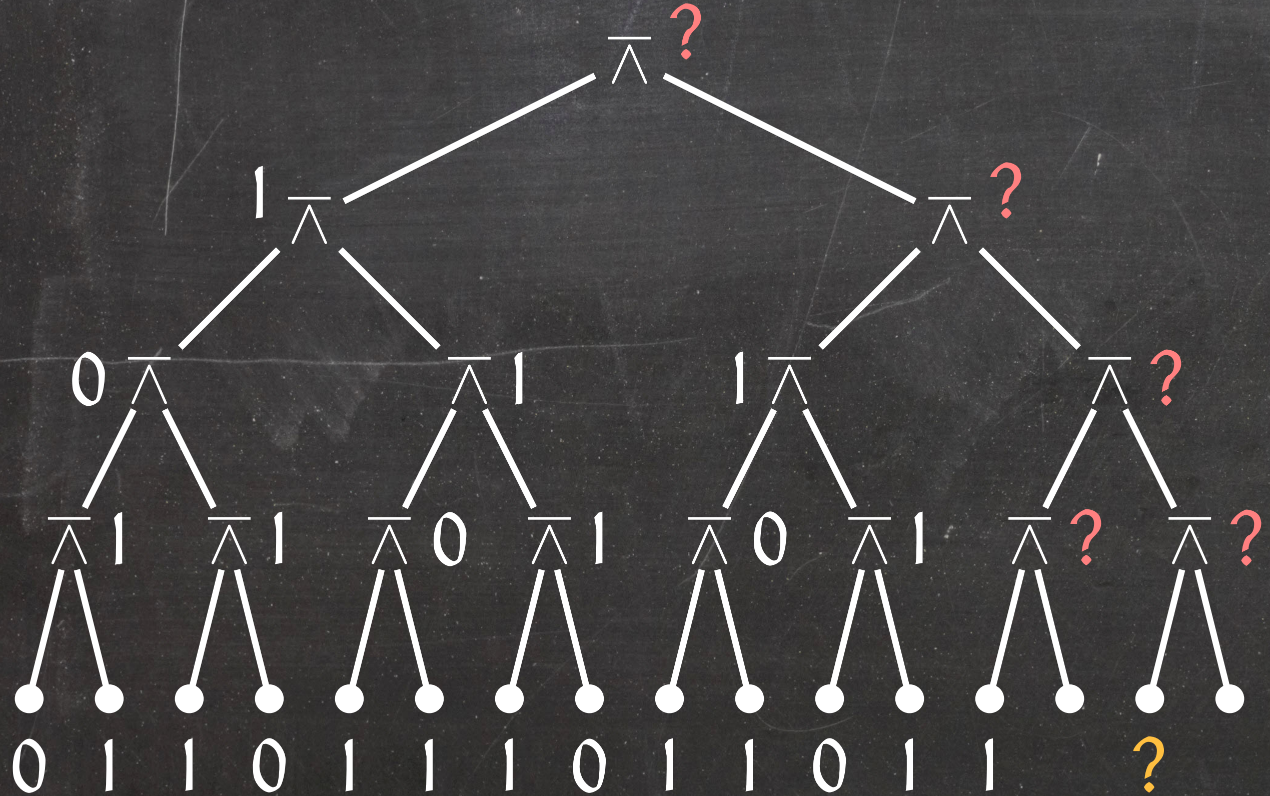

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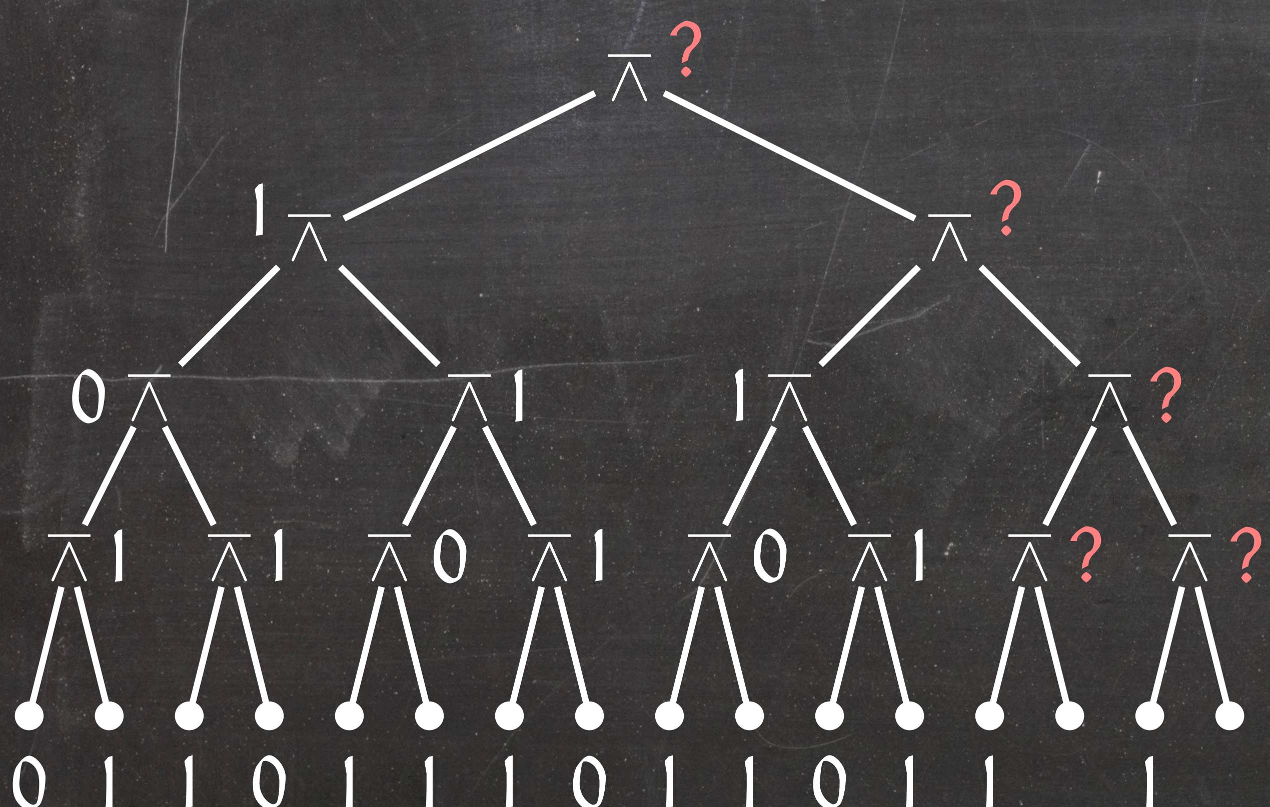

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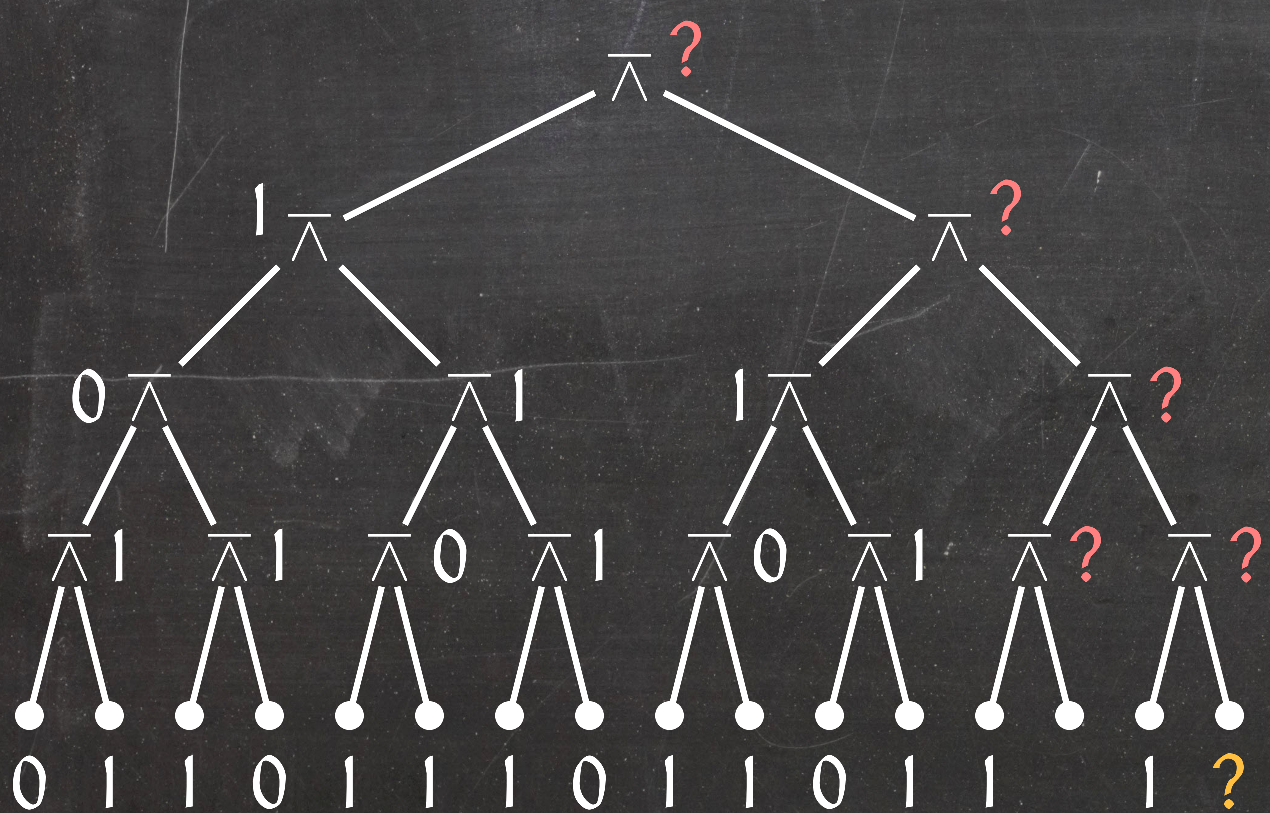

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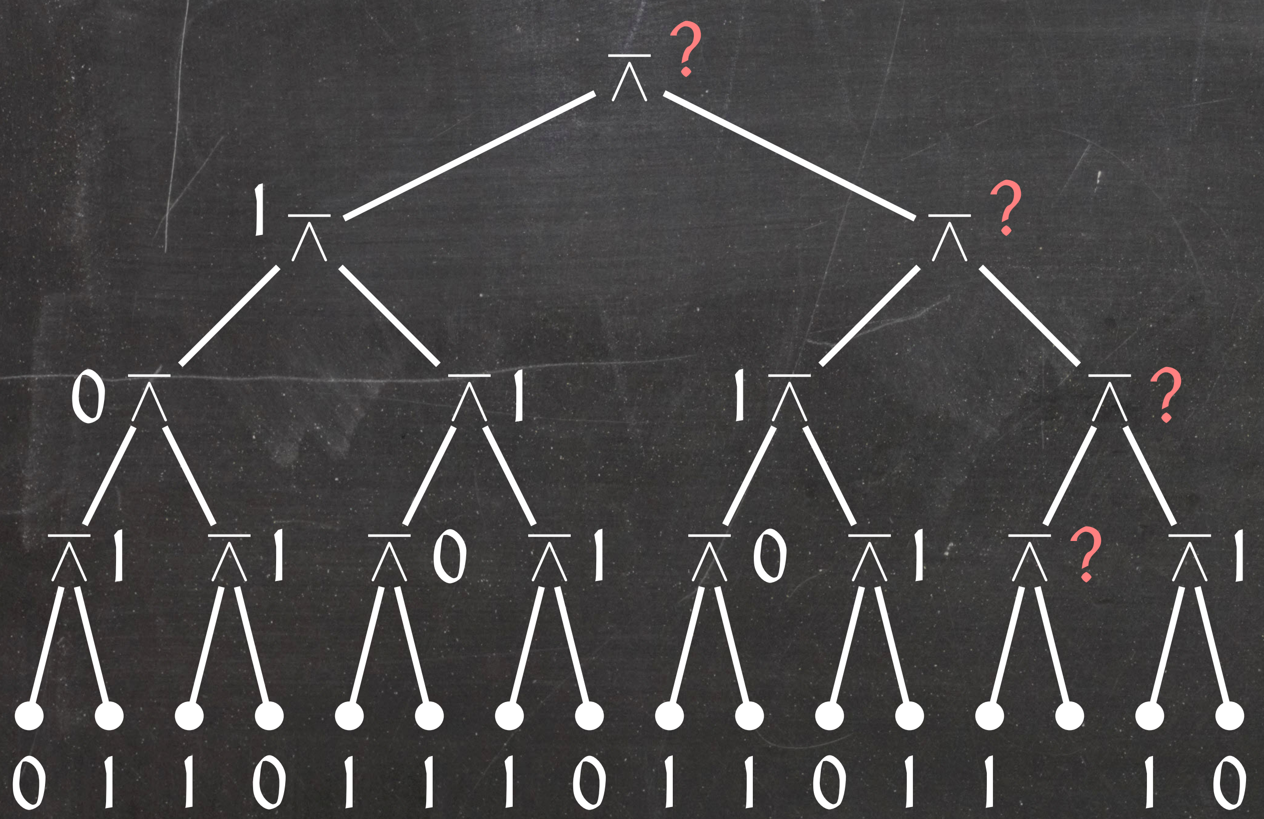

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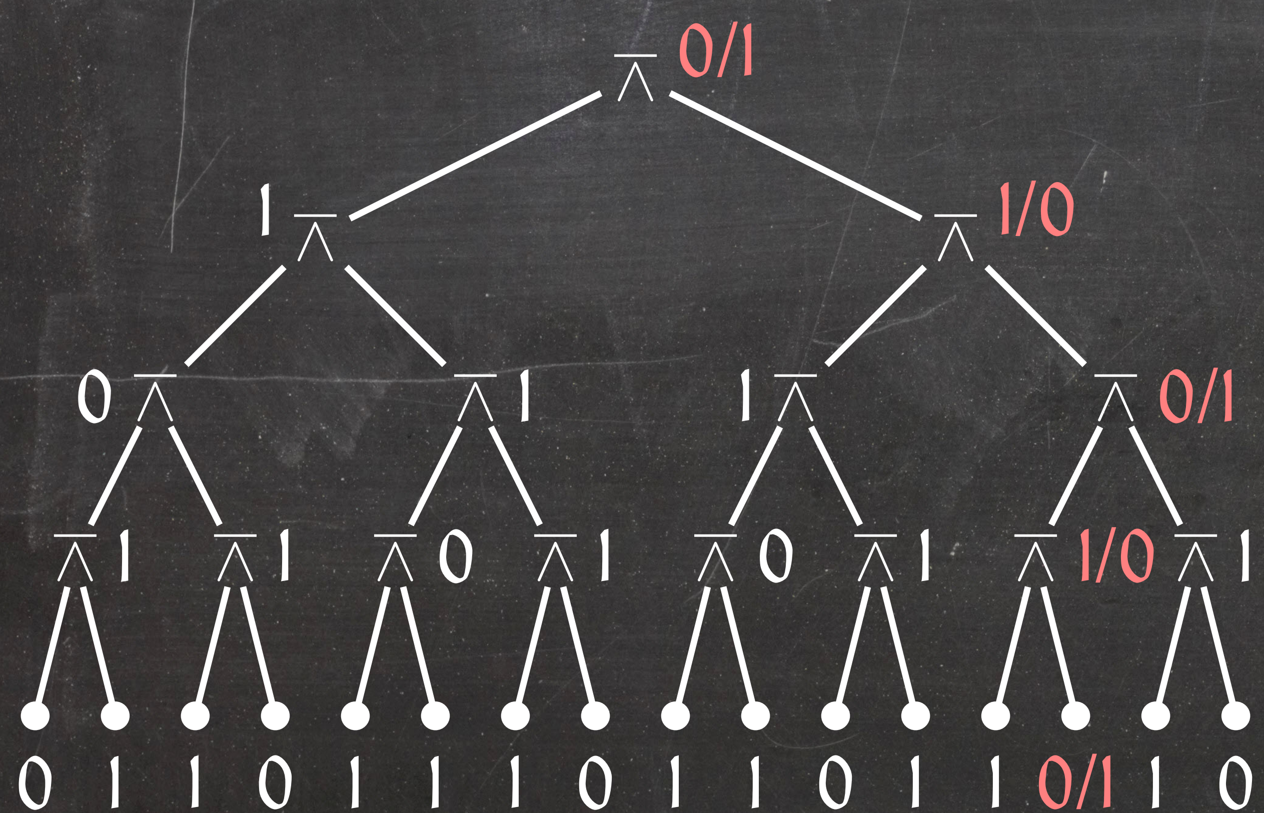

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Game Tree Evaluation: Randomized Algorithm

RandomizedGameValue(v)

```
if v is a leaf
   then return its value
coinFlip = RandomNumber(0, 1)
if coinFlip = 1
   then first    = v.leftChild
        second = v.rightChild
   else first    = v.rightChild
        second = v.leftChild
if not f = GameValue(first)
   then return 1
   else return not GameValue(second)
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$$E[T(n)] \le \max\left(2 \cdot E_1\left[T\left(\frac{n}{2}\right)\right], E_1[T(n)]\right)$$

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$$E_1[T(n)] \in O(n^{0.754}) \Rightarrow E[T(n)] \in O(n^{0.754})$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^{\alpha} - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

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Claim: $E_I[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \leq 0.754$.

Base case: $1 \leq n < 2$.

$T(n) \in O(1) \Rightarrow E_I[T(n)] \leq cn^\alpha - d$ for any d and c sufficiently larger than d.

Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \le cn^{\alpha} - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$E_1[T(n)] \le 2 \cdot E_1\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_1\left[T\left(\frac{n}{2}\right)\right] + a$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^{\alpha} - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$\begin{aligned} E_I[T(n)] &\le 2 \cdot E_I\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_I\left[T\left(\frac{n}{2}\right)\right] + a \\ &\le 2 \cdot \left[c\left(\frac{n}{4}\right)^{\alpha} - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^{\alpha} - d\right] + a \end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_l[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$\begin{aligned}
E_l[T(n)] &\le 2 \cdot E_l\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_l\left[T\left(\frac{n}{2}\right)\right] + a \\
&\le 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \\
&= cn^\alpha\left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2}
\end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_l[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$
\begin{aligned}
E_l[T(n)] &\le 2 \cdot E_l\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_l\left[T\left(\frac{n}{2}\right)\right] + a \\
&\le 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \\
&= cn^\alpha \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2} \\
&\le cn^\alpha \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) - d \quad \forall d \ge \frac{2}{3}a
\end{aligned}
$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$\begin{aligned}
E_I[T(n)] &\le 2 \cdot E_I\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_I\left[T\left(\frac{n}{2}\right)\right] + a \\
&\le 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \\
&= cn^\alpha\left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2} \\
&\le cn^\alpha\left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) - d \quad \forall d \ge \frac{2}{3}a \\
&= cn^\alpha\left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}}\right) - d
\end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \leq 0.754$.

Inductive step: $n \geq 2$.

$$E_I[T(n)] \leq cn^\alpha \left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$E_I[T(n)] \le cn^\alpha \left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d$$

$$= cn^\alpha \left(\frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2} \right) - d$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$E_I[T(n)] \le cn^\alpha \left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d$$

$$= cn^\alpha \left(\frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2} \right) - d$$

$$= cn^\alpha \left(\frac{34 + 2 \cdot \sqrt{33}}{34 + 2 \cdot \sqrt{33}} \right) - d$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$E_I[T(n)] \le cn^\alpha \left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d$$

$$= cn^\alpha \left(\frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2} \right) - d$$

$$= cn^\alpha \left(\frac{34 + 2 \cdot \sqrt{33}}{34 + 2 \cdot \sqrt{33}} \right) - d$$

$$= cn^\alpha - d$$

Summary

Algorithms that are fast on average are often easier to design and faster in practice than worst-case efficient algorithms.

In some applications, worst-case guarantees matter!

Average-case analysis provides a valid performance prediction only if the inputs are uniformly distributed.

Randomized algorithms remove this dependence on the input distribution (but rely on a good random number generator).

There are problems where randomized algorithms are provably faster than the best possible deterministic algorithm.