Average-Case Analysis and Randomization

Textbook Reading

Chapter 7 & Sections 8.4, 9.2

Overview

Design principle

- Do the easy thing and hope it works for most inputs
- Make random choices and hope they're good

Problems

- Sorting (Quick Sort)
- Permuting
- Selection
- Game tree evaluation

Quick Sort Revisited

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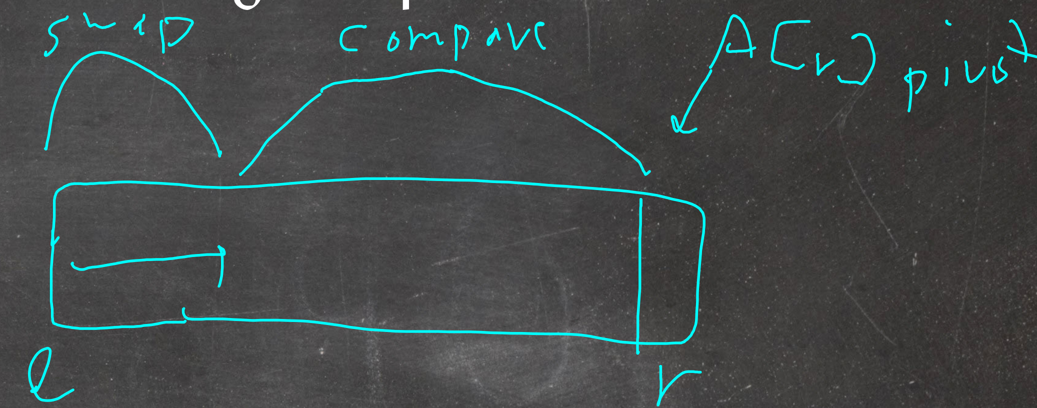

Remedy:

Blindly use the last element as pivot.

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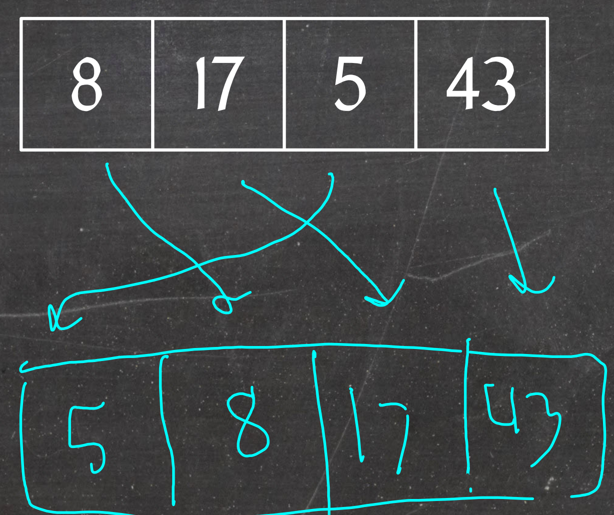

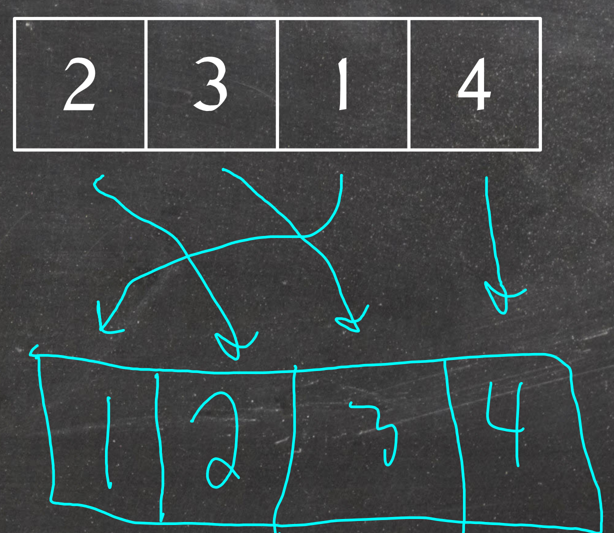

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$\Rightarrow$ The average-case running time of SimpleQuickSort is the same as its expected running time on a uniformly random input permutation.

Partitioning Maintains Uniformity

Lemma: If $A[\ell\,..\,r]$ is a uniform random permutation of the elements in $A[\ell\,..\,r]$, then the two subarrays $A[\ell\,..\,m-1]$ and $A[m+1\,..\,r]$ produced by Partition(A, ℓ, r) are also uniform random permutations of the elements they contain.

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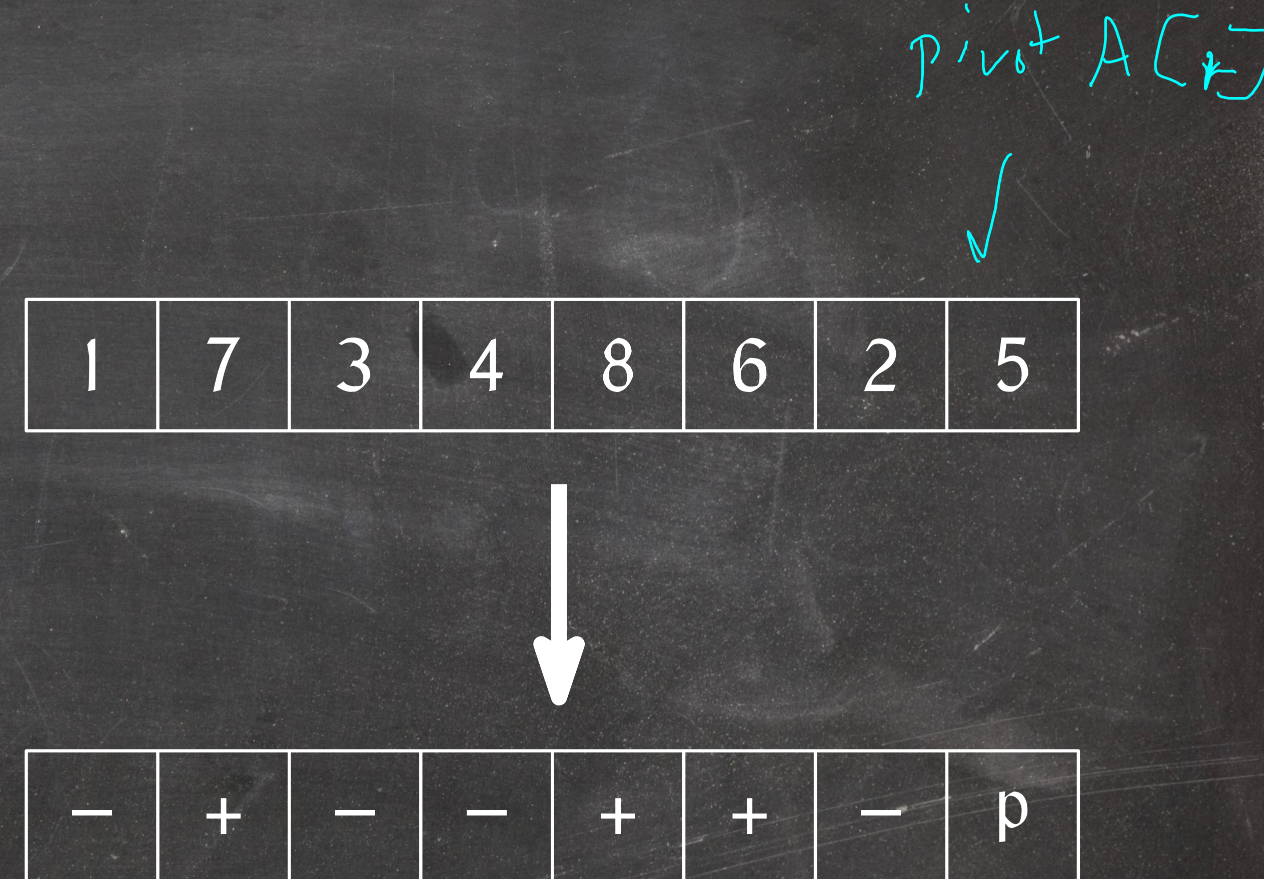

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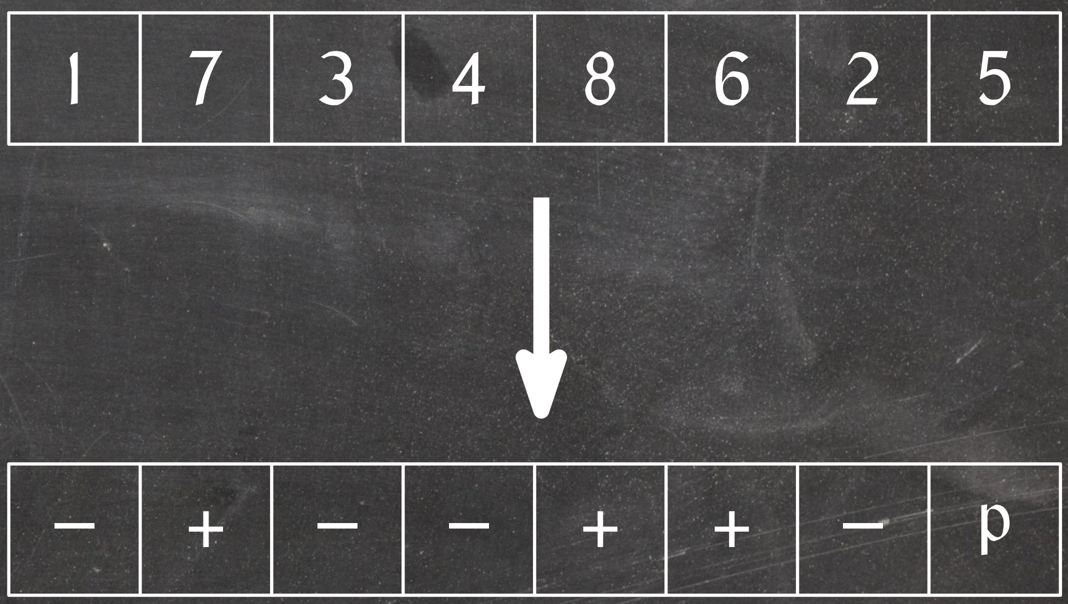

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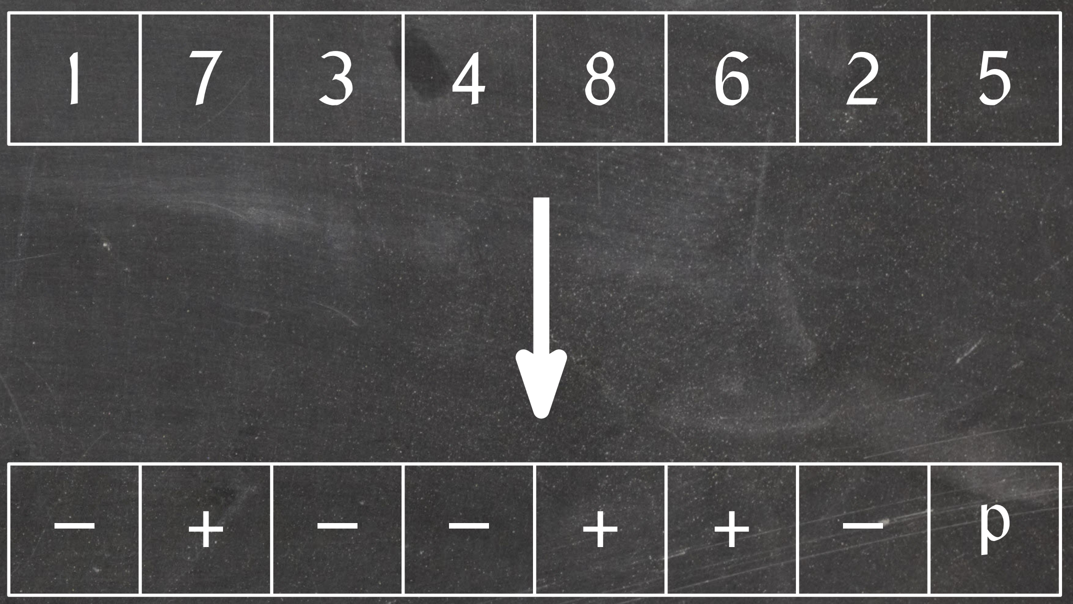

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−	+	−	−	+	+	−	p

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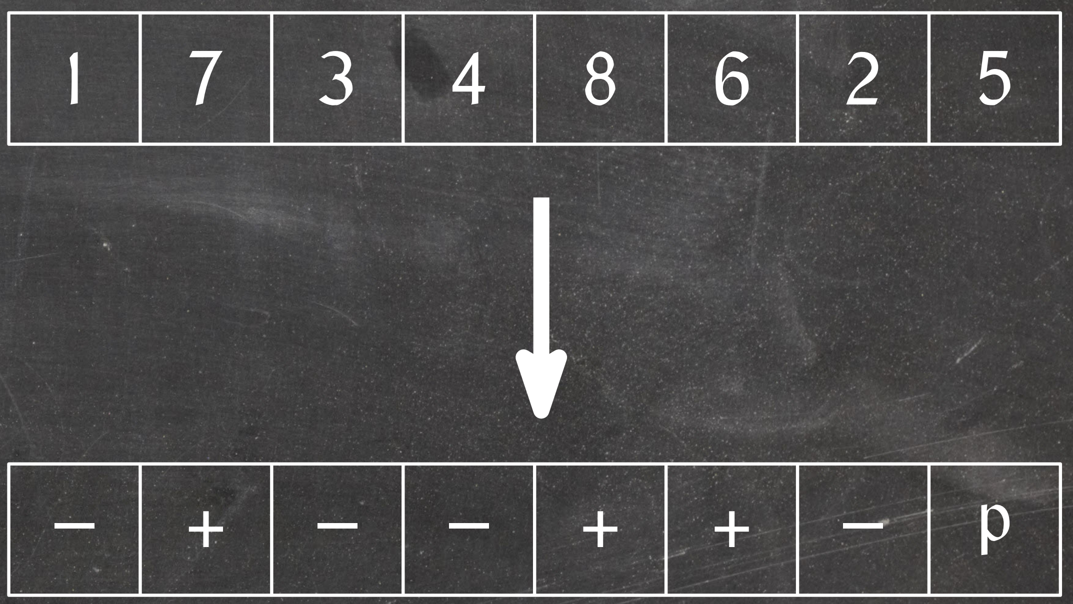

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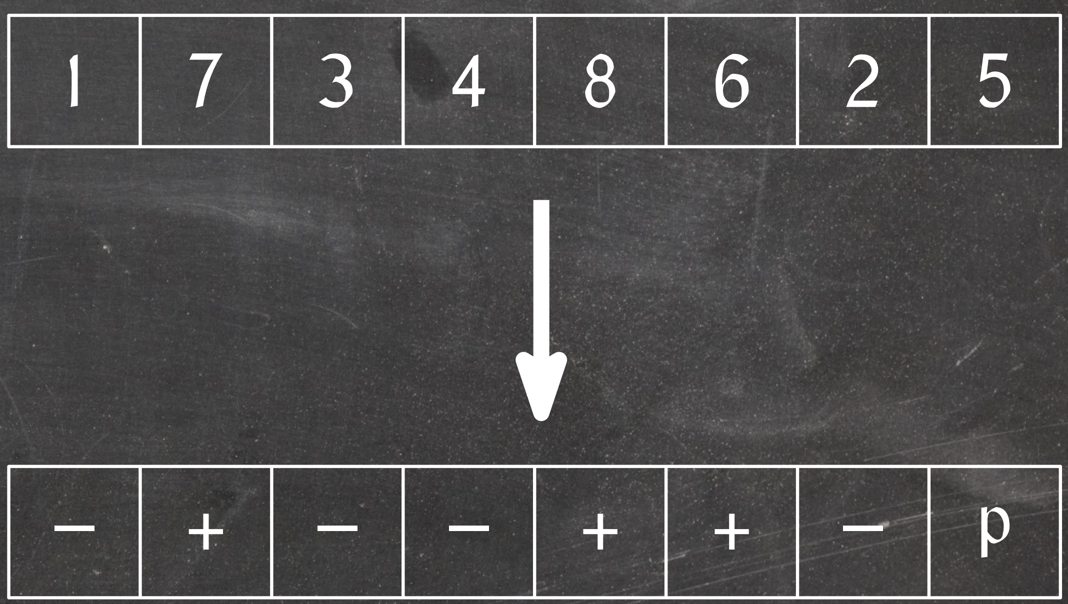
$\Rightarrow$ $A[\ell \,..\, m-1]$ and $A[m+1 \,..\, r]$ are uniform random permutations.



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Observation: The running time of SimpleQuickSort is in $O(n + C)$, where C is the number of comparisons it performs between input elements.

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$\Rightarrow$ It suffices to prove that $E[C] \in O(n \lg n)$.

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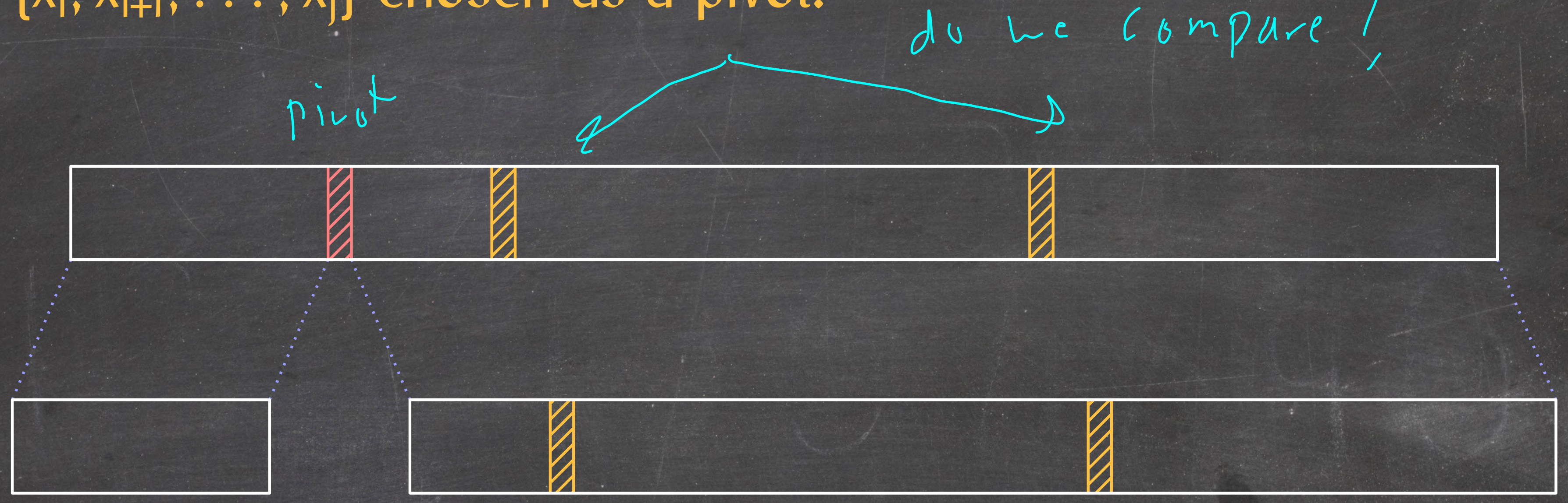

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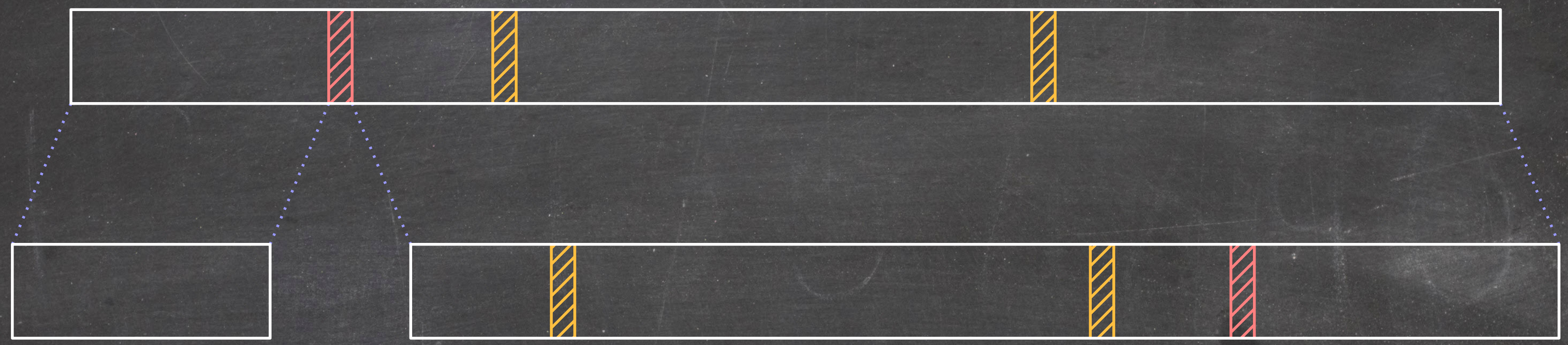

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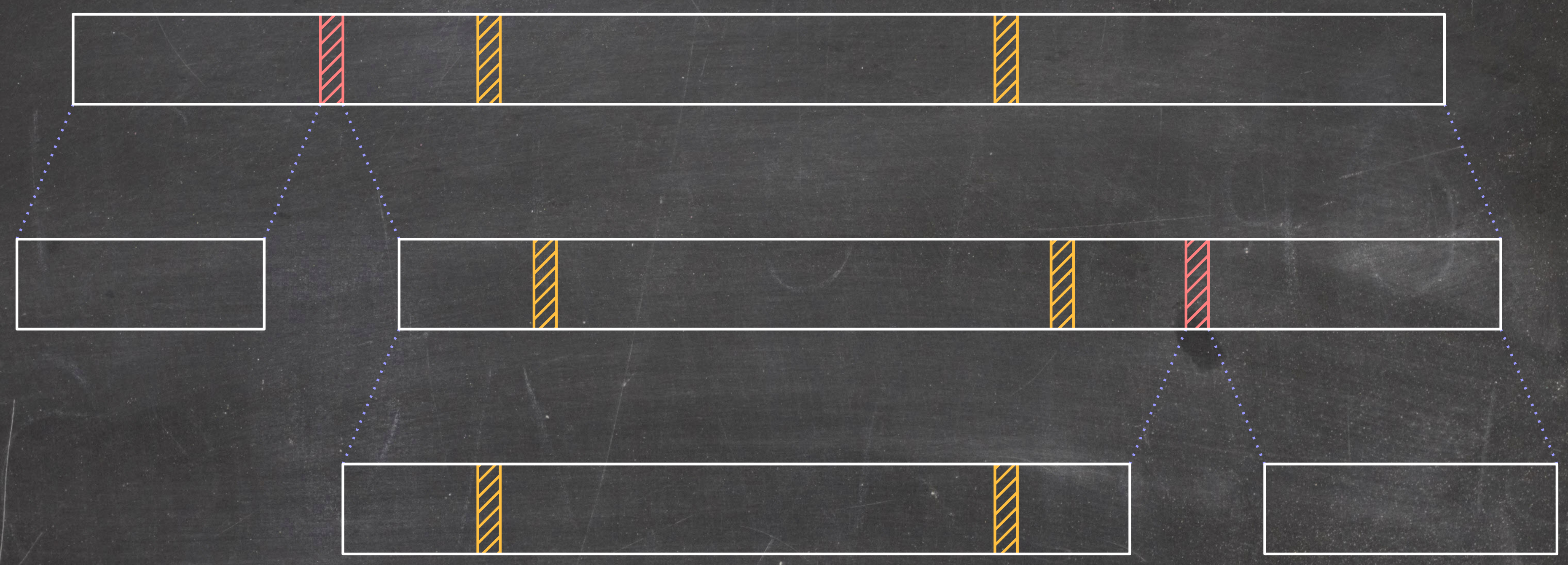

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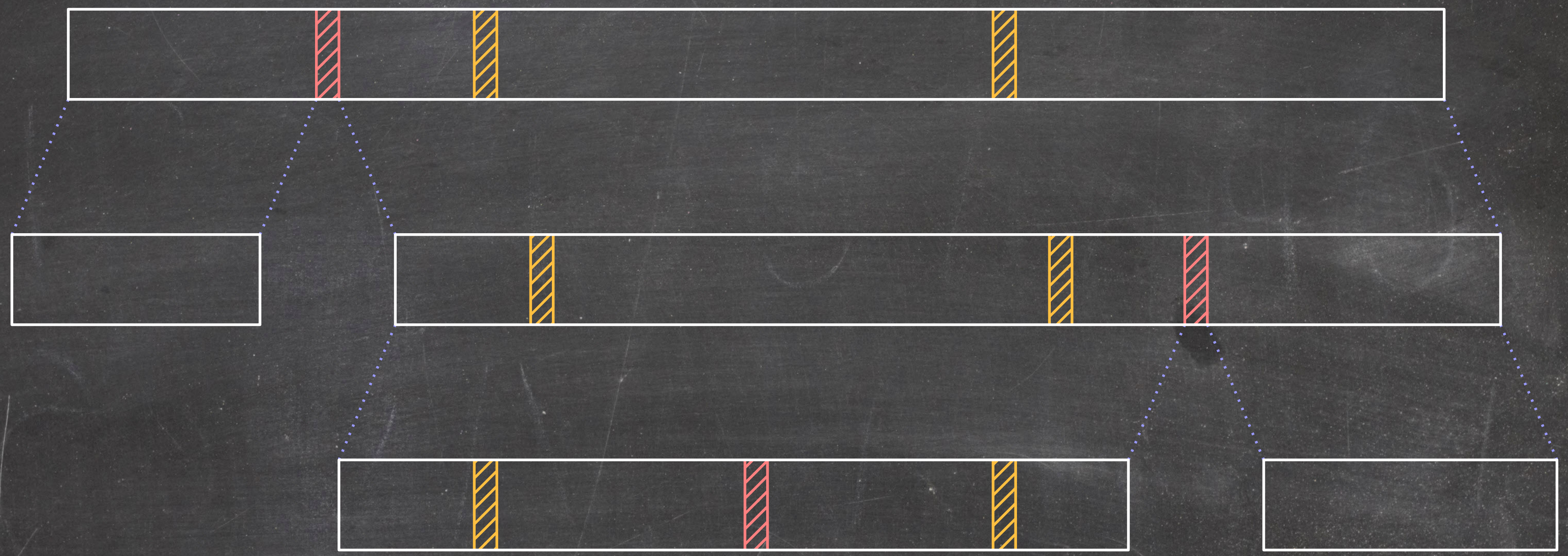

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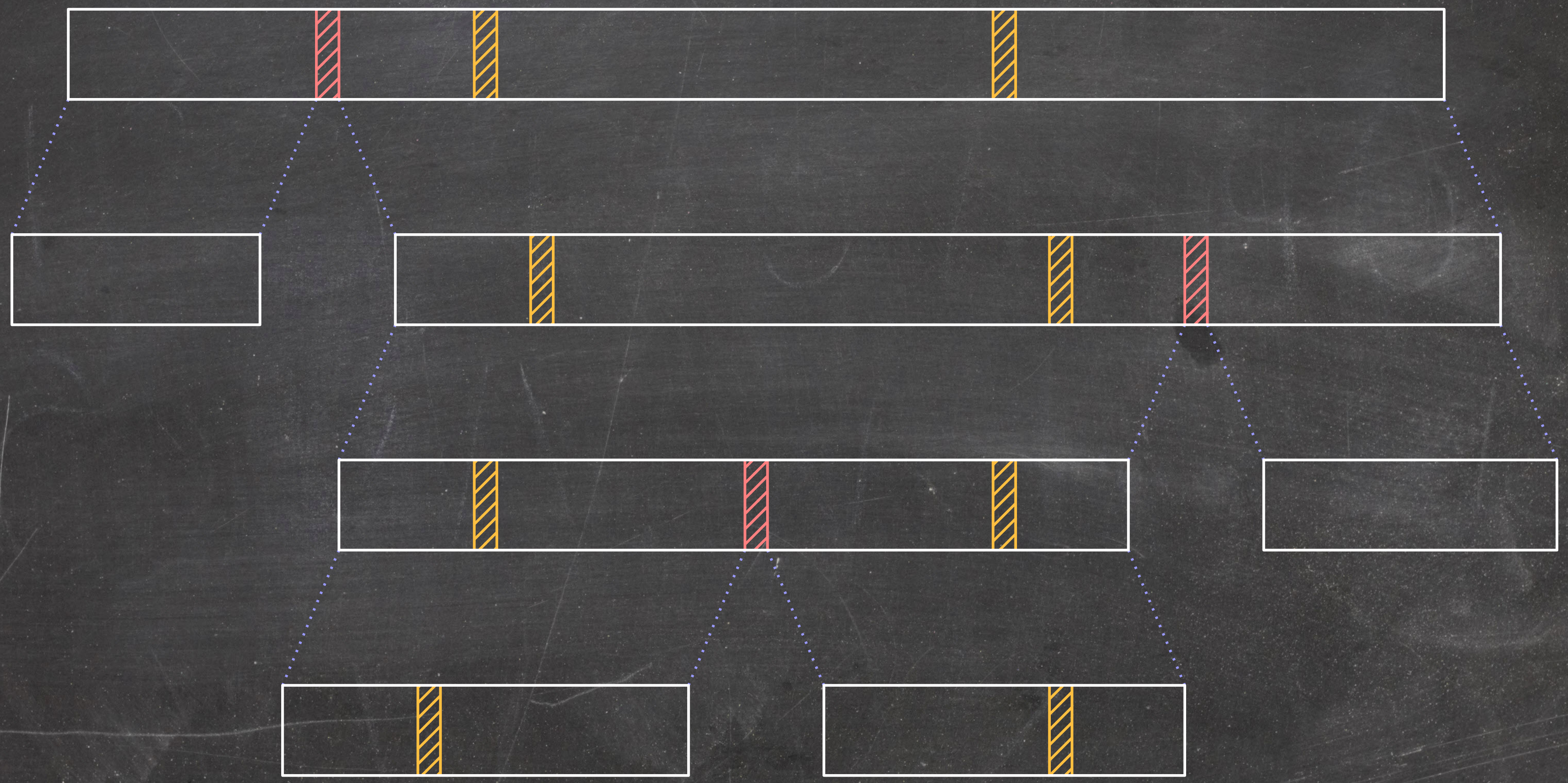

can't be compared

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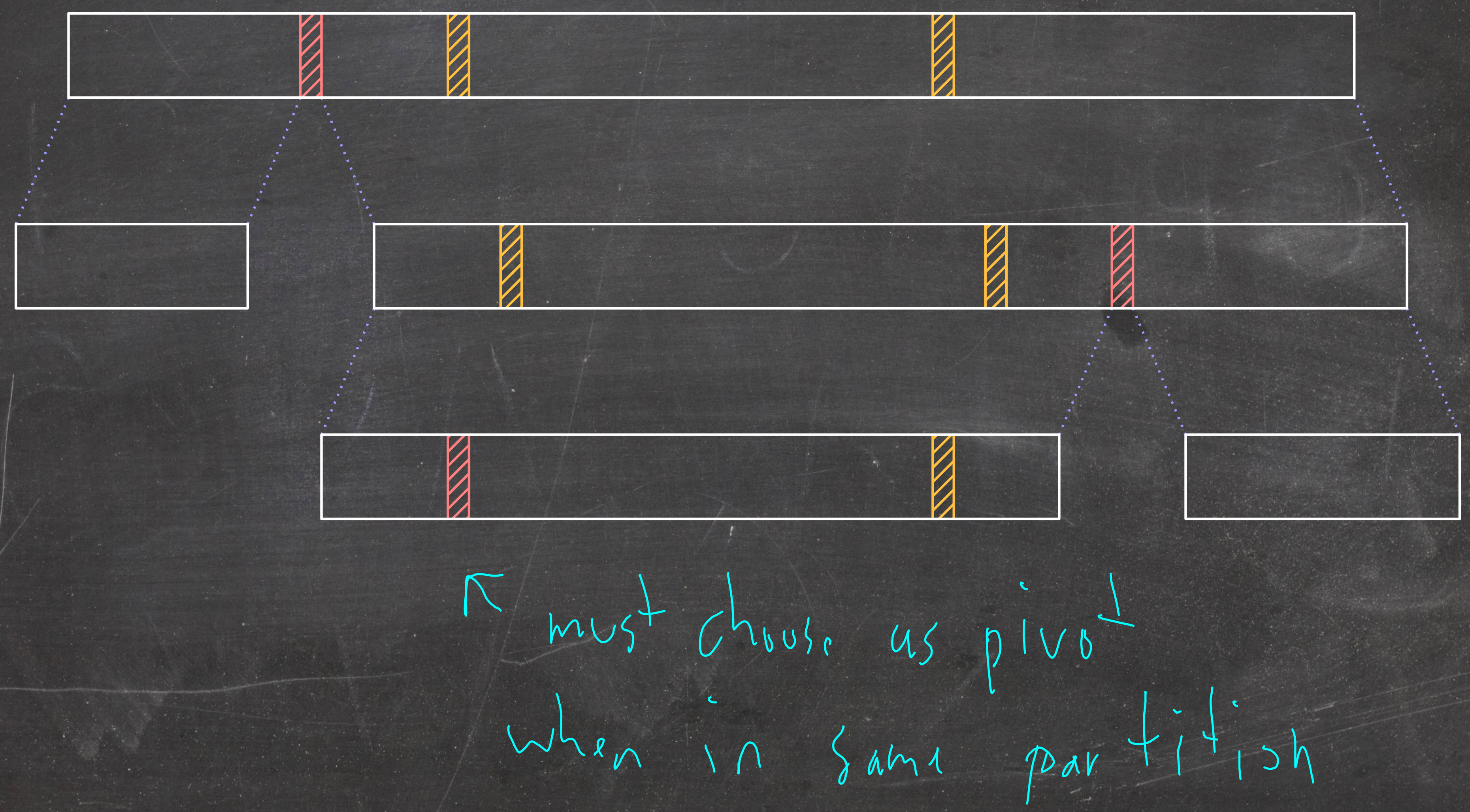

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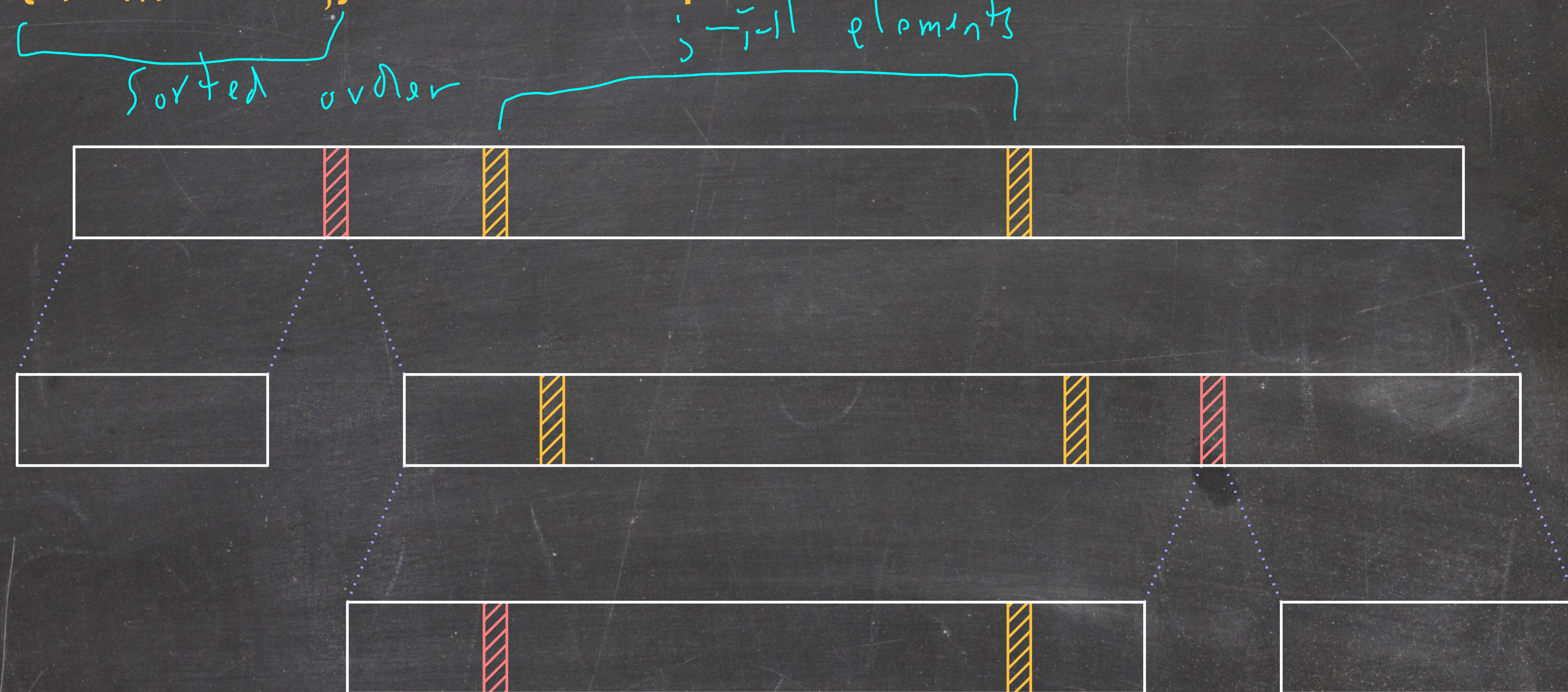

Corollary: $E[C_{ij}] = \dfrac{2}{j - i + 1}$.

Average-Case Analysis of Simple Quick Sort

$$E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}]$$

add up the expectations

Average-Case Analysis of Simple Quick Sort

$$E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

← value we proved

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$$E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

$j = i+1, i+2, \ldots n$

i here

$\frac{2}{j-i+1}$

$= \frac{2}{1+1}, \frac{2}{2+1}, \frac{2}{3+1}, \ldots \frac{2}{n-i+1}$

set to k

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$$= 2(n-1)H_n.$$

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \text{nth Harmonic Number}$$

Average-Case Analysis of Simple Quick Sort

$$\sum_{i=1}^{n} \frac{1}{i}$$

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$$\int_1^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i}$$

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$$\Rightarrow E[C] \leq 2(n-1)H_n \in O(n \lg n)$$

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e.g simple-quicksort

1, 2, ..., n

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Algorithms that are fast on average are often simpler and on average faster than worst-case efficient algorithms.

They are a good choice when we want good performance most of the time and possibly averaged over running the algorithm many times.

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This assumption may not be true in some applications, invalidating the performance prediction we obtain using average-case analysis!

Interpretation of Average-Case Analysis

What exactly is the meaning of the following statement?

"The average-case running time of algorithm A is $T(n)$."

"If every input is equally likely, then we expect to see a running time of $T(n)$ on average."

This assumption may not be true in some applications, invalidating the performance prediction we obtain using average-case analysis!

Example:

SimpleQuickSort takes $\Theta(n^2)$ time on almost sorted inputs.

There are applications where the inputs to be sorted are all almost sorted.

SimpleQuickSort is a poor choice of a sorting algorithm in such applications.

Randomization

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Since a randomized algorithm behaves differently every time it runs, there is no way to force it to exhibit its worst-case running time!

The expected running time of a randomized algorithm is an expectation over the random choices the algorithm makes.

$\Rightarrow$ No more assumptions about the probability distribution. We know the distribution of the choices the algorithm makes.

Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in $O(n \lg n)$.

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So why don't we just ensure the input is a uniform random permutation?

RandomPermutationQuickSort(A)

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RandomPermute(A)
SimpleQuickSort(A, 1, n)
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Corollary: The expected running time of RandomPermutationQuickSort is in O(n lg n).

Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

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If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.

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So why don't we make sure we choose a uniform random pivot, no matter the input permutation?

RandomPivotQuickSort(A, ℓ, r)

```
if r ≤ ℓ
    then return
p = RandomNumber(ℓ, r)
swap A[p] and A[r]
m = Partition(A, ℓ, r)
RandomPivotQuickSort(A, ℓ, m − 1)
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Lemma: The expected running time of RandomPivotQuickSort is in $O(n \lg n)$.

The analysis is 100% identical to that of SimpleQuickSort!

Uniform Random Permutation In Linear Time

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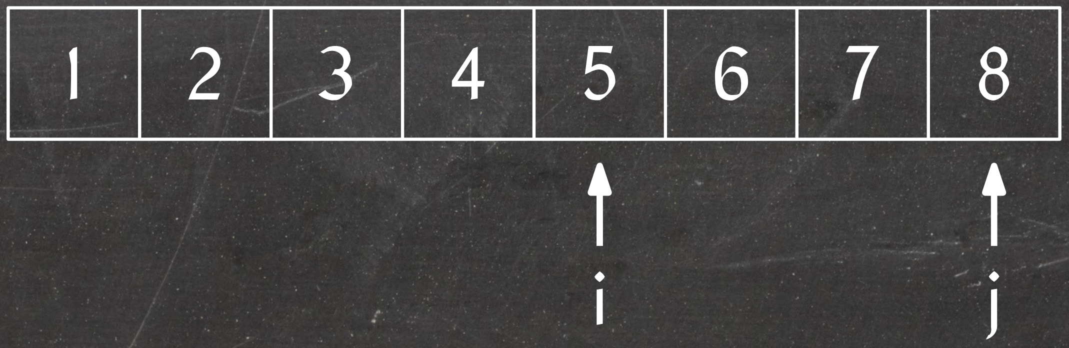

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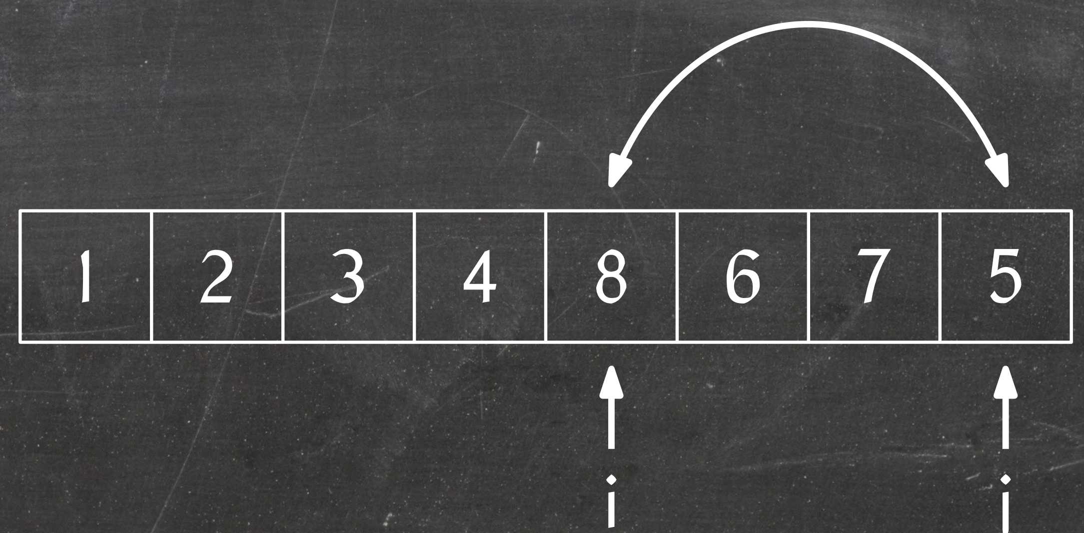

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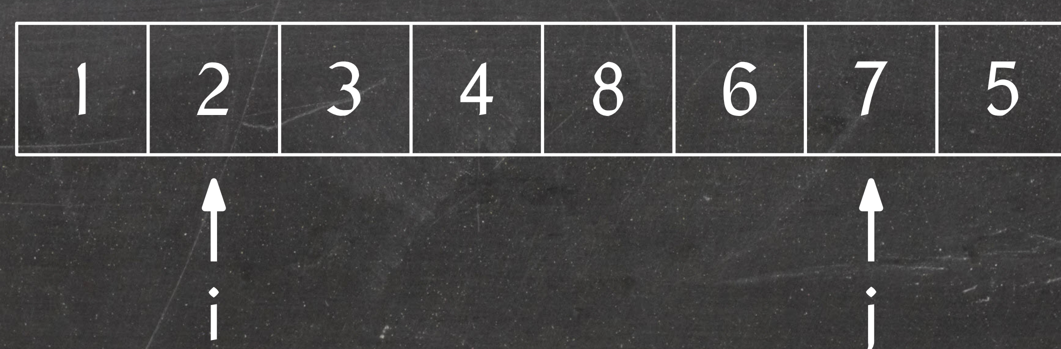

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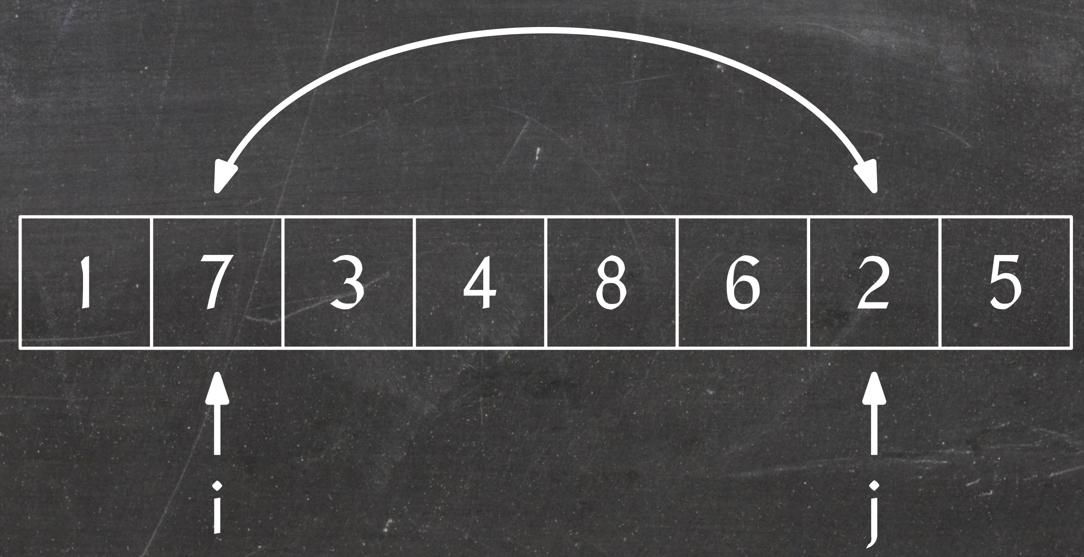

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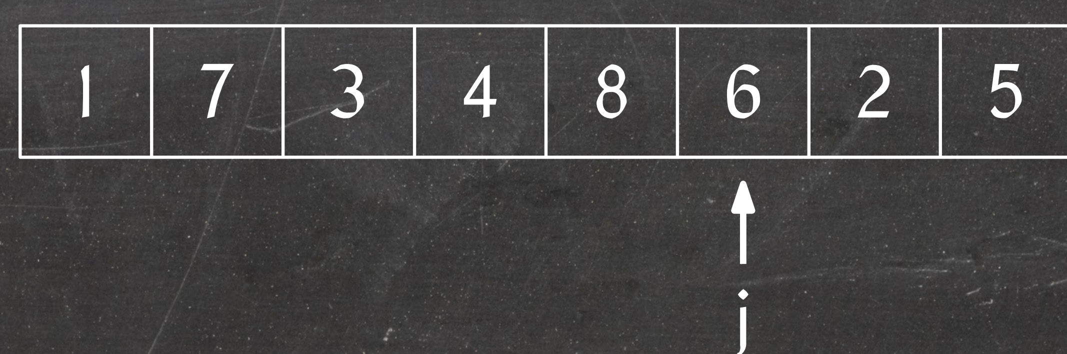

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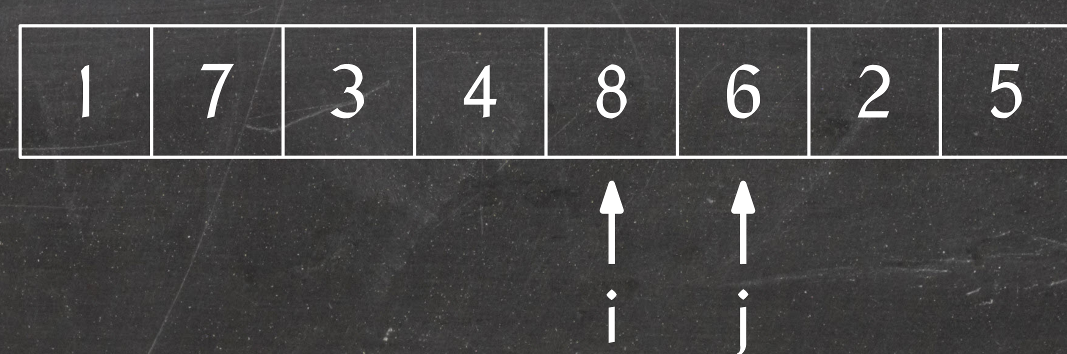

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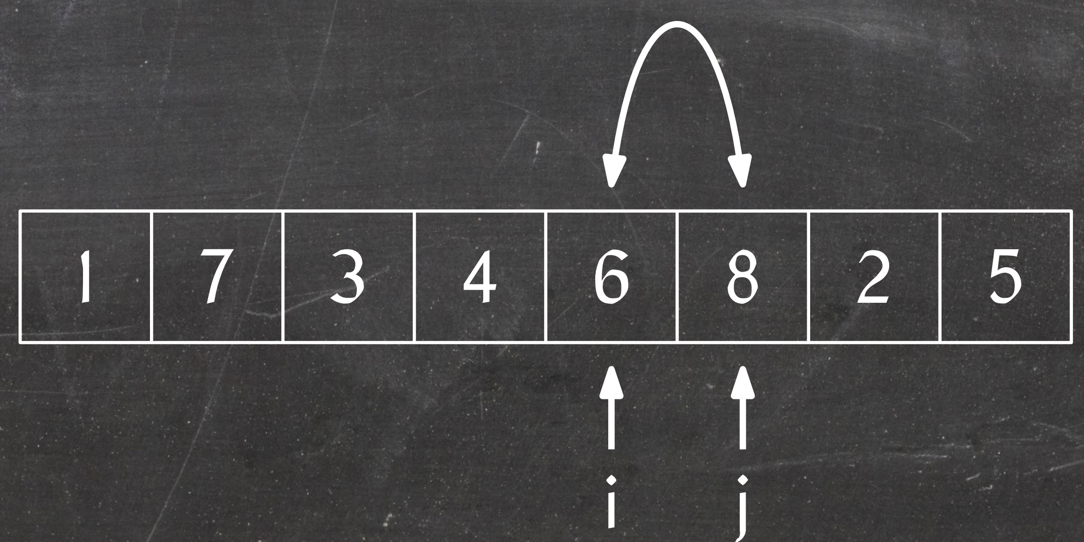

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Induction on n.

If $n = 1$, then it produces the only possible permutation with probability $1 = \frac{1}{1!}$.

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If $n > 1$, then to produce the permutation $\langle x_1, x_2, \ldots, x_n \rangle$ (event E), we need to

- Place x_n into $A[n]$ (event E_1) and
- Place $x_1, x_2, \ldots, x_{n-1}$ into $A[1 \,..\, n-1]$ (event E_2).

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So $P[E] = P[E_1 \cap E_2] = P[E_1] \cdot P[E_2 | E_1] = \frac{1}{n} \cdot \frac{1}{(n-1)!} = \frac{1}{n!}$.

Randomized Selection

recall select(k)
= find kth element from
unsorted array

RandomizedSelection(A, ℓ, r, k)

```
if r ≤ ℓ
    then return A[ℓ]
p = RandomNumber(ℓ, r)
swap A[p] and A[r]
m = Partition(A, ℓ, r)
if m − ℓ = k − 1
    then return A[m]
    else if m − ℓ ≥ k
            then RandomizedSelection(A, ℓ, m − 1, k)
            else RandomizedSelection(A, m + 1, r, k − (m + 1 − ℓ))
```

ideally want median

Randomized Selection

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```
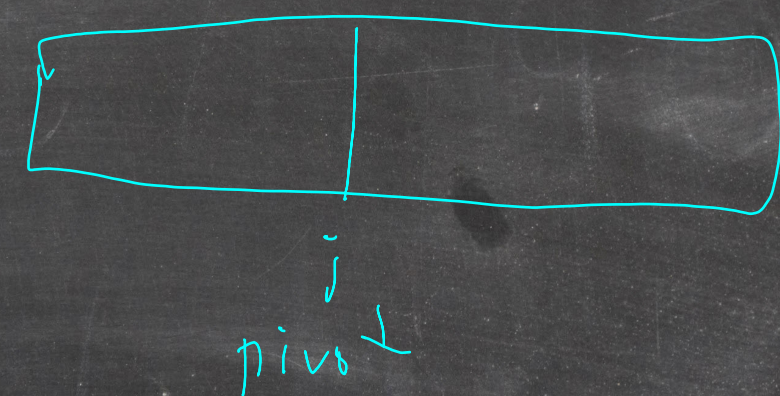

Lemma: The expected running time of RandomizedSelection is in $O(n)$.

Randomized Selection

Observation: If we choose the ith smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(\max(n-i, i-1))].$$



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Base case: $1 \leq n < 4$.

$T(n) \leq c \leq cn$.

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Inductive step: $n \geq 4$.

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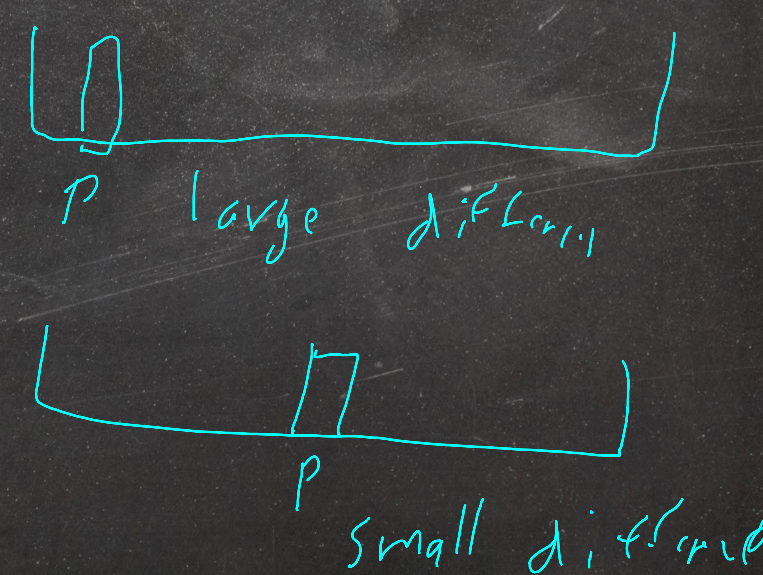

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\end{aligned}
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&\leq cn \quad \forall c \geq 8a.
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Sorting in Linear Time?

Using comparisons only, as Insertion Sort, Merge Sort, Quick Sort do, it is impossible to sort faster than in $\Omega(n \lg n)$ time.

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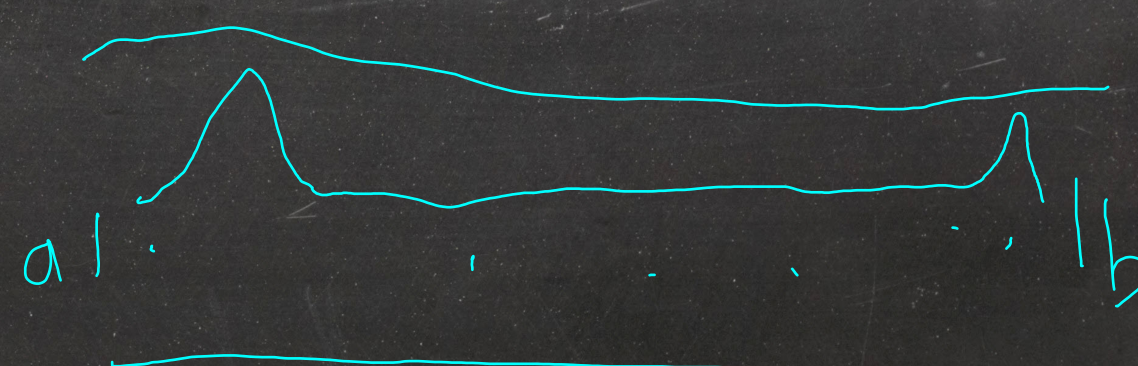
Bucket sort: Sorts n real numbers drawn uniformly at random from an interval $[a, b)$ in expected linear time.



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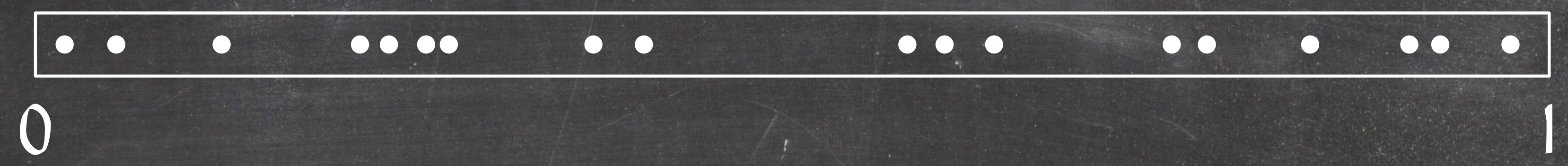

Bucket Sort

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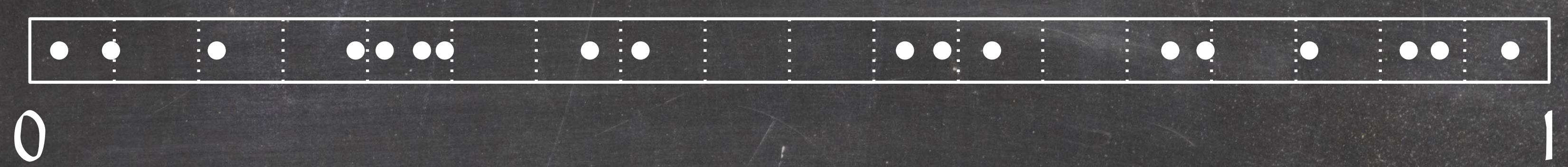
Divide [0, 1) into subintervals of length $\frac{1}{n}$.



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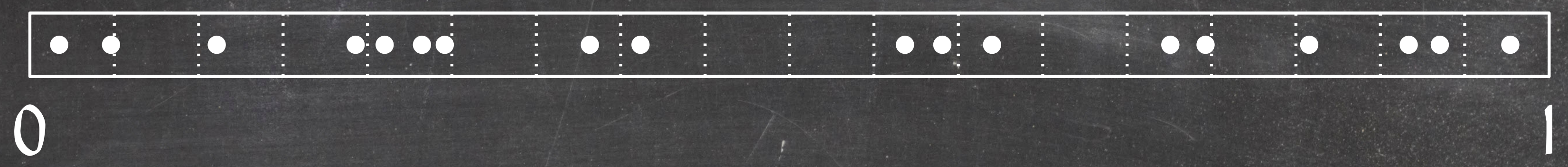

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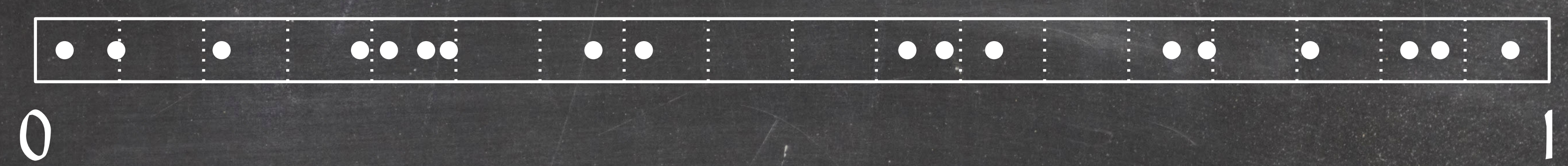

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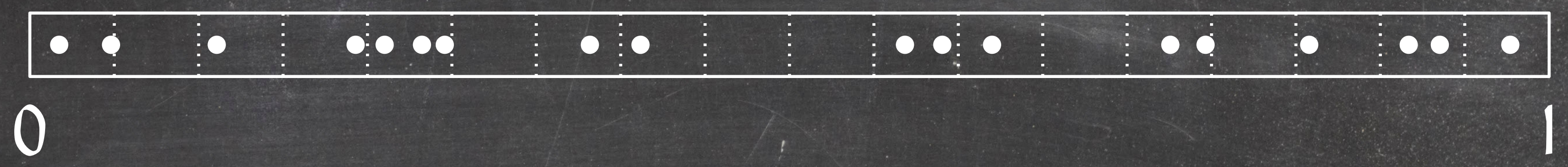

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$\Rightarrow$ Strategy:

- Bucket items according to the subinterval they belong to.
- Sort each bucket, hopefully in constant time.
- Concatenate the sorted buckets.

Bucket Sort

preprocess to range 0,1 before

BucketSort(A)

```
n = |A|
B = an array of n empty singly-linked lists
for i = 1 to n
    do prepend A[i] to list B[1 + ⌊n · A[i]⌋]
for i = 1 to n
    do InsertionSort(B[i])
j = 0
for i = 1 to n
    do for every element x ∈ B[i]
            do A[j] = x
               j = j + 1
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1-indexing

Bucket Sort

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This is where we depart from using comparisons only!

Bucket Sort

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B = an array of n empty singly-linked lists
for i = 1 to n
    do prepend A[i] to list B[1 + ⌊n · A[i]⌋]
for i = 1 to n
    do InsertionSort(B[i])
j = 0
for i = 1 to n
    do for every element x ∈ B[i]
            do A[j] = x
               j = j + 1
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This is where we depart from using comparisons only!

Worst-case running time: $O(n^2)$

Bucket Sort

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This is where we depart from using comparisons only!

Why not Merge Sort?

It only helps in the worst case.

It's more complicated.

It actually hurts when buckets are small, which is what we expect.

Worst-case running time: $O(n^2)$

Bucket Sort

Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right)$

n_i = the number of elements in B[i]

Bucket Sort

Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right)$ ← size of buckets

n_i = the number of elements in B[i]

expected size of buckets

$$E[T(n)] \in O\left(n + \sum_{i=1}^{n} E[n_i]^2\right)$$

Bucket Sort

Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2\right)$

n_i = the number of elements in B[i]

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Corollary: $E[T(n)] \in O(n)$.

$\leq c_1 n + c_2 2 n$

$\leq c_3 (n + 2n)$

$\leq c_3 5n = O(n)$

Bucket Sort

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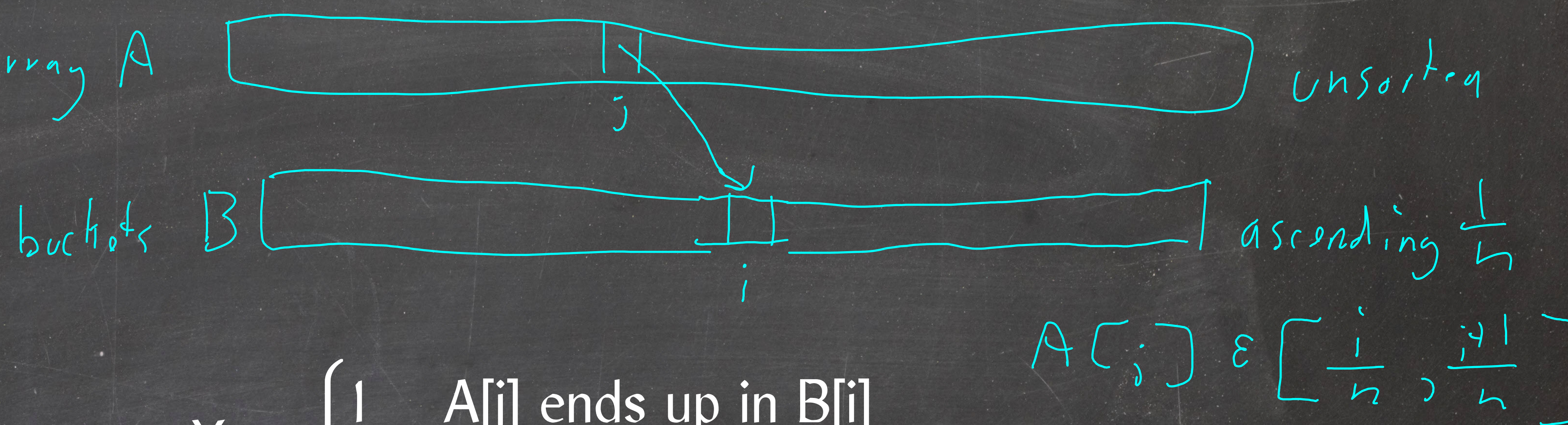
$$X_j = \begin{cases} 1 & \text{A[j] ends up in B[i]} \\ 0 & \text{otherwise} \end{cases}$$

Bucket Sort



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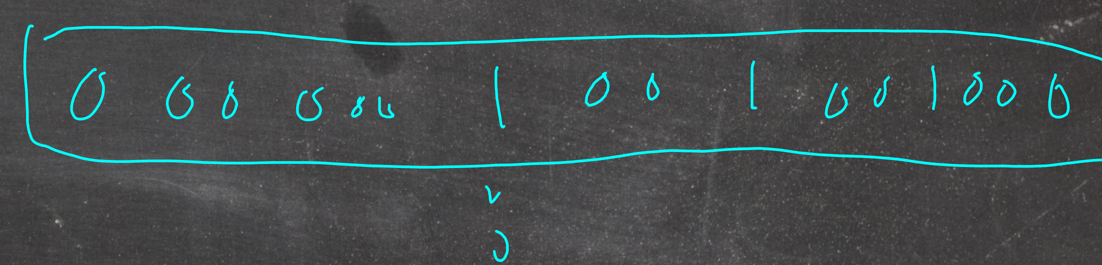
just for bucket i



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$$= \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n}\sum_{\substack{k=1 \\ k\neq j}}^{n} E[X_j]E[X_k]$$

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$$(x_1 + x_2 + x_3 \ldots x_n)^2$$

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↑ more expectations

$$= \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n}\sum_{\substack{k=1 \\ k \neq j}}^{n} E[X_j]E[X_k]$$

$x_1 x_1$ $x_1 x_2$

X_j and X_j are clearly not independent. X_j and X_k are independent.

Bucket Sort

$$E[X_j] = \frac{1}{n} \cdot 1 + \left(1 - \frac{1}{n}\right) \cdot 0 = \frac{1}{n}$$

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$$E[n_i^2] = \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} E[X_j]E[X_k] = n \cdot \frac{1}{n} + \frac{n(n-1)}{n^2} < 2$$

same

different

$1 + <1$

<2

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Bucket Sort relies on the random distribution of the input values.

We can't simply change them without changing the algorithm's output.

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Motwani/Raghavan.
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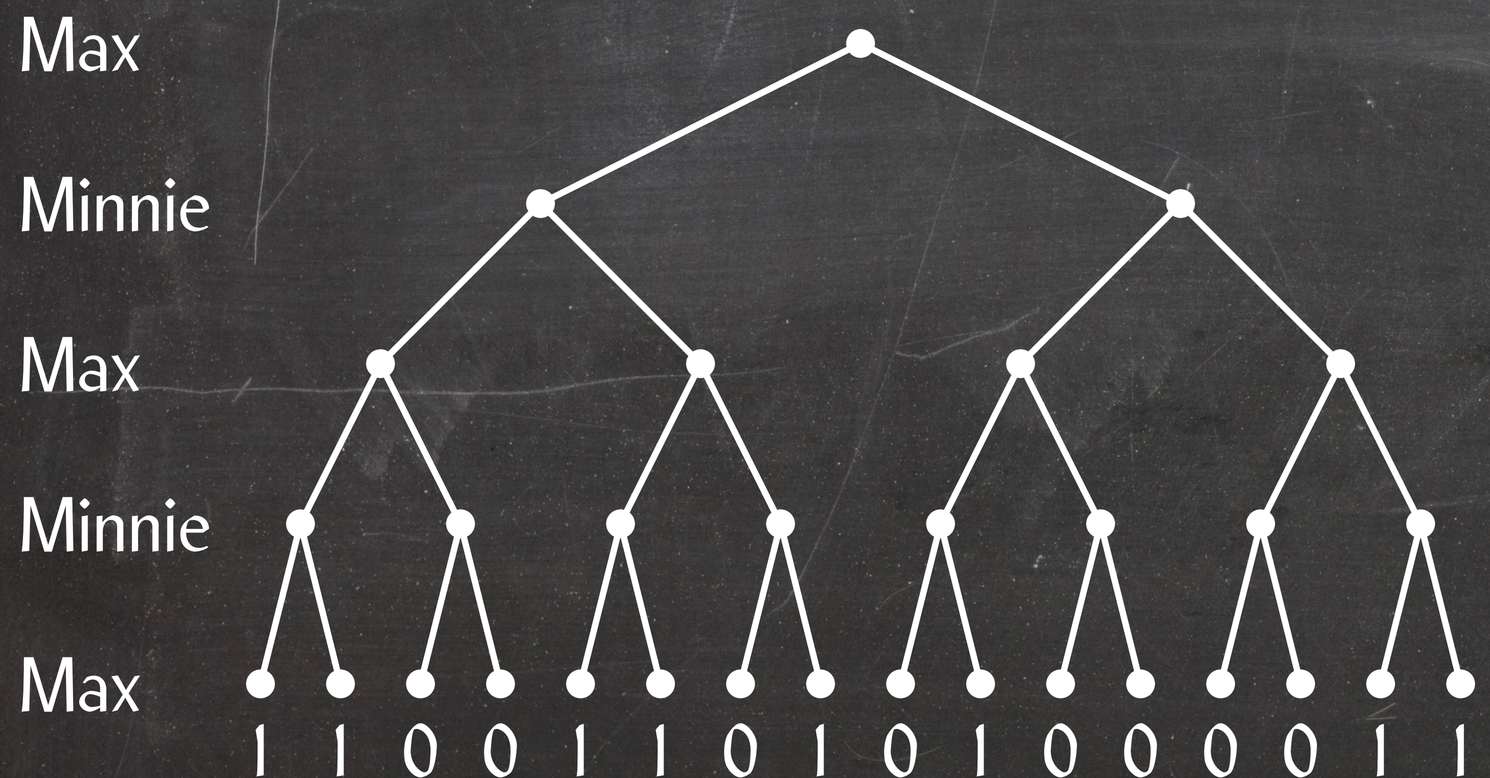

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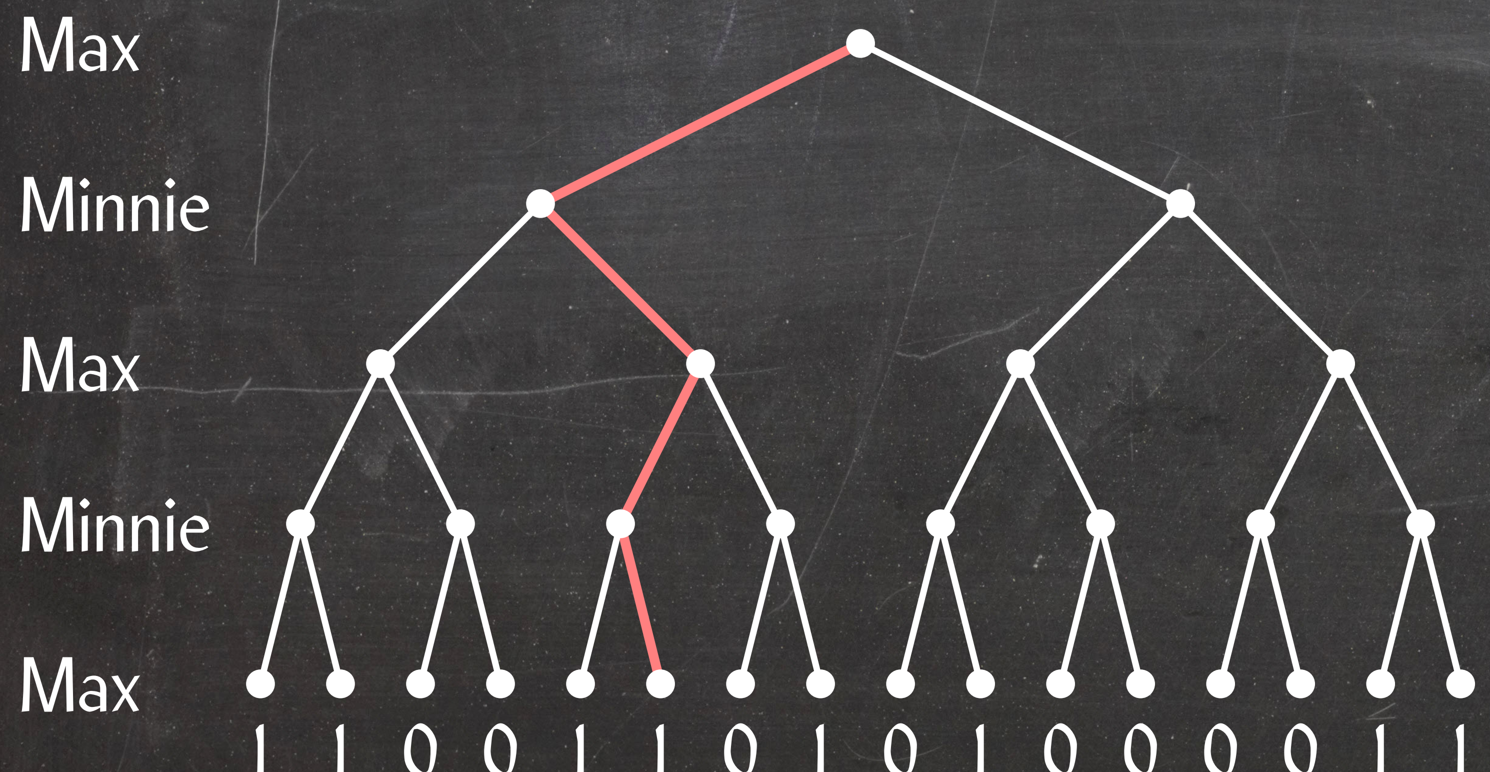

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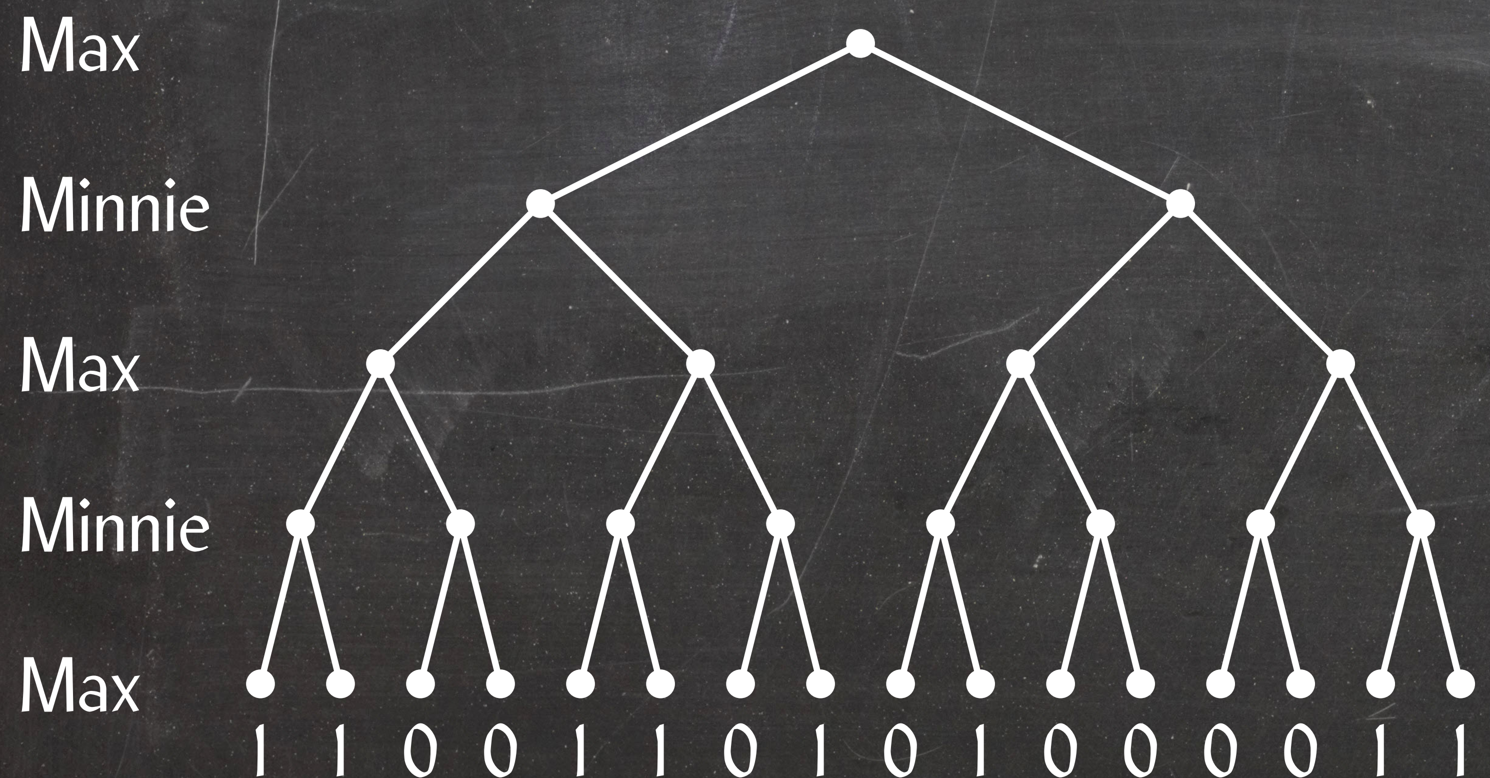

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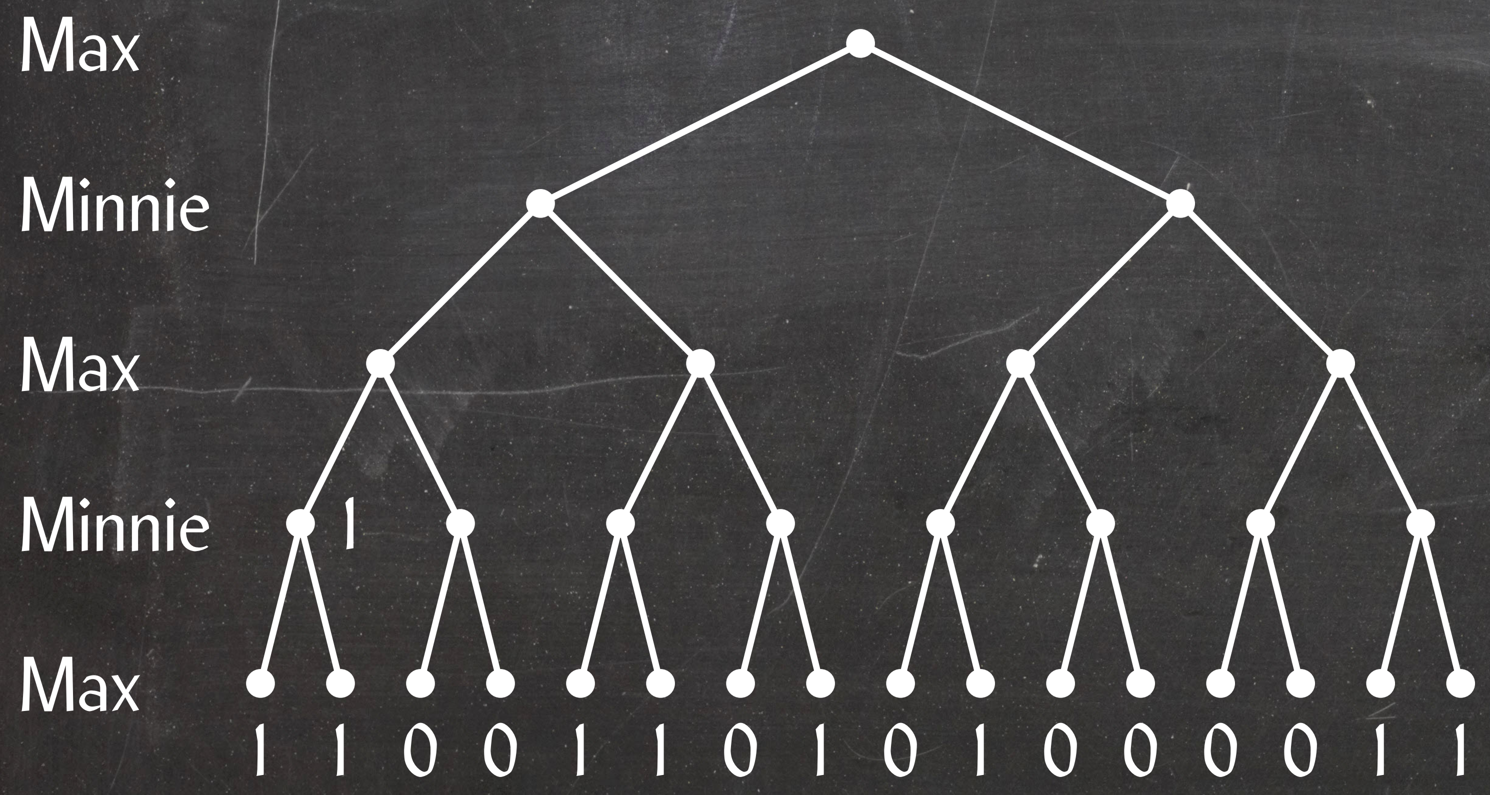

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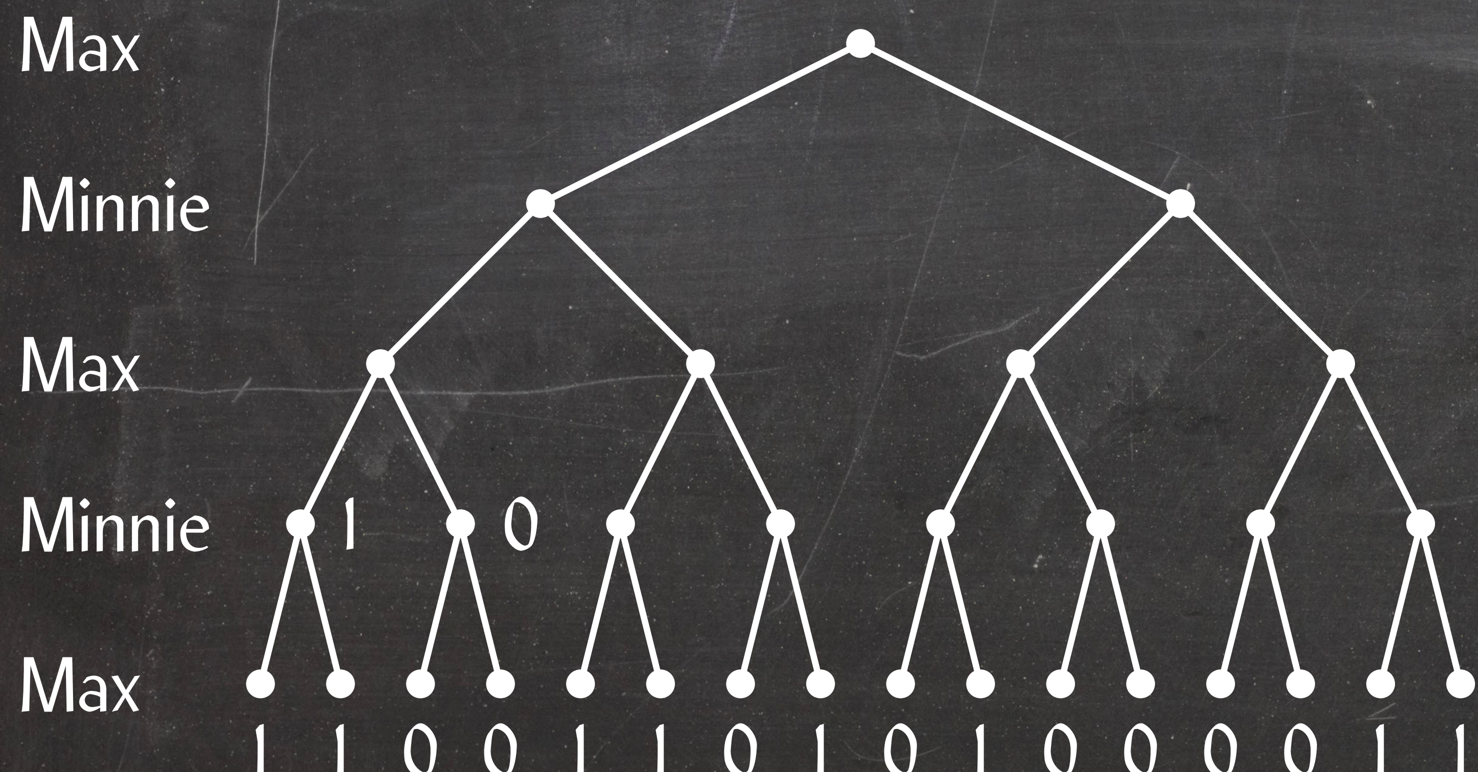

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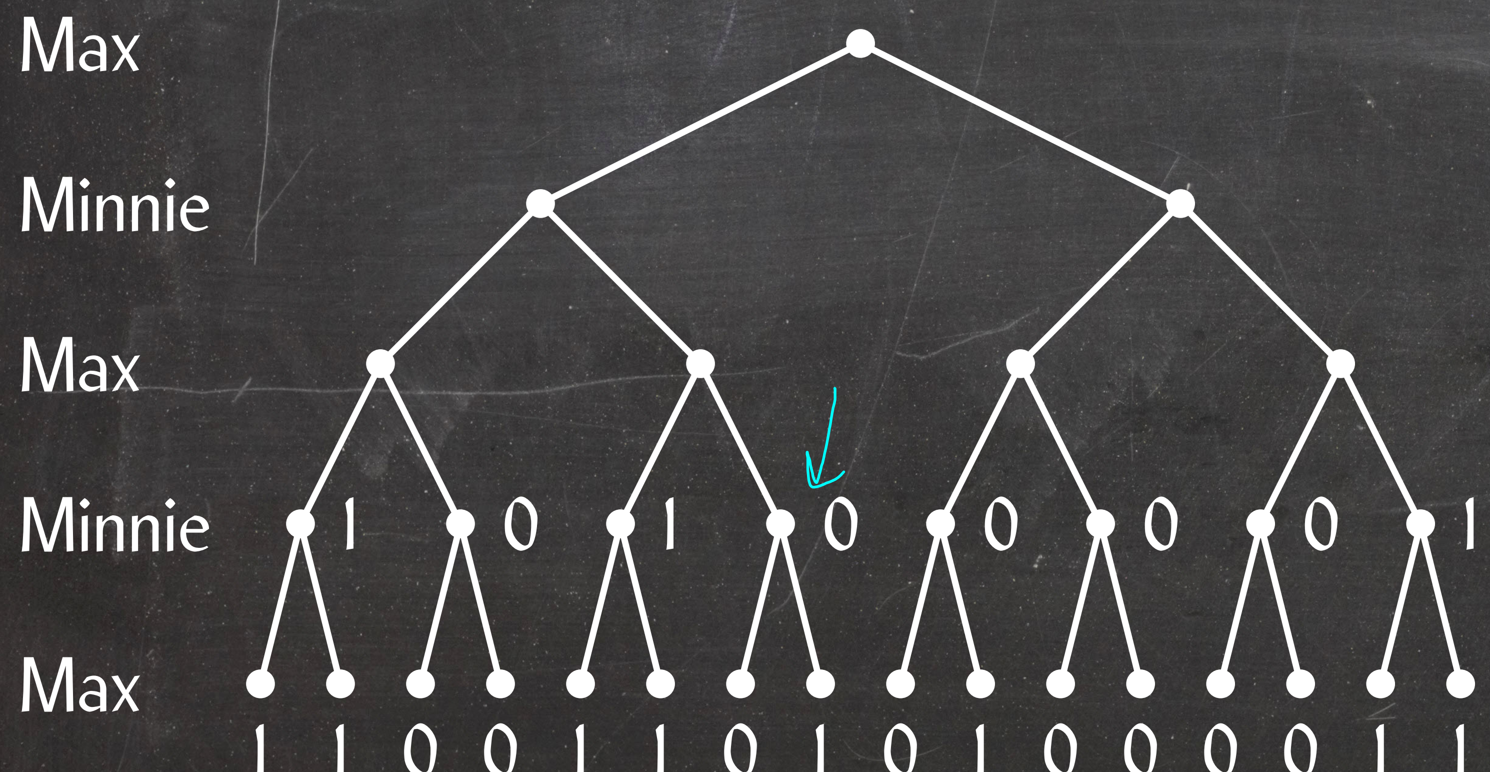

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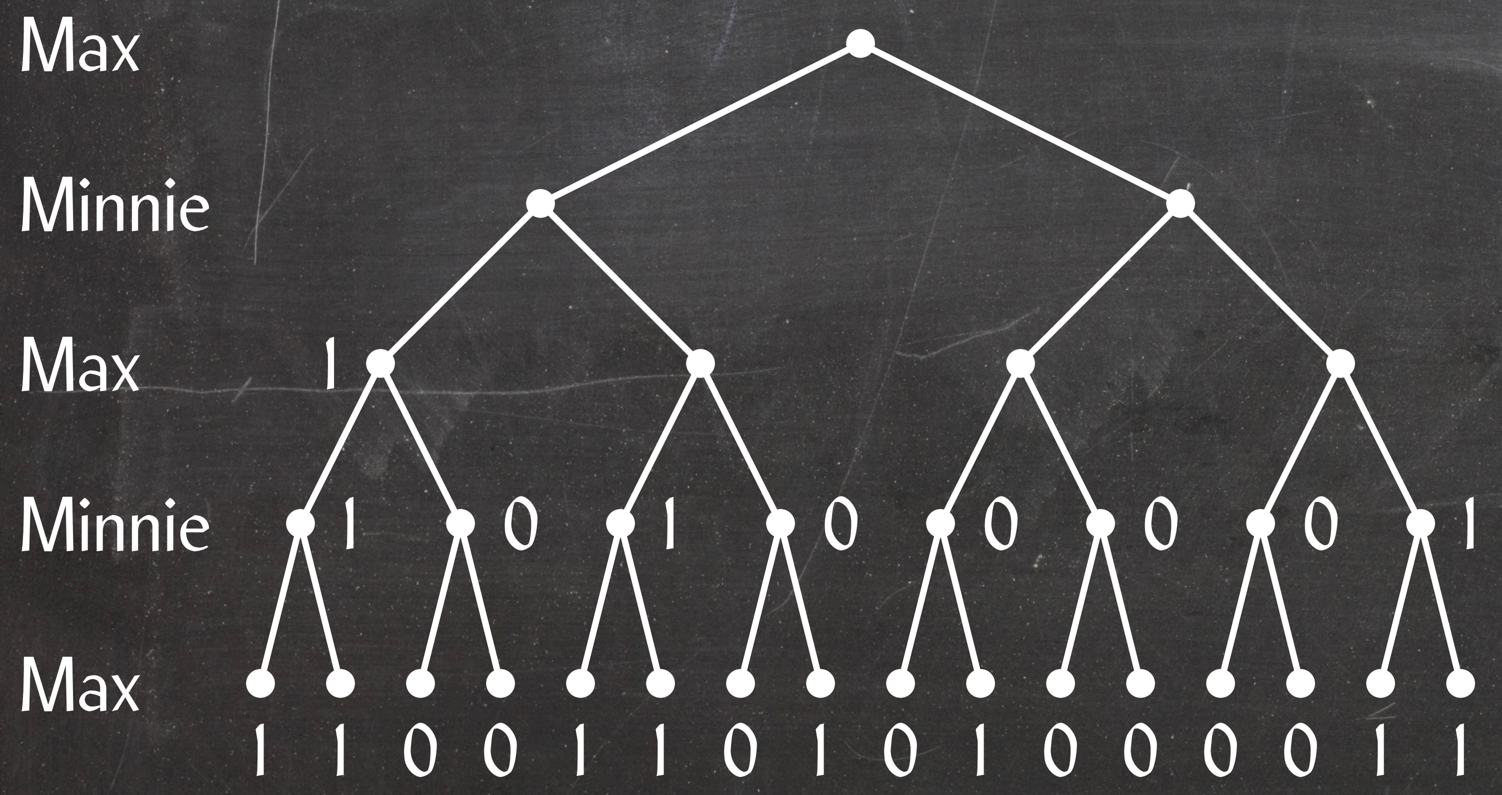

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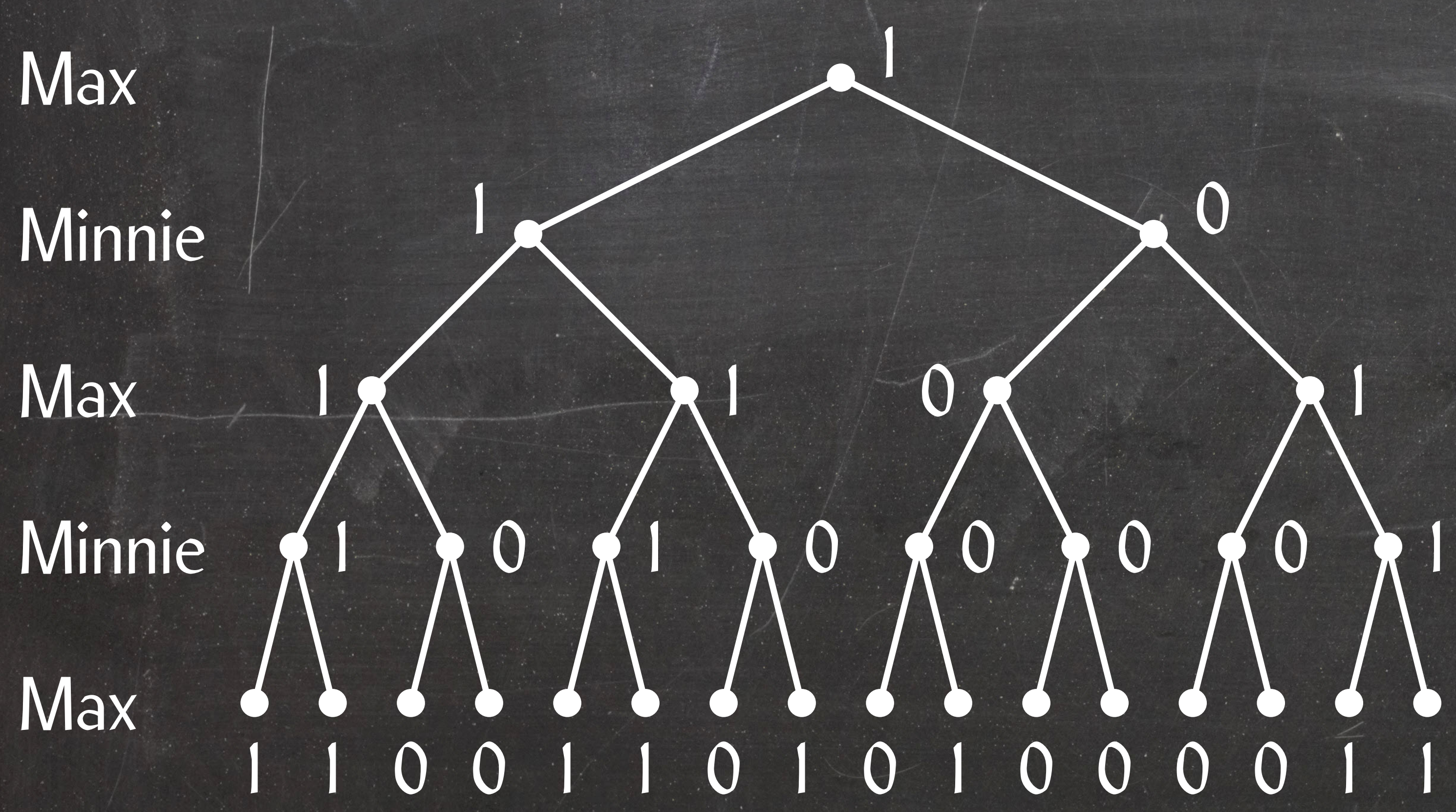

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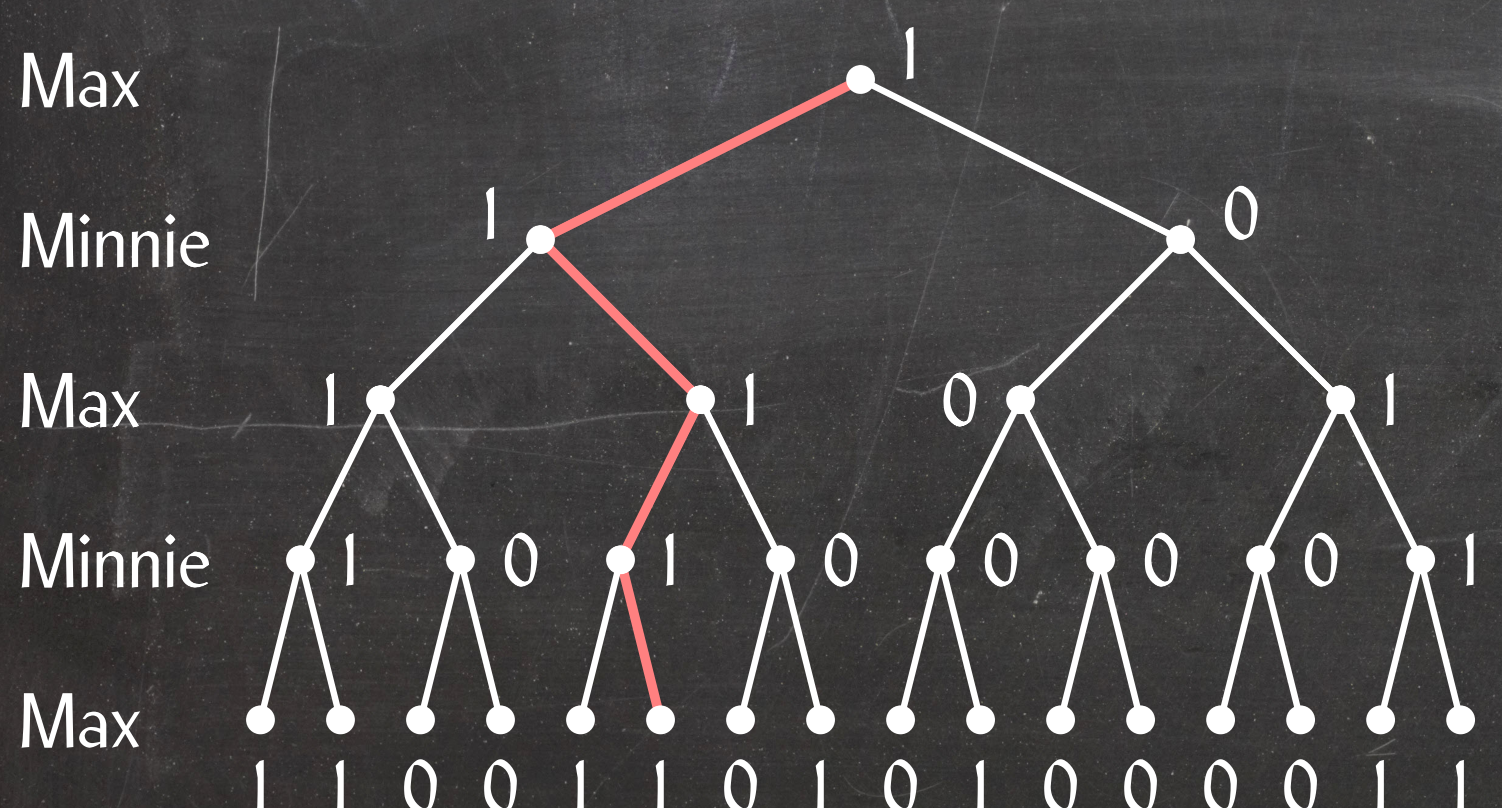

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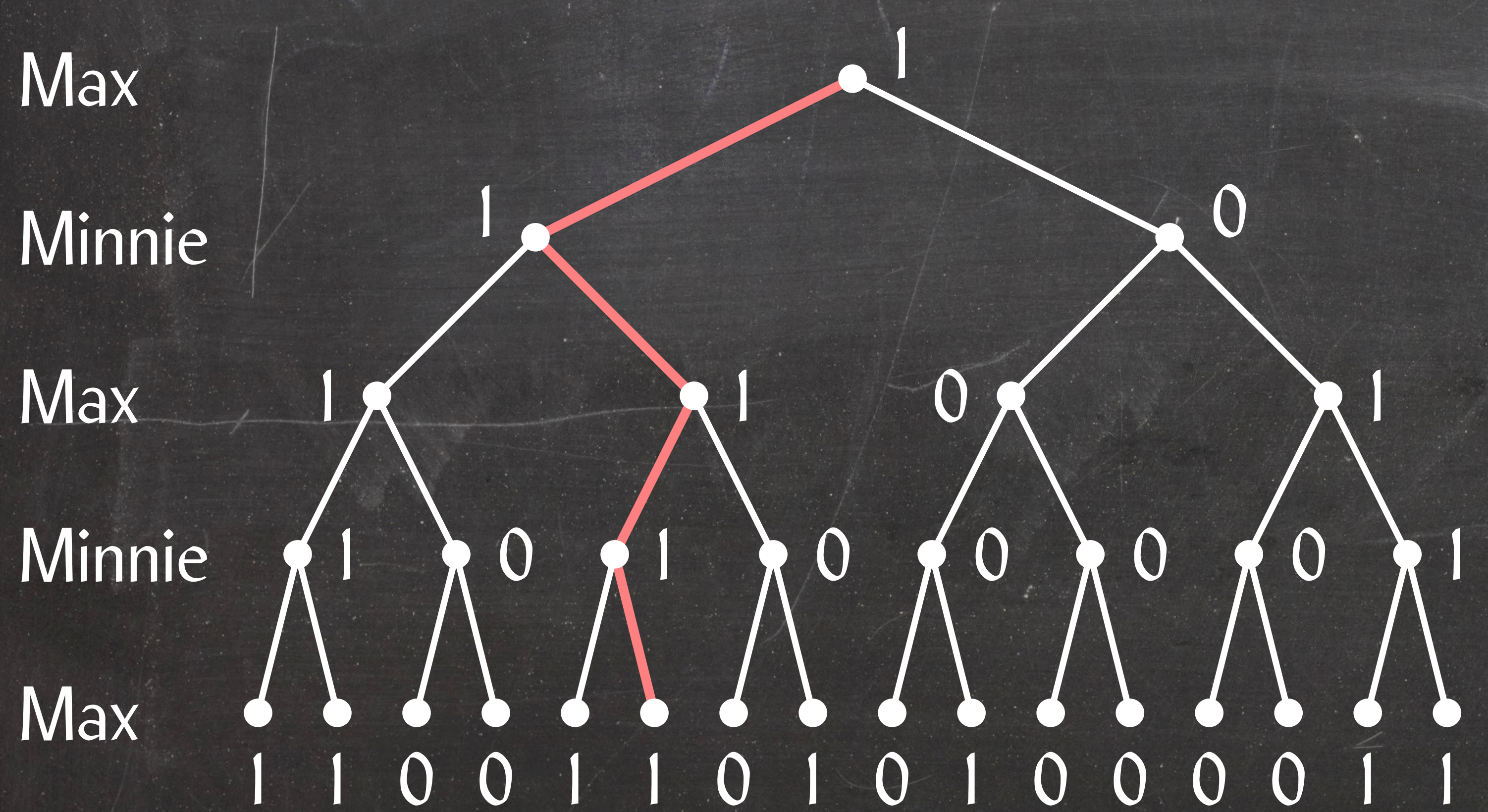

Max-node:

$$\text{label}(v) = \max_{\text{child } w} \text{label}(w)$$

Minnie-node:

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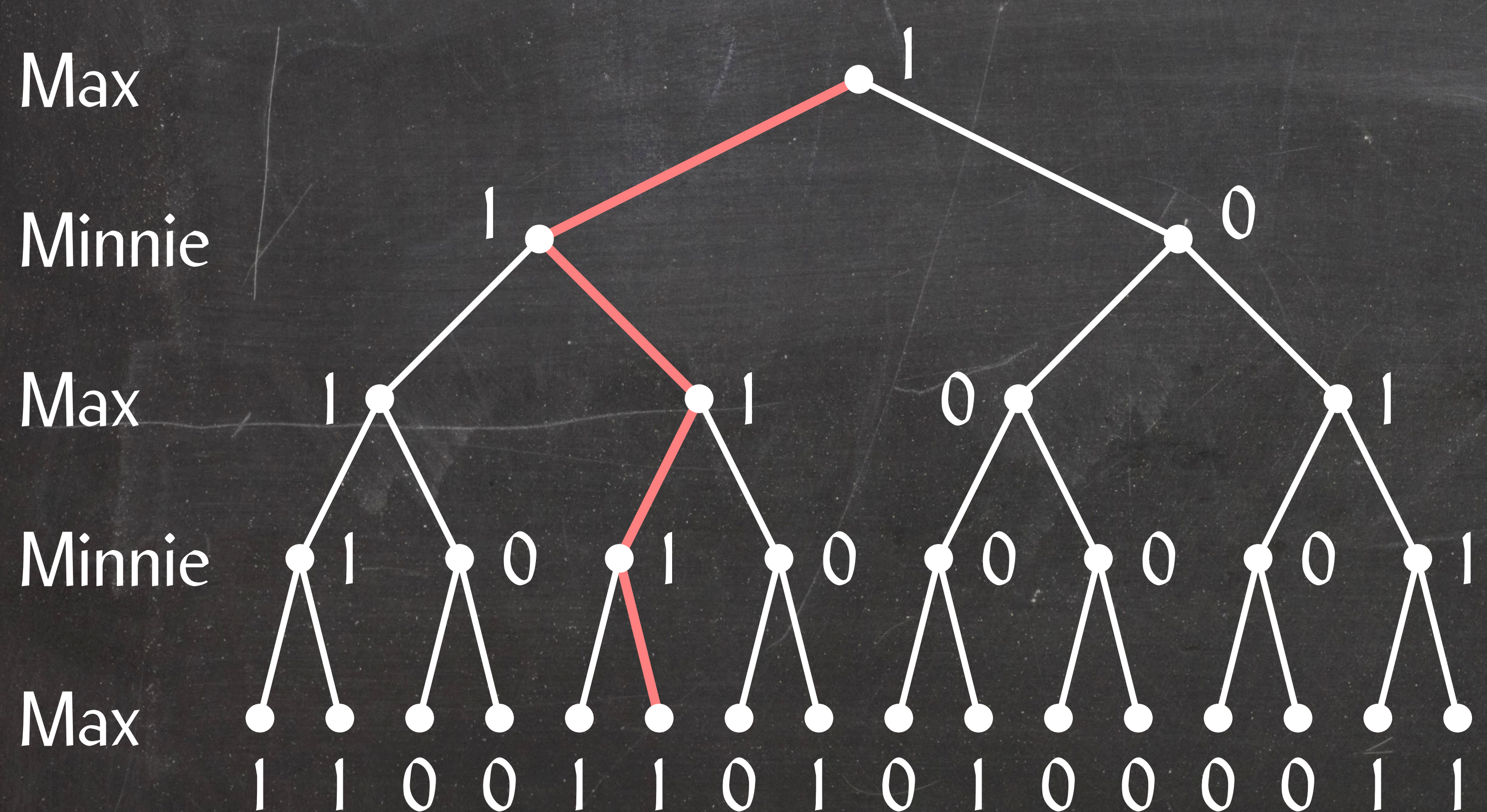

Max-node:

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Minnie-node:

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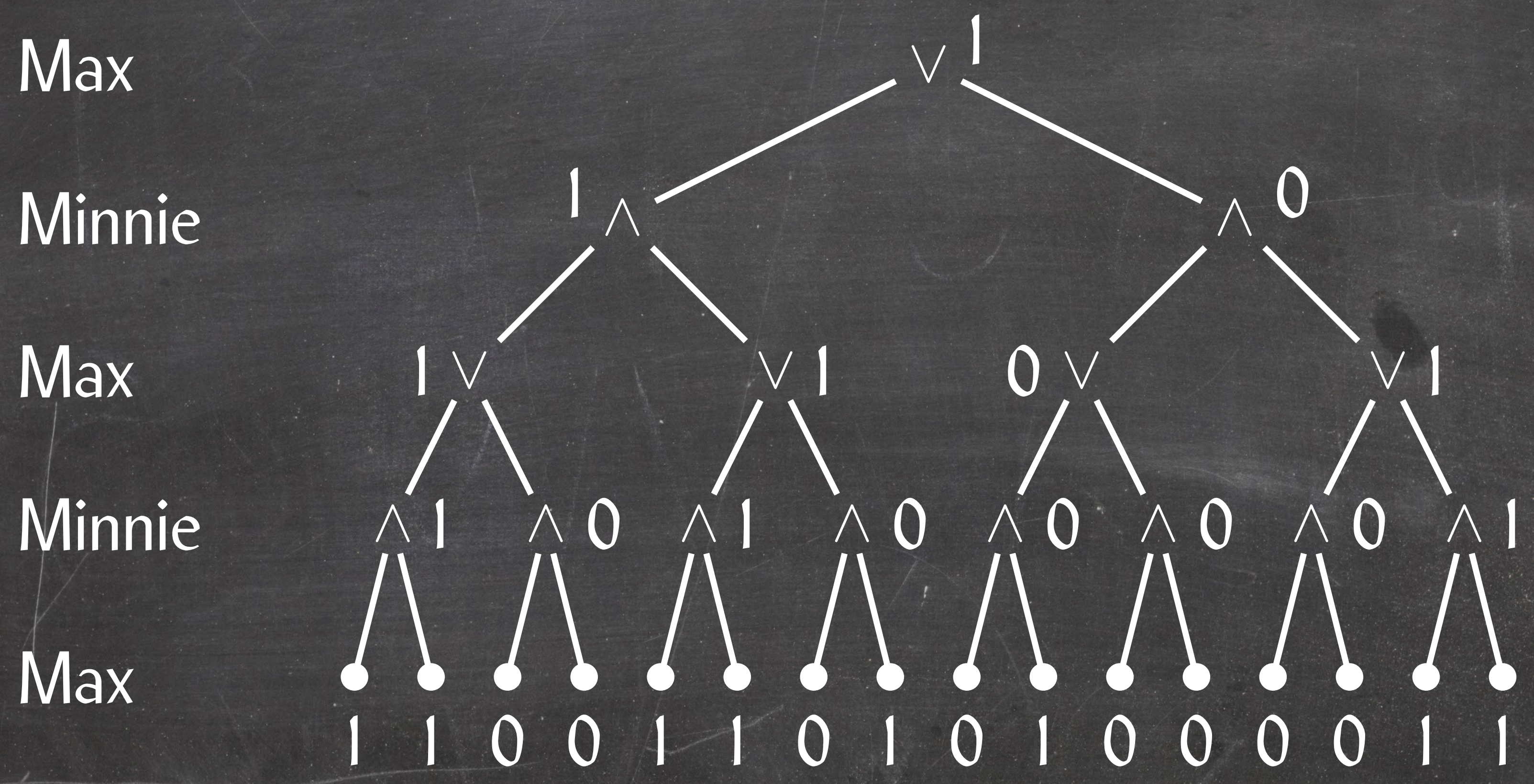
Restrict ourselves to binary game trees of height 2k $\Rightarrow$ n = 2^{2k} leaves



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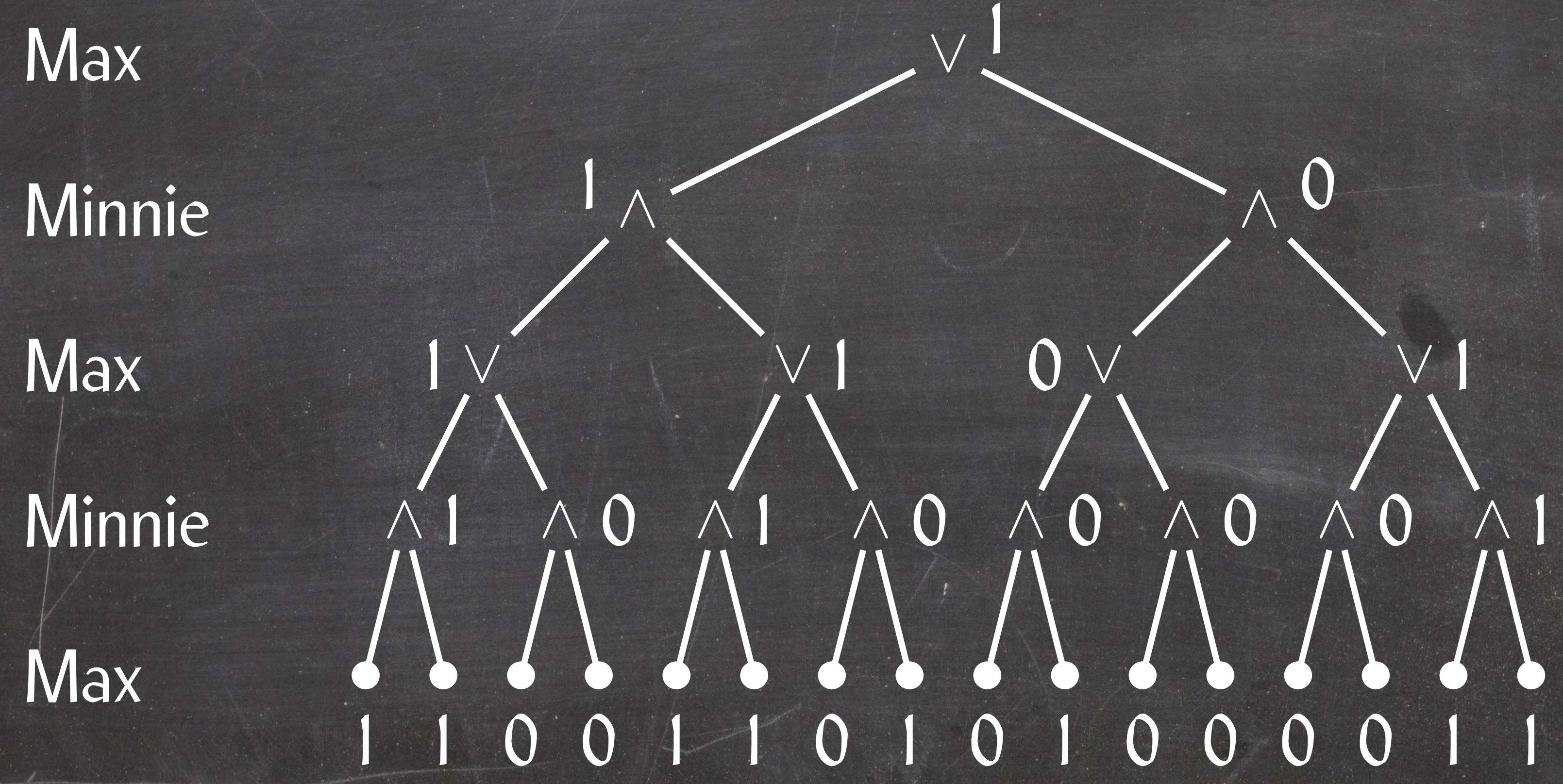

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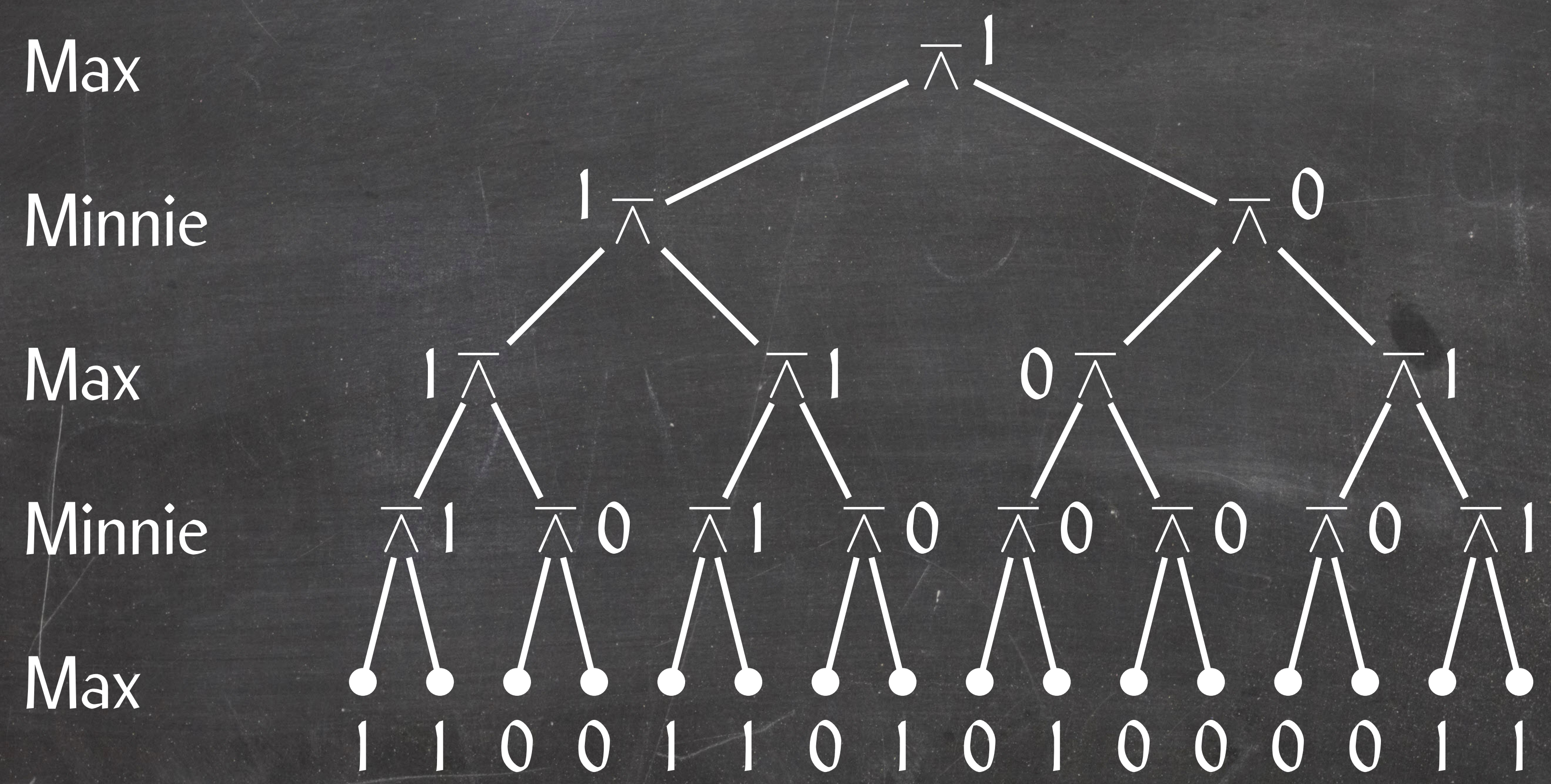

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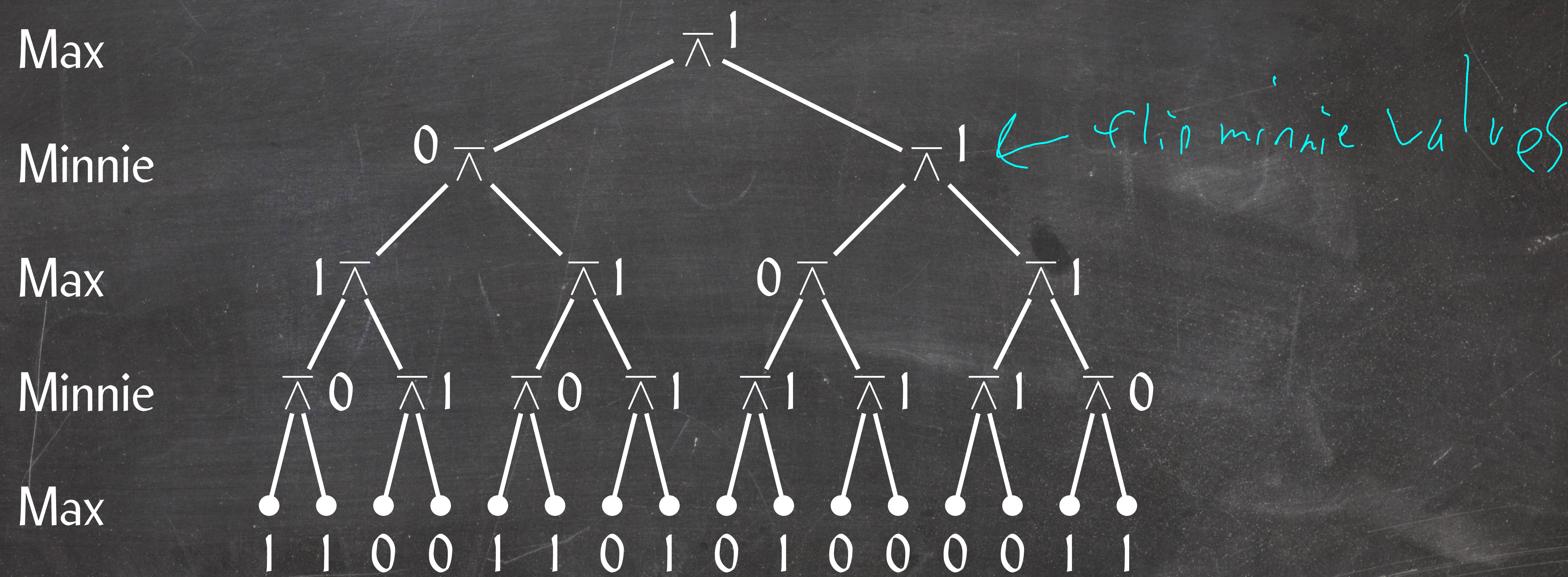

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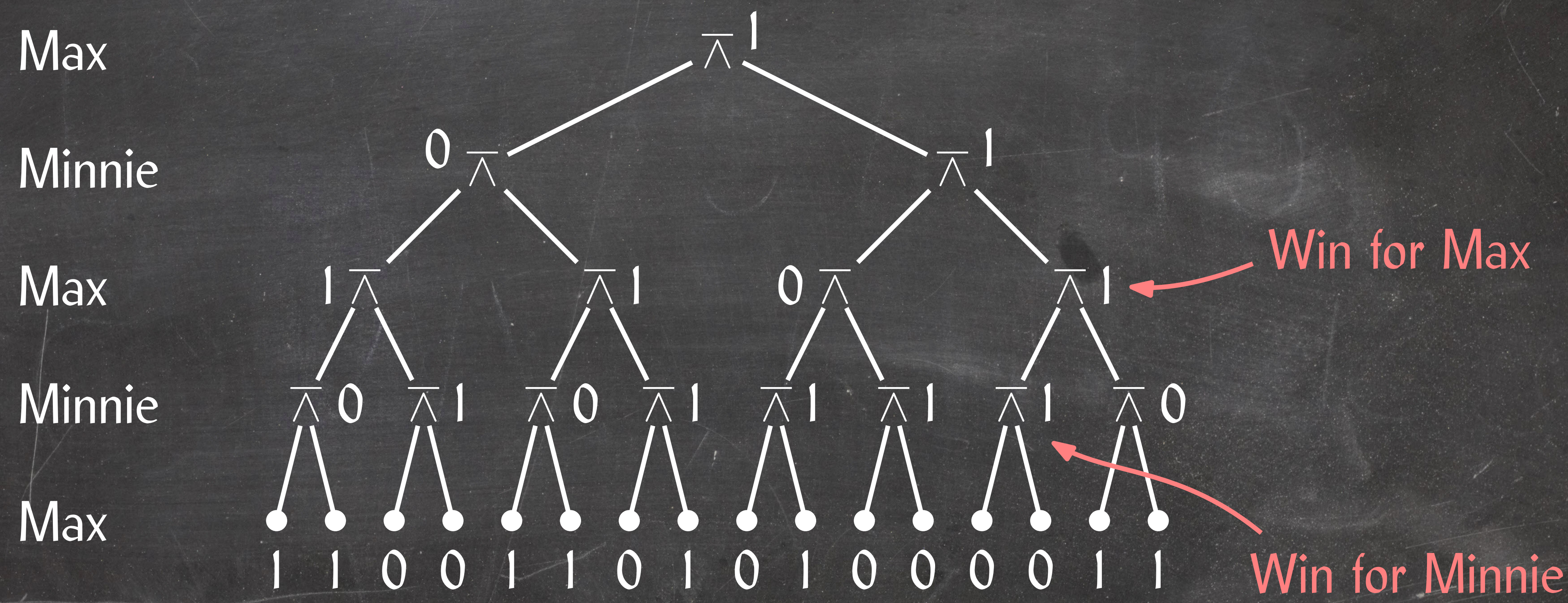

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Game Tree Evaluation: A Deterministic Algorithm

GameValue(v)

```
if v is a leaf
    then return its value
    else return not (GameValue(v.leftChild) and GameValue(v.rightChild))
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- One recursive call per node
- $2n - 1$ nodes

$\Rightarrow$ Running time $O(n)$

Game Tree Evaluation: A Deterministic Algorithm

before: not (value(l) ∧ value(r))

GameValue(v)

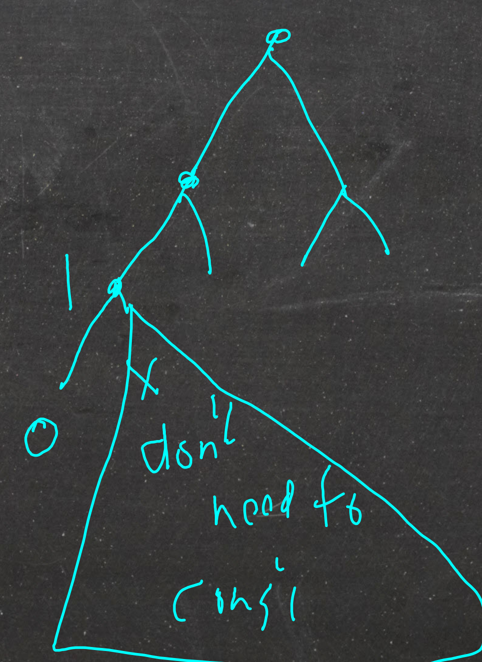
```
if v is a leaf
    then return its value
if not GameValue(v.leftChild)
    then return 1
    else return not GameValue(v.rightChild)
```

if not(value(l)) that is a win

else return not(value(r))

- One recursive call per node
- 2n – 1 nodes

⇒ Running time O(n)



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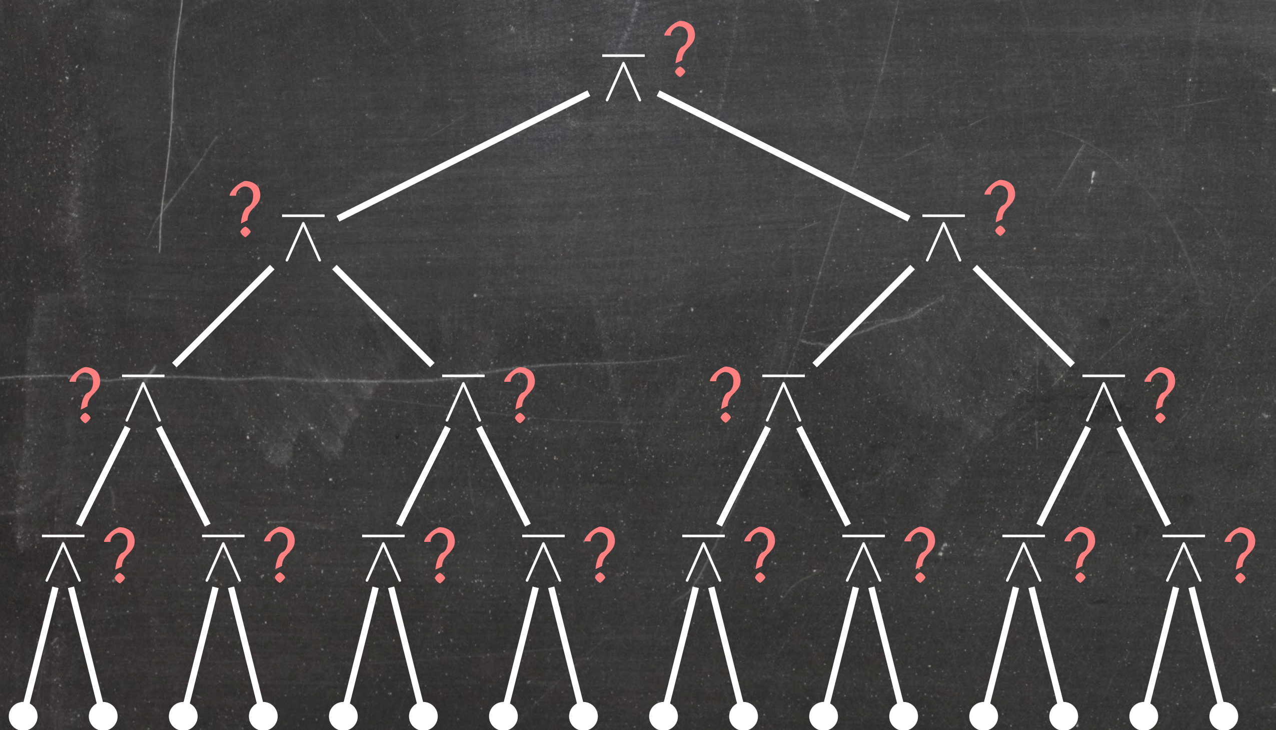

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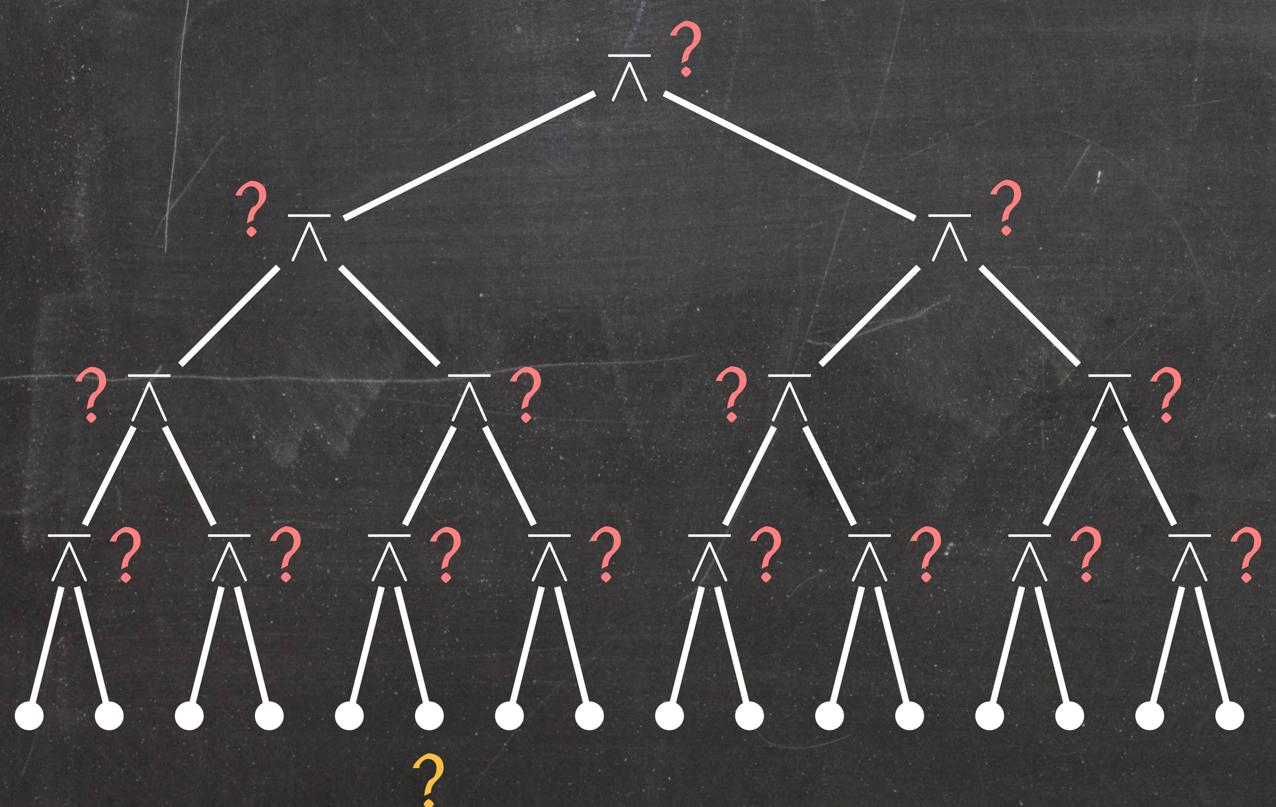

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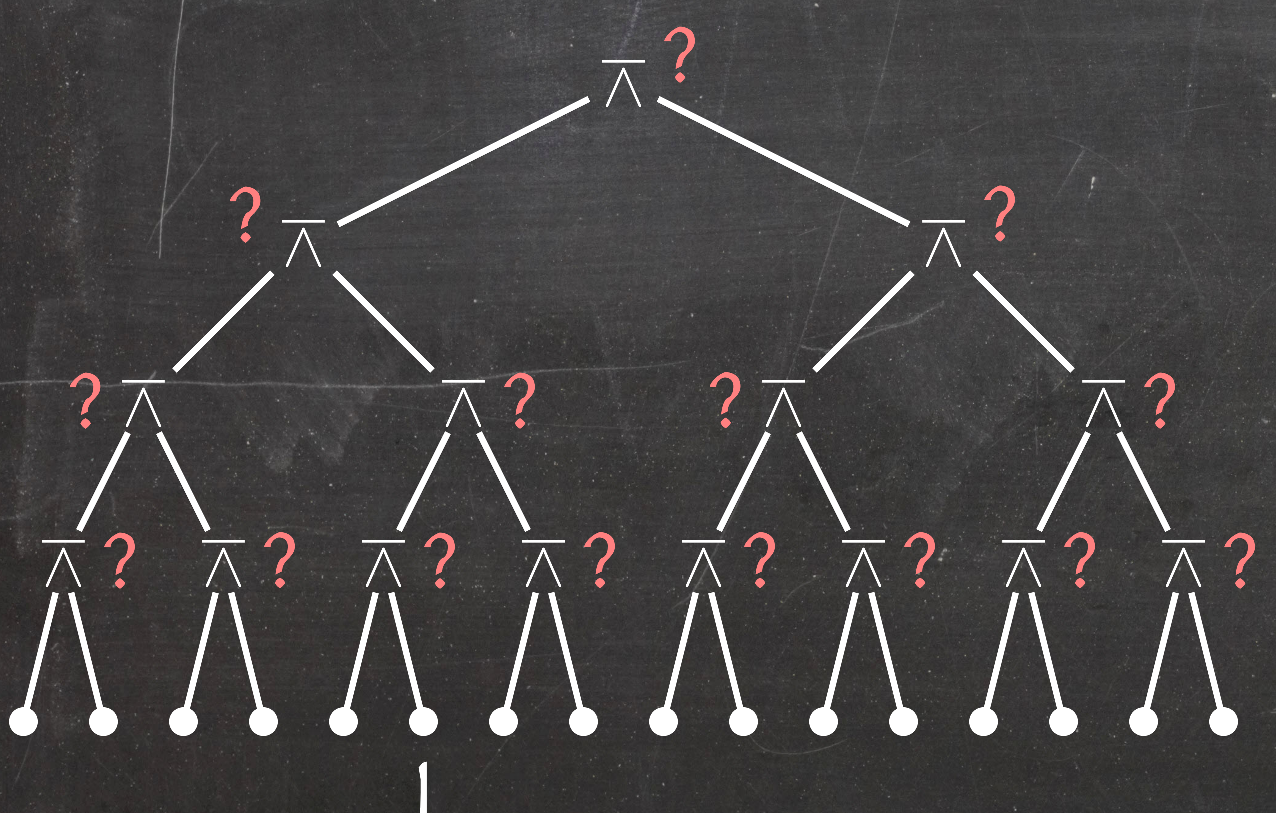

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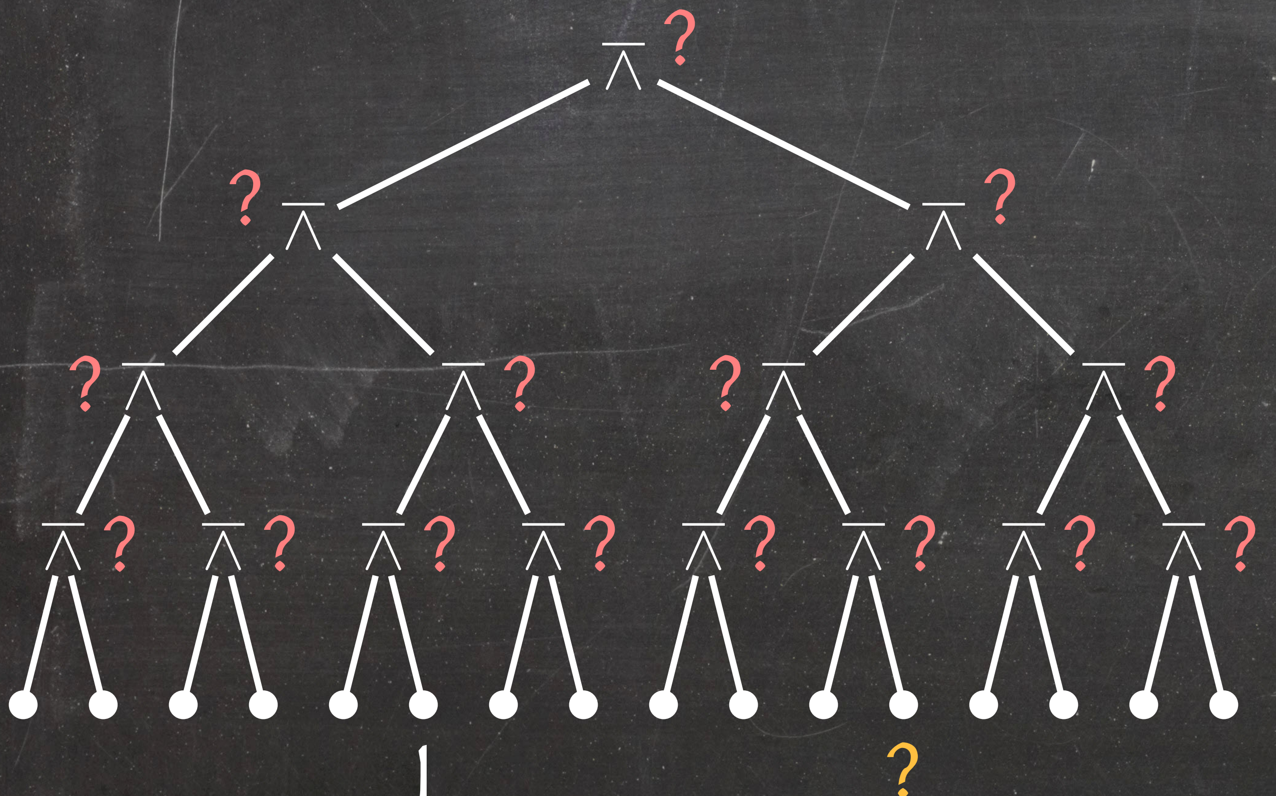

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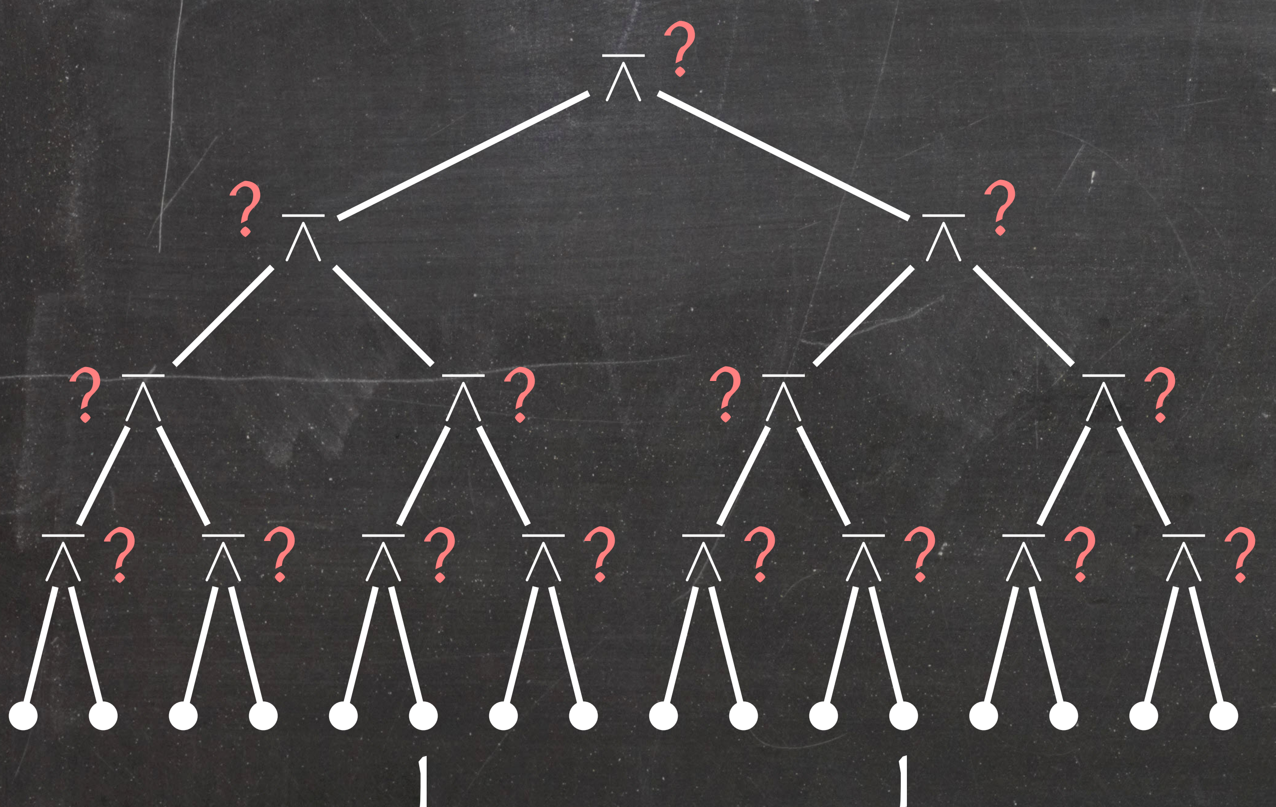

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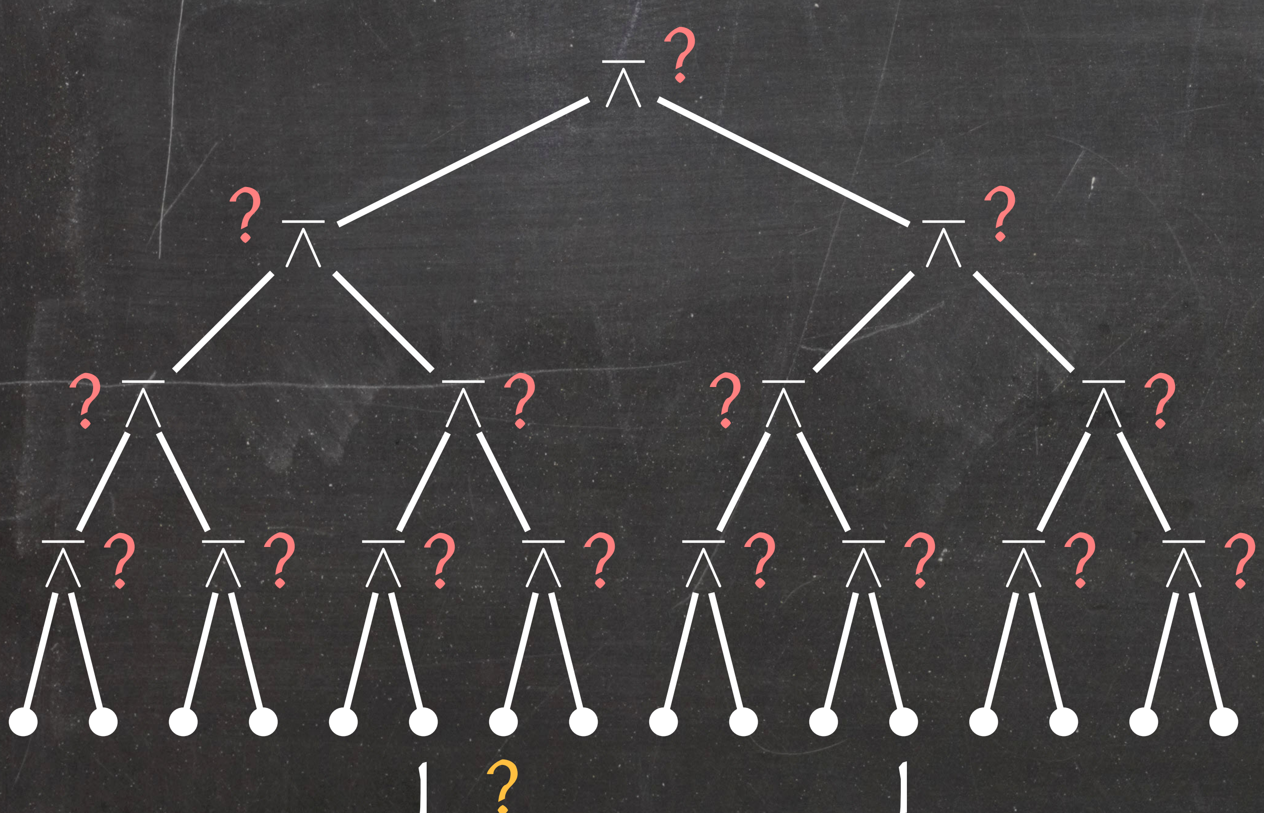

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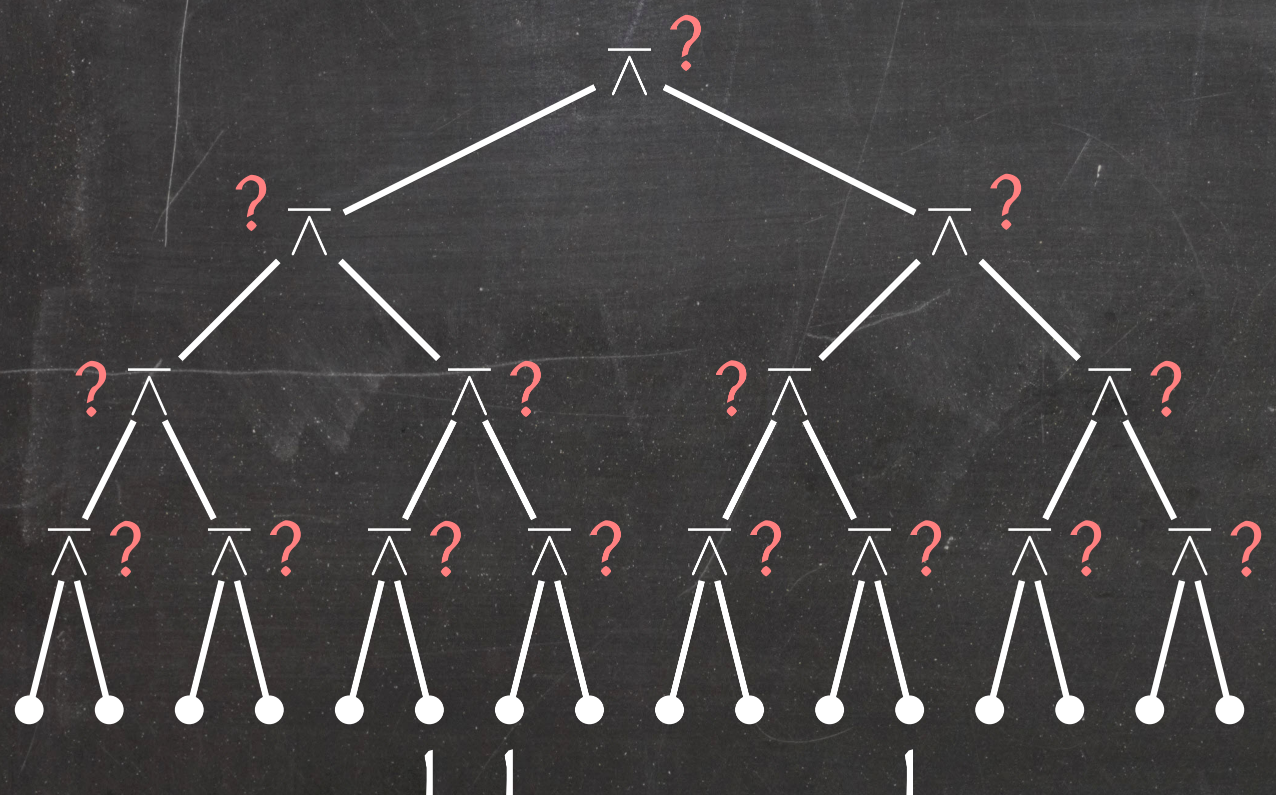

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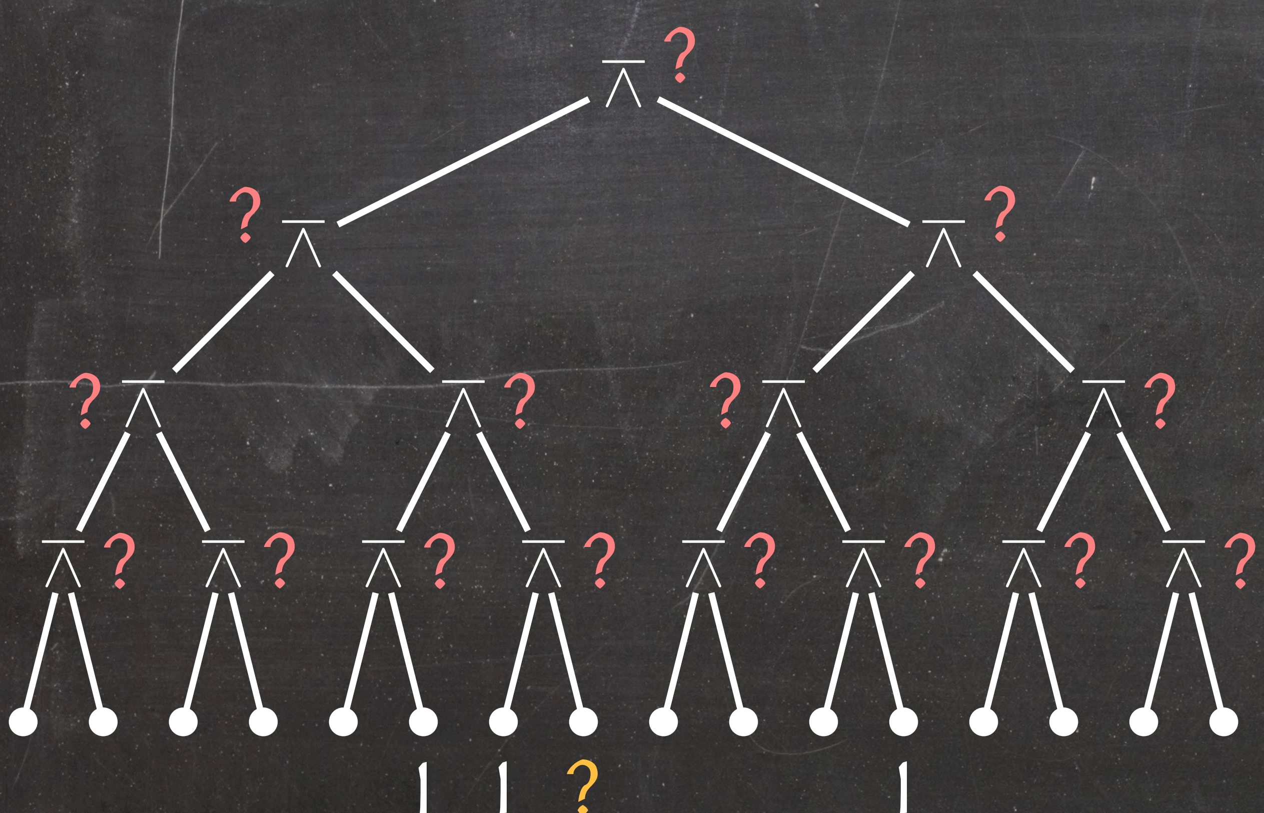

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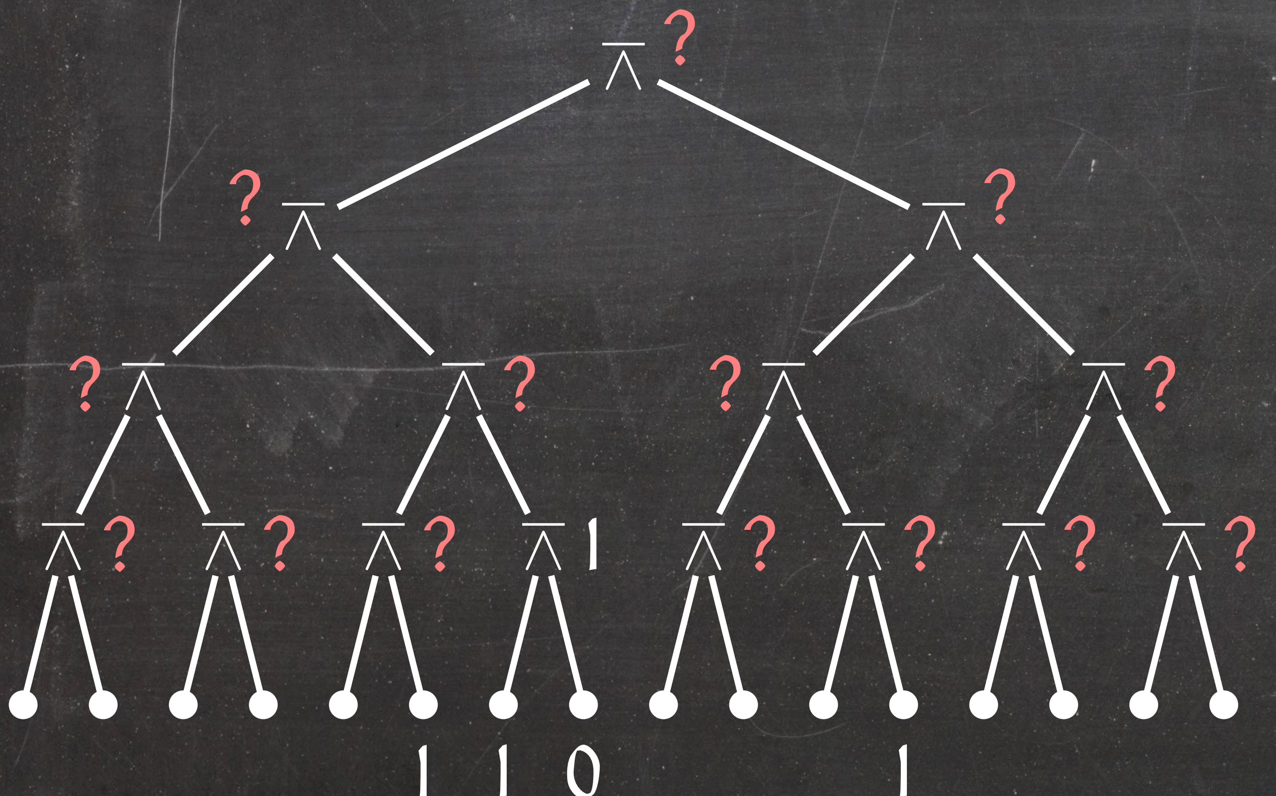

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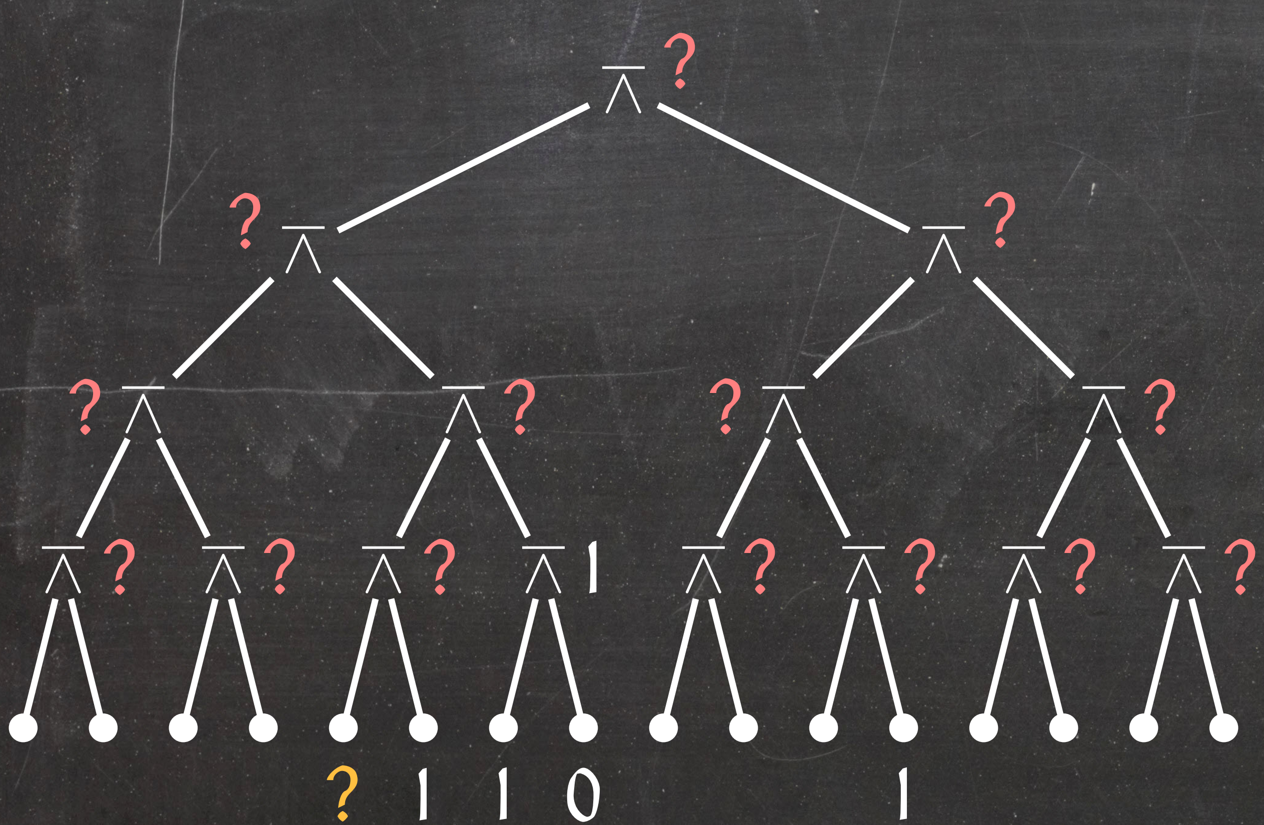

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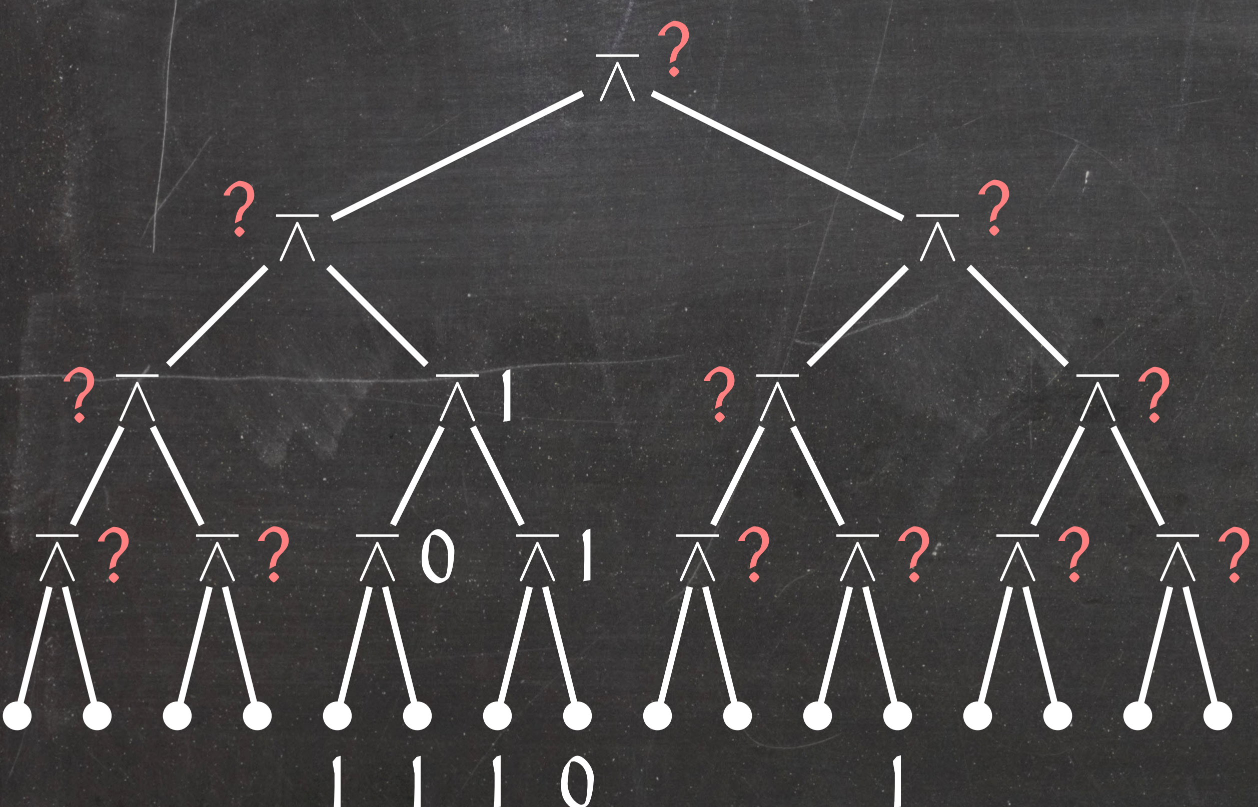

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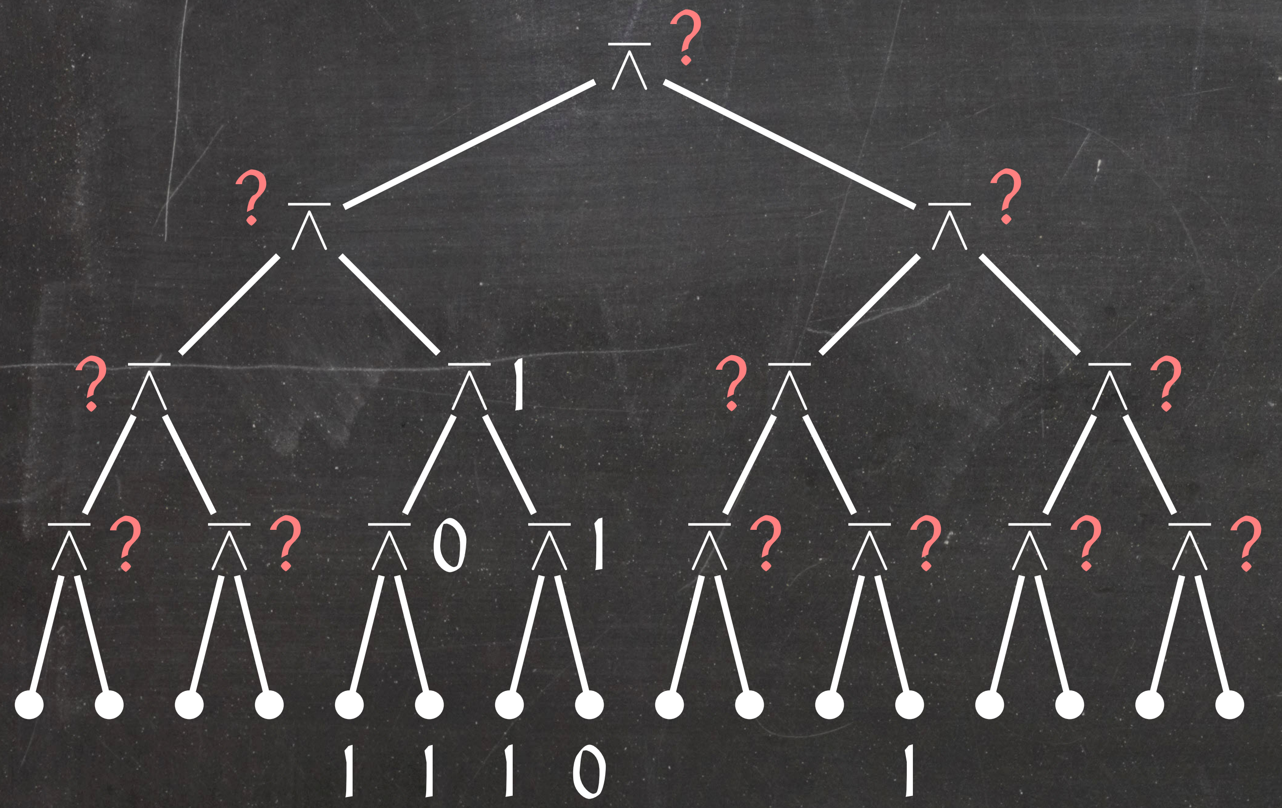

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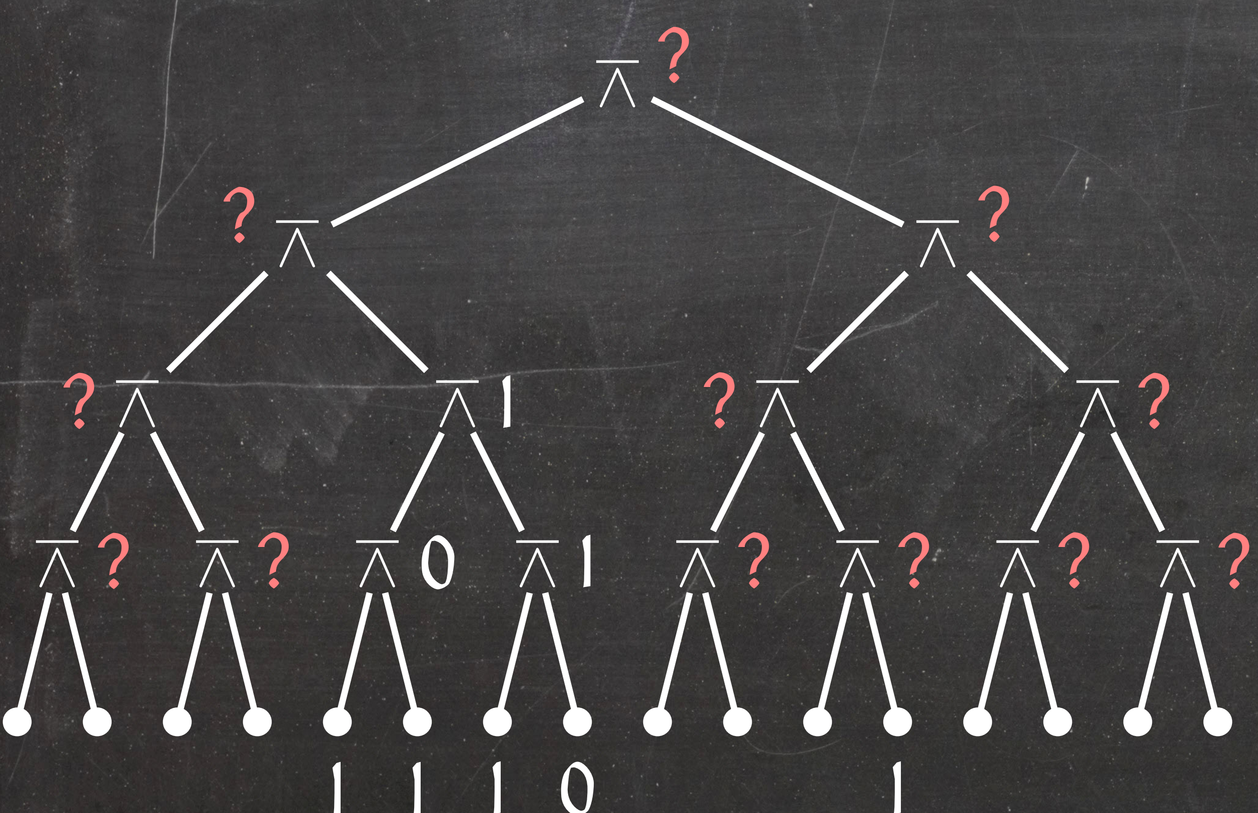

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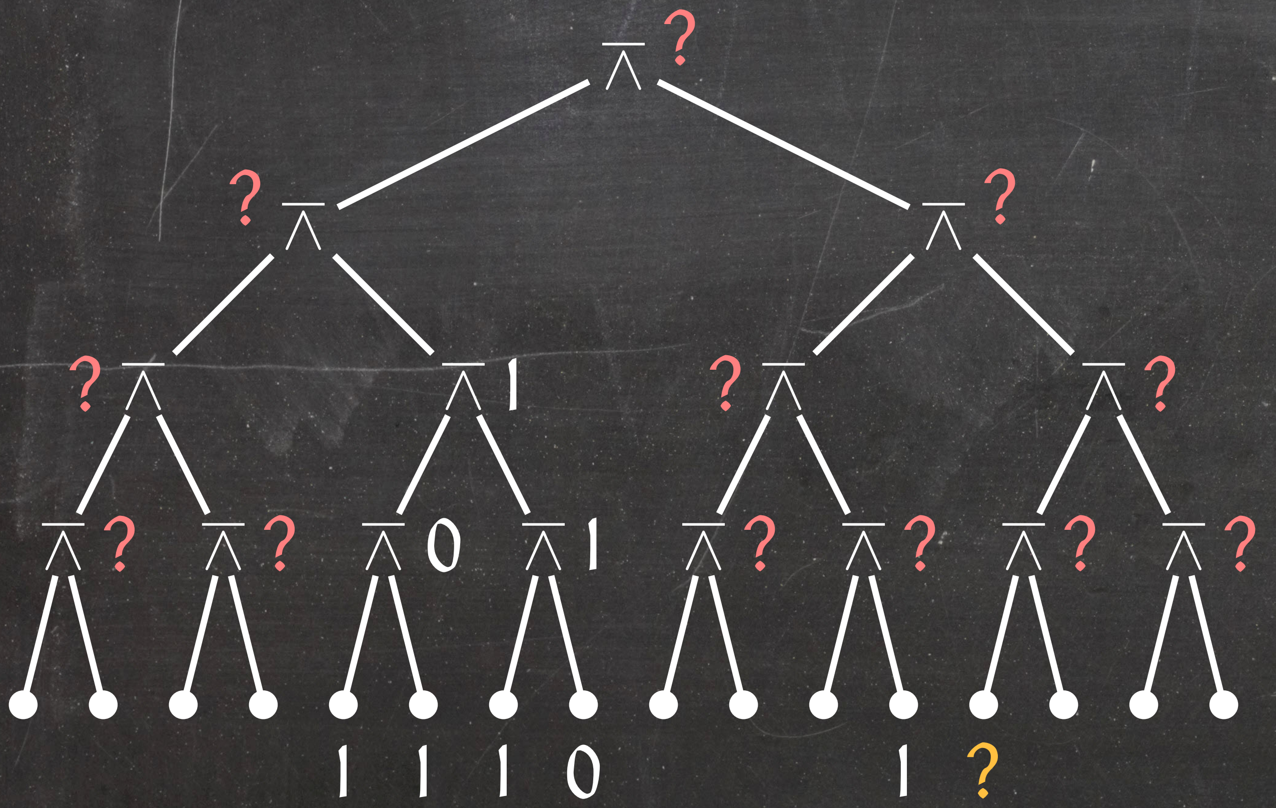

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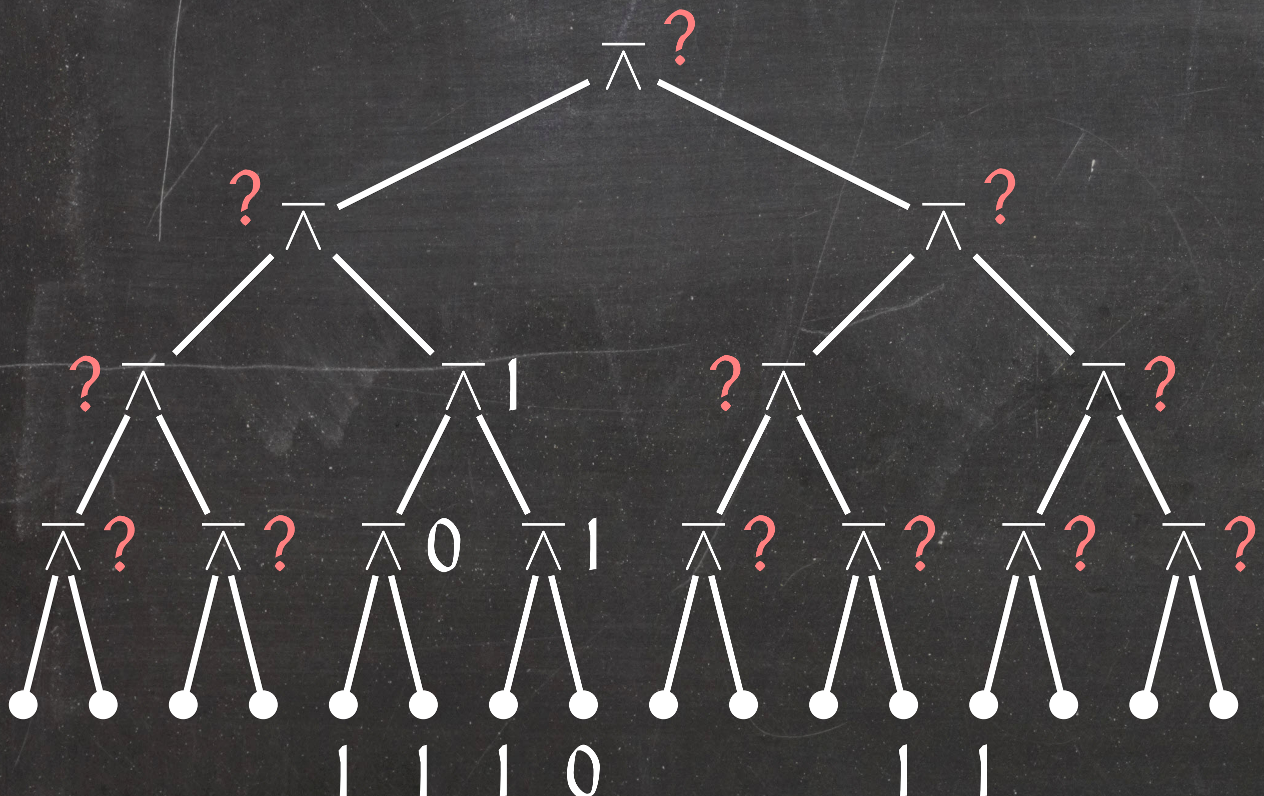

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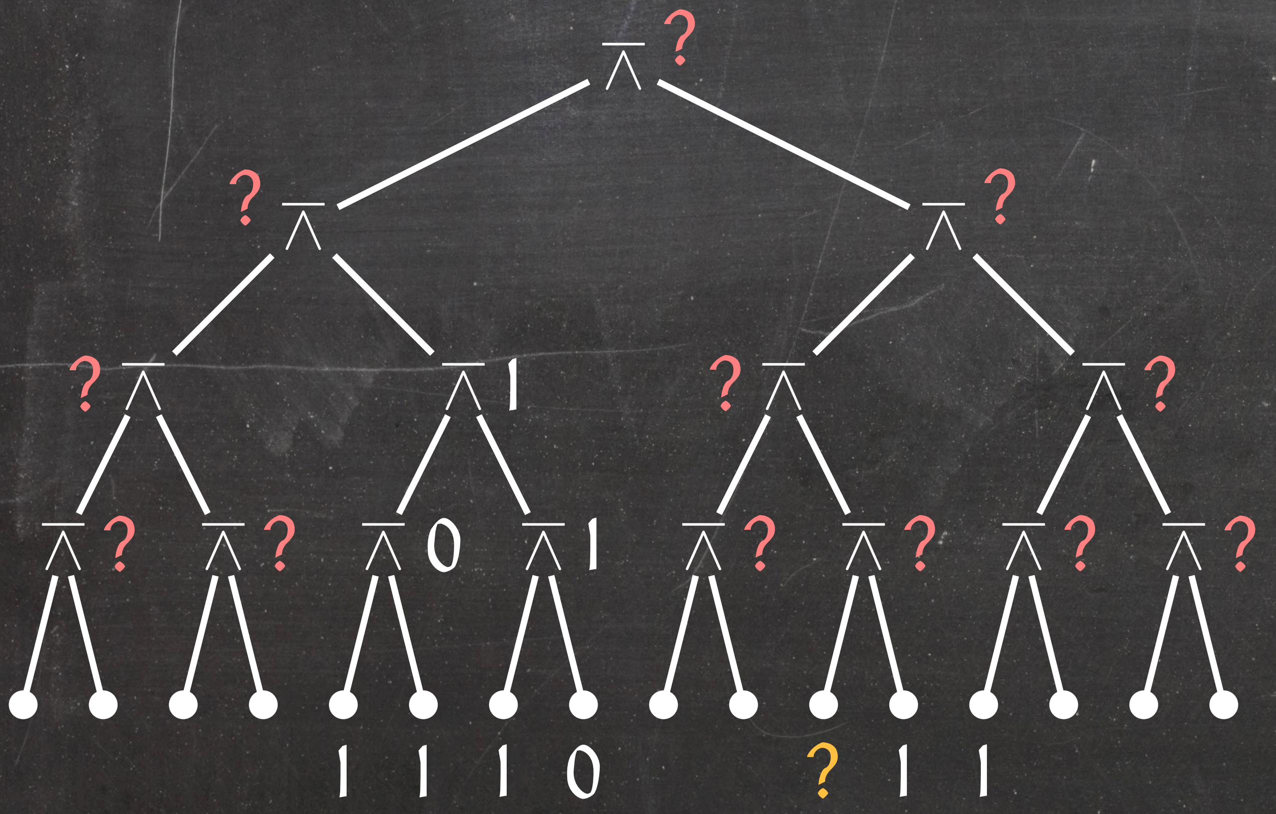

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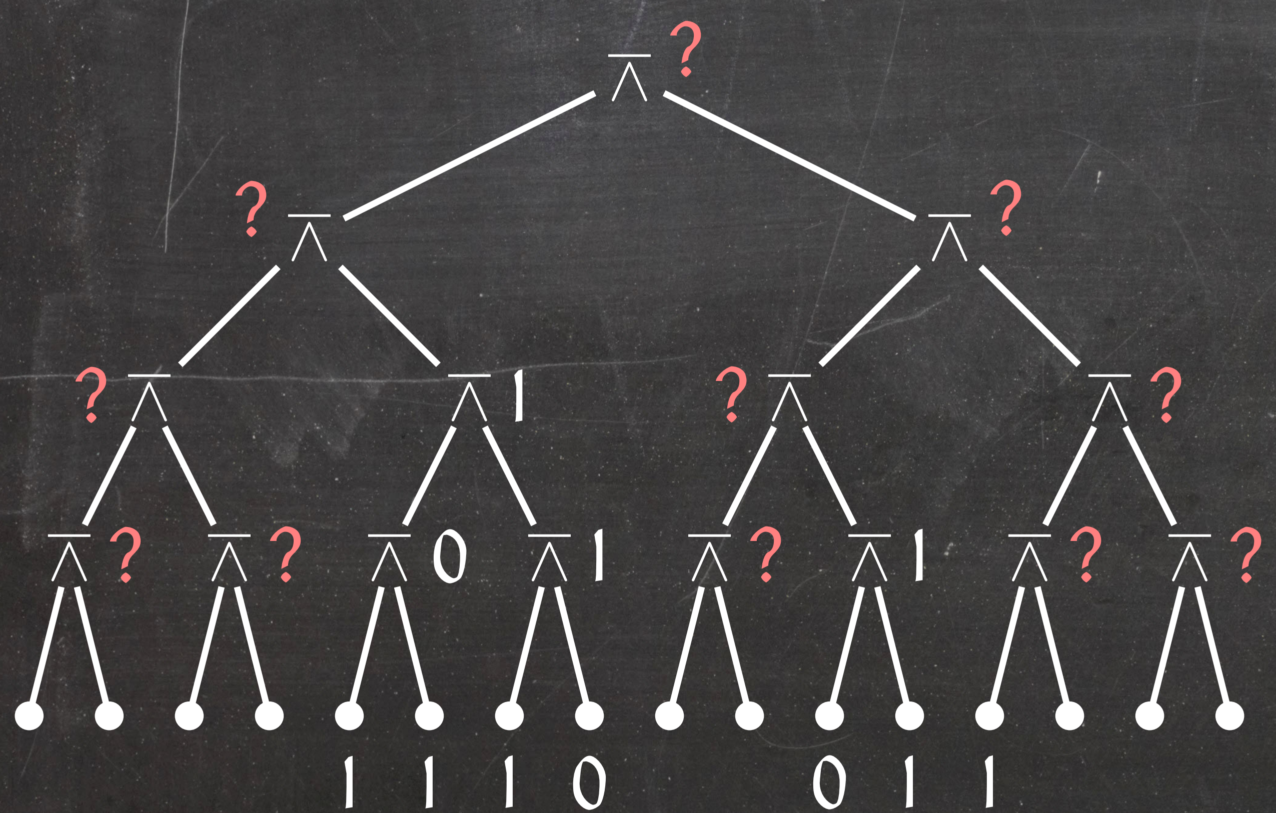

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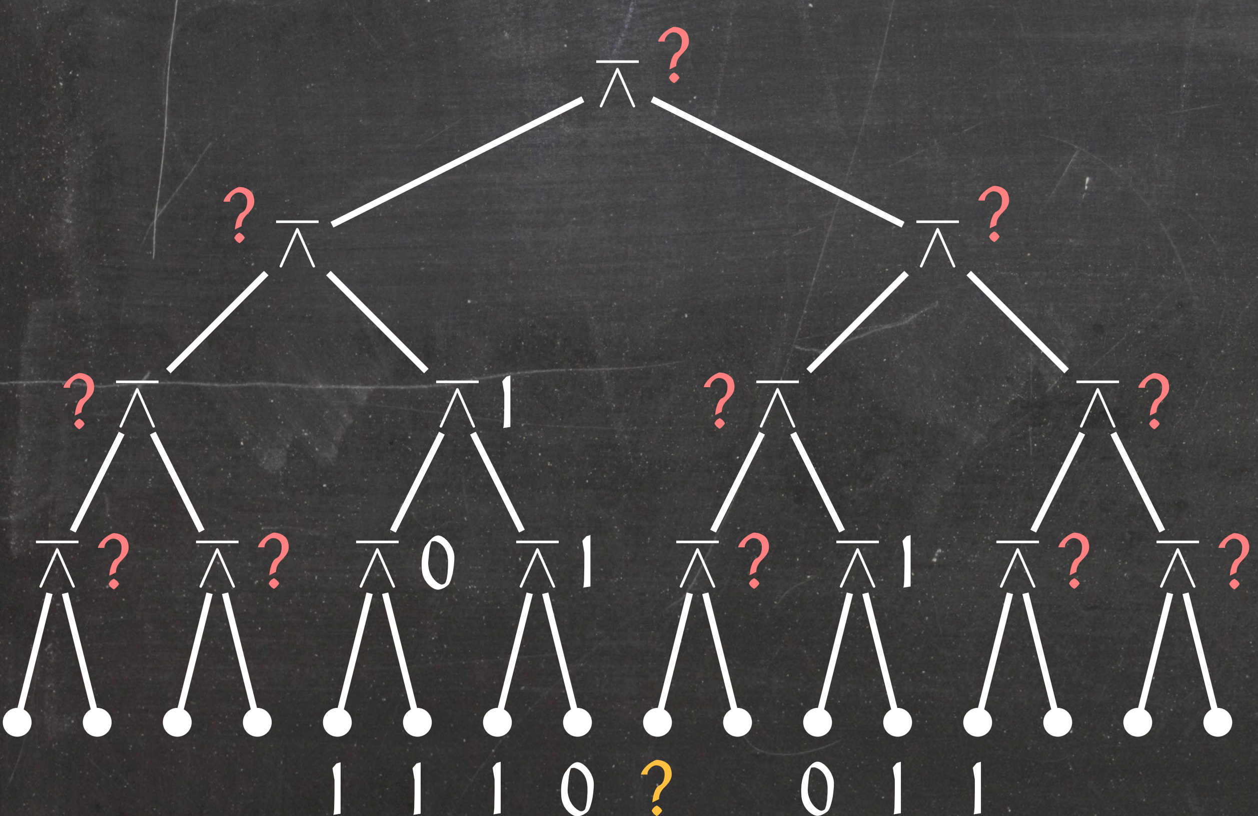

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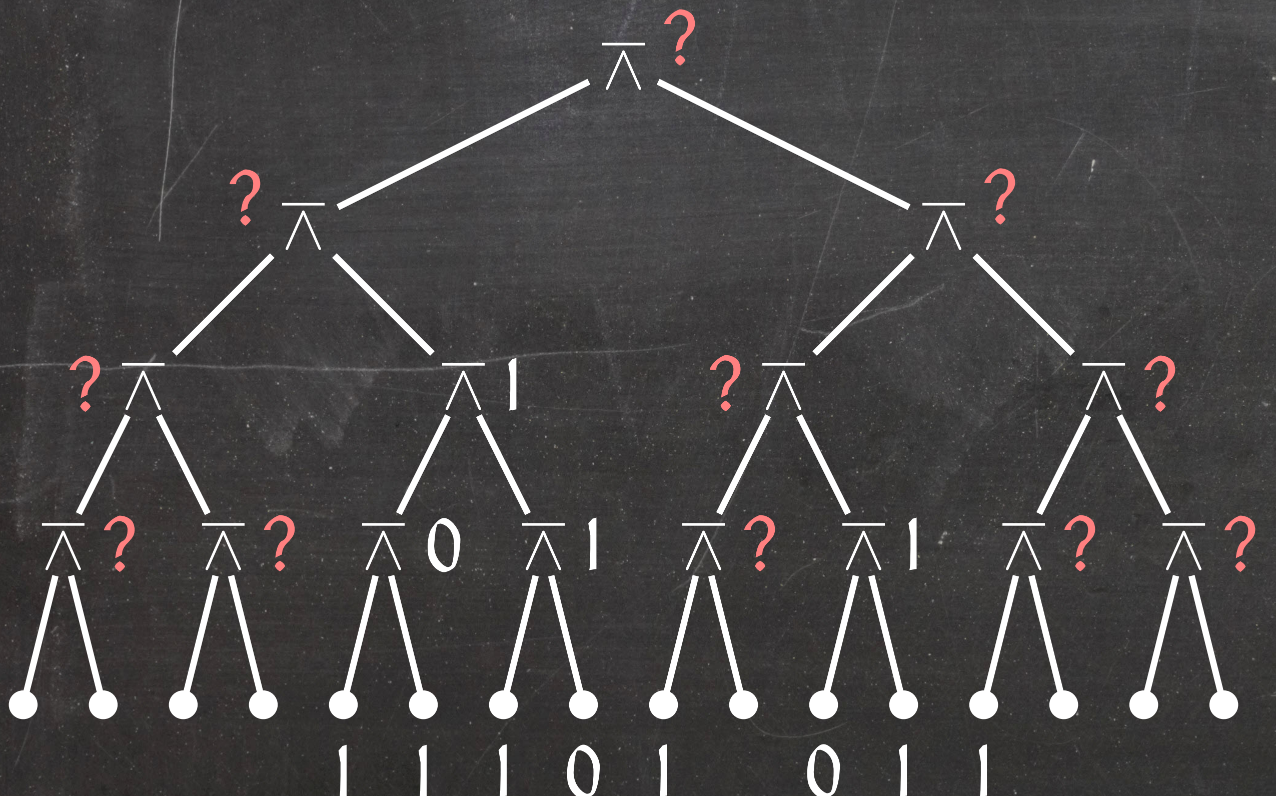

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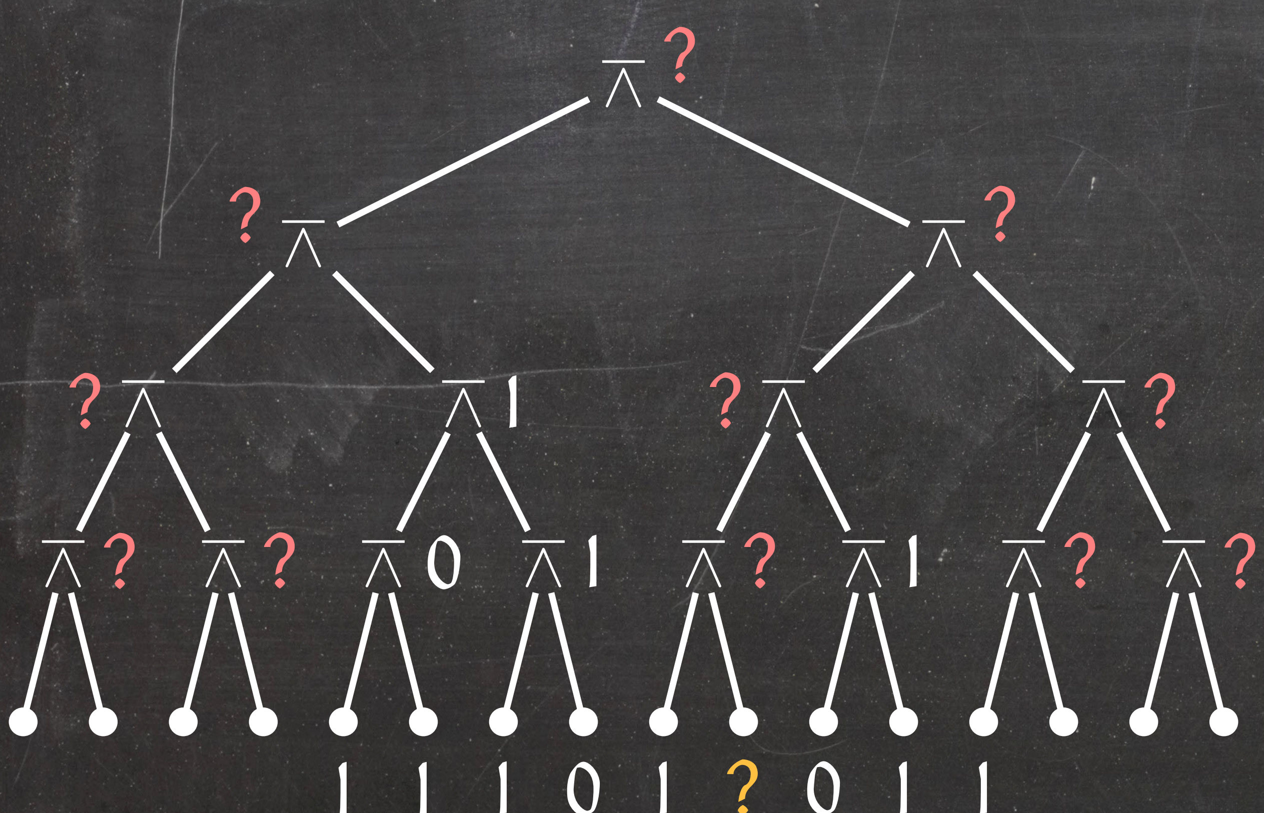

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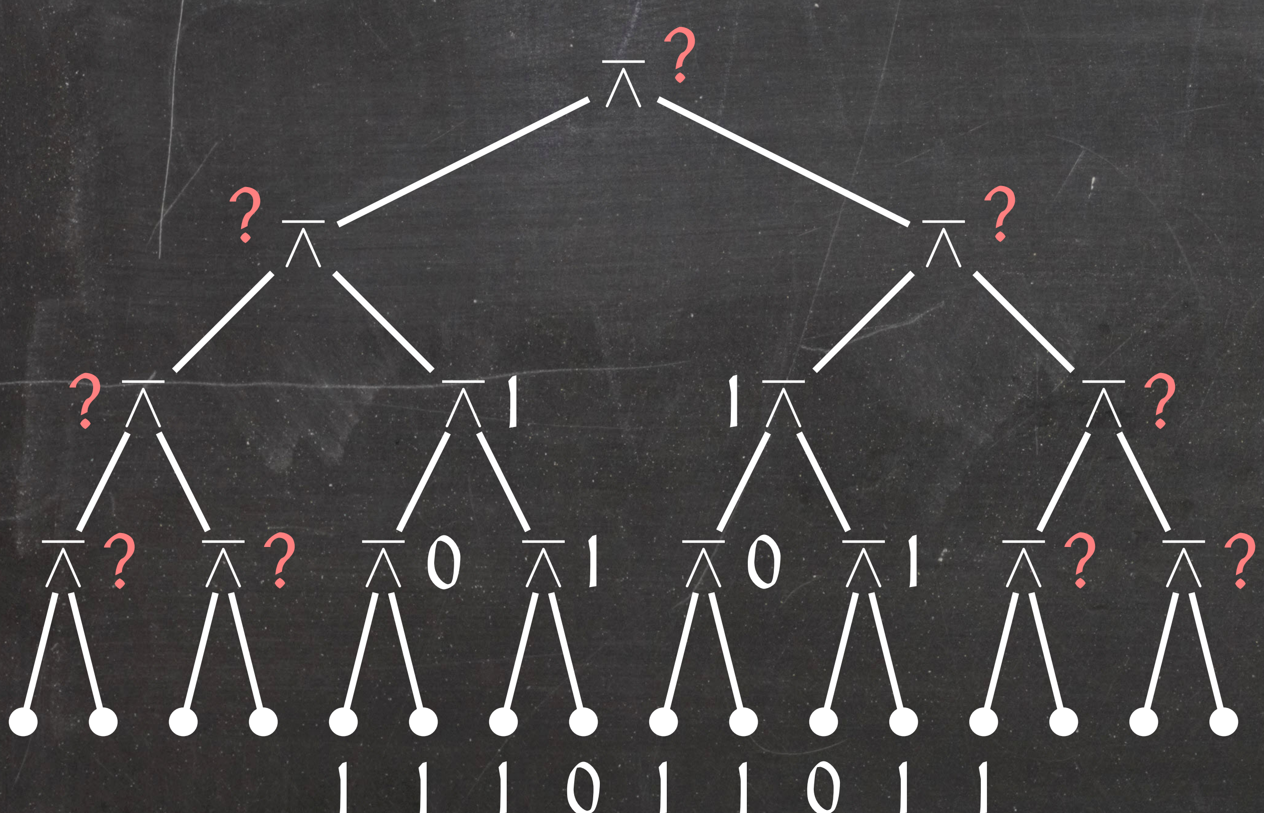

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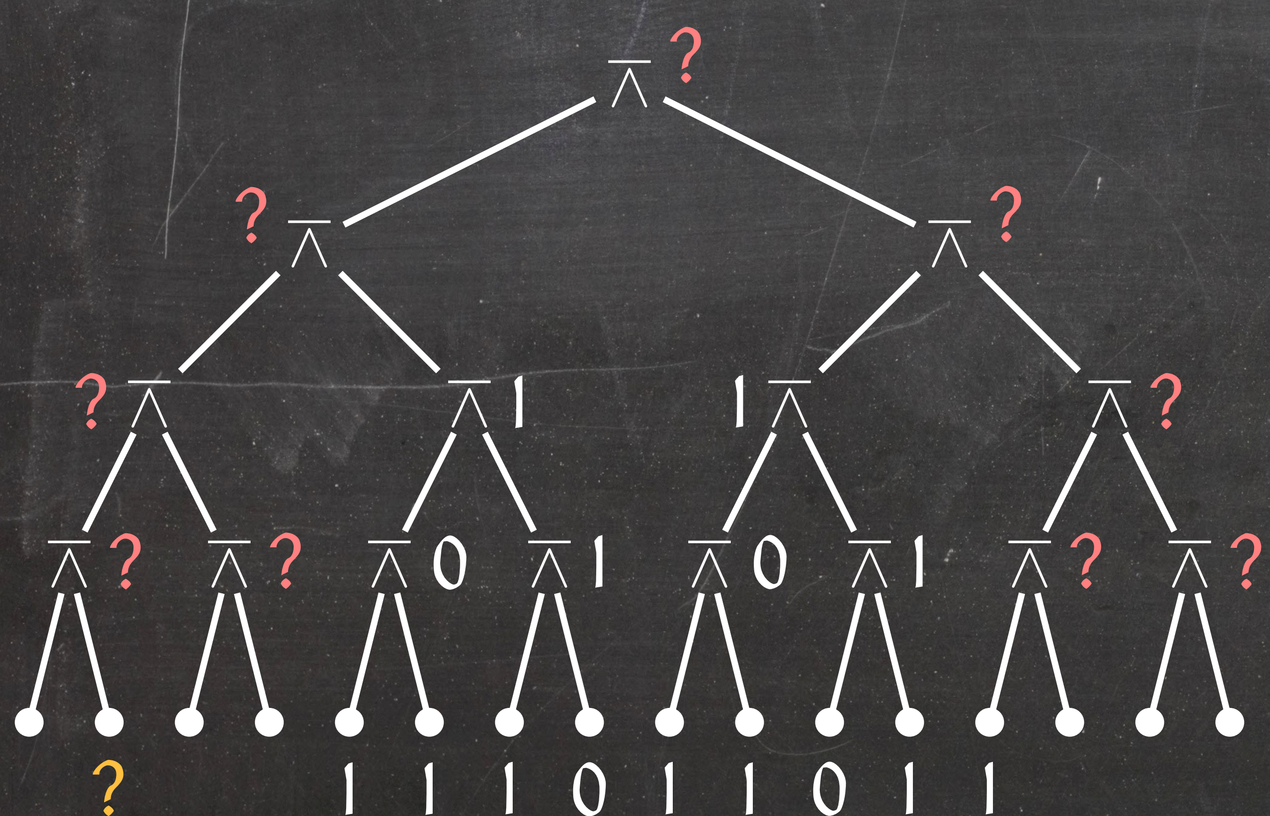

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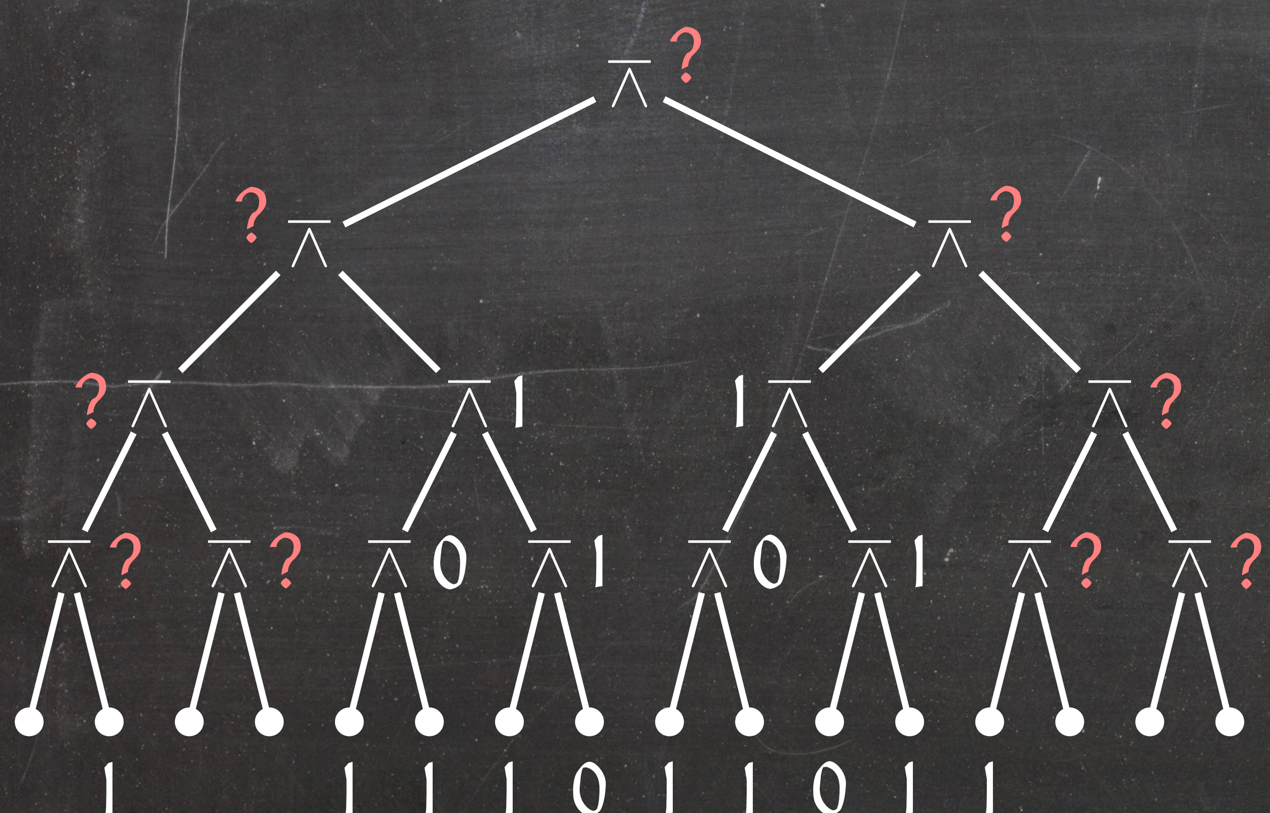

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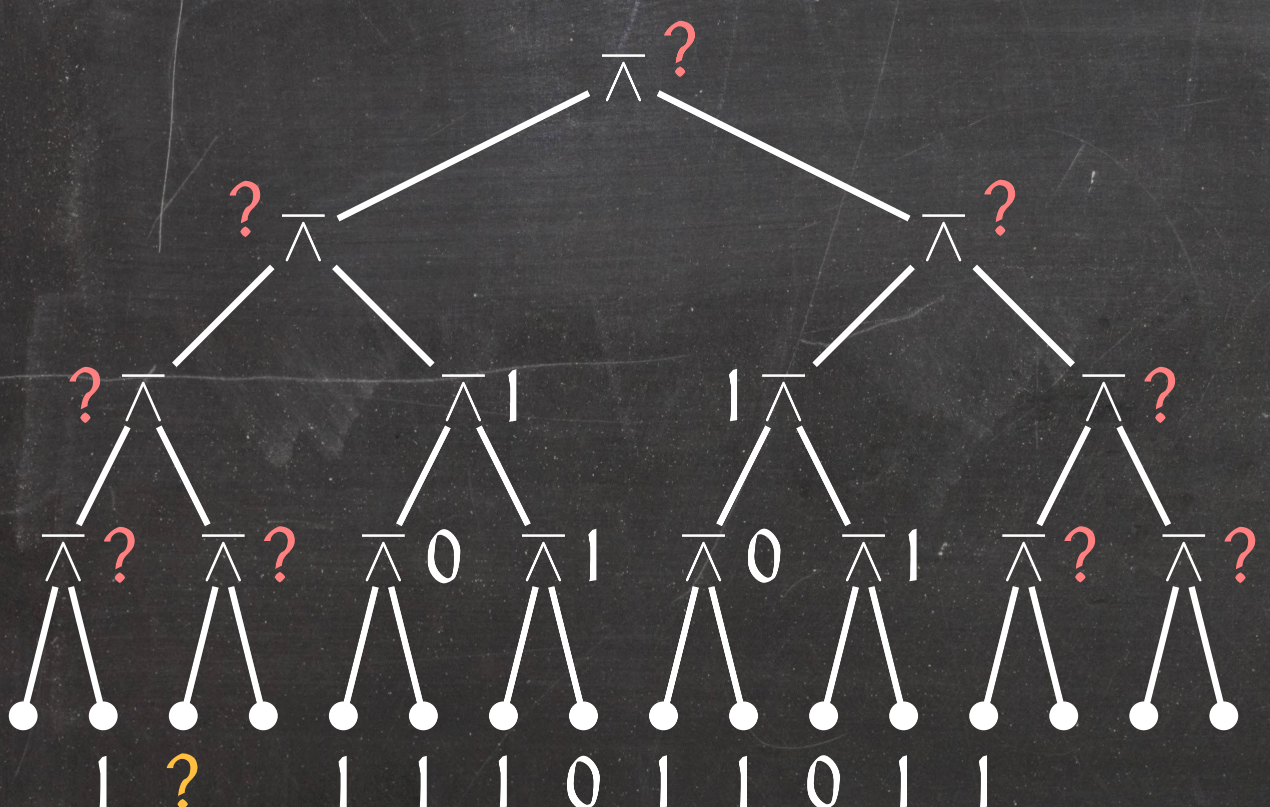

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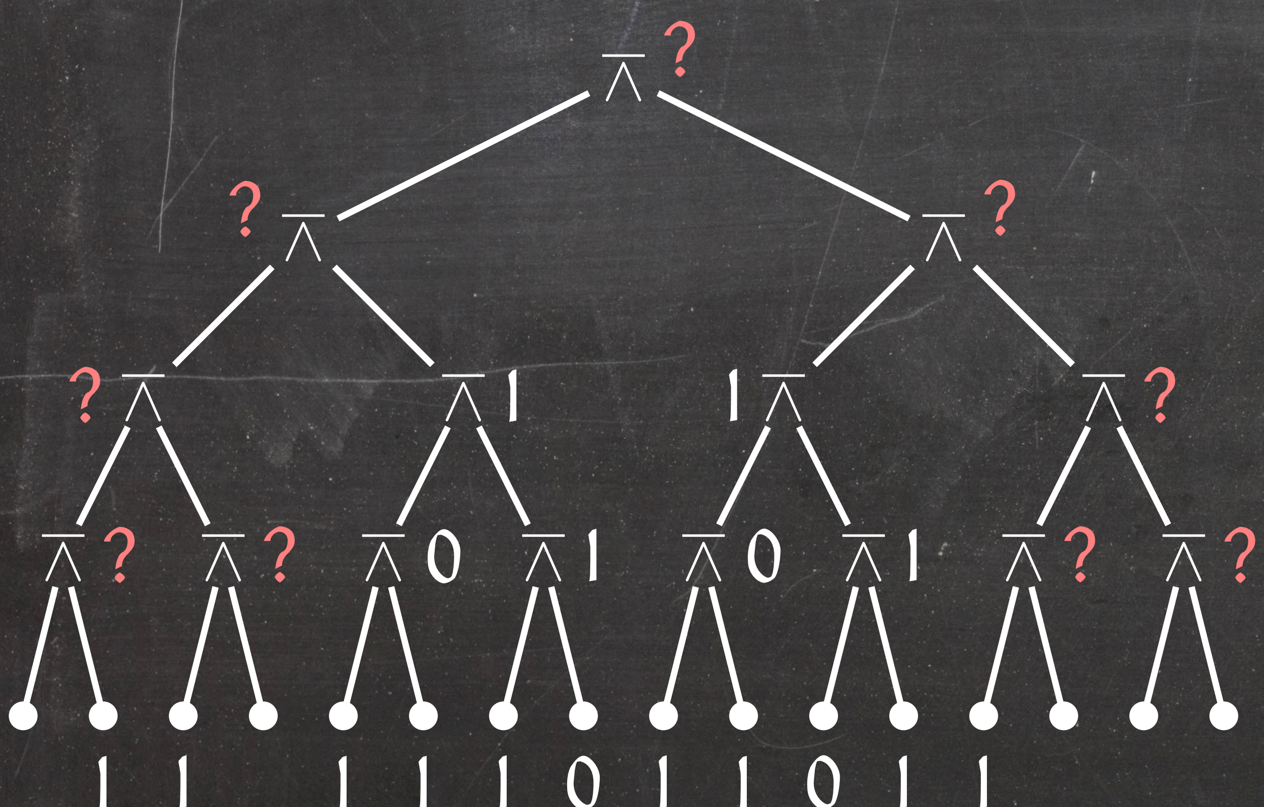

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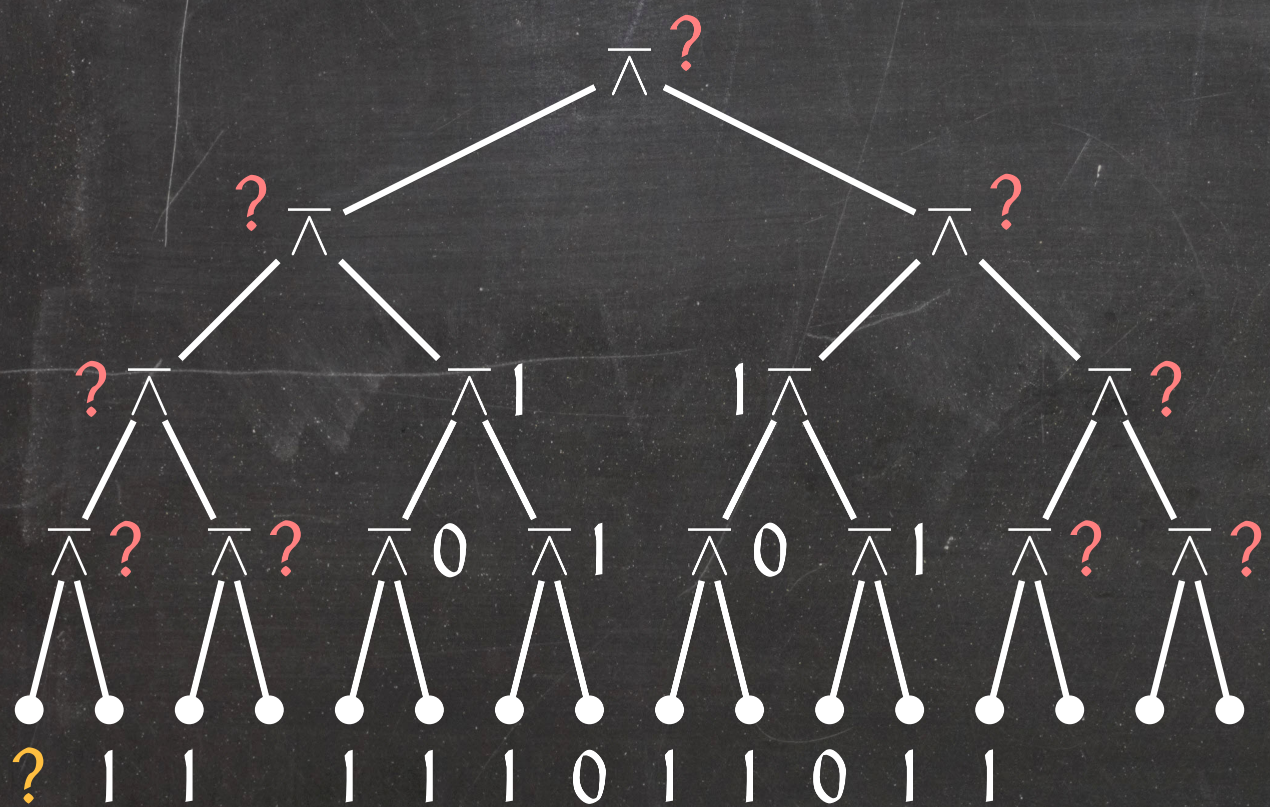

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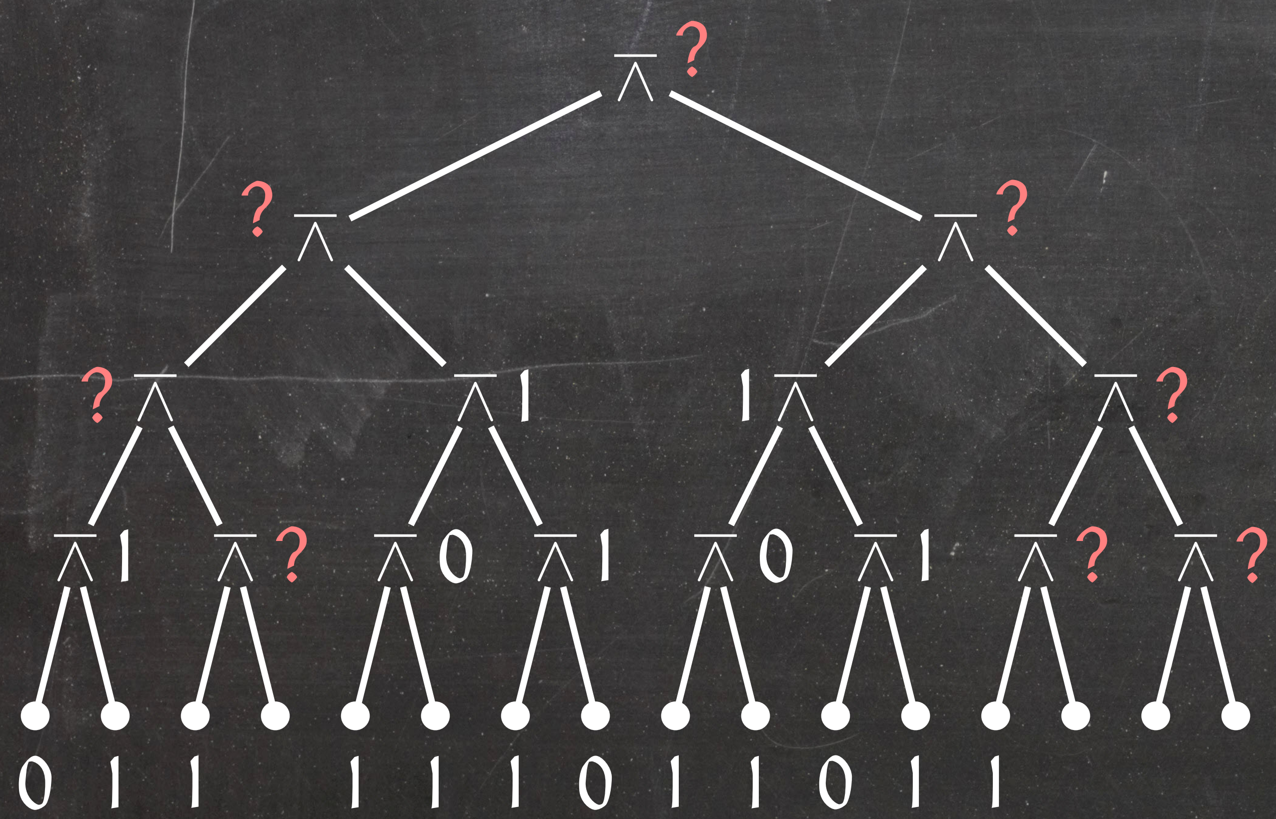

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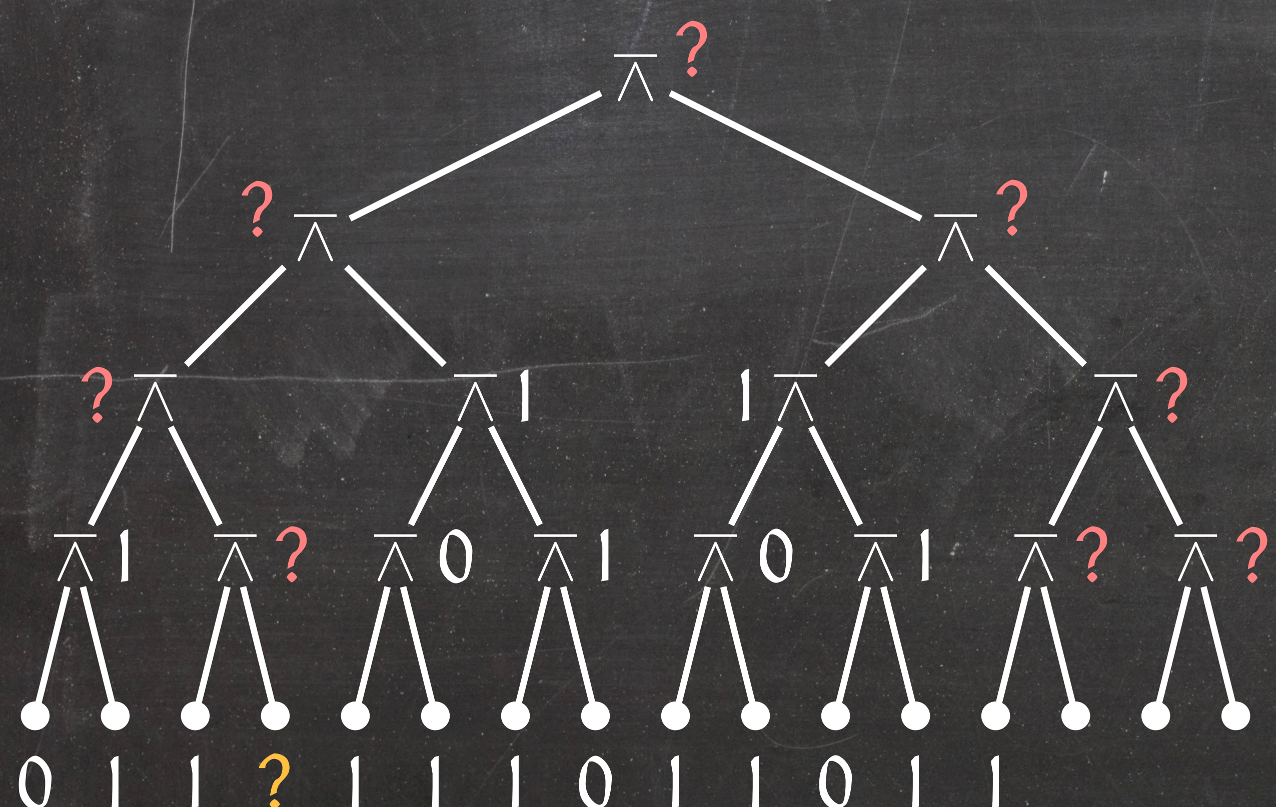

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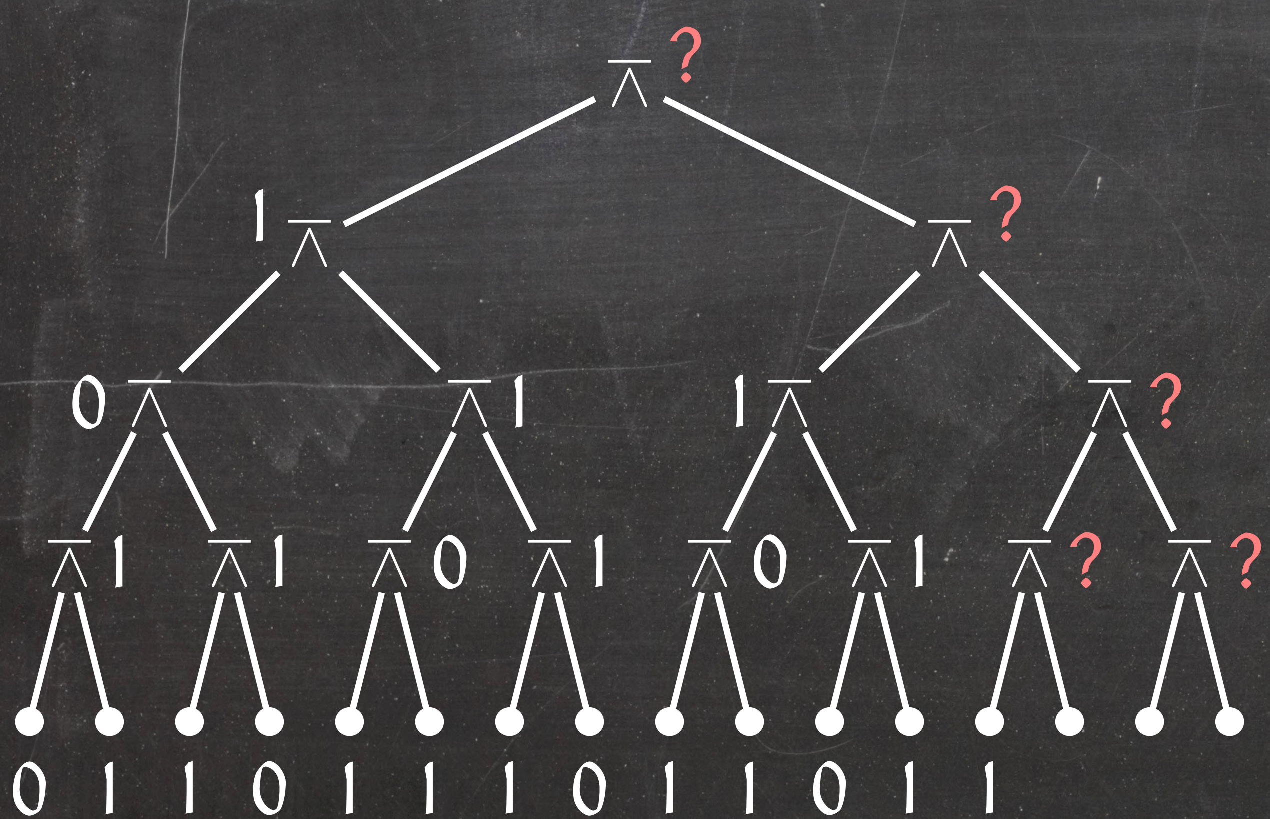

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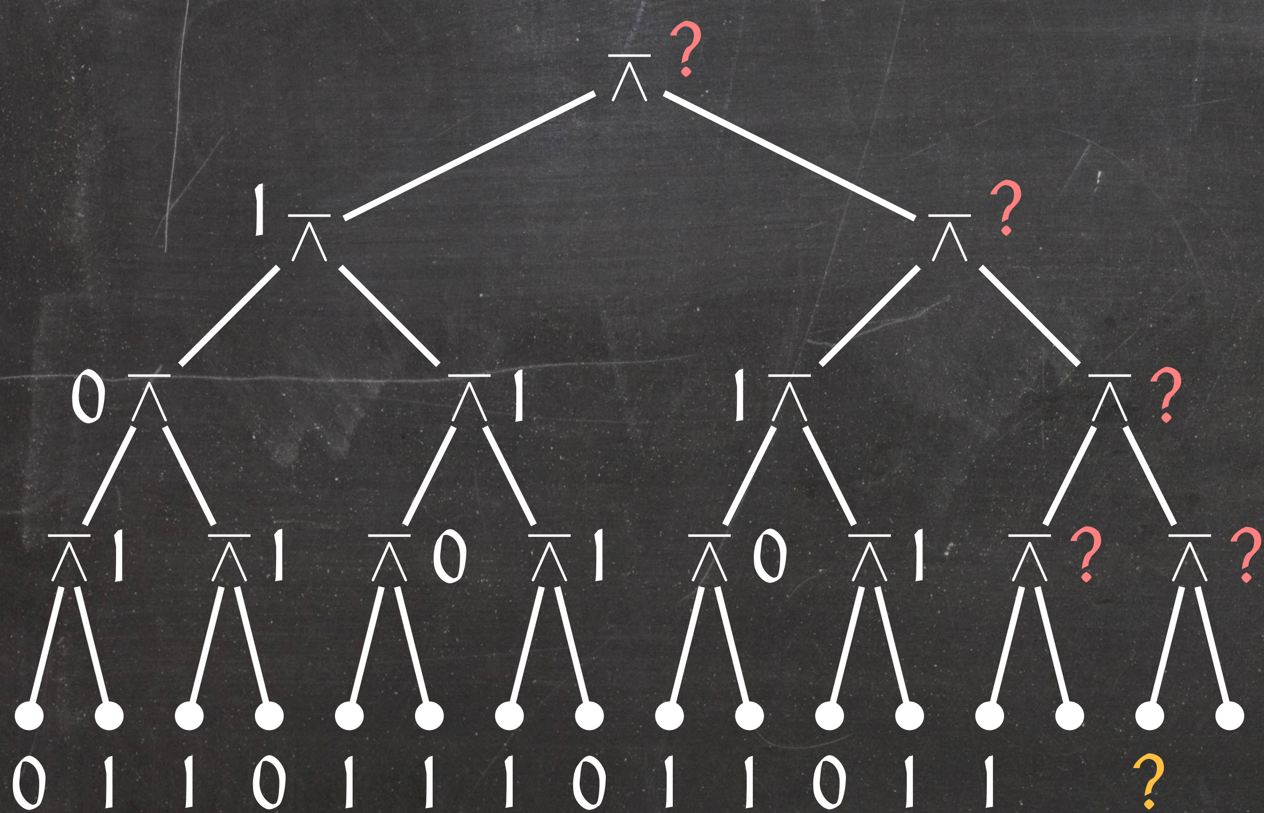

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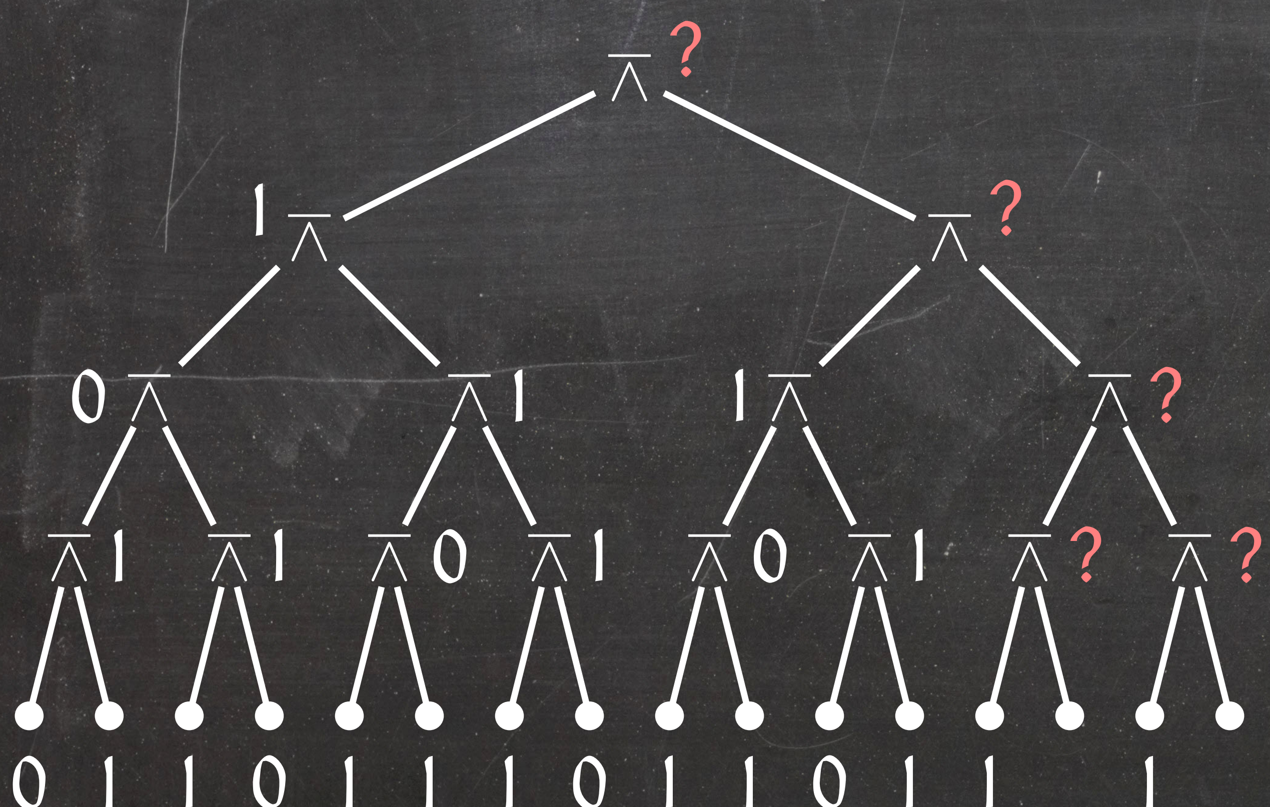

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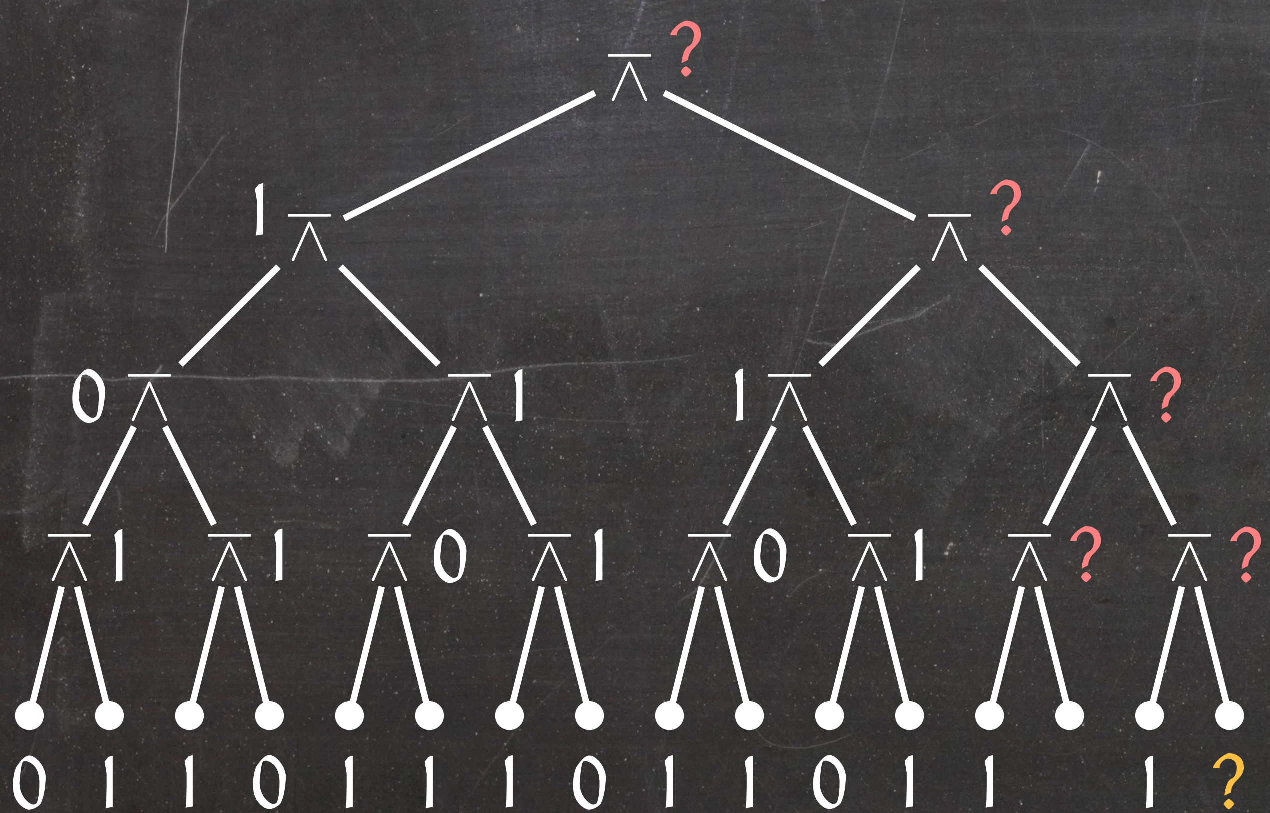

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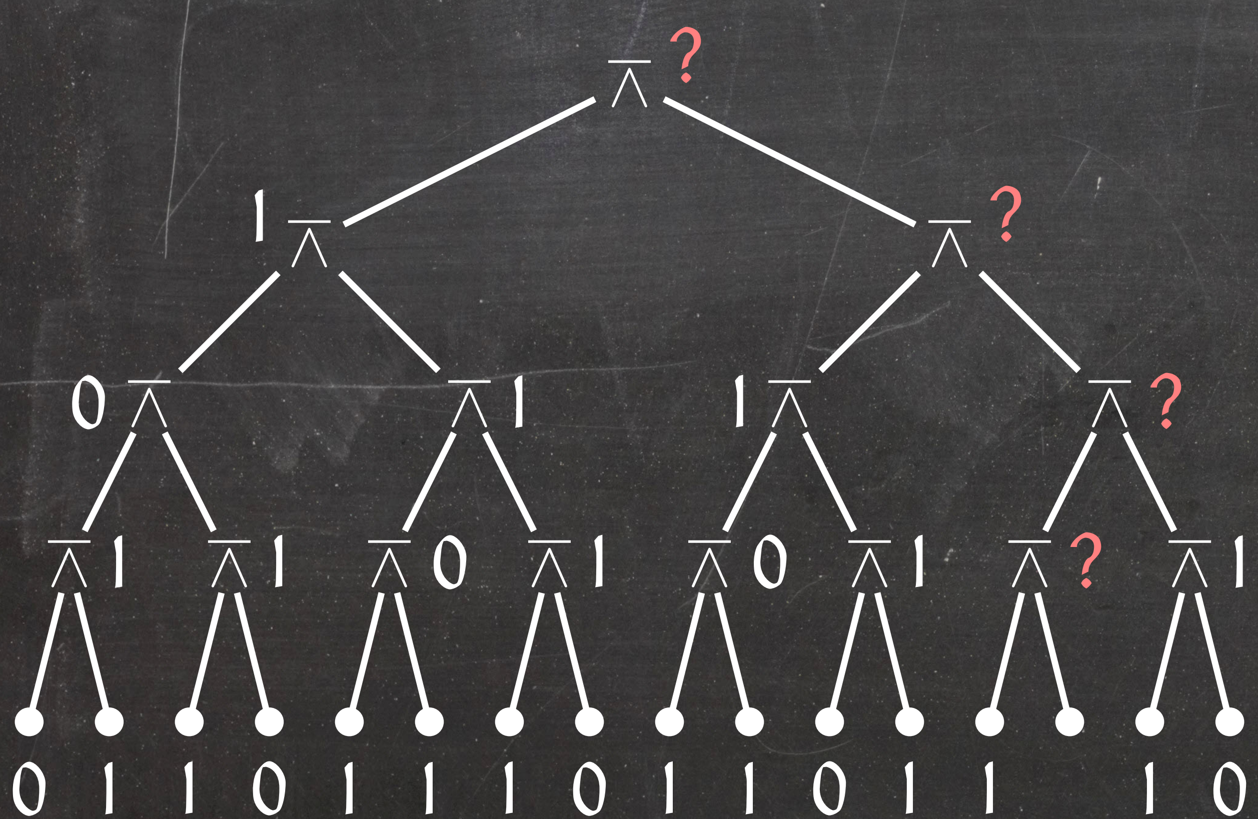

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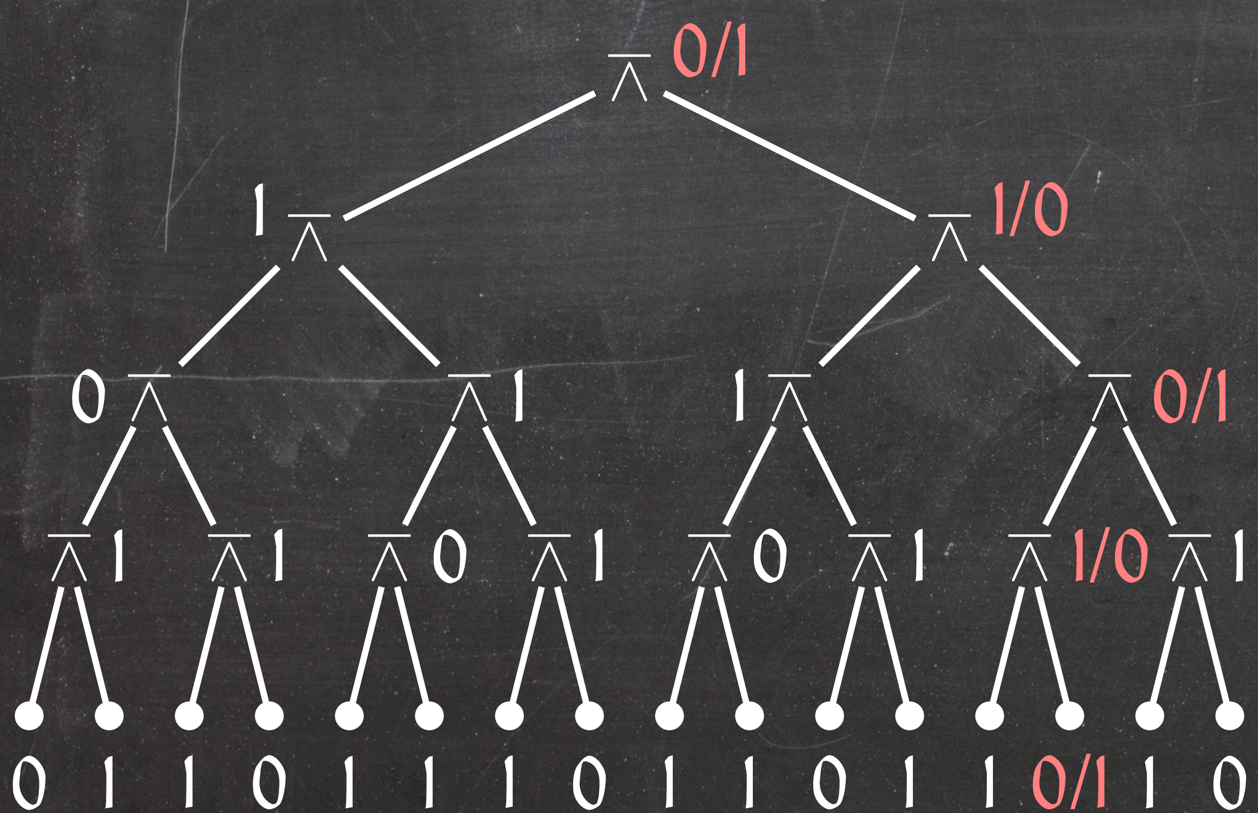

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Game Tree Evaluation: Randomized Algorithm

RandomizedGameValue(v)

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if v is a leaf
    then return its value
coinFlip = RandomNumber(0, 1)
if coinFlip = 1
    then first  = v.leftChild
         second = v.rightChild
    else first  = v.rightChild
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if not f = GameValue(first)
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$$\begin{aligned} E_1[T(n)] &\le \frac{1}{2} \cdot E_0\left[T\left(\frac{n}{2}\right)\right] + \frac{1}{2} \cdot \left(E_1\left[T\left(\frac{n}{2}\right)\right] + E_0\left[T\left(\frac{n}{2}\right)\right]\right) + O(1) \\ &= E_0\left[T\left(\frac{n}{2}\right)\right] + \frac{1}{2} \cdot E_1\left[T\left(\frac{n}{2}\right)\right] + O(1) \end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Lemma: The expected running time of RandomizedGameValue on any input is in $O(n^{0.754})$.

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$$E[T(n)] \le \max\left(2 \cdot E_1\left[T\left(\frac{n}{2}\right)\right], E_1[T(n)]\right)$$

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$$E_1[T(n)] \in O(n^{0.754}) \Rightarrow E[T(n)] \in O(n^{0.754})$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Base case: $1 \le n < 2$.

$T(n) \in O(1) \Rightarrow E_I[T(n)] \le cn^\alpha - d$ for any d and c sufficiently larger than d.

Game Tree Evaluation: Randomized Algorithm

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Inductive step: $n \ge 2$.

$$E_I[T(n)] \le 2 \cdot E_I\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_I\left[T\left(\frac{n}{2}\right)\right] + a$$

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Inductive step: $n \ge 2$.

$$\begin{aligned} E_I[T(n)] &\le 2 \cdot E_I\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_I\left[T\left(\frac{n}{2}\right)\right] + a \\ &\le 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_l[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \leq 0.754$.

Inductive step: $n \geq 2$.

$$
\begin{aligned}
E_l[T(n)] &\leq 2 \cdot E_l\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_l\left[T\left(\frac{n}{2}\right)\right] + a \\
&\leq 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \\
&= cn^\alpha\left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2}
\end{aligned}
$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$\begin{aligned}
E_I[T(n)] &\le 2 \cdot E_I\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_I\left[T\left(\frac{n}{2}\right)\right] + a \\
&\le 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \\
&= cn^\alpha \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2} \\
&\le cn^\alpha \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) - d \quad \forall d \ge \frac{2}{3}a
\end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_l[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

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$$\begin{aligned}
E_l[T(n)] &\le 2 \cdot E_l\left[T\left(\frac{n}{4}\right)\right] + \frac{1}{2} \cdot E_l\left[T\left(\frac{n}{2}\right)\right] + a \\
&\le 2 \cdot \left[c\left(\frac{n}{4}\right)^\alpha - d\right] + \frac{1}{2} \cdot \left[c\left(\frac{n}{2}\right)^\alpha - d\right] + a \\
&= cn^\alpha\left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2} \\
&\le cn^\alpha\left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) - d \quad \forall d \ge \frac{2}{3}a \\
&= cn^\alpha\left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}}\right) - d
\end{aligned}$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

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$$= cn^\alpha \left(\frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2} \right) - d$$

Game Tree Evaluation: Randomized Algorithm

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$$= cn^\alpha \left(\frac{34 + 2 \cdot \sqrt{33}}{34 + 2 \cdot \sqrt{33}} \right) - d$$

Game Tree Evaluation: Randomized Algorithm

Claim: $E_I[T(n)] \le cn^\alpha - d$ for some $c > d > 0$ and all $n \ge 1$, where $\alpha = \lg\left(\frac{1+\sqrt{33}}{4}\right) \le 0.754$.

Inductive step: $n \ge 2$.

$$E_I[T(n)] \le cn^\alpha \left(\frac{2}{\left(\frac{1+\sqrt{33}}{4}\right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}}\right) - d$$

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$$= cn^\alpha - d$$

Summary

Algorithms that are fast on average are often easier to design and faster in practice than worst-case efficient algorithms.

In some applications, worst-case guarantees matter!

Average-case analysis provides a valid performance prediction only if the inputs are uniformly distributed.

Randomized algorithms remove this dependence on the input distribution (but rely on a good random number generator).

There are problems where randomized algorithms are provably faster than the best possible deterministic algorithm.