Average-Case Analysis and Randomization

Textbook Reading
Chapter 7 & Sections 8.4, 9.2
Overview

Design principle
- Do the easy thing and hope it works for most inputs
- Make random choices and hope they're good

Problems
- Sorting (Quick Sort)
- Permuting
- Selection
- Game tree evaluation
Quick Sort Revisited

The problem with deterministic Quick Sort:

The running time is in $O(n \log n)$, but the algorithm for finding the pivot is non-trivial (and slow).
Quick Sort Revisited

The problem with deterministic Quick Sort:

The running time is in $O(n \log n)$, but the algorithm for finding the pivot is non-trivial (and slow).

Remedy:

Blindly use the last element as pivot.

SimpleQuickSort($A, \ell, r$)

1. if $r \leq \ell$
2. then return
3. $m = \text{Partition}(A, \ell, r)$
4. SimpleQuickSort($A, \ell, m - 1$)
5. SimpleQuickSort($A, m + 1, r$)

Partition($A, \ell, r$)

1. $i = \ell - 1$
2. for $j = \ell$ to $r - 1$
3. do if $A[j] \leq A[r]$
4. then $i = i + 1$
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7. return $i + 1$
Average-Case Analysis of Simple Quick Sort

Lemma: The average-case running time of SimpleQuickSort is in $O(n \lg n)$. 
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**Observation:** Simple Quick Sort behaves the same on all inputs whose elements have the same relative order.
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**Problem:** There are infinitely many different inputs of size $n$!

**Observation:** Simple Quick Sort behaves the same on all inputs whose elements have the same relative order.

$$
\begin{array}{cccc}
8 & 17 & 5 & 43 \\
2 & 3 & 1 & 4 \\
\end{array}
$$

$\Rightarrow$ The input to SimpleQuickSort is a permutation $\pi$ of the sorted output sequence $\langle x_1, x_2, \ldots, x_n \rangle$ we expect as the output.
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⇒ The input to SimpleQuickSort is a permutation $\pi$ of the sorted output sequence $\langle x_1, x_2, \ldots, x_n \rangle$ we expect as the output.

⇒ The average-case running time of SimpleQuickSort is the same as its expected running time on a uniformly random input permutation.
Partitioning Maintains Uniformity

**Lemma:** If $A[\ell \ldots r]$ is a uniform random permutation of the elements in $A[\ell \ldots r]$, then the two subarrays $A[\ell \ldots m - 1]$ and $A[m + 1 \ldots r]$ produced by $\text{Partition}(A, \ell, r)$ are also uniform random permutations of the elements they contain.
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The behaviour of Partition depends only on the sequence of $-$s and $+$s!
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The behaviour of Partition depends only on the sequence of $-s$ and $+s$!

The $-s$ are exactly the elements that end up in $A[\ell \ldots m - 1]$, the $+s$ end up in $A[m + 1 \ldots r]$. 

\begin{array}{cccccccc}
1 & 7 & 3 & 4 & 8 & 6 & 2 & 5 \\
\end{array} 

\begin{array}{cccccccc}
- & + & - & - & + & + & - & p \\
\end{array}
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In a uniformly random permutation, any permutation of the $-$s or $+$s is equally likely.
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Each such permutation produces a different permutation of $A[\ell \ldots m - 1]$ or $A[m + 1 \ldots r]$. 
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In a uniformly random permutation, any permutation of the $-$s or $+$s is equally likely.

Each such permutation produces a different permutation of $A[\ell \ldots m - 1]$ or $A[m + 1 \ldots r]$.

$\Rightarrow$ $A[\ell \ldots m - 1]$ and $A[m + 1 \ldots r]$ are uniform random permutations.
Average-Case Analysis of Simple Quick Sort

**Observation:** The running time of SimpleQuickSort is in O(n + C), where C is the number of comparisons it performs between input elements.

**SimpleQuickSort**\((A, ℓ, r)\)

1. if \(r ≤ ℓ\)
2. then return
3. \(m = \text{Partition}(A, ℓ, r)\)
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- There are $O(n)$ recursive calls in total.
- The cost of each recursive call, excluding the call to Partition is constant.
- The cost of Partition is $O(1 + \# \text{ comparisons it performs})$. 
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- There are $O(n)$ recursive calls in total.
- The cost of each recursive call, excluding the call to Partition is constant.
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$\Rightarrow$ It suffices to prove that $E[C] \in O(n \lg n)$. 


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Observation: Two elements $x_i$ and $x_j$ are compared at most once.
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- The pivot is not part of the recursive calls `SimpleQuickSort(A, ℓ, m - 1)` and `SimpleQuickSort(A, m + 1, r)`.

**SimpleQuickSort(A, ℓ, r)**

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Let $C_{ij} = \begin{cases} 1 & \text{if } x_i \text{ and } x_j \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$

```plaintext
\textbf{SimpleQuickSort}(A, ℓ, r)

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3  \( m = \text{Partition}(A, ℓ, r) \)
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6  \( \text{swap } A[i + 1] \) and \( A[r] \)
7  return \( i + 1 \)
```
Average-Case Analysis of Simple Quick Sort

Observation: Two elements $x_i$ and $x_j$ are compared at most once.

- Each call $\text{SimpleQuickSort}(A, \ell, r)$ compares every element in $A[\ell \ldots r-1]$ to the pivot $p$ stored in $A[r]$.
- The pivot is not part of the recursive calls $\text{SimpleQuickSort}(A, \ell, m-1)$ and $\text{SimpleQuickSort}(A, m+1, r)$.

Let $C_{ij} = \begin{cases} 1 & \text{if } x_i \text{ and and } x_j \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow C = \sum_{i,j} C_{ij}$

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\[ C = \sum_{i,j} C_{ij} \]

\[ E[C] = E \left[ \sum_{i,j} C_{ij} \right] = \sum_{i,j} E[C_{ij}] \]

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**Observation:** Two elements $x_i$ and $x_j$ are compared if and only if $x_i$ or $x_j$ is the first element in $\{x_i, x_{i+1}, \ldots, x_j\}$ chosen as a pivot.
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\[ R \text{ must choose as pivot when in same partition} \]
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**Corollary:** $E[C_{ij}] = \frac{2}{j - i + 1}$. 
Average-Case Analysis of Simple Quick Sort

\[ E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}] \]
Average-Case Analysis of Simple Quick Sort

\[ E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[C_{ij}] \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1} \]

\text{value we proved}
Average-Case Analysis of Simple Quick Sort

\[
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\]

\[
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\]

\[
= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}
\]

\[
\text{I have set } \frac{2}{n+i+1} = \frac{2}{2+1} \geq \frac{2}{3+i+1} \geq \ldots \geq \frac{2}{n-1+i+1} = \frac{2}{n+i}
\]
Average-Case Analysis of Simple Quick Sort

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\]

\[
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\]

\[
= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}
\]

\[
< 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}
\]
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\[ < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k} \]

\[ = 2(n - 1)H_n. \]

\[ H_n = \sum_{i=1}^{n} \frac{1}{i} = \text{nth Harmonic Number} \]
Average-Case Analysis of Simple Quick Sort

\[ \sum_{i=1}^{n} \frac{1}{i} \]
Average-Case Analysis of Simple Quick Sort

\[ \int_{1}^{n+1} \frac{d}{x} < \sum_{i=1}^{n} \frac{1}{i} \]
Average-Case Analysis of Simple Quick Sort

\[
\ln(n + 1) = \int_{1}^{n+1} \frac{d}{x} < \sum_{i=1}^{n} \frac{1}{i}
\]
Average-Case Analysis of Simple Quick Sort

\[
\ln(n + 1) = \int_{1}^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_{1}^{n} \frac{dx}{x}
\]
Average-Case Analysis of Simple Quick Sort

\[ \ln(n + 1) = \int_1^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_1^n \frac{dx}{x} = 1 + \ln n \]
Average-Case Analysis of Simple Quick Sort

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\ln(n+1) = \int_{1}^{n+1} \frac{dx}{x} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_{1}^{n} \frac{dx}{x} = 1 + \ln n
\]

\[\Rightarrow E[C] \leq 2(n - 1)H_n \in O(n \lg n)\]
Interpretation of Average-Case Analysis

Algorithms that are fast in the worst case are the gold standard but are difficult to design and often have higher constant factors than algorithms that are efficient on average.
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Worst-case efficiency is desirable if we need performance guarantees every single time we run the algorithm.

\[ \text{e.g. simple}\text{-}\text{quicksort} \]

\[ \begin{array}{c}
1, 2, \ldots, n^2 \\
\end{array} \]
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Algorithms that are fast on average are often simpler and on average faster than worst-case efficient algorithms.

They are a good choice when we want good performance most of the time and possibly averaged over running the algorithm many times.
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What exactly is the meaning of the following statement?

“The average-case running time of algorithm A is $T(n)$.”
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Example:

SimpleQuickSort takes $\Theta(n^2)$ time on almost sorted inputs.

There are applications where the inputs to be sorted are all almost sorted.

SimpleQuickSort is a poor choice of a sorting algorithm in such applications.
Randomization

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Since a randomized algorithm behaves differently every time it runs, there is no way to force it to exhibit its worst-case running time!
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The expected running time of a randomized algorithm is an expectation over the random choices the algorithm makes.
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A randomized algorithm makes no assumptions about the input and ensures randomness by making random choices.

Since a randomized algorithm behaves differently every time it runs, there is no way to force it to exhibit its worst-case running time!

The expected running time of a randomized algorithm is an expectation over the random choices the algorithm makes.

⇒ No more assumptions about the probability distribution. We know the distribution of the choices the algorithm makes.
Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in $O(n \log n)$. 
Randomized Quick Sort, Take 1

The expected running time of SimpleQuickSort on a uniform random permutation is in $O(n \lg n)$.

So why don’t we just ensure the input is a uniform random permutation?

RandomPermutationQuickSort(A)

1. RandomPermute(A)
2. SimpleQuickSort(A, 1, n)
Randomized Quick Sort, Take 1

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We can compute a uniform random permutation in $O(n)$ time in the worst case.
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1 RandomPermute(A)
2 SimpleQuickSort(A, 1, n)

We can compute a uniform random permutation in $O(n)$ time in the worst case.

Corollary: The expected running time of RandomPermutationQuickSort is in $O(n \log n)$.
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.

So why don’t we make sure we choose a uniform random pivot, no matter the input permutation?

RandomPivotQuickSort(A, ℓ, r)

1. if r ≤ ℓ
2. then return
3. p = RandomNumber(ℓ, r)
4. swap A[p] and A[r]
5. m = Partition(A, ℓ, r)
6. RandomPivotQuickSort(A, ℓ, m – 1)
7. RandomPivotQuickSort(A, m + 1, r)
Randomized Quick Sort, Take 2

The key to the analysis of SimpleQuickSort:

If the input is a uniform random permutation, then any element is equally likely to be chosen as pivot.

So why don’t we make sure we choose a uniform random pivot, no matter the input permutation?

RandomPivotQuickSort(A, ℓ, r)

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2. then return
3. p = RandomNumber(ℓ, r)
4. swap A[p] and A[r]
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6. RandomPivotQuickSort(A, ℓ, m – 1)
7. RandomPivotQuickSort(A, m + 1, r)

Lemma: The expected running time of RandomPivotQuickSort is in O(n lg n).

The analysis is 100% identical to that of SimpleQuickSort!
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
2. \( \text{for } j = n \text{ downto } 2 \)
3. \( \text{do } i = \text{RandomNumber}(1, j) \)
4. \( \text{swap } A[i] \text{ and } A[j] \)
Uniform Random Permutation In Linear Time

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Uniform Random Permutation In Linear Time

RandomPermute(A)

1 \ n = |A| 
2 \ for \ j = n \ downto \ 2 
3 \ \ \ do \ i = RandomNumber(1, j) 
4 \ \ \ swap A[i] and A[j]

Observation: RandomPermute takes \text{O}(n)\ time.
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
2. \( \text{for } j = n \text{ downto } 2 \)
3. \( \text{do } i = \text{RandomNumber}(1, j) \)
4. \( \text{swap } A[i] \text{ and } A[j] \)

Observation: RandomPermute takes \( O(n) \) time.

Lemma: RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).
Uniform Random Permutation In Linear Time

\textbf{RandomPermute}(A)

1. \( n = |A| \)
2. \textbf{for } \( j = n \textbf{ downto } 2 \)
3. \textbf{do } \( i = \text{RandomNumber}(1, j) \)
4. \text{swap } A[i] \text{ and } A[j]

\textbf{Observation: } \text{RandomPermute takes } O(n) \text{ time.}

\textbf{Lemma: } \text{RandomPermute produces each permutation of the input array } A \text{ with probability } \frac{1}{n!}.

\text{Induction on } n.
Uniform Random Permutation In Linear Time

**RandomPermute(A)**

1. $n = |A|$
2. for $j = n$ downto 2
3. do $i = \text{RandomNumber}(1, j)$
4. swap $A[i]$ and $A[j]$

**Observation:** RandomPermute takes $O(n)$ time.

**Lemma:** RandomPermute produces each permutation of the input array $A$ with probability $\frac{1}{n!}$.

Induction on $n$.

If $n = 1$, then it produces the only possible permutation with probability $1 = \frac{1}{1!}$. 
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
2. \( \text{for } j = n \text{ downto } 2 \)
3. \( \text{do } i = \text{RandomNumber}(1, j) \)
4. \( \text{swap } A[i] \text{ and } A[j] \)

Observation: RandomPermute takes \( O(n) \) time.

Lemma: RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).

If \( n > 1 \), then to produce the permutation \( \langle x_1, x_2, \ldots, x_n \rangle \) (event \( E \)), we need to

- Place \( x_n \) into \( A[n] \) (event \( E_1 \)) and
- Place \( x_1, x_2, \ldots, x_{n-1} \) into \( A[1 \ldots n-1] \) (event \( E_2 \)).
Uniform Random Permutation In Linear Time

RandomPermute(A)

1. \( n = |A| \)
2. \( \text{for } j = n \text{ downto } 2 \)
3. \( \text{do } i = \text{RandomNumber}(1, j) \)
4. \( \text{swap } A[i] \text{ and } A[j] \)

Observation: RandomPermute takes \( O(n) \) time.

Lemma: RandomPermute produces each permutation of the input array \( A \) with probability \( \frac{1}{n!} \).

If \( n > 1 \), then to produce the permutation \( \langle x_1, x_2, \ldots, x_n \rangle \) (event \( E \)), we need to

- Place \( x_n \) into \( A[n] \) (event \( E_1 \)) and
- Place \( x_1, x_2, \ldots, x_{n-1} \) into \( A[1 \ldots n-1] \) (event \( E_2 \)).

So \( P[E] = P[E_1 \cap E_2] = P[E_1] \cdot P[E_2|E_1] = \frac{1}{n} \cdot \frac{1}{(n-1)!} = \frac{1}{n!} \).
**Randomized Selection**

\[
\text{RandomizedSelection}(A, \ell, r, k) = \begin{cases} 
\text{return } A[\ell] & \text{if } r \leq \ell \\
\text{let } p = \text{RandomNumber}(\ell, r) \\
\text{swap } A[p] \text{ and } A[r] \\
\text{let } m = \text{Partition}(A, \ell, r) \\
\text{if } m - \ell = k - 1 \\
\text{return } A[m] \\
\text{else if } m - \ell \geq k \\
\text{then RandomizedSelection}(A, \ell, m - 1, k) \\
\text{else RandomizedSelection}(A, m + 1, r, k - (m + 1 - \ell))
\end{cases}
\]

\[\text{recall } \text{select}(k) = \text{find } k^{th} \text{ element from unsorted array}\]

- Ideally want median
Randomized Selection

RandomizedSelection(A, ℓ, r, k)

1. if r ≤ ℓ
2. then return A[ℓ]
3. p = RandomNumber(ℓ, r)
4. swap A[p] and A[r]
5. m = Partition(A, ℓ, r)
6. if m − ℓ = k − 1
7. then return A[m]
8. else if m − ℓ ≥ k
9. then RandomizedSelection(A, ℓ, m − 1, k)
10. else RandomizedSelection(A, m + 1, r, k − (m + 1 − ℓ))

Lemma: The expected running time of RandomizedSelection is in \( O(n) \).
Randomized Selection

Observation: If we choose the $i$th smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(\max(n - i, i - 1))].$$
**Randomized Selection**

**Observation**: If we choose the $i$th smallest element as pivot, then

$$E[T(n)] \leq O(n) + E[T(max(n - i, i - 1))]$$

**Corollary**: $E[T(n)] \leq O(n) + \frac{1}{n} \sum_{i=1}^{n} E[T(max(n - i, i - 1))]$. 
Randomized Selection

**Observation:** If we choose the $i$th smallest element as pivot, then

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**Corollary:**

$$E[T(n)] \leq O(n) + \frac{1}{n} \sum_{i=1}^{n} E[T(max(n - i, i - 1))].$$

**Claim:**

$$E[T(n)] \leq cn, \text{ for some } c > 0.$$
Randomized Selection

Observation: If we choose the $i$th smallest element as pivot, then
\[ E[T(n)] \leq O(n) + E[T(\max(n - i, i - 1))]. \]

Corollary: \[ E[T(n)] \leq O(n) + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(n - i, i - 1))]. \]

Claim: \[ E[T(n)] \leq cn, \text{ for some } c > 0. \]

Base case: \[ 1 \leq n < 4. \]
\[ T(n) \leq c \leq cn. \]
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq a n + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))] 
\]
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]
\]

\[
\leq an + \frac{2}{n} \sum_{i=[n/2]}^{n-1} E[T(i)]
\]
**Randomized Selection**

**Inductive step:** $n \geq 4$.

$$E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]$$

$$\leq an + \frac{2}{n} \sum_{i=\lceil n/2 \rceil}^{n-1} E[T(i)]$$

$$\leq an + \frac{2}{n} \sum_{i=\lceil n/2 \rceil}^{n-1} c_i$$
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))]
\]

\[
\leq an + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} E[T(i)]
\]

\[
\leq an + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} ci
\]

\[
= an + \frac{2c}{n} \left( \sum_{i=1}^{n-1} i - \sum_{i=1}^{\lfloor n/2 \rfloor - 1} i \right)
\]
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]
\]

\[
\leq an + \frac{2}{n} \sum_{i=\lceil n/2 \rceil}^{n-1} E[T(i)]
\]

\[
\leq an + \frac{2}{n} \sum_{i=\lceil n/2 \rceil}^{n-1} c_i
\]

\[
= an + \frac{2c}{n} \left( \sum_{i=1}^{n-1} i - \sum_{i=1}^{\lceil n/2 \rceil -1} i \right)
\]

\[
= an + \frac{2c}{n} \left( \frac{n(n-1)}{2} - \frac{\lceil n/2 \rceil -1}{2} \right)
\]
Randomized Selection

Inductive step: $n \geq 4$. 

$$E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]$$

$$\leq an + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} E[T(i)]$$

$$\leq an + \frac{2}{n} \sum_{i=\lfloor n/2 \rfloor}^{n-1} ci$$

$$= an + \frac{2c}{n} \left( \sum_{i=1}^{n-1} i - \sum_{i=1}^{\lfloor n/2 \rfloor - 1} i \right)$$

$$= an + \frac{2c}{n} \left( \frac{n(n-1)}{2} - \frac{\lfloor n/2 \rfloor(\lfloor n/2 \rfloor - 1)}{2} \right)$$

$$\leq an + c \left[ n(n-1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]$$
Randomized Selection

Inductive step: $n \geq 4$.

$$E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))]$$

$$\leq an + \frac{c}{n} \left[ n(n - 1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]$$
Randomized Selection

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\]

\[
= an + \frac{c}{n} \left( \frac{3n^2}{4} + \frac{n}{2} \right)
\]
Randomized Selection

**Inductive step:** \( n \geq 4. \)

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i-1, n-i))]
\]

\[
\leq an + \frac{c}{n} \left[ n(n-1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right]
\]

\[
= an + \frac{c}{n} \left( \frac{3n^2}{4} + \frac{n}{2} \right)
\]

\[
= \left( a + \frac{3c}{4} + \frac{c}{2n} \right) n
\]
Randomized Selection

Inductive step: \( n \geq 4 \).

\[
E[T(n)] \leq an + \frac{1}{n} \sum_{i=1}^{n} E[T(\max(i - 1, n - i))] \\
\leq an + \frac{c}{n} \left[ n(n - 1) - \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \right] \\
= an + \frac{c}{n} \left( \frac{3n^2}{4} + \frac{n}{2} \right) \\
= \left( a + \frac{3c}{4} + \frac{c}{2n} \right) n \\
\leq cn \quad \forall c \geq 8a.
\]
Using comparisons only, as Insertion Sort, Merge Sort, Quick Sort do, it is impossible to sort faster than in $\Omega(n \lg n)$ time.
Sorting in Linear Time?

Using comparisons only, as Insertion Sort, Merge Sort, Quick Sort do, it is impossible to sort faster than in $\Omega(n \lg n)$ time.

By exploiting assumptions about the input and using element values in the algorithm, we can do better:
Sorting in Linear Time?

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**Counting sort**: Sorts $n$ integers between 1 and $n$ in $O(n)$ time.
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**Counting sort:** Sorts $n$ integers between 1 and $n$ in $O(n)$ time.

**Radix sort:** Sorts $n$ integers between 1 and $n^c$ in $O(cn)$ time. This is $O(n)$ if $c$ is a constant.
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**Counting sort:** Sorts $n$ integers between 1 and $n$ in $O(n)$ time.

**Radix sort:** Sorts $n$ integers between 1 and $n^c$ in $O(cn)$ time. This is $O(n)$ if $c$ is a constant.

**Bucket sort:** Sorts $n$ real numbers drawn uniformly at random from an interval $[a, b)$ in expected linear time.
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval $[a, b)$. 
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval \([a, b)\).

We can normalize this to the interval \([0, 1)\).
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval \([a, b)\).

We can normalize this to the interval \([0, 1)\).

Divide \([0, 1)\) into subintervals of length \(\frac{1}{n}\).
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval $[a, b)$.

We can normalize this to the interval $[0, 1)$.

Divide $[0, 1)$ into subintervals of length $\frac{1}{n}$.

How many elements do we expect to end up in each subinterval?
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval \([a, b)\).

We can normalize this to the interval \([0, 1)\).

Divide \([0, 1)\) into subintervals of length \(\frac{1}{n}\).

How many elements do we expect to end up in each subinterval? \(1\)!
Bucket Sort

Assume the inputs are real numbers drawn uniformly at random from some interval \([a, b)\). We can normalize this to the interval \([0, 1)\).

Divide \([0, 1)\) into subintervals of length \(\frac{1}{n}\).

How many elements do we expect to end up in each subinterval? 1!

⇒ Strategy:
- Bucket items according to the subinterval they belong to.
- Sort each bucket, hopefully in constant time.
- Concatenate the sorted buckets.
Bucket Sort

\[ \sqrt{\text{preprocess to range 0,1 before}} \]

BucketSort(A)

1. \( n = |A| \)
2. \( B = \) an array of \( n \) empty singly-linked lists
3. \( \text{for } i = 1 \text{ to } n \)
4. \( \quad \text{do prepend } A[i] \text{ to list } B[1 + \lfloor n \cdot A[i] \rfloor] \)
5. \( \text{for } i = 1 \text{ to } n \)
6. \( \quad \text{do InsertionSort}(B[i]) \quad \text{// indexing} \)
7. \( j = 0 \)
8. \( \text{for } i = 1 \text{ to } n \)
9. \( \quad \text{do for every element } x \in B[i] \)
10. \( \quad \quad \text{do } A[j] = x \)
11. \( \quad \quad j = j + 1 \)
Bucket Sort

BucketSort(A)

1. \( n = |A| \)
2. \( B = \) an array of \( n \) empty singly-linked lists
3. for \( i = 1 \) to \( n \)
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9. do for every element \( x \in B[i] \)
10. do \( A[j] = x \)
11. \( j = j + 1 \)

This is where we depart from using comparisons only!
Bucket Sort

BucketSort(A)

1. \( n = |A| \)
2. \( B = \) an array of \( n \) empty singly-linked lists
3. \( \text{for } i = 1 \text{ to } n \) \( \text{do} \) prepend \( A[i] \) to list \( B[1 + \lfloor n \cdot A[i] \rfloor] \)
4. \( \text{for } i = 1 \text{ to } n \) \( \text{do} \) InsertionSort\( (B[i]) \)
5. \( j = 0 \)
6. \( \text{for } i = 1 \text{ to } n \) \( \text{do} \) for every element \( x \in B[i] \) \( \text{do} \) \( A[j] = x \)
7. \( j = j + 1 \)

Worst-case running time: \( O(n^2) \)

This is where we depart from using comparisons only!
Bucket Sort

BucketSort(A)

1. n = |A|
2. B = an array of n empty singly-linked lists
3. for i = 1 to n
   4. do prepend A[i] to list B[1 + ⌊n · A[i]⌋]
5. for i = 1 to n
   6. do InsertionSort(B[i])
7. j = 0
8. for i = 1 to n
   9. do for every element x ∈ B[i]
      10. do A[j] = x
      11. j = j + 1

Worst-case running time: $O(n^2)$

This is where we depart from using comparisons only!

Why not Merge Sort?
Bucket Sort

BucketSort(A)

1. n = |A|
2. B = an array of n empty singly-linked lists
3. for i = 1 to n
   do prepend A[i] to list B[1 + ⌊n \cdot A[i]⌋]
4. for i = 1 to n
   do InsertionSort(B[i])
5. j = 0
6. for i = 1 to n
   do for every element x ∈ B[i]
      do A[j] = x
      j = j + 1

Worst-case running time: \(O(n^2)\)

This is where we depart from using comparisons only!

Why not Merge Sort?
- It only helps in the worst case.
- It's more complicated.
- It actually hurts when buckets are small, which is what we expect.
Bucket Sort

Running time: \( T(n) \in \mathcal{O}\left(n + \sum_{i=1}^{n} n_i^2\right)\)

\(n_i = \) the number of elements in \(B[i]\)
Bucket Sort

Running time: \( T(n) \in O \left( n + \sum_{i=1}^{n} n_i^2 \right) \)

\( n_i = \) the number of elements in \( B[i] \)

\( E[T(n)] \in O \left( n + \sum_{i=1}^{n} E[n_i]^2 \right) \)
Bucket Sort

Running time: $T(n) \in O\left(n + \sum_{i=1}^{n} n_i^2 \right)$

$n_i = \text{the number of elements in } B[i]$

$E[T(n)] \in O\left(n + \sum_{i=1}^{n} E[n_i]^2 \right)$

Lemma: $E[n_i^2] < 2$. 
Bucket Sort

Running time: $T(n) \in O \left( n + \sum_{i=1}^{n} n_i^2 \right)$

$n_i$ = the number of elements in $B[i]$

$E[T(n)] \in O \left( n + \sum_{i=1}^{n} E[n_i]^2 \right)$

Lemma: $E[n_i^2] < 2$.

Corollary: $E[T(n)] \in O(n)$.

\[ \leq cn + c_2 \cdot n \]
\[ \leq c_3 (n + 2n) \]
\[ \leq c_5 \cdot 5n = O(n) \]
Bucket Sort

Lemma: $E[n_i^2] < 2.$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

$$X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in } B[i] \\
0 & \text{otherwise}
\end{cases}$$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

Let $X_j = \begin{cases} 1 & \text{A}[j] \text{ ends up in } \text{B}[i] \\ 0 & \text{otherwise} \end{cases}$

Then $n_i = \sum_{j=1}^{n} X_j$ just for bucket $i$.

$A[i] \in \left[ \frac{i}{n}, \frac{i+1}{n} \right]$. Unsorted ascending $\frac{1}{n}$. ...
Bucket Sort

Lemma:  $E[n_i^2] < 2$. 

$$X_j = \begin{cases} 1 & \text{A}[j] \text{ ends up in B}[i] \\ 0 & \text{otherwise} \end{cases}$$

$$n_i = \sum_{j=1}^{n} X_j$$

$$E[n_i^2] = E \left[ \left( \sum_{j=1}^{n} X_j \right)^2 \right]$$
Bucket Sort

Lemma: $E[n_i^2] < 2.$

\[ X_j = \begin{cases} 1 & \text{A}[j] \text{ ends up in } B[i] \\ 0 & \text{otherwise} \end{cases} \]

\[ n_i = \sum_{j=1}^{n} X_j \]

\[ E[n_i^2] = E \left[ \left( \sum_{j=1}^{n} X_j \right)^2 \right] = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \right] \]
Bucket Sort

Lemma: \( E[n_i^2] < 2. \)

\[
X_j = \begin{cases} 
1 & \text{A}[j] \text{ ends up in B}[i] \\
0 & \text{otherwise}
\end{cases}
\]

\[
n_i = \sum_{j=1}^{n} X_j
\]

\[
E[n_i^2] = E \left[ \left( \sum_{j=1}^{n} X_j \right)^2 \right] = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[X_j X_k]
\]
Bucket Sort

Lemma: \( \mathbb{E}[n_i^2] < 2. \)

\[
X_j = \begin{cases} 
  1 & \text{A}[j] \text{ ends up in } B[i] \\
  0 & \text{otherwise}
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\]

\[
= \sum_{j=1}^{n} \mathbb{E}[X_j^2] + \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E}[X_j] \mathbb{E}[X_k]
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\[
= \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n} \sum_{k \neq j}^{n} E[X_j] E[X_k]
\]

\(X_j \text{ and } X_j \text{ are clearly not independent.}\)

\(X_j \text{ and } X_k \text{ are independent.}\)
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\[ E[X_j] = \frac{1}{n} \cdot 1 + \left(1 - \frac{1}{n}\right) \cdot 0 = \frac{1}{n} \]
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E[n_i^2] = \sum_{j=1}^{n} E[X_j^2] + \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} E[X_j]E[X_k] = n \cdot \frac{1}{n} + \frac{n(n-1)}{n^2} < 2
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Randomized Bucket Sort?

For Quick Sort, we were able to eliminate assumptions about the input distribution using randomization.
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Does that work for Bucket Sort?
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We can't simply change them without changing the algorithm's output.
Consider a game where two players, Max and Minnie, take turns. Assume the game cannot end in a draw.
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We label a win for Max with 1 and a win for Minnie with 0.
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Max (Minnie) has a winning strategy if he can win the game no matter how Minnie (Max) plays.
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We can model all possible games as a game tree:
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Restrict ourselves to binary game trees of height $2k \Rightarrow n = 2^{2k}$ leaves
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Max

Minnie

Max

Minnie

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$(a \land b) \lor (c \land d) = (a \land b) \land (c \land d)$
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Game Tree Evaluation: A Deterministic Algorithm

\[
\text{GameValue}(v)
\]

1. if \( v \) is a leaf
2. then return its value
3. else return not (GameValue(\(v.\text{leftChild}\)) \text{ and } \text{GameValue}(\(v.\text{rightChild}\)))
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- One recursive call per node
- \(2n - 1\) nodes

⇒ Running time \(O(n)\)
Game Tree Evaluation: A Deterministic Algorithm

GameValue(v)
1 if v is a leaf
2 then return its value
3 if not GameValue(v.leftChild)
4 then return 1
5 else return not GameValue(v.rightChild)

• One recursive call per node
• 2n – 1 nodes
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Game Tree Evaluation: A Lower Bound

**Observation:** Any deterministic algorithm has to inspect every leaf in the worst case and thus takes $\Omega(n)$ time in the worst case.
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Game Tree Evaluation: Randomized Algorithm

RandomizedGameValue(v)

1. if v is a leaf
2. then return its value
3. coinFlip = RandomNumber(0, 1)
4. if coinFlip = 1
5. then first = v.leftChild
6. second = v.rightChild
7. else first = v.rightChild
8. second = v.leftChild
9. if not f = GameValue(first)
10. then return 1
11. else return not GameValue(second)
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8. second = v.leftChild
9. if not f = GameValue(first)
10. then return 1
11. else return not GameValue(second)

Lemma: The expected running time of RandomizedGameValue on any input is in $O(n^{0.754})$. 
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Game Tree Evaluation: Randomized Algorithm

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\[ = E_0 \left[ T \left( \frac{n}{2} \right) \right] + \frac{1}{2} \cdot E_1 \left[ T \left( \frac{n}{2} \right) \right] + O(1) \]

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**Game Tree Evaluation: Randomized Algorithm**

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\[
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\]

\[
E[T(n)] \leq \max \left( 2 \cdot E_1 \left[ T \left( \frac{n}{2} \right) \right], E_1[T(n)] \right)
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Game Tree Evaluation: Randomized Algorithm

Lemma: The expected running time of RandomizedGameValue on any input is in \( O(n^{0.754}) \).

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\[ E_i[T(n)] \in O(n^{0.754}) \Rightarrow E[T(n)] \in O(n^{0.754}) \]
Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

$$\alpha = \log \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754.$$
Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where
$\alpha = \log \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754$.

Base case: $1 \leq n < 2$.

$T(n) \in \mathcal{O}(1) \Rightarrow E_1[T(n)] \leq cn^\alpha - d$ for any $d$ and $c$ sufficiently larger than $d$. 
Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

$$\alpha = \log \left(\frac{1 + \sqrt{33}}{4}\right) \leq 0.754.$$  

Inductive step: $n \geq 2$.

$$E_1[T(n)] \leq 2 \cdot E_1[T\left(\frac{n}{4}\right)] + \frac{1}{2} \cdot E_1[T\left(\frac{n}{2}\right)] + a$$
Game Tree Evaluation: Randomized Algorithm

Claim: \( E_1[T(n)] \leq cn^\alpha - d \) for some \( c > d > 0 \) and all \( n \geq 1 \), where
\[
\alpha = \lg \left( \frac{1+\sqrt{33}}{4} \right) \leq 0.754.
\]

**Inductive step:** \( n \geq 2 \).

\[
E_1[T(n)] \leq 2 \cdot E_1 \left[ T \left( \frac{n}{4} \right) \right] + \frac{1}{2} \cdot E_1 \left[ T \left( \frac{n}{2} \right) \right] + a
\]

\[
\leq 2 \cdot \left[ c \left( \frac{n}{4} \right)^\alpha - d \right] + \frac{1}{2} \cdot \left[ c \left( \frac{n}{2} \right)^\alpha - d \right] + a
\]
Game Tree Evaluation: Randomized Algorithm

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= cn^\alpha \left( \frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha} \right) + a - \frac{5d}{2}
$$
Game Tree Evaluation: Randomized Algorithm

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\]

Inductive step: $n \geq 2$.

\[
E_1[T(n)] \leq 2 \cdot E_1[T\left(\frac{n}{4}\right)] + \frac{1}{2} \cdot E_1[T\left(\frac{n}{2}\right)] + a
\]
\[
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= cn^\alpha \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) + a - \frac{5d}{2}
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\[
\leq cn^\alpha \left(\frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha}\right) - d \quad \forall d \geq \frac{2}{3}a
\]
Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

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Inductive step: $n \geq 2$.

$$E_1[T(n)] \leq 2 \cdot E_1 \left[ T \left( \frac{n}{4} \right) \right] + \frac{1}{2} \cdot E_1 \left[ T \left( \frac{n}{2} \right) \right] + a$$

$$\leq 2 \cdot \left[ c \left( \frac{n}{4} \right)^\alpha - d \right] + \frac{1}{2} \cdot \left[ c \left( \frac{n}{2} \right)^\alpha - d \right] + a$$

$$= cn^\alpha \left( \frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha} \right) + a - \frac{5d}{2}$$

$$\leq cn^\alpha \left( \frac{2}{4^\alpha} + \frac{1}{2 \cdot 2^\alpha} \right) - d \quad \forall d \geq \frac{2}{3}a$$

$$= cn^\alpha \left( \frac{2}{\left( \frac{1+\sqrt{33}}{4} \right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d$$
Game Tree Evaluation: Randomized Algorithm

Claim: \( E_1[T(n)] \leq cn^\alpha - d \) for some \( c > d > 0 \) and all \( n \geq 1 \), where \( \alpha = \lg \left( \frac{1+\sqrt{33}}{4} \right) \leq 0.754 \).

Inductive step: \( n \geq 2 \).

\[
E_1[T(n)] \leq cn^\alpha \left( \frac{2}{\left( \frac{1+\sqrt{33}}{4} \right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d
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Inductive step: \( n \geq 2 \).

\[
E_1[T(n)] \leq cn^\alpha \left( \frac{2}{\left( \frac{1+\sqrt{33}}{4} \right)^2} + \frac{1}{2 \cdot \frac{1+\sqrt{33}}{4}} \right) - d
\]

\[
= cn^\alpha \left( \frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2} \right) - d
\]
Game Tree Evaluation: Randomized Algorithm

Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

$\alpha = \log \left( \frac{1 + \sqrt{33}}{4} \right) \leq 0.754.$

Inductive step: $n \geq 2$.

$$E_1[T(n)] \leq cn^\alpha \left( \frac{2}{\left( \frac{1 + \sqrt{33}}{4} \right)^2} + \frac{1}{2 \cdot \frac{1 + \sqrt{33}}{4}} \right) - d$$

$$= cn^\alpha \left( \frac{32 + 2 \cdot (1 + \sqrt{33})}{(1 + \sqrt{33})^2} \right) - d$$

$$= cn^\alpha \left( \frac{34 + 2 \cdot \sqrt{33}}{34 + 2 \cdot \sqrt{33}} \right) - d$$
Claim: $E_1[T(n)] \leq cn^\alpha - d$ for some $c > d > 0$ and all $n \geq 1$, where

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Inductive step: $n \geq 2$.

$$E_1[T(n)] \leq cn^\alpha \left( \frac{2}{\left( \frac{1 + \sqrt{33}}{4} \right)^2} + \frac{1}{2 \cdot \frac{1 + \sqrt{33}}{4}} \right) - d$$

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$$= cn^\alpha \left( \frac{34 + 2 \cdot \sqrt{33}}{34 + 2 \cdot \sqrt{33}} \right) - d$$

$$= cn^\alpha - d$$
Summary

Algorithms that are fast on average are often easier to design and faster in practice than worst-case efficient algorithms.

In some applications, worst-case guarantees matter!

Average-case analysis provides a valid performance prediction only if the inputs are uniformly distributed.

Randomized algorithms remove this dependence on the input distribution (but rely on a good random number generator).

There are problems where randomized algorithms are provably faster than the best possible deterministic algorithm.