NP-Hardness

Textbook Reading
Chapter 34
Overview

- Computational (in)tractability
- Decision problems and optimization problems
- Decision problems and formal languages
- The class P
- Decision and verification
- The class NP
- NP hardness and NP completeness
- Polynomial-time reductions

**NP-complete problems:**
- Satisfiability
- Vertex cover
- Hamiltonian cycle
- Subset sum
Computational (In)tractability

A problem is considered *computationally tractable* if it has a polynomial-time solution. If no such solution exists, the problem is considered *computationally intractable*. 
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\textbf{Tractable problems:}

- Sorting
- Shortest paths
- Minimum spanning tree
- Sequence alignment
- ...
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**Tractable problems:**
- Sorting
- Shortest paths
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**(Probably) intractable problems:**
- Satisfiability
- Vertex cover
- Hamiltonian cycle
- Subset sum
- ...

A decision problem asks a yes/no question:

- Is this input sequence sorted?
- Does there exist a path from v to w in G?
- Does G contain a cycle?
- Are there two points in S that have distance less than d from each other?
- ...
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- Are there two points in $S$ that have distance less than $d$ from each other?
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Every optimization problem has a corresponding decision problem:

- Does $G$ have a spanning tree of weight at most $w$?
- Is there a path from $v$ to $w$ in $G$ of length at most $\ell$?
- Are there $k$ non-overlapping intervals in $S$?
- ...

Decision Problems & Optimization Problems
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- Does G have a spanning tree of weight at most w?
- Is there a path from v to w in G of length at most ℓ?
- Are there k non-overlapping intervals in S?
- ...

To turn an optimization problem into a decision problem, we provide a threshold for the cost/weight/... of the solution.
Yes/no answers usually aren’t that useful in practice.

However, if we can provide evidence that the decision version of an optimization problem is intractable, then so is the optimization problem itself, by the following lemma:

Lemma. If an optimization problem can be solved in polynomial time, then so can its decision version.
Decision Is No Harder Than Optimization

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**Lemma.** If an optimization problem can be solved in polynomial time, then so can its decision version.

**Decision algorithm:**

- Solve the optimization problem.
- Compare the value of its solution to the given threshold.
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Decision Problems & Formal Languages

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**Language → decision problem:**
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**Decision problem → language:**
- Define a binary encoding of the input instances of the decision problem.
- Every instance is now a string over the alphabet \( \Sigma = \{0, 1\} \).
- Let \( L \) be the set of all such strings that encode yes-instances of the decision problem.
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Consider the transformations

- Problem $P \rightarrow$ language $L \rightarrow$ problem $P'$
- Language $L \rightarrow$ problem $P \rightarrow$ language $L'$

Then $P = P'$ and $L = L'$. 
The Complexity Class P

Given a string $x \in \Sigma^*$, a decision algorithm $D$ is said to accept $x$ if it answers yes given input $x$; it rejects $x$ if it answers no given input $x$. 
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Algorithm $D$ is said to decide a language $L \subseteq \Sigma^*$ if it accepts all strings in $L$ and rejects all other strings.

In other words, the output of $D$ is the answer to the question “Does $x$ belong to $L$?”
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The complexity class \( \mathbf{P} \) is the set of all languages that can be decided in polynomial time.

Formally, a language \( L \) belongs to \( \mathbf{P} \) if and only if there exists an algorithm \( D \) that decides \( L \) and the running time of \( D \) on any input \( x \in \Sigma^* \) is in \( \mathcal{O}(|x|^c) \) for some constant \( c \).
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Informally, $P$ is the set of all tractable decision problems, since

- We observed that decision problems and formal languages are the same thing and
- We consider a problem tractable if it can be solved in polynomial time.
Verification

Consider an algorithm $V$ that decides a language $L' \subseteq \Sigma^* \times \Sigma^*$, that is, its input is a pair $(x, y)$ such that $x, y \in \Sigma^*$.

Algorithm $V$ is said to verify a language $L$ if

- for every $x \in L$, there exists a $y \in \Sigma^*$ such that $(x, y) \in L'$ and
- for every $x \notin L$, there is no $y \in \Sigma^*$ such that $(x, y) \in L'$.
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Thus, given an input $(x, y)$ consisting of an element $x \in L$ and an appropriate “proof” $y \in \Sigma^*$ that shows that $x \in L$, $V$ answers yes.

For a string $x \notin L$, we can provide whatever “proof” $y$ of its membership in $L$ we want; $V$ will reject every such pair $(x, y)$. 
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Thus, we can think of \( V \) as a “proof checker” that verifies whether any given proof of \( x \)'s membership in \( L \) is in fact correct.

\( V \) does not decide whether \( x \in L \). \( V \) may answer no even if \( x \in L \) if the provided proof of its membership in \( L \) is incorrect.
Verifying is Easier Than Deciding

Verifying a language may be easier than deciding it.
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Given a sequence $S = \langle x_1, x_2, \ldots, x_n \rangle$ of numbers, the element uniqueness problem asks us to decide whether there exist indices $i \neq j$ such that $x_i = x_j$. 
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Let $L$ be the language of all sequences where two such indices exist.
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Verifying \( L \) can be done in constant time!

- Let \( L' = \{(S, (i, j)) \mid x_i = x_j, i \neq j\} \)
- Given some pair \( (S, (i, j)) \), we can decide in constant time whether \( (S, (i, j)) \in L' \) by comparing \( x_i \) and \( x_j \).
- This algorithm verifies \( L \) because \( x \in L \) if and only if there exists a pair \( (i, j) \) such that \( (S, (i, j)) \in L' \).
The complexity class **NP** is the set of all languages that can be verified in polynomial time.

Formally, a language $L$ belongs to NP if and only if there exists a language $L' \in P$ and a constant $c$ such that $x \in L$ if and only if $(x, y) \in L'$ for some $y \in \Sigma^*$, $|y| \leq |x|^c$. 
Lemma: $P \subseteq NP$. (Every language that can be decided in polynomial time can be verified in polynomial time.)

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Nobody knows the answer, but . . .

Given that we know verifying some languages is easier than deciding them, it is likely that $P \subset NP$. 
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Nobody knows the answer, but . . .

Given that we know verifying some languages is easier than deciding them, it is likely that $P \subset NP$.

We will show that there exist languages that cannot be decided (decision problems that cannot be solved) in polynomial time unless $P = NP$!
NP-Hardness and NP-Completeness

A language L is \textbf{NP-hard} if $L \in P$ implies that $P = NP$. 
NP-Hardness and NP-Completeness

A language $L$ is **NP-hard** if $L \in P$ implies that $P = NP$.

A language $L$ is **NP-complete** if

- $L \in NP$ and
- $L$ is NP-hard.

Intuitively, NP-complete languages are the hardest languages in NP.
NP-Hardness and NP-Completeness

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Neither NP-hard nor NP-complete
Polynomial-Time Reductions

An algorithm $R$ reduces a language $L_1 \subseteq \Sigma^*$ to a language $L_2 \subseteq \Sigma^*$ if, for all $x \in \Sigma^*$,

$$x \in L_1 \iff R(x) \in L_2.$$

$R$ is a polynomial-time reduction if its running time is polynomial in $|x|$. 

$\begin{array}{c}
\text{x} \\
\rightarrow \\
R \\
\rightarrow \\
R(x)
\end{array}$
Proving NP-Hardness Using Polynomial-Time Reductions

Lemma: If there exists a polynomial-time reduction $R$ from a language $L_1$ to a language $L_2 \in P$, then $L_1 \in P$. 
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\[ x \rightarrow R \rightarrow R(x) \quad y \rightarrow D \rightarrow \text{yes/no} \]

(Is $y \in L_2$?)
Proving NP-Hardness Using Polynomial-Time Reductions

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Diagram:

- $x \xrightarrow{R} R(x) \xrightarrow{D} \text{yes/no}$
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$x \in L_1 \iff R(x) \in L_2$
Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction $R$ from a language $L_1$ to a language $L_2 \in \mathbb{P}$, then $L_1 \in \mathbb{P}$.

\[
x \in L_1 \iff R(x) \in L_2 \\
R(x) \in L_2 \iff D(R(x)) = \text{yes}
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Proving NP-Hardness Using Polynomial-Time Reductions

Lemma: If there exists a polynomial-time reduction \( R \) from a language \( L_1 \) to a language \( L_2 \in P \), then \( L_1 \in P \).

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x \in L_1 \iff R(x) \in L_2 \quad R(x) \in L_2 \iff D(R(x)) = \text{yes}
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\[
x \in L_1 \iff D'(x) = \text{yes}
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Lemma: If there exists a polynomial-time reduction $R$ from a language $L_1$ to a language $L_2 \in P$, then $L_1 \in P$.

\[ T_R(|x|) \leq c|x|^a, \text{ for some } a, c. \]
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\[ \Rightarrow |R(x)| \leq c|x|^a, \text{ for some } a, c. \]
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Proving NP-Hardness Using Polynomial-Time Reductions 

$x \in L_1 \iff R(x) \in L_2 \iff R(x) \in L_2 \iff D(R(x)) = yes$

$x \in L_1 \iff D'(x) = yes$

$T_R(|x|) \leq c|x|^a$, for some $a, c$. 

$|R(x)| \leq c|x|^a$, for some $a, c$. 

$T_D(|R(x)|) \leq c'|R(x)|^{a'} \leq c'(c|x|^a)^{a'}$, for some $a', c'$. 

Diagram:

- $x$ (input)
- $R$ (reduction)
- $R(x)$ (reduced input)
- $D$ (decider)
- $yes/no$ (output)
**Lemma:** If there exists a polynomial-time reduction \( R \) from a language \( L_1 \) to a language \( L_2 \in P \), then \( L_1 \in P \).

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T_R(|x|) \leq c|x|^a, \text{ for some } a, c.
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\[
\Rightarrow T_D(|R(x)|) \leq c'(|R(x)|)^{a'} \leq c'(c|x|^a)^{a'}, \text{ for some } a', c'.
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\[
\Rightarrow T_{D'}(|x|) = T_R(|x|) + T_D(|R(x)|) \leq c|x|^a + c'(c|x|^a)^{a'} \in O(|x|^{aa'}).
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Where Do We Get Our First NP-Hard Problem From?

To prove that a language $L$ is NP-hard, we need an NP-hard language $L'$ that we can reduce to $L$ in polynomial time.

How do we prove a language $L$ is NP-hard when we haven't shown any other language to be NP-hard yet?

Enter Satisfiability, the mother of all NP-hard problems . . .
Satisfiability (SAT)

A Boolean formula:

\[ F = (x_1 \lor (x_2 \land \bar{x}_3)) \land (\bar{x}_1 \lor x_4) \]
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- A **truth assignment** assigns a value **true** or **false** to each variable in \( F \).
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- A truth assignment assigns a value **true** or **false** to each variable in \(F\).
- A truth assignment **satisfies** \(F\) if it makes \(F\) true. Example:
  - \(x_1 = x_2 = x_3 = x_4 = \text{true}\) satisfies \(F\).
  - \(x_1 = x_2 = x_3 = x_4 = \text{false}\) does not.
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- \(F\) is **satisfiable** if it has a satisfying truth assignment.
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- \(F\) is satisfiable if it has a satisfying truth assignment.

The satisfiability problem (SAT): Given a Boolean formula \(F\), decide whether \(F\) is satisfiable.
3-SAT

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions.

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_2 \lor x_3) \land (\bar{x}_1 \lor \bar{x}_2 \lor \bar{x}_3) \]

The disjunctions are also called clauses.

A formula is in 3-CNF if each of its clauses consists of three literals.

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_2 \lor x_3 \lor x_4) \land (\bar{x}_1 \lor \bar{x}_2 \lor \bar{x}_3) \]

3-SAT: Decide whether a given formula in 3-CNF is satisfiable.
Things We Don't Have Time To Prove

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Thus, $SAT \in P \Rightarrow L \in P$ for all $L \in NP$, that is, $P = NP$. 
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Thus, $SAT \in P \Rightarrow L \in P$ for all $L \in NP$, that is, $P = NP$.

In other words, SAT is NP-hard.
Things We Don't Have Time To Prove

Any polynomial-time verification algorithm for a language $L$ can be turned into a polynomial-time reduction from $L$ to SAT.

Thus, $\text{SAT} \in \text{P} \Rightarrow L \in \text{P}$ for all $L \in \text{NP}$, that is, $\text{P} = \text{NP}$.

In other words, $\text{SAT}$ is $\text{NP}$-hard.

Any Boolean formula $F$ can be turned, in polynomial time, into a Boolean formula $F'$ in 3-CNF, of size $|F'| \in O(|F|)$, and such that $F$ is satisfiable if and only if $F'$ is.
Any polynomial-time verification algorithm for a language $L$ can be turned into a polynomial-time reduction from $L$ to SAT.

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Any Boolean formula $F$ can be turned, in polynomial time, into a Boolean formula $F'$ in 3-CNF, of size $|F'| \in O(|F|)$, and such that $F$ is satisfiable if and only if $F'$ is.

Thus, **3-SAT is NP-hard**.
Examples of Polynomial-Time Reductions

3-SAT

- Vertex cover
- Subset sum

Hamiltonian cycle
**Vertex Cover**

A vertex cover of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every edge in $E$ has at least one endpoint in $S$. 
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Optimization problem: Given a graph $G$, find the smallest possible vertex cover of $G$. 
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**Optimization problem:** Given a graph $G$, find the smallest possible vertex cover of $G$.

**Decision problem:** Given a graph $G$ and an integer $k$, decide whether $G$ has a vertex cover of size $k$. 
Vertex Cover is NP-Hard

Reduction from 3-SAT:

• Given any formula $F$, we build a graph $G_F$ that has a small vertex cover if and only if $F$ is satisfiable.

• $G_F$ will be built from subgraphs, called widgets, that guarantee certain properties of $G_F$.

• It will be obvious that this construction can be carried out in polynomial time.
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Variable widget for variable $x_i$:

- Two vertices $x_i$ and $\bar{x}_i$
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Variable widget for variable $x_i$:
- Two vertices $x_i$ and $\bar{x}_i$
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Clause widget for clause $C_j$:
- Three literal vertices $\lambda_{j,1}$, $\lambda_{j,2}$, and $\lambda_{j,3}$
- Three edges $(\lambda_{j,1}, \lambda_{j,2})$, $(\lambda_{j,2}, \lambda_{j,3})$, and $(\lambda_{j,3}, \lambda_{j,1})$
Vertex Cover is NP-Hard

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]
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- Connect every literal node \( \lambda_{i,j} \) to its corresponding node \( x_k \) or \( \bar{x}_k \)
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**Observation:** Any vertex cover of \( G_F \) of size \( n + 2m \) contains one vertex per variable widget and two vertices per clause widget.

\[ x_1 \quad \bar{x}_1 \quad x_2 \quad \bar{x}_2 \quad x_3 \quad \bar{x}_3 \quad x_4 \quad \bar{x}_4 \]
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Truth assignment

Vertex cover
Vertex Cover is NP-Hard

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Vertex Cover is NP-Complete

Since Vertex Cover is NP-hard, we only have to verify that it is in NP:
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Let $\text{VC} = \{(G, k) \mid G \text{ has a vertex cover of size } k\}$.

To prove that $\text{VC} \in \text{NP}$, we have to prove that there exists a language $\text{VC}' \in \text{P}$ such that $(G, k) \in \text{VC}$ if and only if $(G, k, y) \in \text{VC}'$ for some $y$ with $|y| \in O((|G|, k)^c)$. 
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Since Vertex Cover is NP-hard, we only have to verify that it is in NP:

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Let \( \text{VC}' = \{(G, k, C) \mid C \text{ is a vertex cover of } G \text{ of size } k\} \).

\( \text{VC}' \in \text{P}: \)

- We can test in polynomial time whether every vertex in \( C \) belongs to \( G \).
- We can test in polynomial time whether \( |C| = k \).
- We can test in polynomial time whether every edge of \( G \) has at least one endpoint in \( C \).
A Hamiltonian cycle of a graph $G$ is a simple cycle that contains all vertices of $G$ and whose edges are edges of $G$. 

**Hamiltonian Cycle**
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Hamiltonian Cycle is NP-Complete

Hamiltonian Cycle Problem: Decide whether a given graph $G$ is Hamiltonian.
Hamiltonian Cycle is NP-Complete

**Hamiltonian Cycle Problem:** Decide whether a given graph \( G \) is Hamiltonian.

**Exercise:** Verify that Hamiltonian Cycle is in NP.
Hamiltonian Cycle is NP-Complete

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To prove: Hamiltonian Cycle is NP-hard.
Hamiltonian Cycle is NP-Complete

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To prove: Hamiltonian Cycle is NP-hard.

Reduction from Vertex Cover: Given a vertex cover instance $(G, k)$, we build a graph $G'$ that has a Hamiltonian cycle if and only if $G$ has a vertex cover of size $k$. 
Hamiltonian Cycle is NP-Complete

Hamiltonian Cycle Problem: Decide whether a given graph $G$ is Hamiltonian.

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To prove: Hamiltonian Cycle is NP-hard.

Reduction from Vertex Cover: Given a vertex cover instance $(G, k)$, we build a graph $G'$ that has a Hamiltonian cycle if and only if $G$ has a vertex cover of size $k$. Again, it is trivial to verify that the construction takes polynomial time.
Edge Widgets

We build $G'$ from edge widgets.
Observation: Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.
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Hamiltonian Cycle is NP-Complete

$$k = 2$$

$G$
Hamiltonian Cycle is NP-Complete

$G$, $k = 2$, $G'$
Hamiltonian Cycle is NP-Complete

$G \quad k = 2 \quad G'$
Hamiltonian Cycle is NP-Complete

G

k = 2

G'

G

G'
Hamiltonian Cycle is NP-Complete
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$G$

$k = 2$

$G'$
Hamiltonian Cycle is NP-Complete

\[ G \]

\[ k = 2 \]

\[ G' \]
Hamiltonian Cycle is NP-Complete

$G$

$k = 2$

$G'$
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**Hamiltonian Cycle is NP-Complete**

Hamiltonian cycle

Vertex cover

$G$

$k = 2$

$G'$
Hamiltonian Cycle is NP-Complete

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Hamiltonian Cycle is NP-Complete
Subset Sum

Given:
• A set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct numbers
• A parameter $t$

Question:
Is there a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = t$?
**Subset Sum**

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**Question:**
Is there a subset \( S' \subseteq S \) such that \( \sum_{x \in S'} x = t \)?

**Example:**
\[ S = \{1, 2, 8, 13\} \]

\( S \) has a subset \( S' \) whose elements sum to 22, namely \( S' = \{1, 8, 13\} \), but there is no subset whose elements sum to 12.
Subset Sum is NP-Complete

**Exercise:** Verify that Subset Sum is in NP.
Subset Sum is NP-Complete

**Exercise:** Verify that Subset Sum is in NP.

**To prove:** Subset Sum is NP-hard.
**Subset Sum is NP-Complete**

**Exercise:** Verify that **Subset Sum** is in NP.

**To prove:** **Subset Sum** is NP-hard.

**Reduction from 3-SAT:**

Given a formula $F$ in 3-CNF, we construct a set $S_F$ of $2n + 2m$ numbers with $n + m$ digits in base-10 notation and a number $t = \sum_{i=0}^{n-1} 10^{i+m} + \sum_{i=0}^{m-1} 4 \cdot 10^i$:

\[
\begin{align*}
\underbrace{11 \cdots 1}_{\text{n variable digits}} & \quad \underbrace{44 \cdots 4}_{\text{m clause digits}}
\end{align*}
\]
Subset Sum is NP-Complete

Exercise: Verify that Subset Sum is in NP.

To prove: Subset Sum is NP-hard.

Reduction from 3-SAT:
Given a formula $F$ in 3-CNF, we construct a set $S_F$ of $2n + 2m$ numbers with $n + m$ digits in base-10 notation and a number $t = \sum_{i=0}^{n-1} 10^{i+m} + \sum_{i=0}^{m-1} 4 \cdot 10^i$:

There will be a subset $S' \subseteq S_F$ such that $\sum_{x \in S'} x = t$ if and only if $F$ is satisfiable.
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

Literal numbers:
- \( x_1 \)
- \( \bar{x}_1 \)
- \( x_2 \)
- \( \bar{x}_2 \)
- \( x_3 \)
- \( \bar{x}_3 \)
- \( x_4 \)
- \( \bar{x}_4 \)
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

Literal numbers
\[
\begin{align*}
&x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3, x_4, \bar{x}_4 \\
\end{align*}
\]

Slack numbers
\[
\begin{align*}
&s_1, s'_1, s_2, s'_2, s_3, s'_3, s_4, s'_4 \\
\end{align*}
\]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4) \]

Literal numbers

- \( x_1 \)
- \( \overline{x}_1 \)
- \( x_2 \)
- \( \overline{x}_2 \)
- \( x_3 \)
- \( \overline{x}_3 \)
- \( x_4 \)
- \( \overline{x}_4 \)

Slack numbers

- \( s_1 \)
- \( s'_1 \)
- \( s_2 \)
- \( s'_2 \)
- \( s_3 \)
- \( s'_3 \)
- \( s_4 \)
- \( s'_4 \)

---

| \( t \) | 1 | 1 | 1 | 1 | 4 | 4 | 4 | 4 |
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4) \]
**Subset Sum is NP-Complete**

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

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\[ t \]

\[ 1 \quad 1 \quad 1 \quad 1 \quad 4 \quad 4 \quad 4 \quad 4 \]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

Literal numbers

\[ \begin{align*}
  x_1 & : 1 \\
  \bar{x}_1 & : 1 \\
  x_2 & : 1 \\
  \bar{x}_2 & : 1 \\
  x_3 & : 1 \\
  \bar{x}_3 & : 1 \\
  x_4 & : 1 \\
  \bar{x}_4 & : 1
\end{align*} \]

Slack numbers

\[ \begin{align*}
  s_1 & : 1 \\
  s_1' & : 1 \\
  s_2 & : 1 \\
  s_2' & : 1 \\
  s_3 & : 1 \\
  s_3' & : 1 \\
  s_4 & : 1 \\
  s_4' & : 1 \\
\end{align*} \]

\[ t \quad 1 \quad 1 \quad 1 \quad 1 \quad 4 \quad 4 \quad 4 \quad 4 \]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4) \]

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 Slack numbers
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

- Literal numbers
  - \( x_1 \)
  - \( \bar{x}_1 \)
  - \( x_2 \)
  - \( \bar{x}_2 \)
  - \( x_3 \)
  - \( \bar{x}_3 \)
  - \( x_4 \)
  - \( \bar{x}_4 \)

- Slack numbers
  - \( s_1 \)
  - \( s'_1 \)
  - \( s_2 \)
  - \( s'_2 \)
  - \( s_3 \)
  - \( s'_3 \)
  - \( s_4 \)
  - \( s'_4 \)

| \( t \) | 1 | 1 | 1 | 1 | 4 | 4 | 4 | 4 |
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

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Slack numbers

| \(t\) | 1 | 1 | 1 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

Lemma: \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
**Subset Sum is NP-Complete**

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (x_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

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<tr>
<td>( \bar{x}_4 )</td>
<td>( s'_4 )</td>
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</tbody>
</table>

| \( t \) | 1 | 1 | 1 | 1 | 4 | 4 | 4 | 4 |
**Subset Sum is NP-Complete**

\[
F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4)
\]

\[
x_1 = x_2 = x_3 = x_4 = \text{true}
\]

**Truth assignment**

**Subset \( S' \)**

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

Lemma: \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Truth assignment**

**Subset \( S' \)**

**Literal numbers**

**Slack numbers**

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
**Subset Sum is NP-Complete**

\[ F = (x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3 \lor x_4) \land (\overline{x}_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Truth assignment**

\[ S' \]

**Literal numbers**

<table>
<thead>
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<th>( x_1 )</th>
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<th>( x_3 )</th>
<th>( x_4 )</th>
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**Slack numbers**

<table>
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<tbody>
<tr>
<td>2</td>
<td>1 1 1 1 4 2 1 2</td>
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</table>

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
**Subset Sum is NP-Complete**

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).

**Truth assignment**

\[ \begin{align*}
  x_1 &= 1 \\
  x_2 &= 1 \\
  x_3 &= 1 \\
  x_4 &= 1 \\
  \end{align*} \]

**Literal numbers**

\[ \begin{array}{cccc}
  x_1 & 1 & 1 & 1 \\
  x_2 & 1 & 1 \\
  x_3 & 1 & 1 & 1 \\
  x_4 & 1 & 1 & 1 \\
\end{array} \]

**Slack numbers**

\[ \begin{array}{cccc}
  s'_1 & 2 \\
  s'_2 & 2 \\
  s'_3 & 1 \\
  s'_4 & 2 \\
\end{array} \]

\[ t \quad 1 \quad 1 \quad 1 \quad 1 \quad 4 \quad 4 \quad 4 \quad 4 \]
**Subset Sum is NP-Complete**

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

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\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Truth assignment**

\[ \text{Subset } S' \]

**Literal numbers**

\[ \begin{array}{c|c|c|c|c}
\hline
x_1 & 1 & 1 & 1 & 1 \\
\hline
x_2 & 1 & 1 &   &   \\
\hline
x_3 &   & 1 & 1 & 1 \\
\hline
x_4 &   & 1 & 1 & 1 \\
\hline
\end{array} \]

**Slack numbers**

\[ \begin{array}{c|c|c|c|c|c}
\hline
s'_1 &   &   &   &   & 2 \\
\hline
s'_2 &   &   &   &   & 2 \\
\hline
s'_3 &   &   & 1 &   &   \\
\hline
s'_4 &   &   &   & 2 &   \\
\hline
\end{array} \]

**Lemma:** If \( F \) is satisfiable, then there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).

\[ \begin{array}{c|c|c|c|c|c|c|c|c}
\hline
& t & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 \\
\hline
\end{array} \]
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
Subset Sum is NP-Complete

\[ F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4) \]

\[ x_1 = x_2 = x_3 = x_4 = \text{true} \]

**Lemma:** \( F \) is satisfiable if and only if there is a subset \( S' \subseteq S_F \) such that \( \sum_{x \in S'} x = t \).
Summary

Many important problems are NP-hard or NP-complete.

Examples:

• Satisfiability
• Vertex cover
• Subset sum
• Hamiltonian cycle
• Clique
• Independent set
• …

These problems are unlikely to be solvable in polynomial time.

Techniques to cope with NP-hardness:

• Parameterized algorithms
• Approximation algorithms
• Heuristics