

# NP-Hardness

Textbook Reading

Chapter 34

# Overview

- Computational (in)tractability
- Decision problems and optimization problems
- Decision problems and formal languages
- The class P
- Decision and verification
- The class NP
- NP hardness and NP completeness
- Polynomial-time reductions

## NP-complete problems:

- Satisfiability
- Vertex cover
- Hamiltonian cycle
- Subset sum

# Computational (In)Tractability

A problem is considered **computationally tractable** if it has a polynomial-time solution.  
If no such solution exists, the problem is considered **computationally intractable**.

# Computational (In)Tractability

A problem is considered **computationally tractable** if it has a polynomial-time solution.

If no such solution exists, the problem is considered **computationally intractable**.

## Tractable problems:

- Sorting
- Shortest paths
- Minimum spanning tree
- Sequence alignment
- ...

# Computational (In)Tractability

A problem is considered **computationally tractable** if it has a polynomial-time solution.

If no such solution exists, the problem is considered **computationally intractable**.

## Tractable problems:

- Sorting
- Shortest paths
- Minimum spanning tree
- Sequence alignment
- ...

## (Probably) intractable problems:

- Satisfiability
- Vertex cover
- Hamiltonian cycle
- Subset sum
- ...

# Decision Problems & Optimization Problems

A **decision problem** asks a yes/no question:

- Is this input sequence sorted?
- Does there exist a path from  $v$  to  $w$  in  $G$ ?
- Does  $G$  contain a cycle?
- Are there two points in  $S$  that have distance less than  $d$  from each other?
- ...

# Decision Problems & Optimization Problems

A **decision problem** asks a yes/no question:

- Is this input sequence sorted?
- Does there exist a path from  $v$  to  $w$  in  $G$ ?
- Does  $G$  contain a cycle?
- Are there two points in  $S$  that have distance less than  $d$  from each other?
- ...

Every **optimization problem** has a corresponding decision problem:

- Does  $G$  have a spanning tree of weight at most  $w$ ? *MST*
- Is there a path from  $v$  to  $w$  in  $G$  of length at most  $l$ ? *shortest paths*
- Are there  $k$  non-overlapping intervals in  $S$ ? *interval scheduling*
- ...

# Decision Problems & Optimization Problems

A **decision problem** asks a yes/no question:

- Is this input sequence sorted?
- Does there exist a path from  $v$  to  $w$  in  $G$ ?
- Does  $G$  contain a cycle?
- Are there two points in  $S$  that have distance less than  $d$  from each other?
- ...

Every **optimization problem** has a corresponding decision problem:

- Does  $G$  have a spanning tree of weight at most  $w$ ?
- Is there a path from  $v$  to  $w$  in  $G$  of length at most  $\ell$ ?
- Are there  $k$  non-overlapping intervals in  $S$ ?
- ...

To turn an optimization problem into a decision problem, we provide a threshold for the cost/weight/... of the solution.



# Decision Is No Harder Than Optimization

Yes/no answers usually aren't that useful in practice.

However, if we can provide evidence that the decision version of an optimization problem is intractable, then so is the optimization problem itself, by the following lemma:

**Lemma.** If an optimization problem can be solved in polynomial time, then so can its decision version.

# Decision Is No Harder Than Optimization

Yes/no answers usually aren't that useful in practice.

However, if we can provide evidence that the decision version of an optimization problem is intractable, then so is the optimization problem itself, by the following lemma:

**Lemma.** If an optimization problem can be solved in polynomial time, then so can its decision version.

## Decision algorithm:

- Solve the optimization problem.
- Compare the value of its solution to the given threshold.

# Decision Problems & Formal Languages

A (formal) language over an alphabet  $\Sigma$  is a set of strings formed using letters from  $\Sigma$ :  
 $L \subseteq \Sigma^*$ .

# Decision Problems & Formal Languages

A (formal) language over an alphabet  $\Sigma$  is a set of strings formed using letters from  $\Sigma$ :  
 $L \subseteq \Sigma^*$ .

Formal languages and decision problems are “the same thing”.

# Decision Problems & Formal Languages

A (formal) language over an alphabet  $\Sigma$  is a set of strings formed using letters from  $\Sigma$ :  
 $L \subseteq \Sigma^*$ .

Formal languages and decision problems are “the same thing”.

Language  $\rightarrow$  decision problem:

- Given a language  $L$ , decide whether a given string  $x \in \Sigma^*$  belongs to  $L$ .

# Decision Problems & Formal Languages

A (formal) language over an alphabet  $\Sigma$  is a set of strings formed using letters from  $\Sigma$ :  
 $L \subseteq \Sigma^*$ .

Formal languages and decision problems are “the same thing”.

Language  $\rightarrow$  decision problem:

- Given a language  $L$ , decide whether a given string  $x \in \Sigma^*$  belongs to  $L$ .

Decision problem  $\rightarrow$  language:

- Define a binary encoding of the input instances of the decision problem.
- Every instance is now a string over the alphabet  $\Sigma = \{0, 1\}$ .
- Let  $L$  be the set of all such strings that encode yes-instances of the decision problem.

# Decision Problems & Formal Languages

A (formal) language over an alphabet  $\Sigma$  is a set of strings formed using letters from  $\Sigma$ :  
 $L \subseteq \Sigma^*$ .

Formal languages and decision problems are “the same thing”.

Language  $\rightarrow$  decision problem:

- Given a language  $L$ , decide whether a given string  $x \in \Sigma^*$  belongs to  $L$ .

Decision problem  $\rightarrow$  language:

- Define a binary encoding of the input instances of the decision problem.
- Every instance is now a string over the alphabet  $\Sigma = \{0, 1\}$ .
- Let  $L$  be the set of all such strings that encode yes-instances of the decision problem.

Consider the transformations

- Problem  $P \rightarrow$  language  $L \rightarrow$  problem  $P'$
- Language  $L \rightarrow$  problem  $P \rightarrow$  language  $L'$

Then  $P = P'$  and  $L = L'$ .

# The Complexity Class P

Given a string  $x \in \Sigma^*$ , a decision algorithm D is said to **accept** x if it answers **yes** given input x; it **rejects** x if it answers **no** given input x.



# The Complexity Class P

Given a string  $x \in \Sigma^*$ , a decision algorithm  $D$  is said to **accept**  $x$  if it answers **yes** given input  $x$ ; it **rejects**  $x$  if it answers **no** given input  $x$ .

Algorithm  $D$  is said to **decide** a language  $L \subseteq \Sigma^*$  if it accepts all strings in  $L$  and rejects all other strings.

In other words, the output of  $D$  is the answer to the question "Does  $x$  belong to  $L$ ?"

# The Complexity Class P

Given a string  $x \in \Sigma^*$ , a decision algorithm  $D$  is said to **accept**  $x$  if it answers **yes** given input  $x$ ; it **rejects**  $x$  if it answers **no** given input  $x$ .

Algorithm  $D$  is said to **decide** a language  $L \subseteq \Sigma^*$  if it accepts all strings in  $L$  and rejects all other strings.

In other words, the output of  $D$  is the answer to the question “Does  $x$  belong to  $L$ ?”

The complexity class **P** is the set of all languages that can be decided in polynomial time.

Formally, a language  $L$  belongs to **P** if and only if there exists an algorithm  $D$  that decides  $L$  and the running time of  $D$  on any input  $x \in \Sigma^*$  is in  $O(|x|^c)$  for some constant  $c$ .

# The Complexity Class P

Given a string  $x \in \Sigma^*$ , a decision algorithm  $D$  is said to **accept**  $x$  if it answers **yes** given input  $x$ ; it **rejects**  $x$  if it answers **no** given input  $x$ .

Algorithm  $D$  is said to **decide** a language  $L \subseteq \Sigma^*$  if it accepts all strings in  $L$  and rejects all other strings.

In other words, the output of  $D$  is the answer to the question "Does  $x$  belong to  $L$ ?"

The complexity class **P** is the set of all languages that can be decided in polynomial time.

Formally, a language  $L$  belongs to **P** if and only if there exists an algorithm  $D$  that decides  $L$  and the running time of  $D$  on any input  $x \in \Sigma^*$  is in  $O(|x|^c)$  for some constant  $c$ .

A decision problem  $\in P$  iff it can be solved in polynomial time

Informally, **P** is the set of all tractable decision problems, since

- We observed that decision problems and formal languages are the same thing and
- We consider a problem tractable if it can be solved in polynomial time.

# Verification

Consider an algorithm  $V$  that decides a language  $L' \subseteq \Sigma^* \times \Sigma^*$ , that is, its input is a pair  $(x, y)$  such that  $x, y \in \Sigma^*$ .

Algorithm  $V$  is said to **verify** a language  $L$  if

- for every  $x \in L$ , there exists a  $y \in \Sigma^*$  such that  $(x, y) \in L'$  and
- for every  $x \notin L$ , there is no  $y \in \Sigma^*$  such that  $(x, y) \in L'$ .

# Verification

Consider an algorithm  $V$  that decides a language  $L' \subseteq \Sigma^* \times \Sigma^*$ , that is, its input is a pair  $(x, y)$  such that  $x, y \in \Sigma^*$ .

Algorithm  $V$  is said to **verify** a language  $L$  if

- for every  $x \in L$ , there exists a  $y \in \Sigma^*$  such that  $(x, y) \in L'$  and
- for every  $x \notin L$ , there is no  $y \in \Sigma^*$  such that  $(x, y) \in L'$ .

Thus, given an input  $(x, y)$  consisting of an element  $x \in L$  and an appropriate “proof”  $y \in \Sigma^*$  that shows that  $x \in L$ ,  $V$  answers yes.

For a string  $x \notin L$ , we can provide whatever “proof”  $y$  of its membership in  $L$  we want;  $V$  will reject every such pair  $(x, y)$ .

# Verification

Consider an algorithm  $V$  that decides a language  $L' \subseteq \Sigma^* \times \Sigma^*$ , that is, its input is a pair  $(x, y)$  such that  $x, y \in \Sigma^*$ .

Algorithm  $V$  is said to **verify** a language  $L$  if

- for every  $x \in L$ , there exists a  $y \in \Sigma^*$  such that  $(x, y) \in L'$  and
- for every  $x \notin L$ , there is no  $y \in \Sigma^*$  such that  $(x, y) \in L'$ .

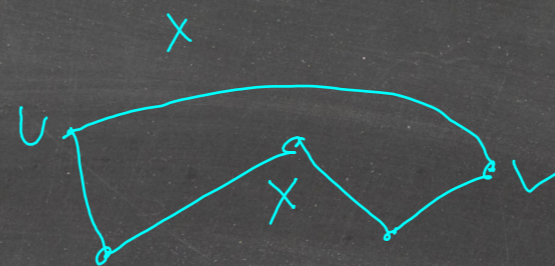
Thus, given an input  $(x, y)$  consisting of an element  $x \in L$  and an appropriate “proof”  $y \in \Sigma^*$  that shows that  $x \in L$ ,  $V$  answers yes.

For a string  $x \notin L$ , we can provide whatever “proof”  $y$  of its membership in  $L$  we want;  $V$  will reject every such pair  $(x, y)$ .

Thus, we can think of  $V$  as a “proof checker” that verifies whether any given proof of  $x$ 's membership in  $L$  is in fact correct.

# Verification

e.g.



Consider an algorithm  $V$  that decides a language  $L' \subseteq \Sigma^* \times \Sigma^*$ , that is, its input is a pair  $(x, y)$  such that  $x, y \in \Sigma^*$ .

Q: is there a path from  $u \rightarrow v$  of length  $\geq 1$ ?

yes  $y$  is used to verify

Algorithm  $V$  is said to **verify** a language  $L$  if

- for every  $x \in L$ , there exists a  $y \in \Sigma^*$  such that  $(x, y) \in L'$  and
- for every  $x \notin L$ , there is no  $y \in \Sigma^*$  such that  $(x, y) \in L'$ .

Thus, given an input  $(x, y)$  consisting of an element  $x \in L$  and an appropriate “proof”  $y \in \Sigma^*$  that shows that  $x \in L$ ,  $V$  answers yes.

For a string  $x \notin L$ , we can provide whatever “proof”  $y$  of its membership in  $L$  we want;  $V$  will reject every such pair  $(x, y)$ .

Thus, we can think of  $V$  as a “proof checker” that verifies whether any given proof of  $x$ 's membership in  $L$  is in fact correct.

$V$  does not decide whether  $x \in L$ .  $V$  may answer no even if  $x \in L$  if the provided proof of its membership in  $L$  is incorrect.

# Verifying is Easier Than Deciding

Verifying a language may be easier than deciding it.



# Verifying is Easier Than Deciding

Verifying a language may be easier than deciding it.

Given a sequence  $S = \langle x_1, x_2, \dots, x_n \rangle$  of numbers, the **element uniqueness** problem asks us to decide whether there exist indices  $i \neq j$  such that  $x_i = x_j$ .

# Verifying is Easier Than Deciding

Verifying a language may be easier than deciding it.

Given a sequence  $S = \langle x_1, x_2, \dots, x_n \rangle$  of numbers, the **element uniqueness** problem asks us to decide whether there exist indices  $i \neq j$  such that  $x_i = x_j$ .

Let  $L$  be the language of all sequences where two such indices exist.

# Verifying is Easier Than Deciding

Verifying a language may be easier than deciding it.

Given a sequence  $S = \langle x_1, x_2, \dots, x_n \rangle$  of numbers, the **element uniqueness** problem asks us to decide whether there exist indices  $i \neq j$  such that  $x_i = x_j$ .

Let  $L$  be the language of all sequences where two such indices exist.

It can be shown that, using comparisons only, it takes  $\Omega(n \lg n)$  time in the worst case to decide whether a given sequence  $S$  belongs to  $L$ .

# Verifying is Easier Than Deciding

Verifying a language may be easier than deciding it.

Given a sequence  $S = \langle x_1, x_2, \dots, x_n \rangle$  of numbers, the **element uniqueness** problem asks us to decide whether there exist indices  $i \neq j$  such that  $x_i = x_j$ .

Let  $L$  be the language of all sequences where two such indices exist.

It can be shown that, using comparisons only, it takes  $\Omega(n \lg n)$  time in the worst case to decide whether a given sequence  $S$  belongs to  $L$ .

Verifying  $L$  can be done in constant time!

- Let  $L' = \{(S, (i, j)) \mid x_i = x_j, i \neq j\}$
- Given some pair  $(S, (i, j))$ , we can decide in constant time whether  $(S, (i, j)) \in L'$  by comparing  $x_i$  and  $x_j$ .
- This algorithm verifies  $L$  because  $x \in L$  if and only if there exists a pair  $(i, j)$  such that  $(S, (i, j)) \in L'$ .

original input  $S$   
new input  $(S, (i, j))$

# The Complexity Class NP

NP stands for nondeterministic polynomial

The complexity class **NP** is the set of all languages that can be verified in polynomial time.

Formally, a language  $L$  belongs to NP if and only if there exists a language  $L' \in P$  and a constant  $c$  such that  $x \in L$  if and only if  $(x, y) \in L'$  for some  $y \in \Sigma^*$ ,  $|y| \leq |x|^c$ .

polynomial time verifier

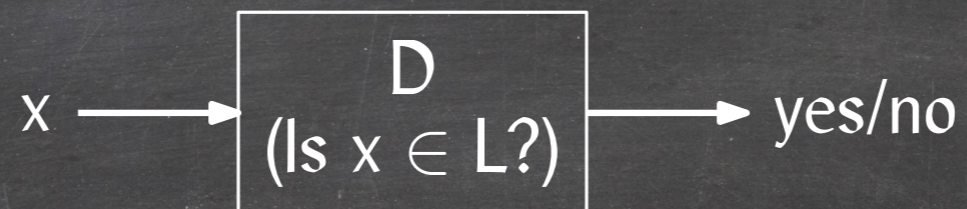
polynomial in  $x$  size proof

# P versus NP

**Lemma:**  $P \subseteq NP$ . (Every language that can be decided in polynomial time can be verified in polynomial time.)

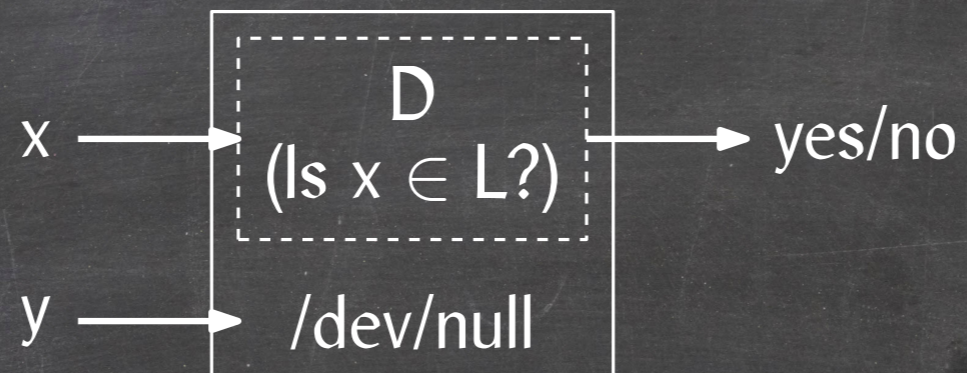
# P versus NP

**Lemma:**  $P \subseteq NP$ . (Every language that can be decided in polynomial time can be verified in polynomial time.)



# P versus NP

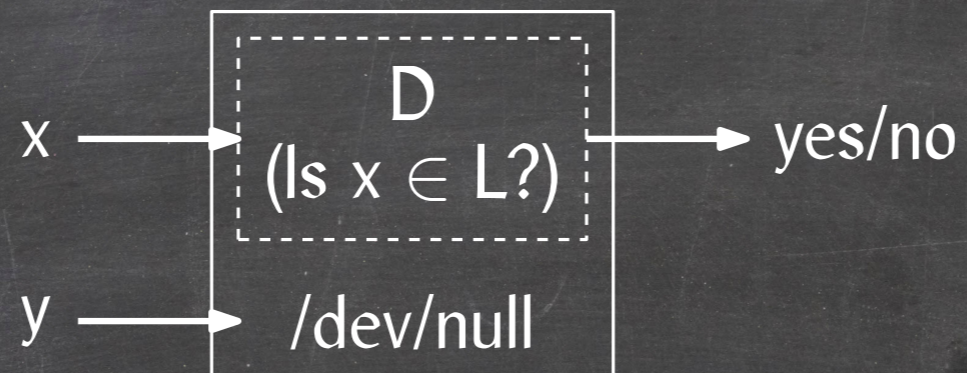
**Lemma:**  $P \subseteq NP$ . (Every language that can be decided in polynomial time can be verified in polynomial time.)





# P versus NP

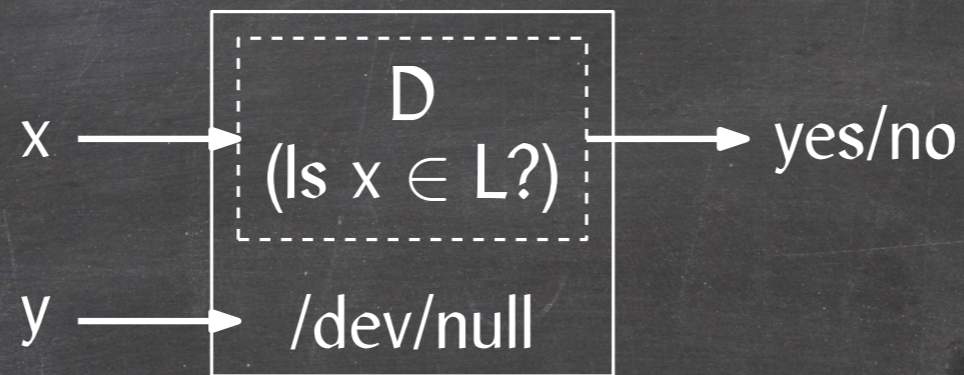
**Lemma:**  $P \subseteq NP$ . (Every language that can be decided in polynomial time can be verified in polynomial time.)



Is  $P = NP$  or is  $P \subset NP$ ?

# P versus NP

**Lemma:**  $P \subseteq NP$ . (Every language that can be decided in polynomial time can be verified in polynomial time.)



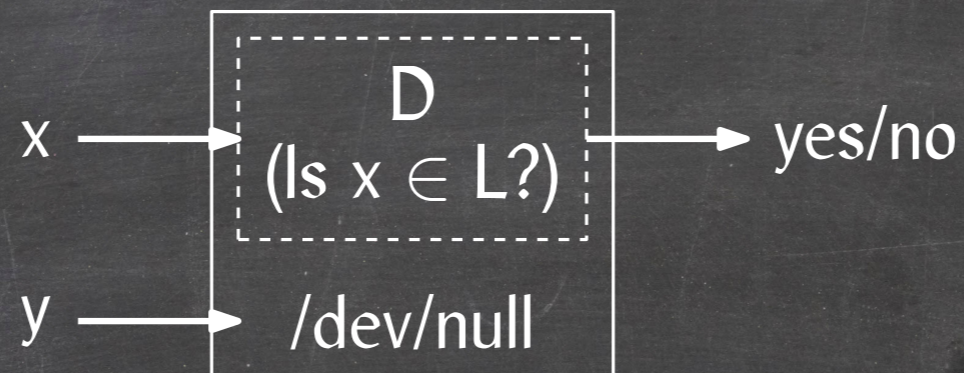
Is  $P = NP$  or is  $P \subset NP$ ?

Nobody knows the answer, but ...

Given that we know verifying some languages is easier than deciding them, it is likely that  $P \subset NP$ .

# P versus NP

**Lemma:**  $P \subseteq NP$ . (Every language that can be decided in polynomial time can be verified in polynomial time.)



Is  $P = NP$  or is  $P \subset NP$ ?

Nobody knows the answer, but ...

Given that we know verifying some languages is easier than deciding them, it is likely that  $P \subset NP$ .

We will show that there exist languages that cannot be decided (decision problems that cannot be solved) in polynomial time unless  $P = NP$ !

# NP-Hardness and NP-Completeness

A language  $L$  is **NP-hard** if  $L \in P$  implies that  $P = NP$ .

# NP-Hardness and NP-Completeness

A language  $L$  is **NP-hard** if  $L \in P$  implies that  $P = NP$ .

A language  $L$  is **NP-complete** if

- $L \in NP$  and
- $L$  is NP-hard.

Intuitively, NP-complete languages are the hardest languages in NP.

# NP-Hardness and NP-Completeness

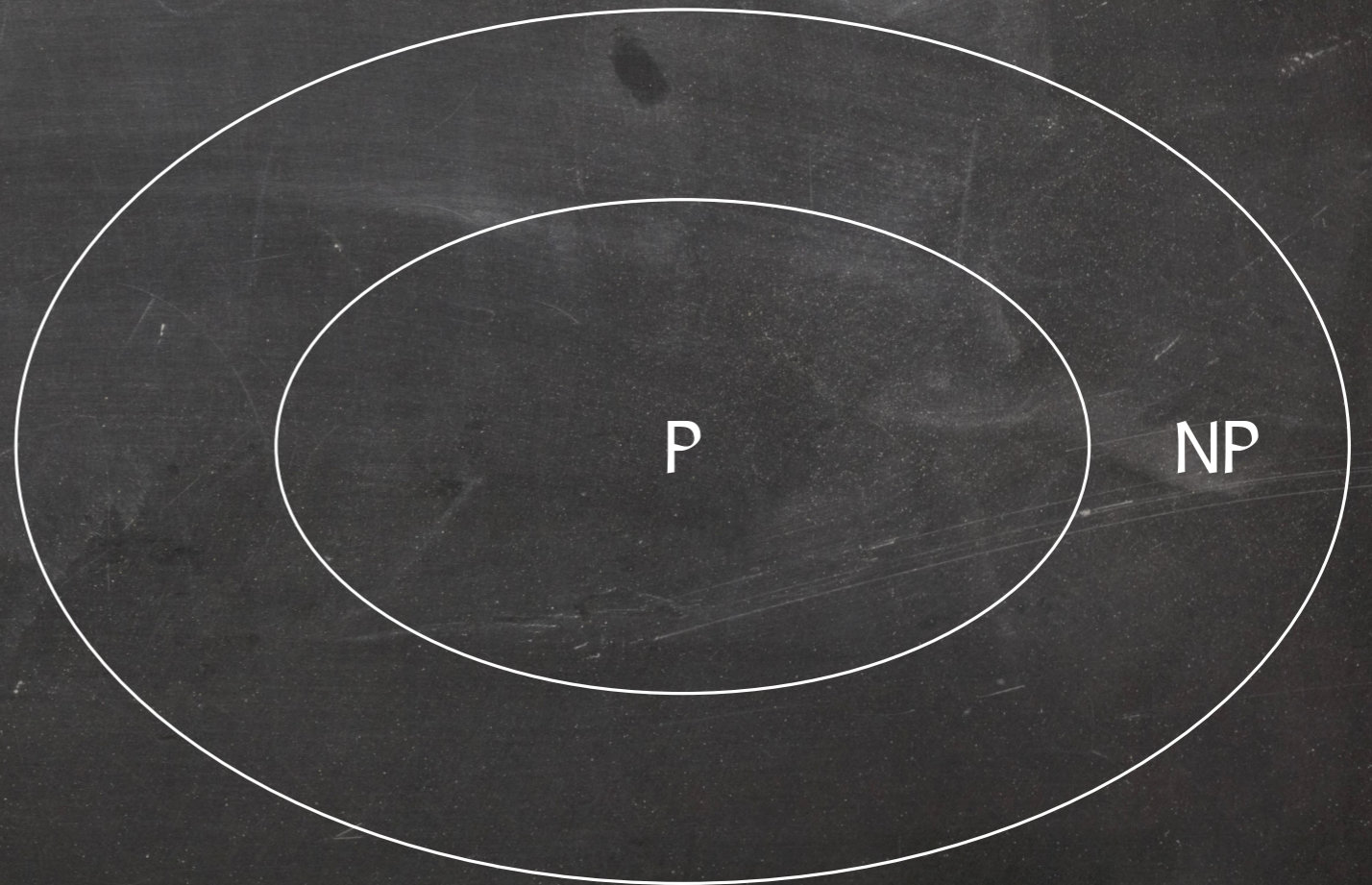
A language  $L$  is **NP-hard** if  $L \in P$  implies that  $P = NP$ .

A language  $L$  is **NP-complete** if

- $L \in NP$  and
- $L$  is NP-hard.

Intuitively, NP-complete languages are the hardest languages in NP.

Assume  $P \neq NP$ .



# NP-Hardness and NP-Completeness

A language  $L$  is **NP-hard** if  $L \in P$  implies that  $P = NP$ .

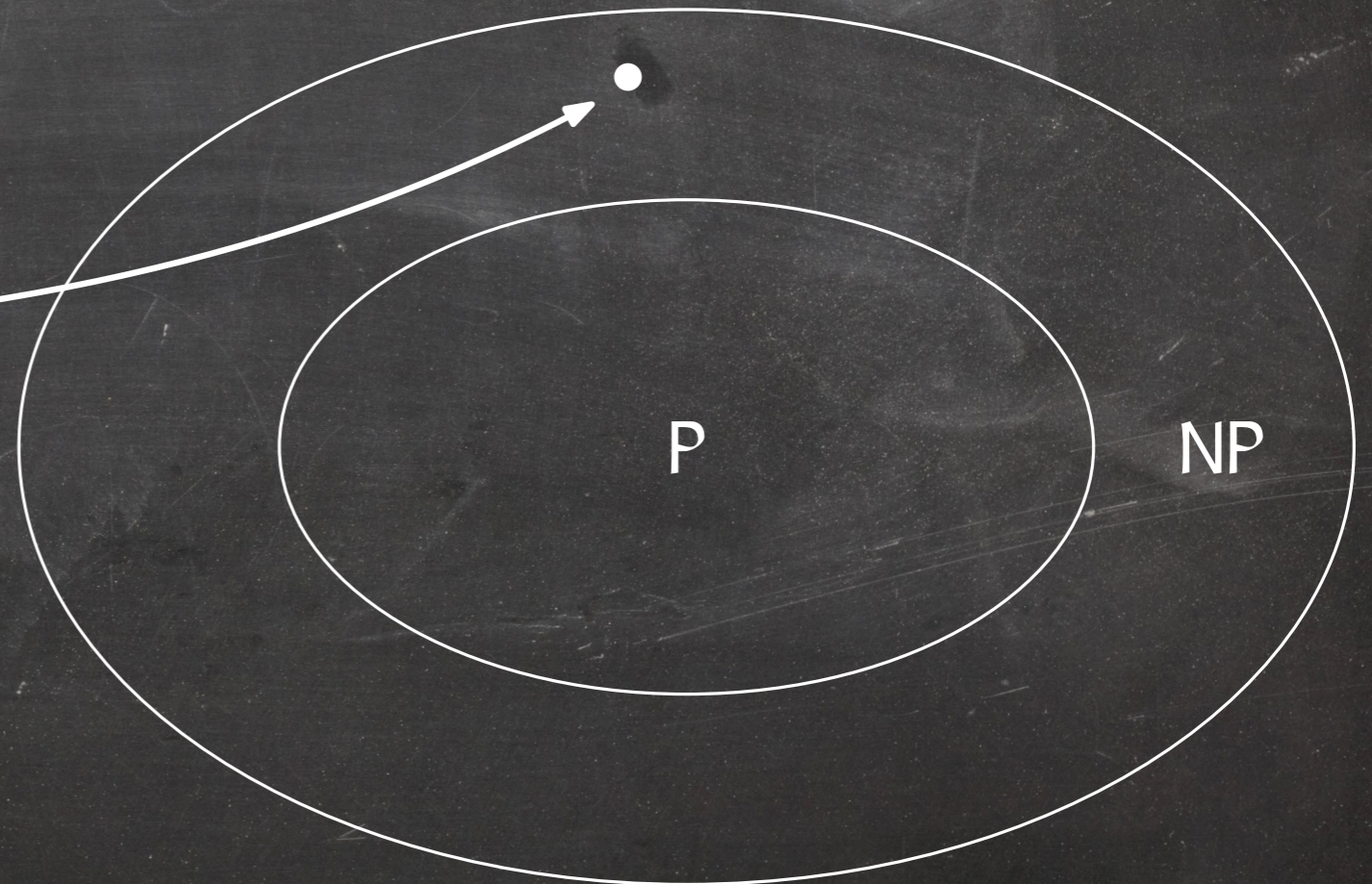
A language  $L$  is **NP-complete** if

- $L \in NP$  and
- $L$  is NP-hard.

Intuitively, NP-complete languages are the hardest languages in NP.

Assume  $P \neq NP$ .

NP-complete if NP-hard



# NP-Hardness and NP-Completeness

A language  $L$  is **NP-hard** if  $L \in P$  implies that  $P = NP$ .

A language  $L$  is **NP-complete** if

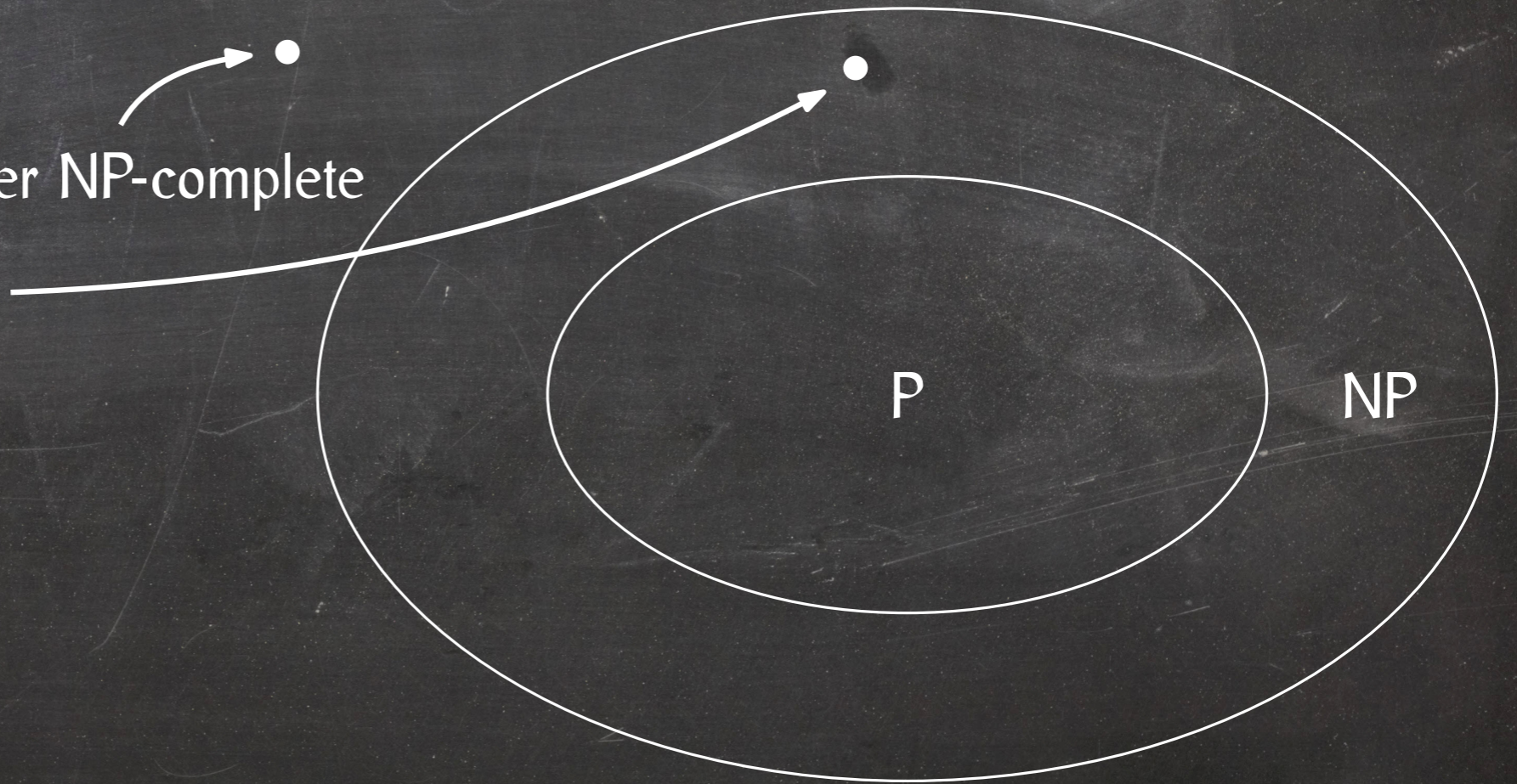
- $L \in NP$  and
- $L$  is NP-hard.

Intuitively, NP-complete languages are the hardest languages in NP.

Assume  $P \neq NP$ .

Maybe NP-hard but never NP-complete

NP-complete if NP-hard





# NP-Hardness and NP-Completeness

A language  $L$  is **NP-hard** if  $L \in P$  implies that  $P = NP$ .

A language  $L$  is **NP-complete** if

- $L \in NP$  and
- $L$  is NP-hard.

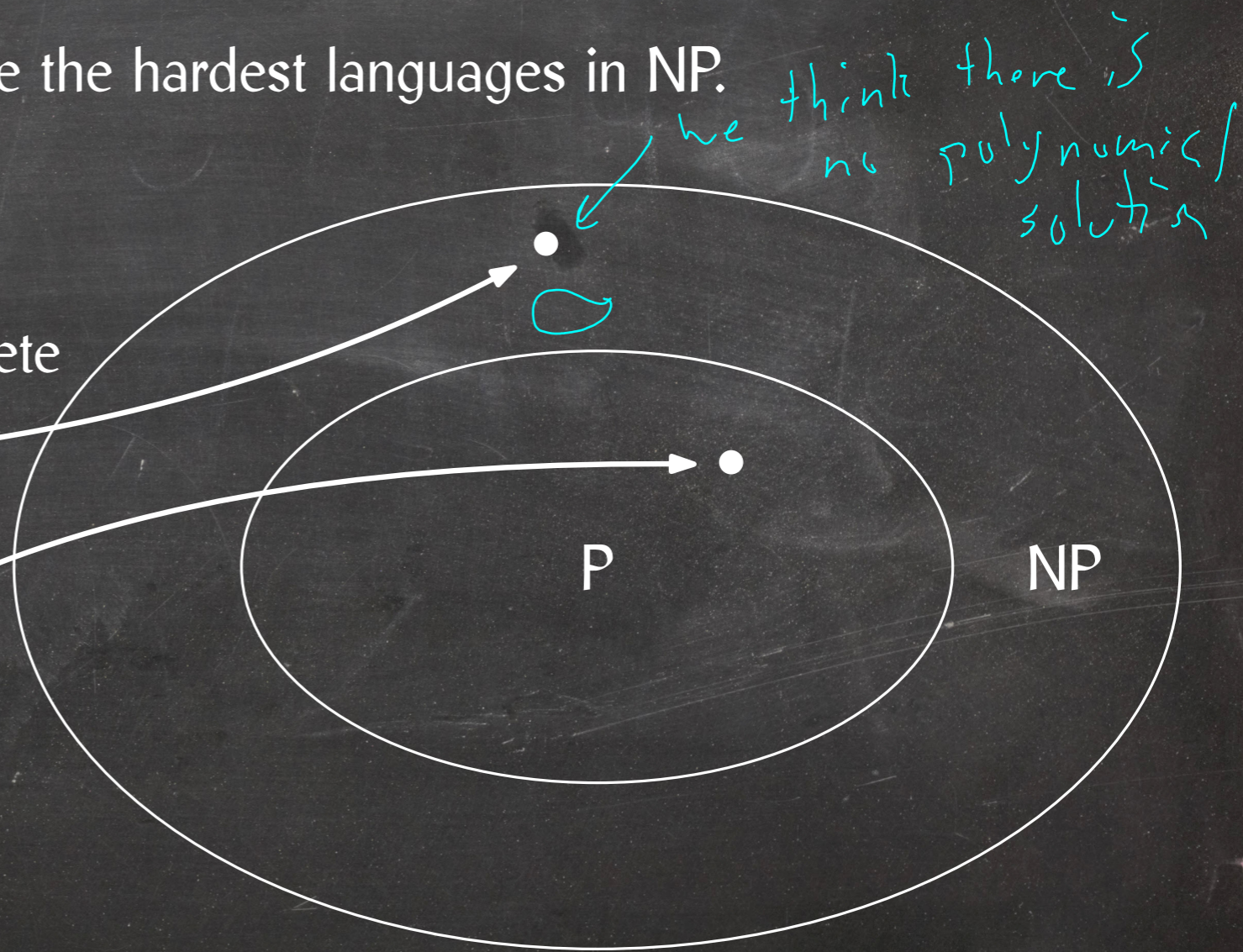
Intuitively, NP-complete languages are the hardest languages in NP.

Assume  $P \neq NP$ .

Maybe NP-hard but never NP-complete

NP-complete if NP-hard

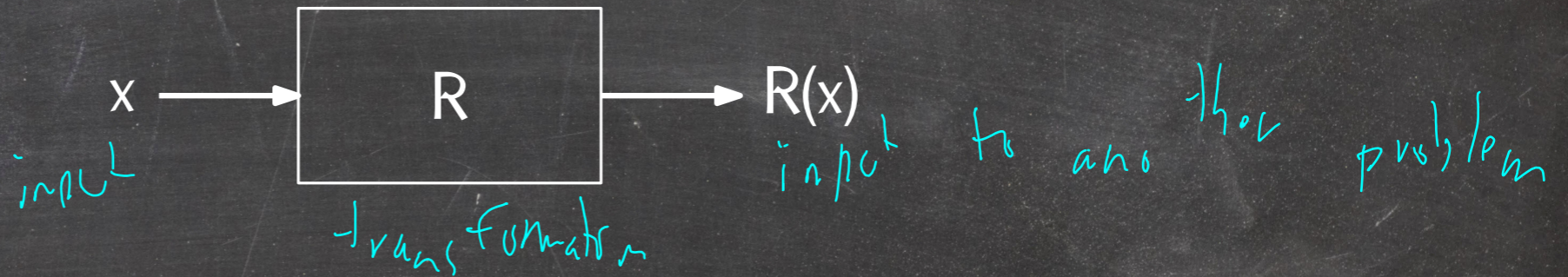
Neither NP-hard nor NP-complete



# Polynomial-Time Reductions

An algorithm  $R$  **reduces** a language  $L_1 \subseteq \Sigma^*$  to a language  $L_2 \subseteq \Sigma^*$  if, for all  $x \in \Sigma^*$ ,

$$x \in L_1 \Leftrightarrow R(x) \in L_2.$$



$R$  is a **polynomial-time** reduction if its running time is polynomial in  $|x|$ .

# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .

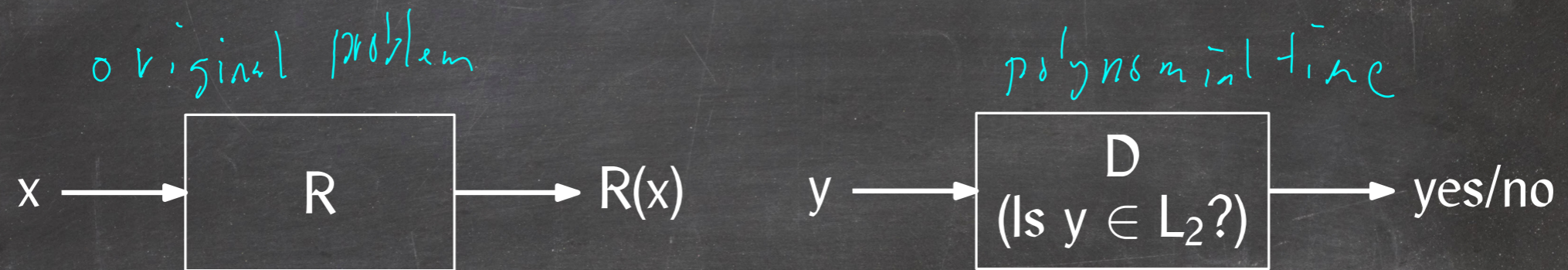
# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



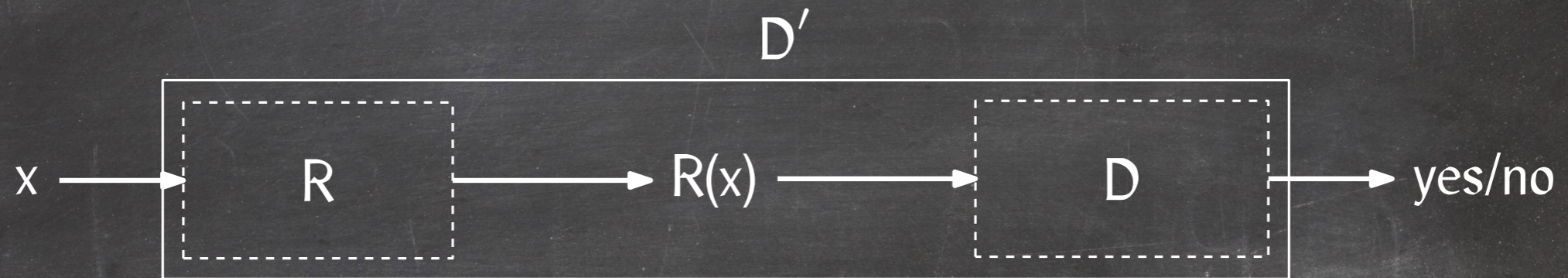
# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



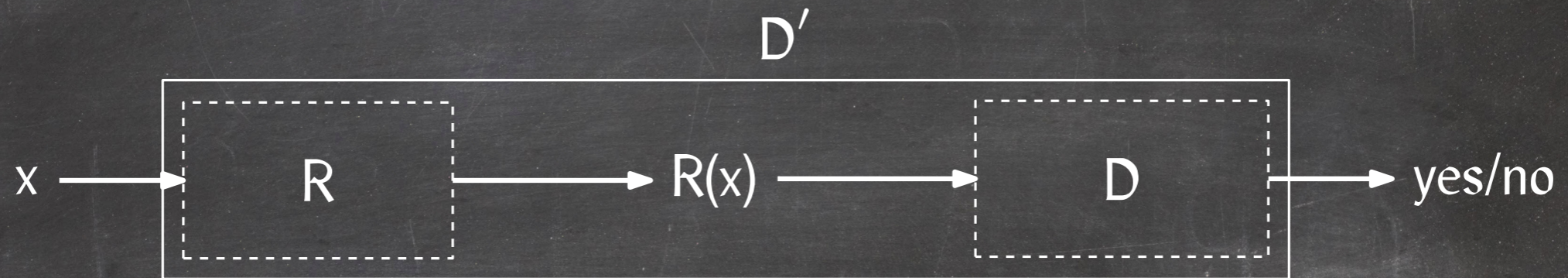
# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



# Proving NP-Hardness Using Polynomial-Time Reductions

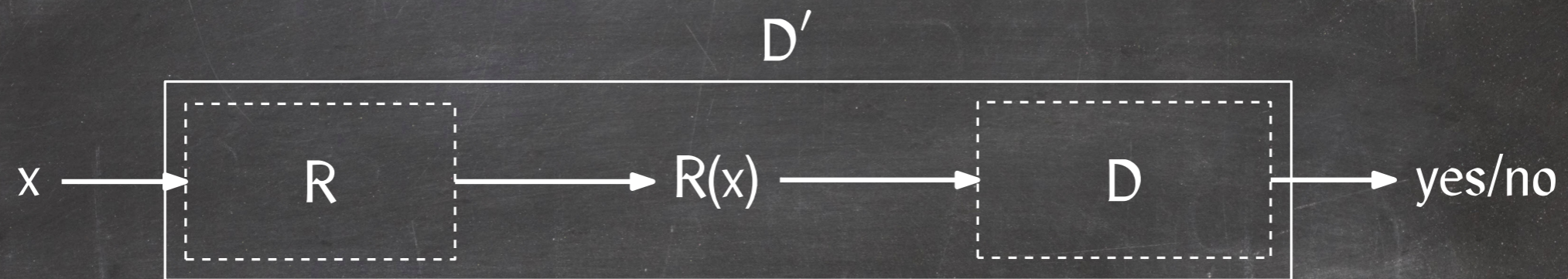
**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



$$x \in L_1 \Leftrightarrow R(x) \in L_2$$

# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



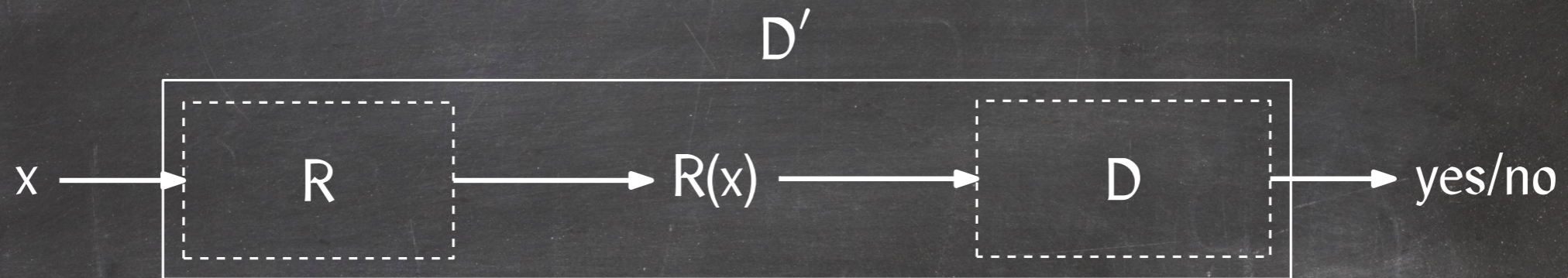
$$x \in L_1 \Leftrightarrow R(x) \in L_2$$

$$R(x) \in L_2 \Leftrightarrow D(R(x)) = \text{yes}$$



# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .

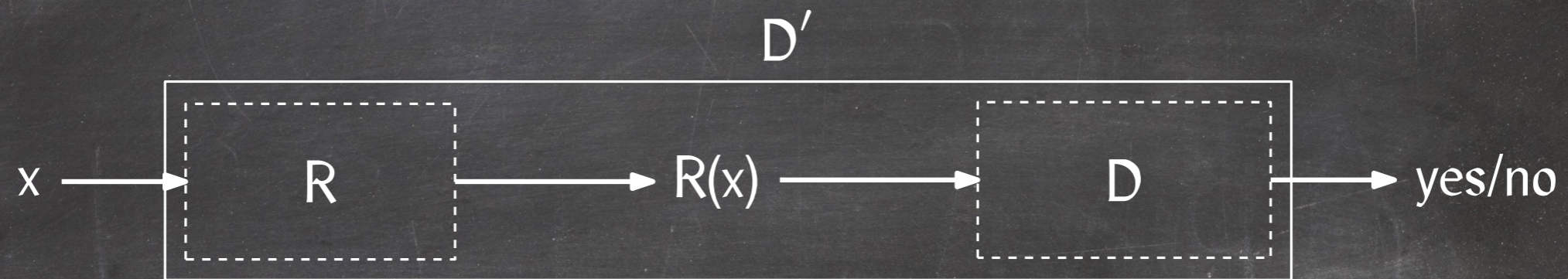


$$x \in L_1 \Leftrightarrow R(x) \in L_2 \quad R(x) \in L_2 \Leftrightarrow D(R(x)) = \text{yes}$$

$$x \in L_1 \Leftrightarrow D'(x) = \text{yes}$$

# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



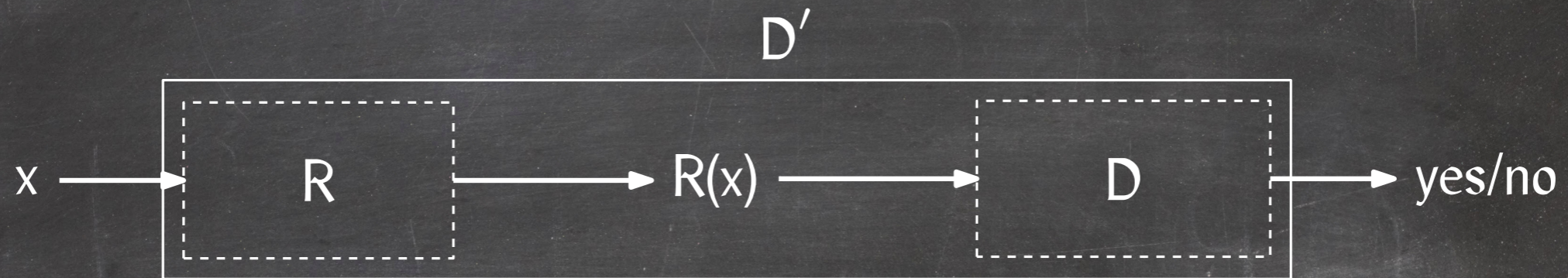
$$x \in L_1 \Leftrightarrow R(x) \in L_2 \quad R(x) \in L_2 \Leftrightarrow D(R(x)) = \text{yes}$$

$$x \in L_1 \Leftrightarrow D'(x) = \text{yes}$$

$$T_R(|x|) \leq c|x|^a, \text{ for some } a, c.$$

# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



$$x \in L_1 \Leftrightarrow R(x) \in L_2 \quad R(x) \in L_2 \Leftrightarrow D(R(x)) = \text{yes}$$

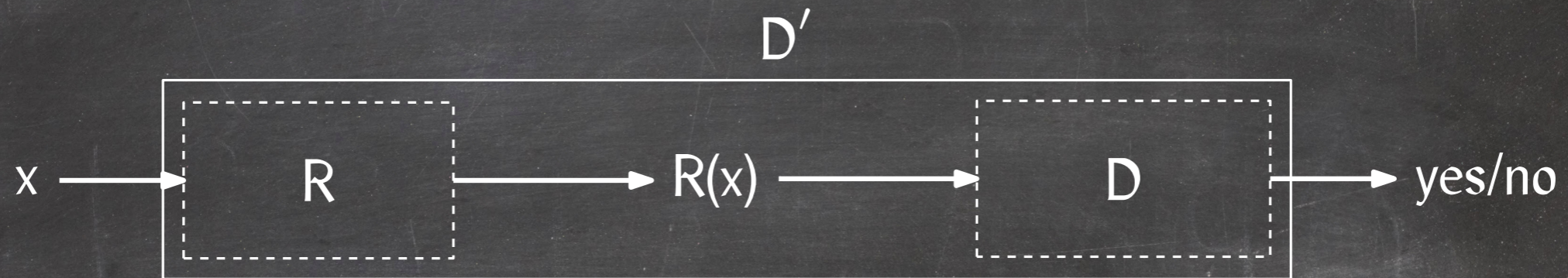
$$x \in L_1 \Leftrightarrow D'(x) = \text{yes}$$

$$T_R(|x|) \leq c|x|^a, \text{ for some } a, c.$$

$$\Rightarrow |R(x)| \leq c|x|^a, \text{ for some } a, c.$$

# Proving NP-Hardness Using Polynomial-Time Reductions

**Lemma:** If there exists a polynomial-time reduction  $R$  from a language  $L_1$  to a language  $L_2 \in P$ , then  $L_1 \in P$ .



$$x \in L_1 \Leftrightarrow R(x) \in L_2 \quad R(x) \in L_2 \Leftrightarrow D(R(x)) = \text{yes}$$

$$x \in L_1 \Leftrightarrow D'(x) = \text{yes}$$

$$T_R(|x|) \leq c|x|^a, \text{ for some } a, c.$$

$$\Rightarrow |R(x)| \leq c|x|^a, \text{ for some } a, c.$$

$$\Rightarrow T_D(|R(x)|) \leq c'|R(x)|^{a'} \leq c'(c|x|^a)^{a'}, \text{ for some } a', c'.$$



# Proving NP-Hardness Using Polynomial-Time Reductions

**Corollary:** If there exists a polynomial-time reduction from an NP-hard language  $L_1$  to a language  $L_2$ , then  $L_2$  is also NP-hard.

# Proving NP-Hardness Using Polynomial-Time Reductions

**Corollary:** If there exists a polynomial-time reduction from an NP-hard language  $L_1$  to a language  $L_2$ , then  $L_2$  is also NP-hard.



# Proving NP-Hardness Using Polynomial-Time Reductions

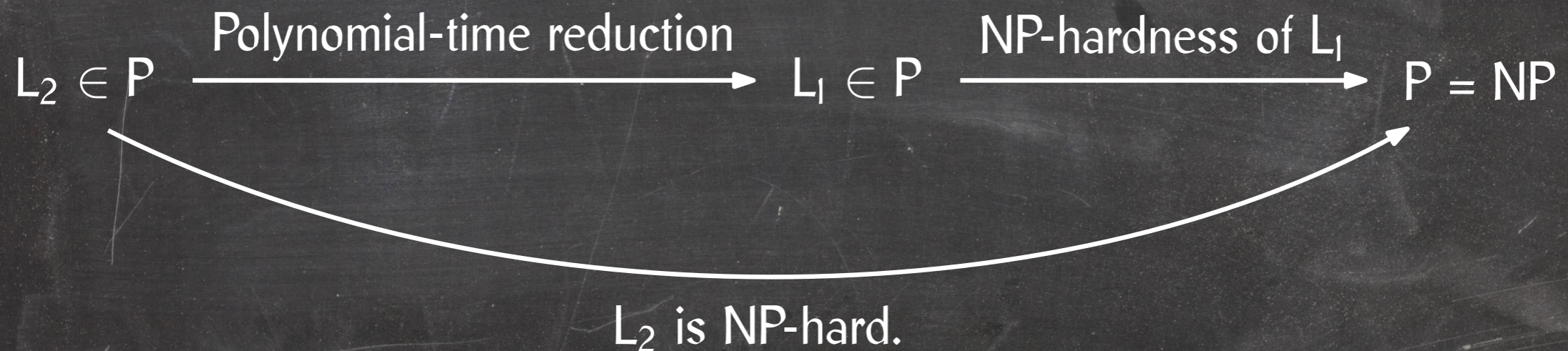
**Corollary:** If there exists a polynomial-time reduction from an NP-hard language  $L_1$  to a language  $L_2$ , then  $L_2$  is also NP-hard.





# Proving NP-Hardness Using Polynomial-Time Reductions

**Corollary:** If there exists a polynomial-time reduction from an NP-hard language  $L_1$  to a language  $L_2$ , then  $L_2$  is also NP-hard.



# Where Do We Get Our First NP-Hard Problem From?

To prove that a language  $L$  is NP-hard, we need an NP-hard language  $L'$  that we can reduce to  $L$  in polynomial time.

How do we prove a language  $L$  is NP-hard when we haven't shown any other language to be NP-hard yet?

Enter **Satisfiability**, the mother of all NP-hard problems ...

# Satisfiability (SAT)

A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

*or*      *and*

*negation*

# Satisfiability (SAT)

A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

- $x_1, x_2, x_3, x_4$  are **Boolean variables**, which can be **true** or **false**.

# Satisfiability (SAT)

A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

- $x_1, x_2, x_3, x_4$  are **Boolean variables**, which can be **true** or **false**.
- $x_1, \bar{x}_1, x_2, \bar{x}_3, x_4$  are **literals** (a Boolean variable or its negation).

# Satisfiability (SAT)

A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

- $x_1, x_2, x_3, x_4$  are **Boolean variables**, which can be **true** or **false**.
- $x_1, \bar{x}_1, x_2, \bar{x}_3, x_4$  are **literals** (a Boolean variable or its negation).
- A **truth assignment** assigns a value **true** or **false** to each variable in  $F$ .

# Satisfiability (SAT)

A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

T   T   F   F   T  
T   T   F   F   T  
T   T   F   F   T   = T

- $x_1, x_2, x_3, x_4$  are **Boolean variables**, which can be **true** or **false**.
- $x_1, \bar{x}_1, x_2, \bar{x}_3, x_4$  are **literals** (a Boolean variable or its negation).
- A **truth assignment** assigns a value **true** or **false** to each variable in  $F$ .
- A truth assignment **satisfies**  $F$  if it makes  $F$  true. Example:
  - $x_1 = x_2 = x_3 = x_4 = \text{true}$  satisfies  $F$ .
  - $x_1 = x_2 = x_3 = x_4 = \text{false}$  does not.

# Satisfiability (SAT)

## A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

- $x_1, x_2, x_3, x_4$  are **Boolean variables**, which can be **true** or **false**.
- $x_1, \bar{x}_1, x_2, \bar{x}_3, x_4$  are **literals** (a Boolean variable or its negation).
- A **truth assignment** assigns a value **true** or **false** to each variable in  $F$ .
- A truth assignment **satisfies**  $F$  if it makes  $F$  true. Example:
  - $x_1 = x_2 = x_3 = x_4 = \text{true}$  satisfies  $F$ .
  - $x_1 = x_2 = x_3 = x_4 = \text{false}$  does not.
- $F$  is **satisfiable** if it has a satisfying truth assignment.



# Satisfiability (SAT)

## A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

- $x_1, x_2, x_3, x_4$  are **Boolean variables**, which can be **true** or **false**.
- $x_1, \bar{x}_1, x_2, \bar{x}_3, x_4$  are **literals** (a Boolean variable or its negation).
- A **truth assignment** assigns a value **true** or **false** to each variable in  $F$ .
- A truth assignment **satisfies**  $F$  if it makes  $F$  true. Example:
  - $x_1 = x_2 = x_3 = x_4 = \text{true}$  satisfies  $F$ .
  - $x_1 = x_2 = x_3 = x_4 = \text{false}$  does not.
- $F$  is **satisfiable** if it has a satisfying truth assignment.

**The satisfiability problem (SAT):** Given a Boolean formula  $F$ , decide whether  $F$  is satisfiable.

# 3-SAT

A formula is in **conjunctive normal form** (CNF) if it is a conjunction of disjunctions.

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

The disjunctions are also called **clauses**.

A formula is in **3-CNF** if each of its clauses consists of three literals.

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

**3-SAT:** Decide whether a given formula in 3-CNF is satisfiable.

# Things We Don't Have Time To Prove

Any polynomial-time verification algorithm for a language  $L$  can be turned into a polynomial-time reduction from  $L$  to SAT.

# Things We Don't Have Time To Prove

Any polynomial-time verification algorithm for a language  $L$  can be turned into a polynomial-time reduction from  $L$  to SAT.

Thus,  $\text{SAT} \in \text{P} \Rightarrow L \in \text{P}$  for all  $L \in \text{NP}$ , that is,  $\text{P} = \text{NP}$ .

# Things We Don't Have Time To Prove

Any polynomial-time verification algorithm for a language  $L$  can be turned into a polynomial-time reduction from  $L$  to SAT.

Thus,  $\text{SAT} \in P \Rightarrow L \in P$  for all  $L \in \text{NP}$ , that is,  $P = \text{NP}$ .

In other words, **SAT is NP-hard**.

# Things We Don't Have Time To Prove

Any polynomial-time verification algorithm for a language  $L$  can be turned into a polynomial-time reduction from  $L$  to SAT.

Thus,  $\text{SAT} \in \text{P} \Rightarrow L \in \text{P}$  for all  $L \in \text{NP}$ , that is,  $\text{P} = \text{NP}$ .

In other words, **SAT is NP-hard**.

Any Boolean formula  $F$  can be turned, in polynomial time, into a Boolean formula  $F'$  in 3-CNF, of size  $|F'| \in O(|F|)$ , and such that  $F$  is satisfiable if and only if  $F'$  is.

# Things We Don't Have Time To Prove

Any polynomial-time verification algorithm for a language  $L$  can be turned into a polynomial-time reduction from  $L$  to SAT.

Thus,  $\text{SAT} \in \text{P} \Rightarrow L \in \text{P}$  for all  $L \in \text{NP}$ , that is,  $\text{P} = \text{NP}$ .

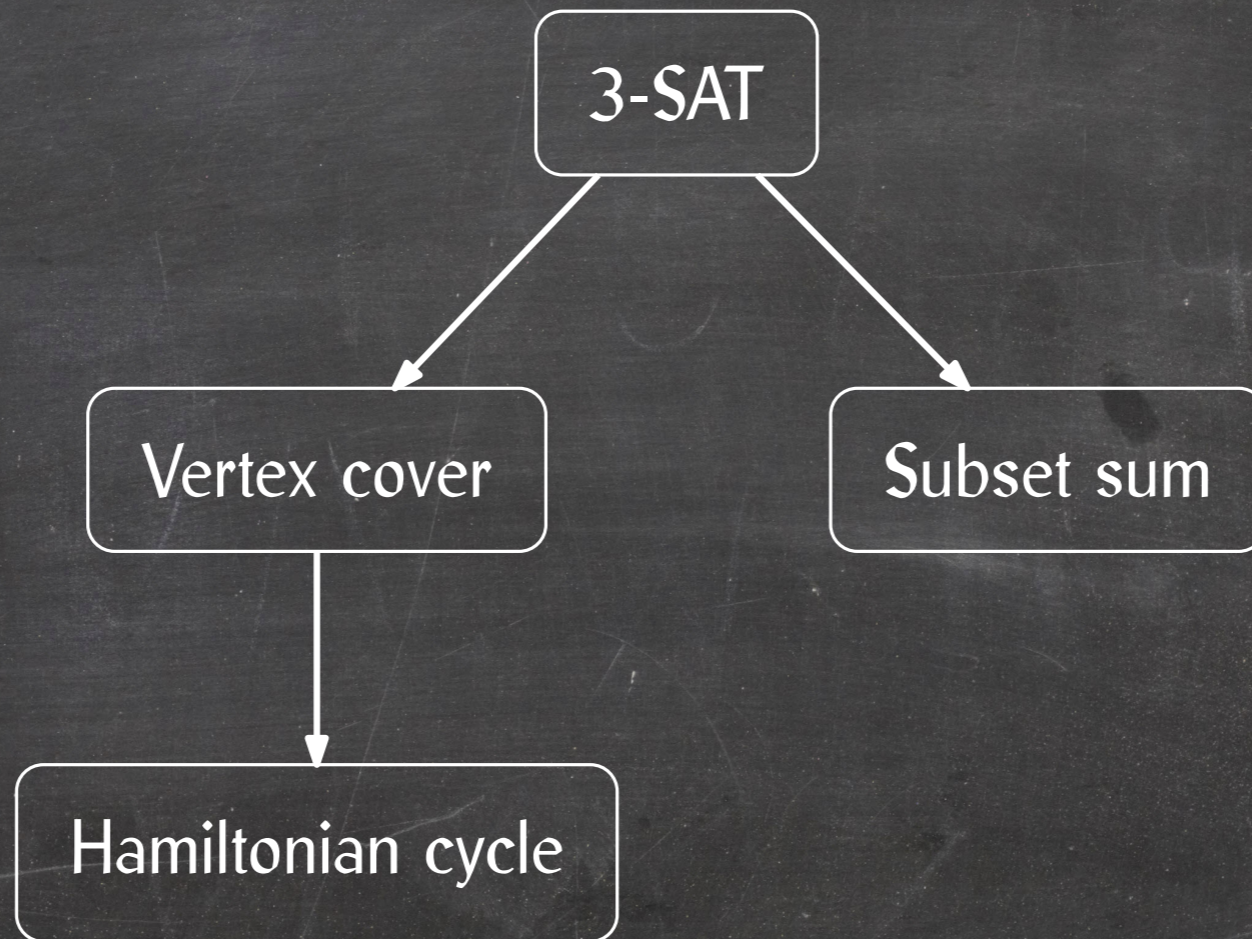
In other words, **SAT is NP-hard**.

Any Boolean formula  $F$  can be turned, in polynomial time, into a Boolean formula  $F'$  in 3-CNF, of size  $|F'| \in O(|F|)$ , and such that  $F$  is satisfiable if and only if  $F'$  is.

Thus, **3-SAT is NP-hard**.

# Examples of Polynomial-Time Reductions

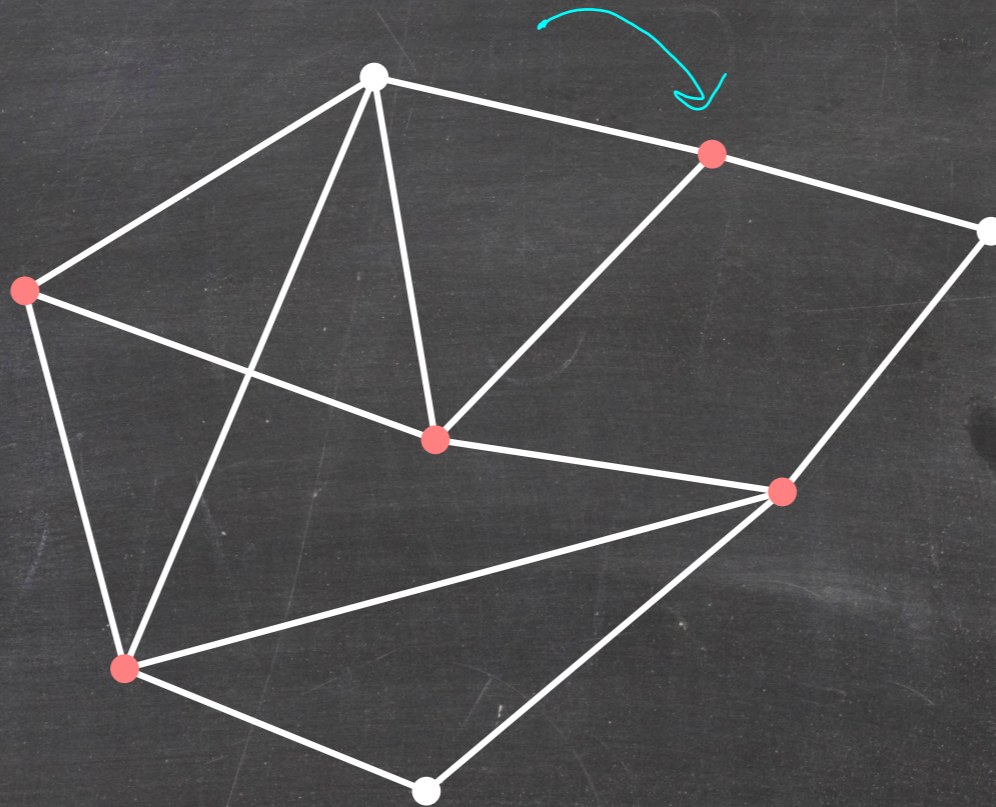
SAT  $\rightarrow$  NP-hard  
 $\downarrow$





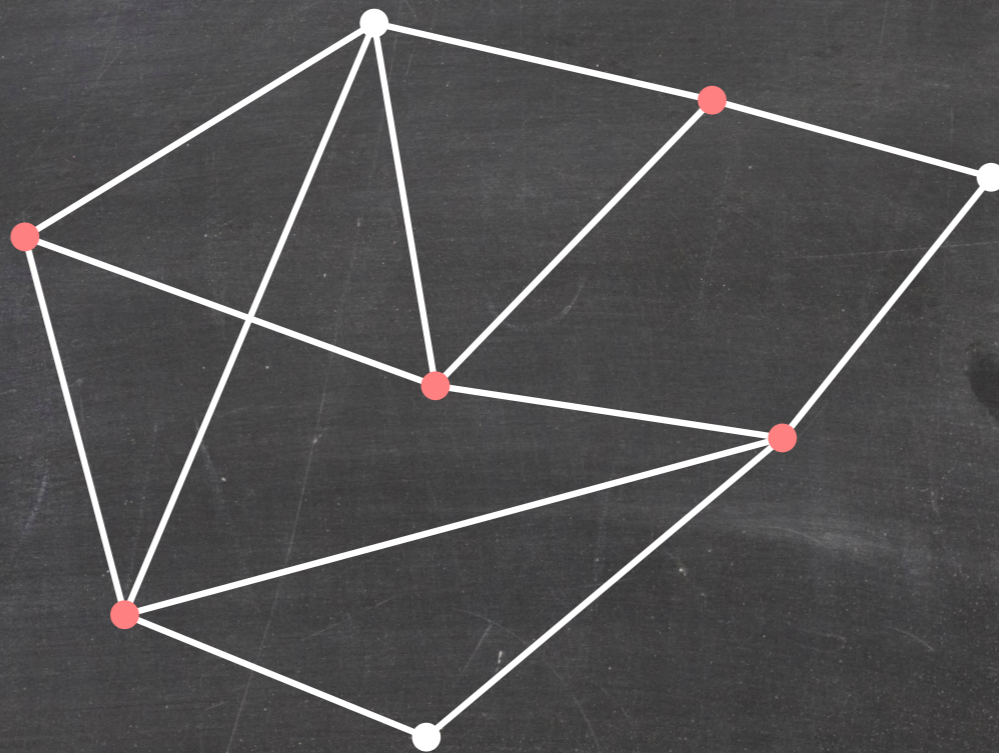
# Vertex Cover

A **vertex cover** of a graph  $G = (V, E)$  is a subset  $S \subseteq V$  such that every edge in  $E$  has at least one endpoint in  $S$ .



# Vertex Cover

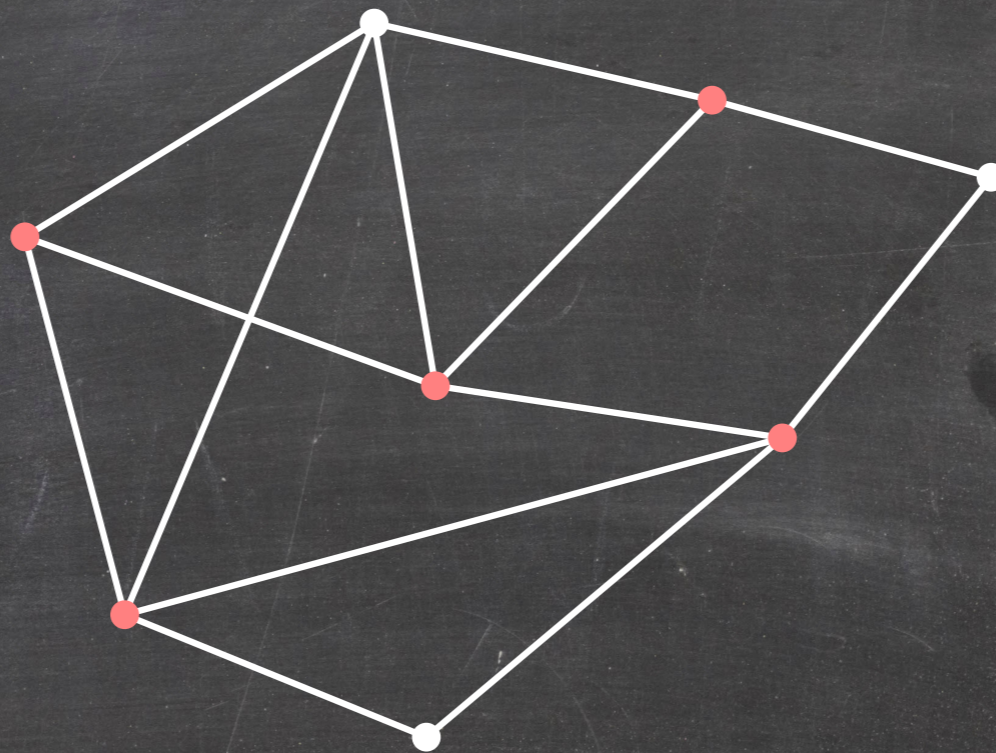
A **vertex cover** of a graph  $G = (V, E)$  is a subset  $S \subseteq V$  such that every edge in  $E$  has at least one endpoint in  $S$ .



**Optimization problem:** Given a graph  $G$ , find the smallest possible vertex cover of  $G$ .

# Vertex Cover

A **vertex cover** of a graph  $G = (V, E)$  is a subset  $S \subseteq V$  such that every edge in  $E$  has at least one endpoint in  $S$ .



**Optimization problem:** Given a graph  $G$ , find the smallest possible vertex cover of  $G$ .

**Decision problem:** Given a graph  $G$  and an integer  $k$ , decide whether  $G$  has a vertex cover of size  $k$ .

# Vertex Cover is NP-Hard

## Reduction from 3-SAT:

- Given any formula  $F$ , we build a graph  $G_F$  that has a small vertex cover if and only if  $F$  is satisfiable.
- $G_F$  will be built from subgraphs, called **widgets**, that guarantee certain properties of  $G_F$ .
- It will be obvious that this construction can be carried out in polynomial time.

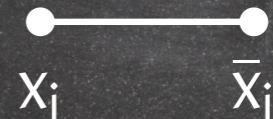
# Vertex Cover is NP-Hard

## Reduction from 3-SAT:

- Given any formula  $F$ , we build a graph  $G_F$  that has a small vertex cover if and only if  $F$  is satisfiable.
- $G_F$  will be built from subgraphs, called **widgets**, that guarantee certain properties of  $G_F$ .
- It will be obvious that this construction can be carried out in polynomial time.

## Variable widget for variable $x_i$ :

- Two vertices  $x_i$  and  $\bar{x}_i$
- One edge  $(x_i, \bar{x}_i)$



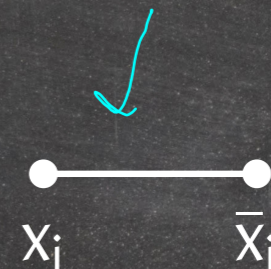
# Vertex Cover is NP-Hard

## Reduction from 3-SAT:

- Given any formula  $F$ , we build a graph  $G_F$  that has a small vertex cover if and only if  $F$  is satisfiable.
- $G_F$  will be built from subgraphs, called **widgets**, that guarantee certain properties of  $G_F$ .
- It will be obvious that this construction can be carried out in polynomial time.

## Variable widget for variable $x_i$ :

- Two vertices  $x_i$  and  $\bar{x}_i$
- One edge  $(x_i, \bar{x}_i)$



## Clause widget for clause $C_j$ :

- Three literal vertices  $\lambda_{j,1}$ ,  $\lambda_{j,2}$ , and  $\lambda_{j,3}$
- Three edges  $(\lambda_{j,1}, \lambda_{j,2})$ ,  $(\lambda_{j,2}, \lambda_{j,3})$ , and  $(\lambda_{j,3}, \lambda_{j,1})$



# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :



# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable



# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause



$x_1$        $\bar{x}_1$

$x_2$        $\bar{x}_2$

$x_3$        $\bar{x}_3$

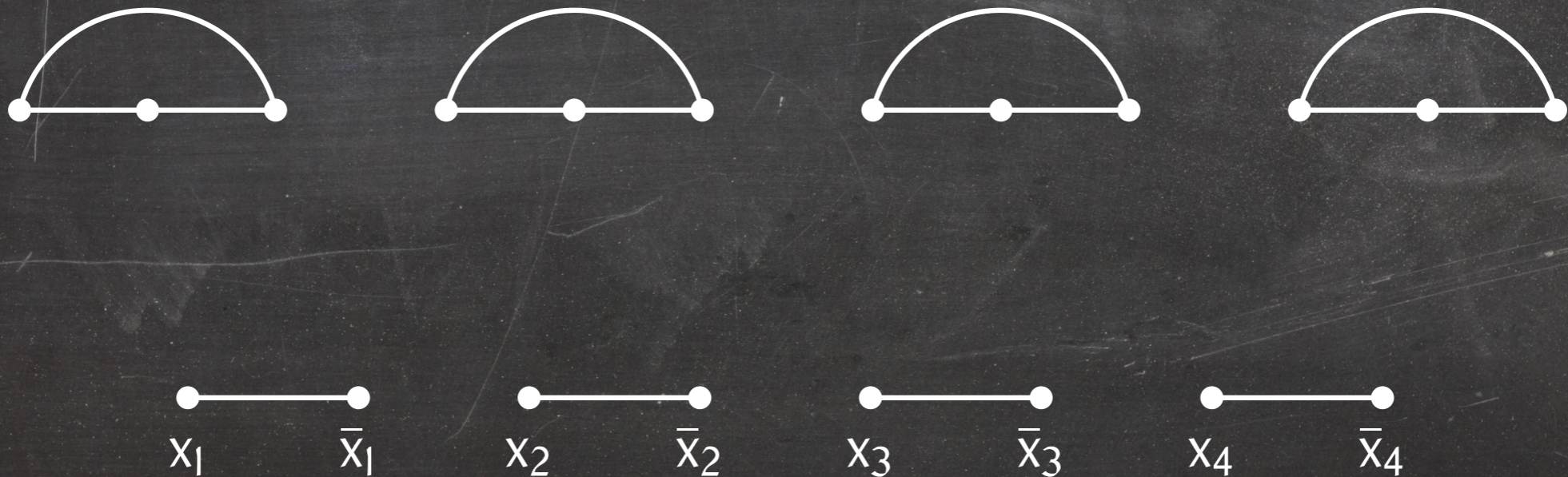
$x_4$        $\bar{x}_4$

# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$

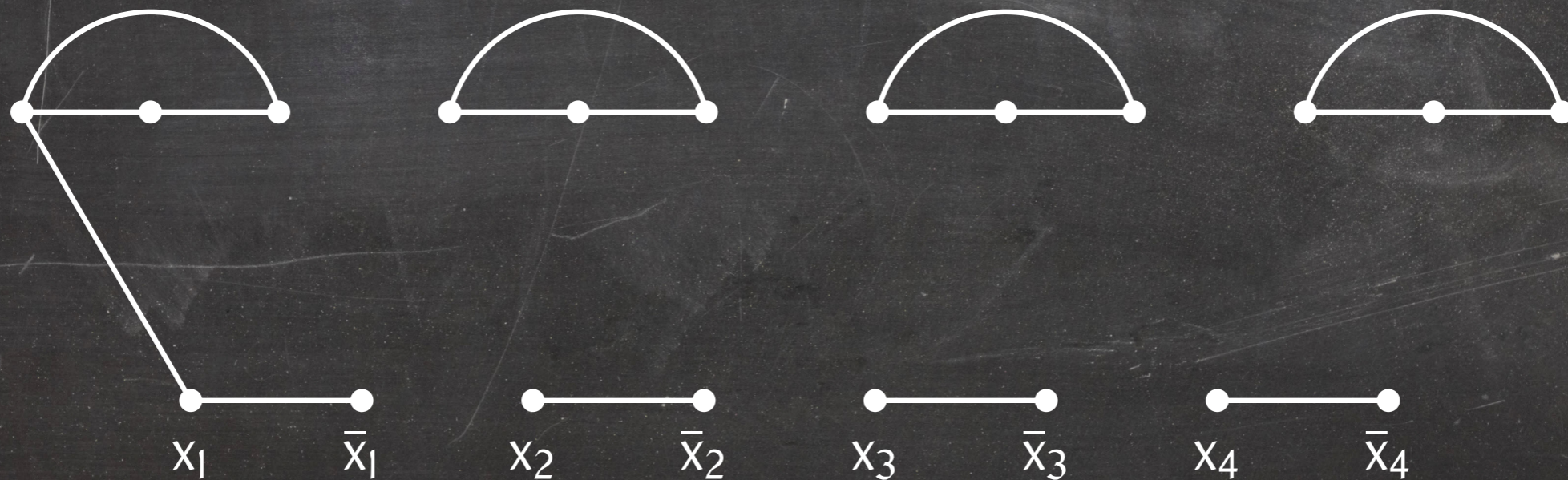


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$

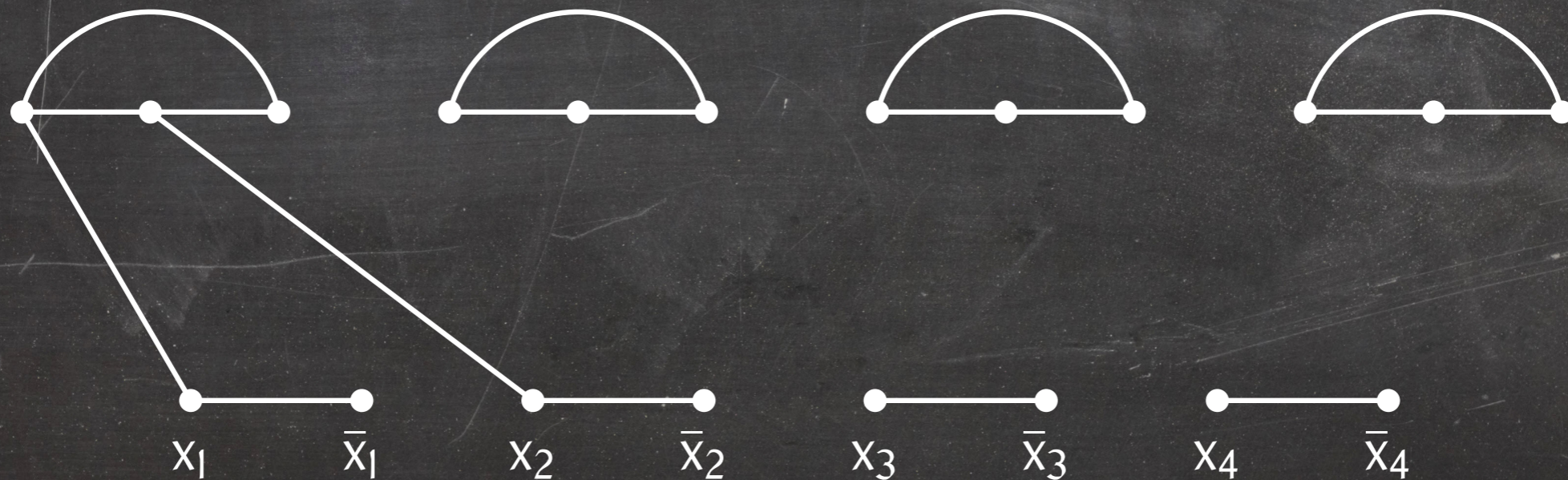


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$

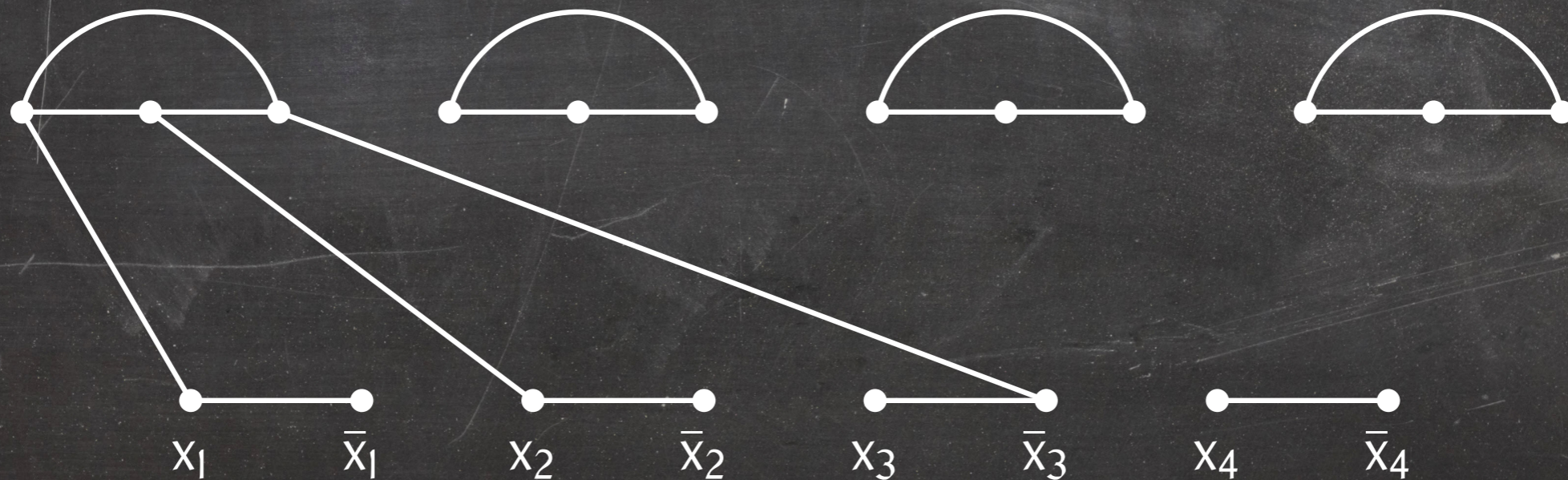


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$

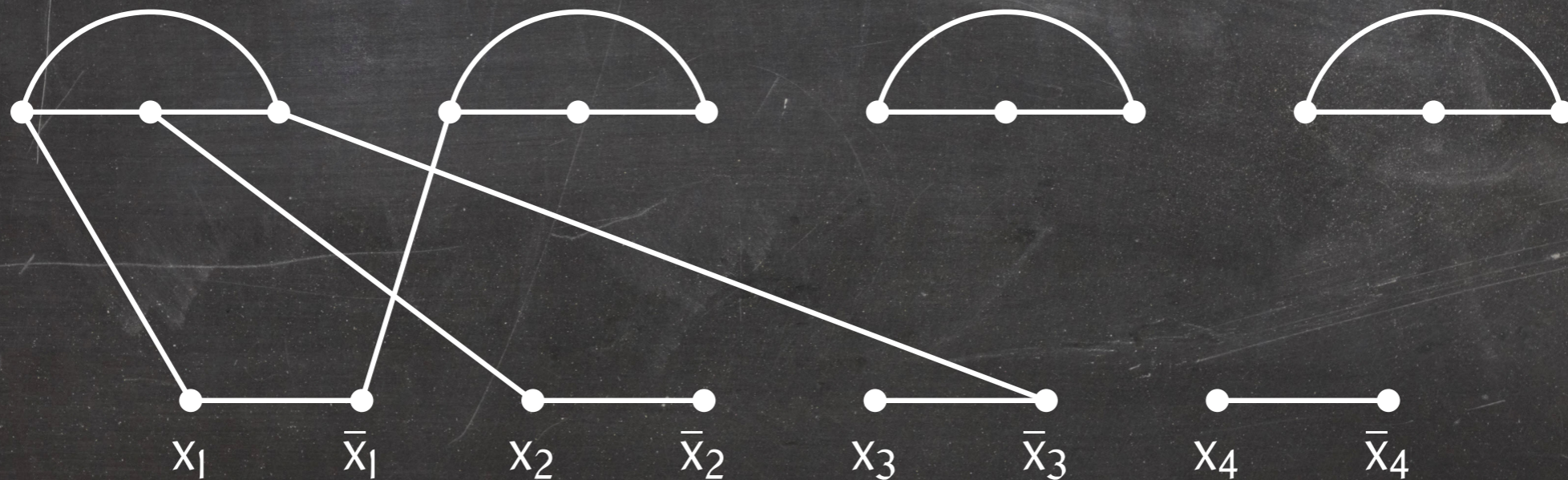


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$

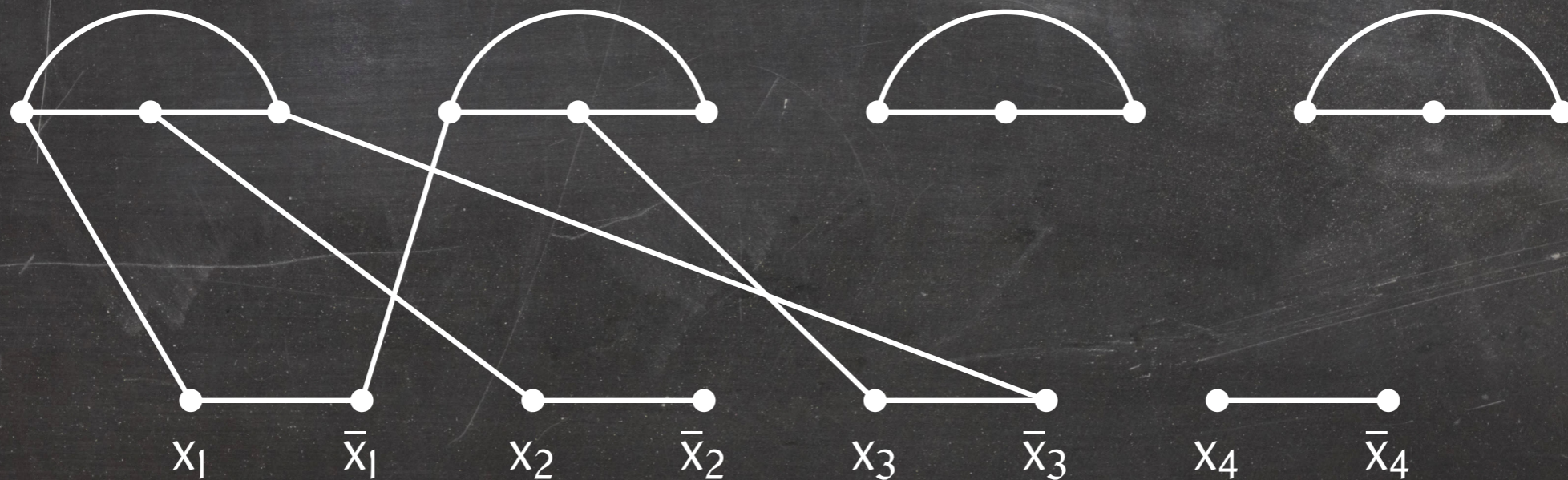


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$



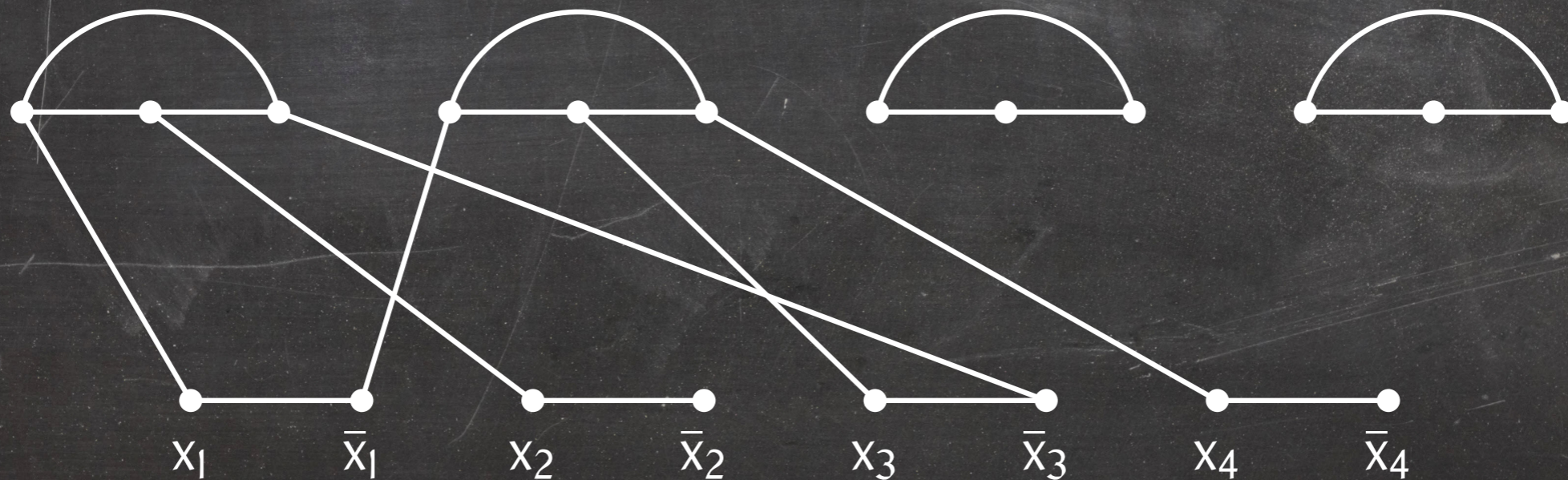


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$

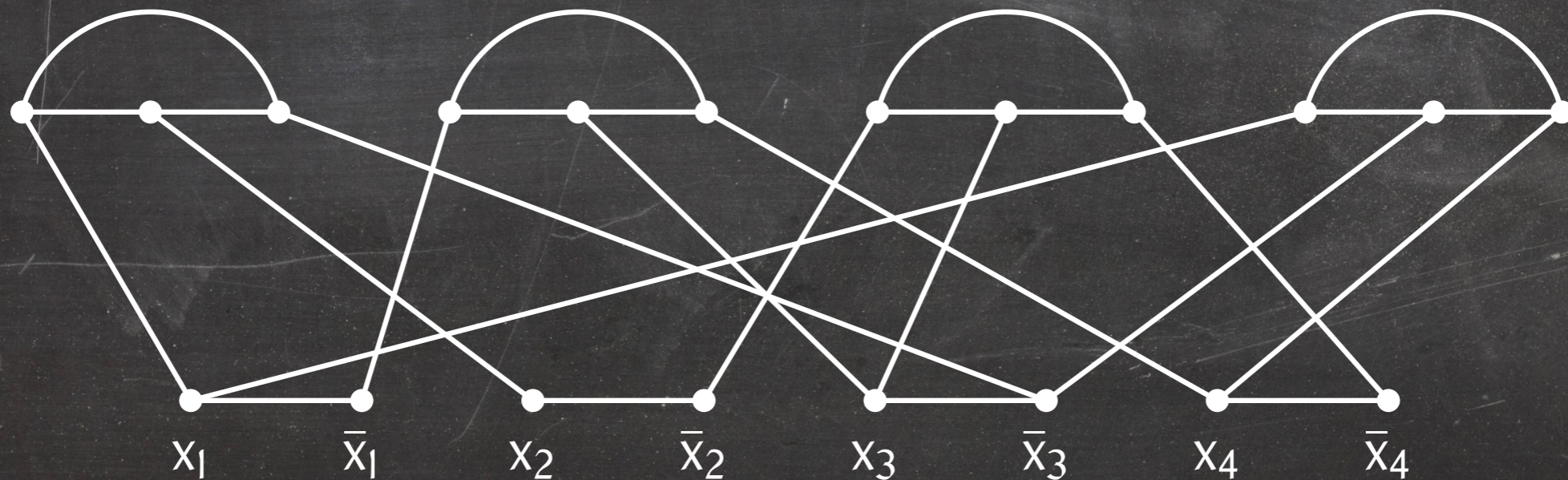


# Vertex Cover is NP-Hard

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$G_F$ :

- One variable widget per variable
- One clause widget per clause
- Connect every literal node  $\lambda_{i,j}$  to its corresponding node  $x_k$  or  $\bar{x}_k$



# Vertex Cover is NP-Hard

$n$  = number of variables

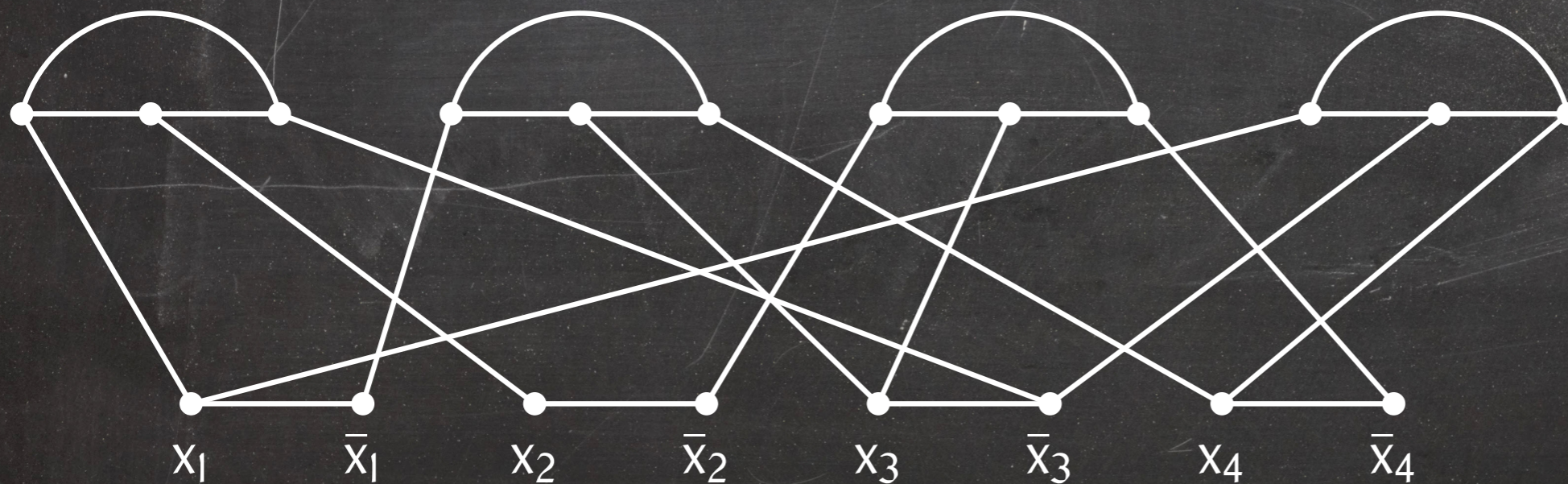
$m$  = number of clauses

# Vertex Cover is NP-Hard

$n$  = number of variables

$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.



# Vertex Cover is NP-Hard

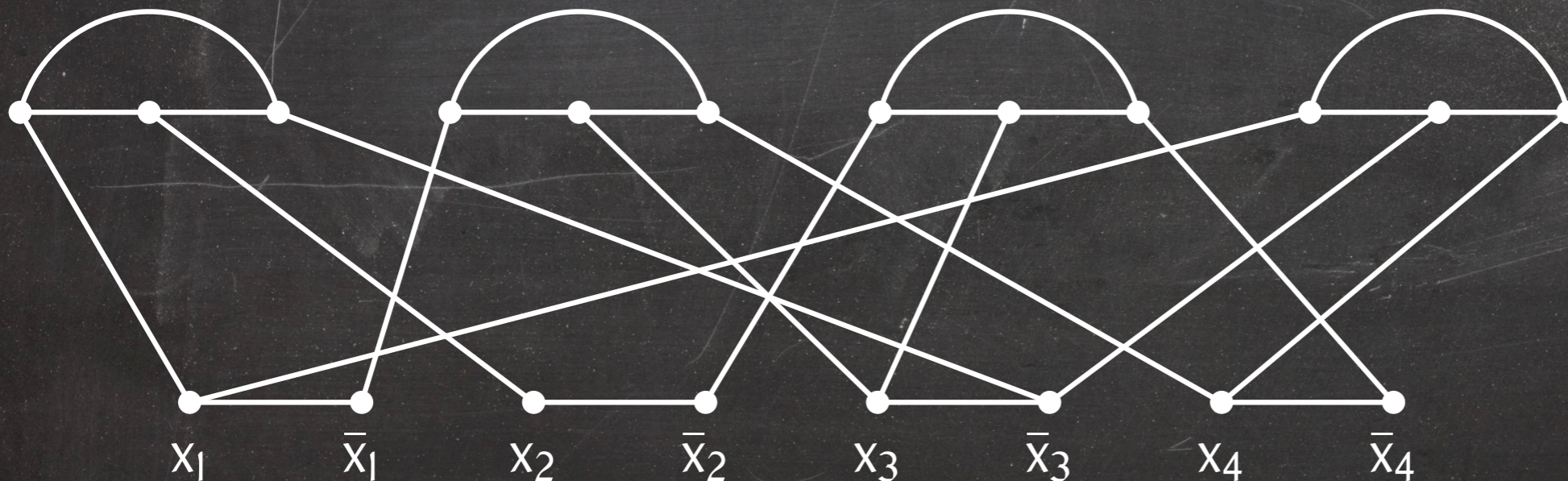
$n$  = number of variables

$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$



# Vertex Cover is NP-Hard

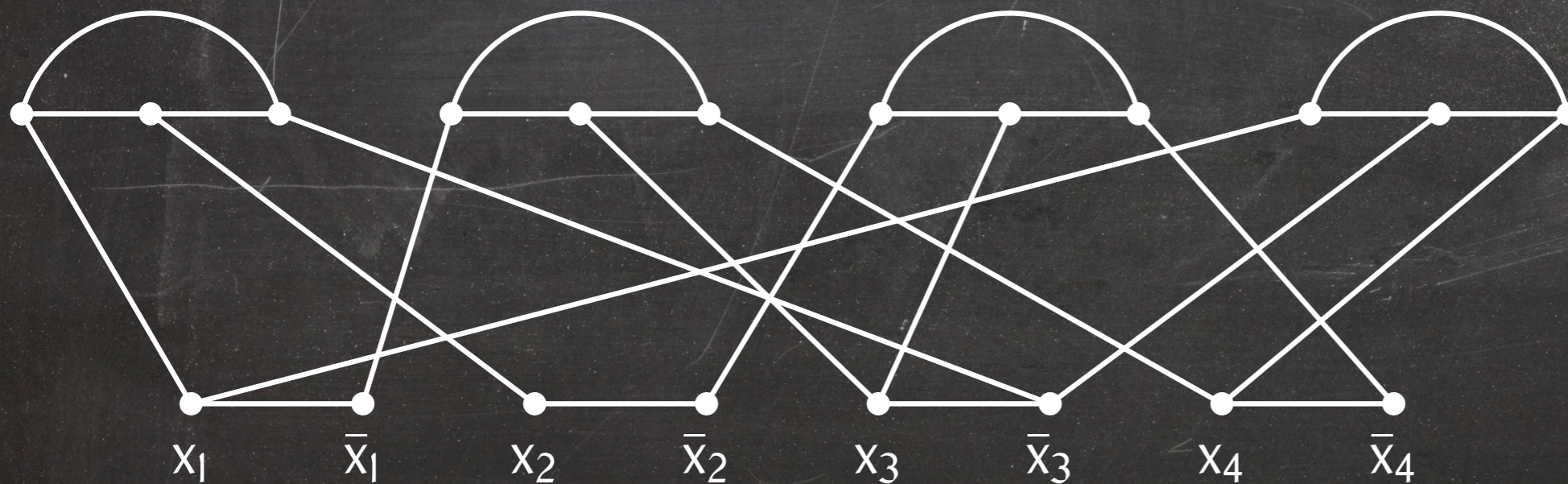
$n$  = number of variables

$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$



Truth assignment



Vertex cover

# Vertex Cover is NP-Hard

$n$  = number of variables

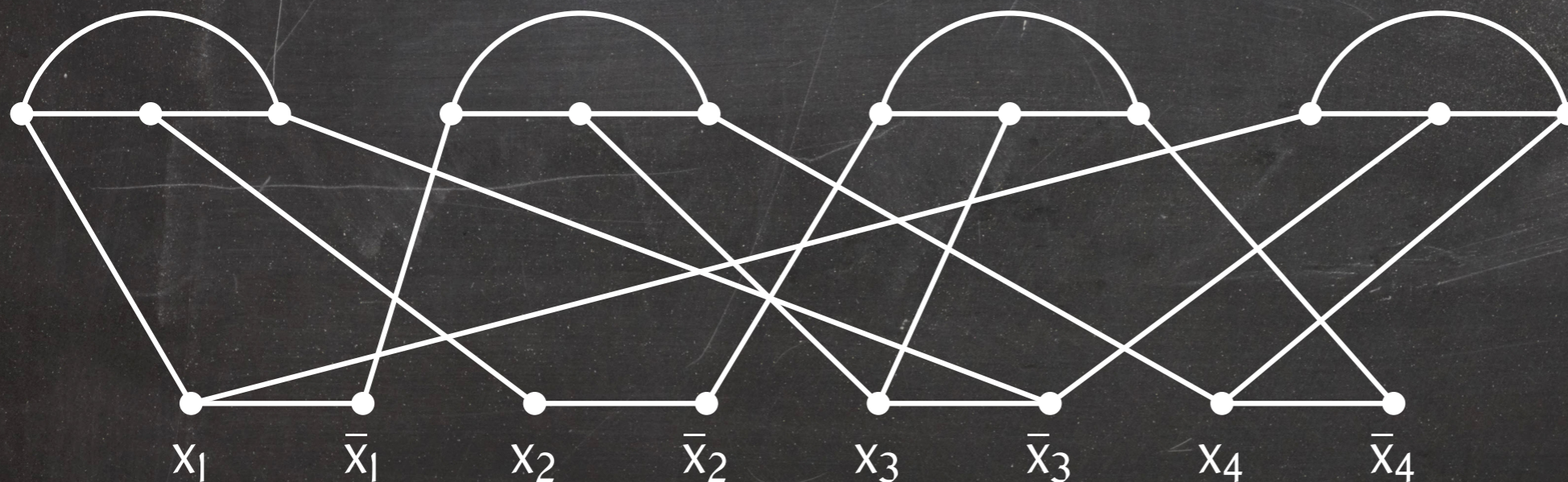
$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment



Vertex cover

# Vertex Cover is NP-Hard

$n$  = number of variables

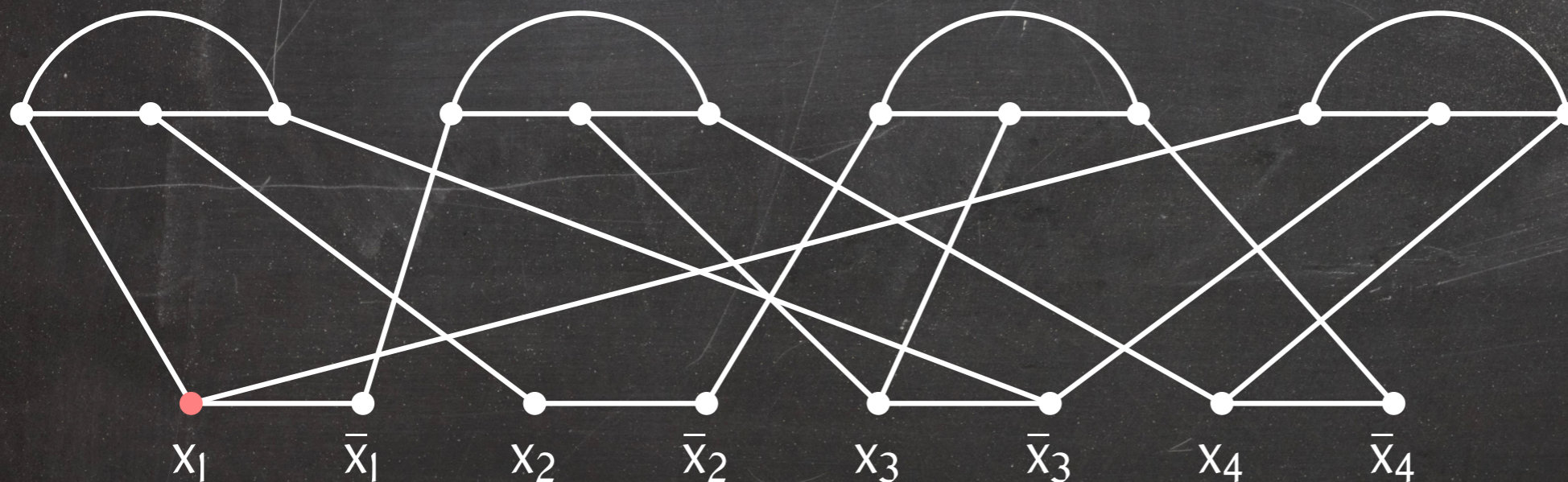
$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment



Vertex cover



# Vertex Cover is NP-Hard

$n$  = number of variables

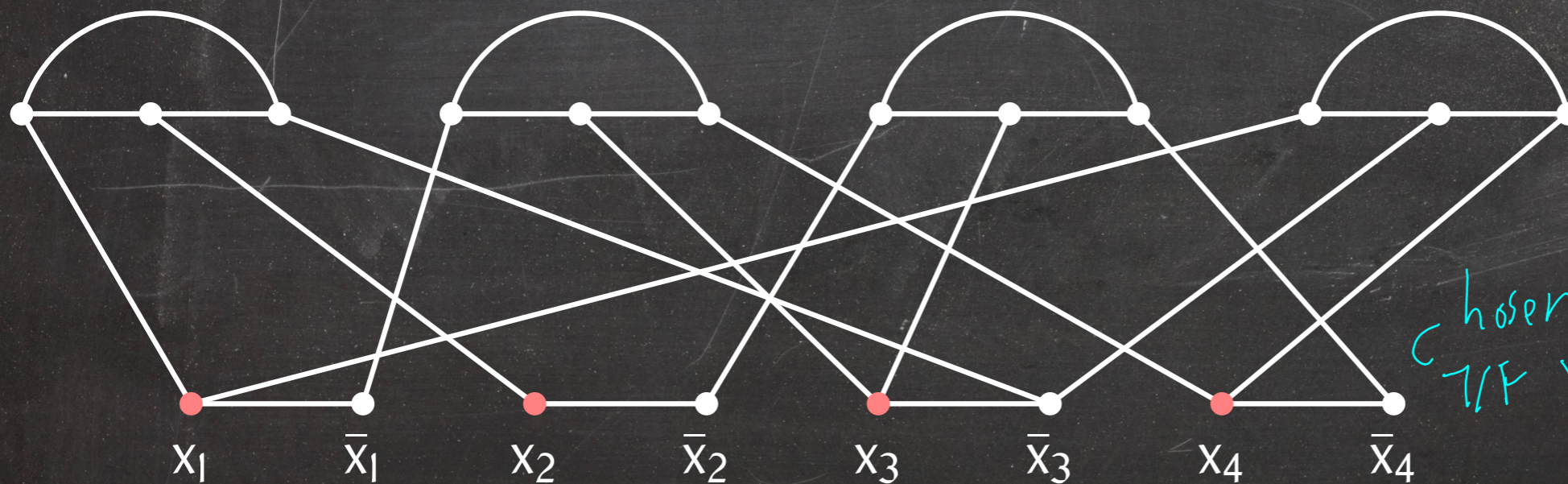
$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment

*chosen  
T/F values*

Vertex cover

# Vertex Cover is NP-Hard

$n$  = number of variables

*one per edge widget  
chosen T/F*

$m$  = number of clauses

*two per clause widget  
need at least 2 to cover clause*

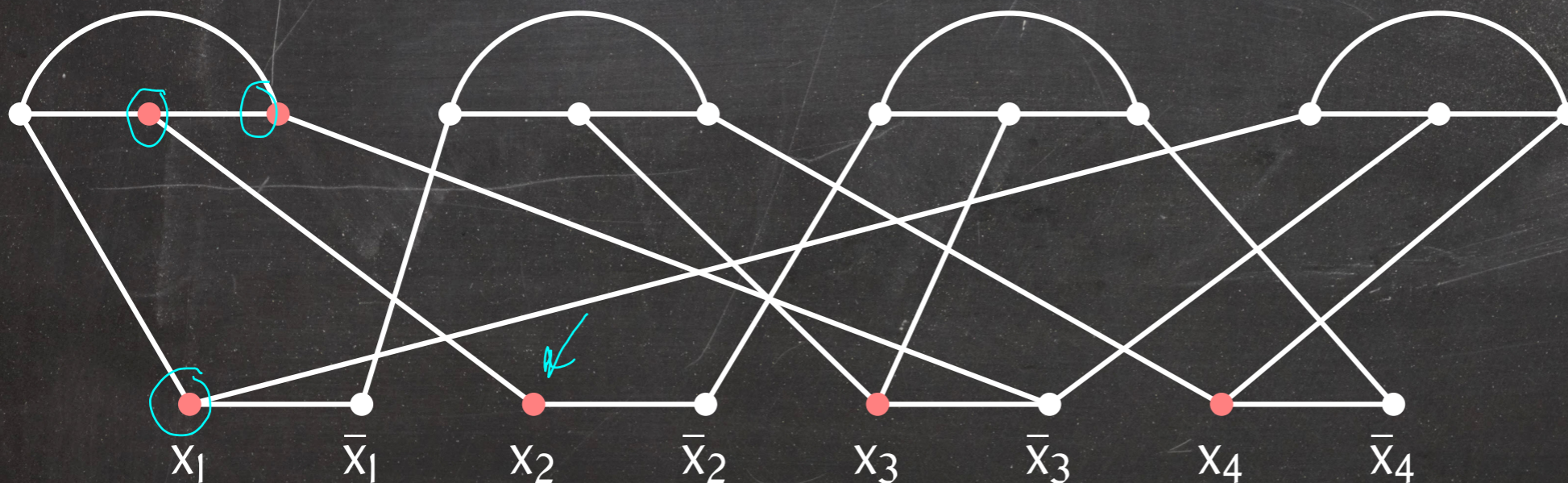
**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

*pick  $x_1$  -  $x_1 \vee \bar{x}_3 \vee x_4$*

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment



Vertex cover

# Vertex Cover is NP-Hard

$n$  = number of variables

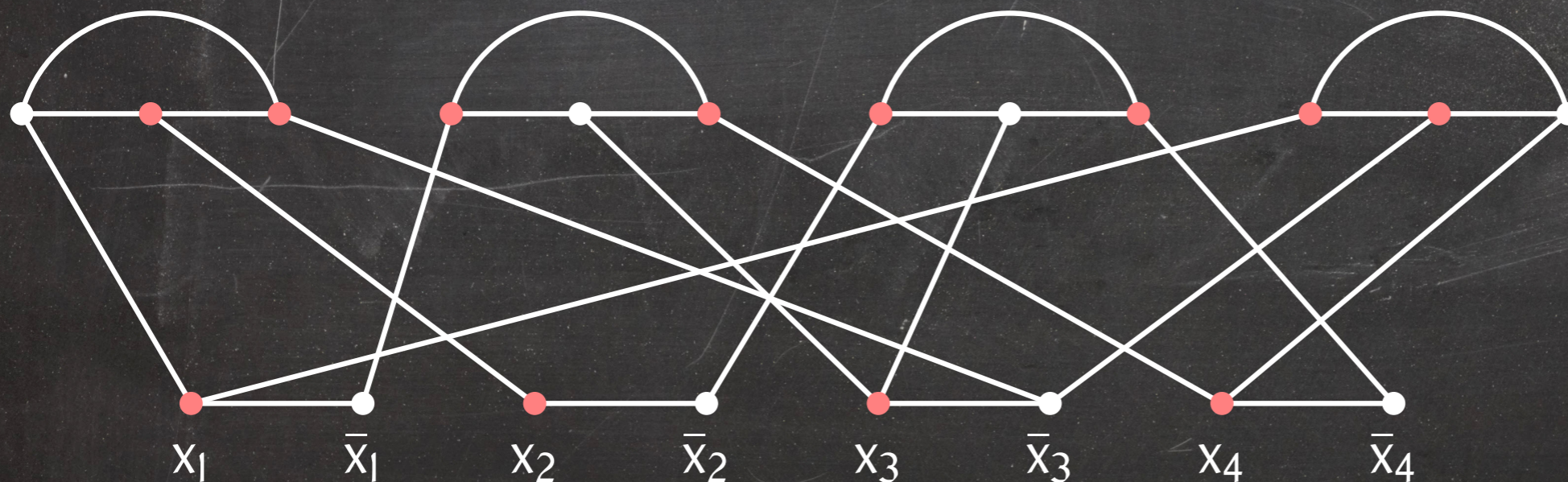
$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment



Vertex cover

# Vertex Cover is NP-Hard

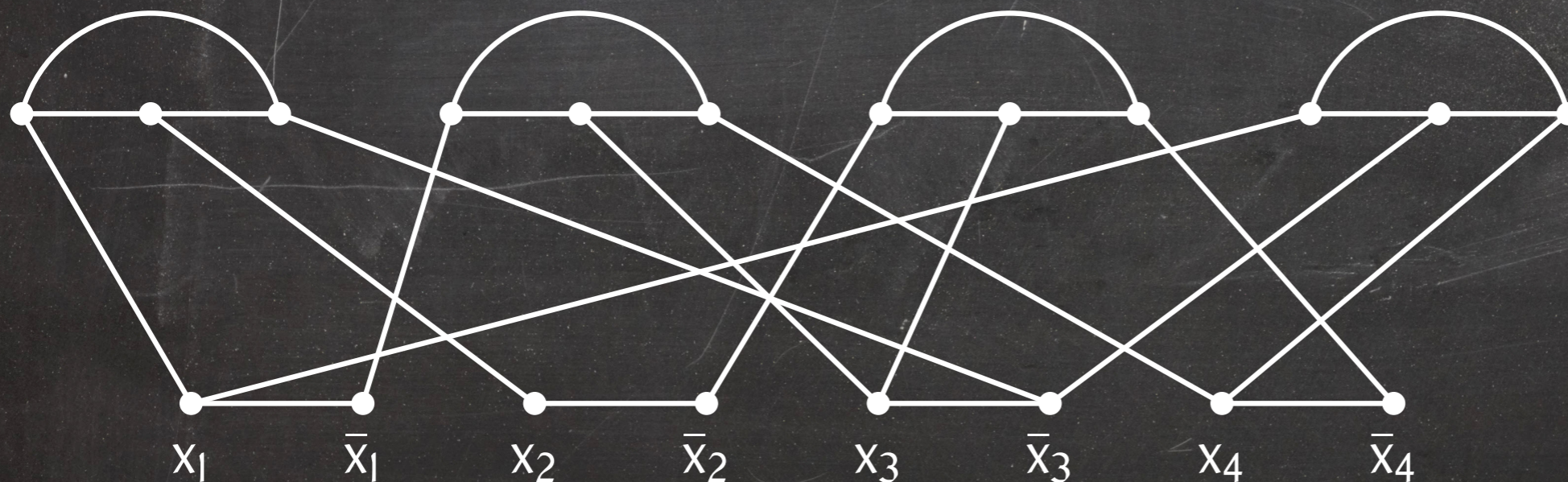
$n$  = number of variables

$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$



Truth assignment

Vertex cover

# Vertex Cover is NP-Hard

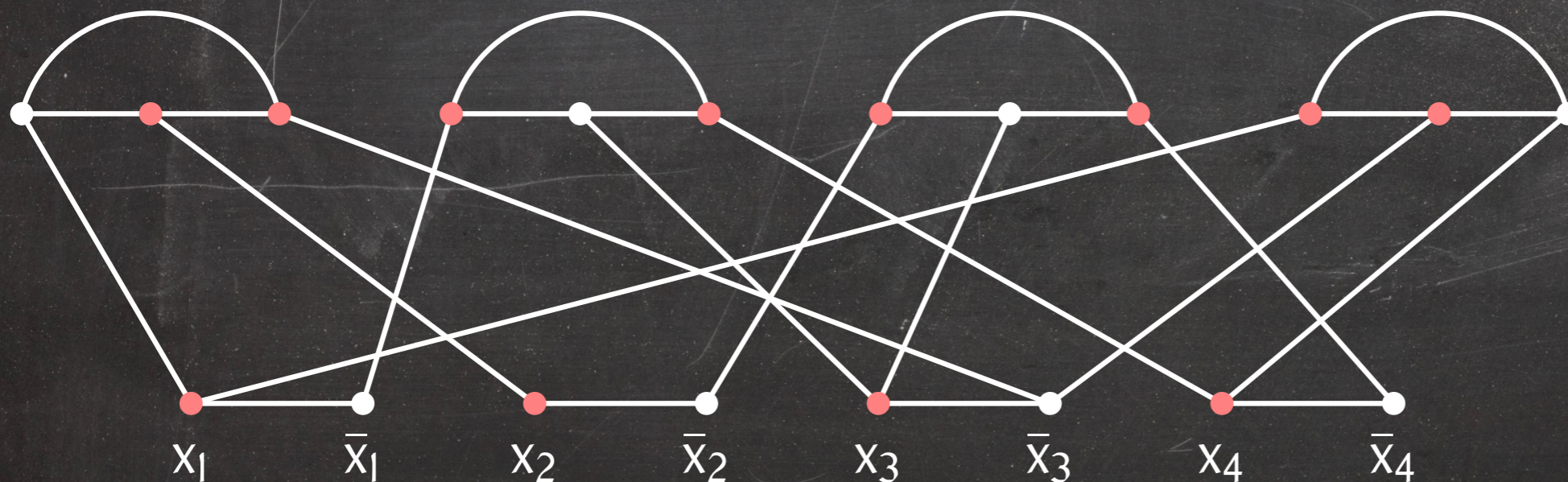
$n$  = number of variables

$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$



Truth assignment

Vertex cover

# Vertex Cover is NP-Hard

$n$  = number of variables

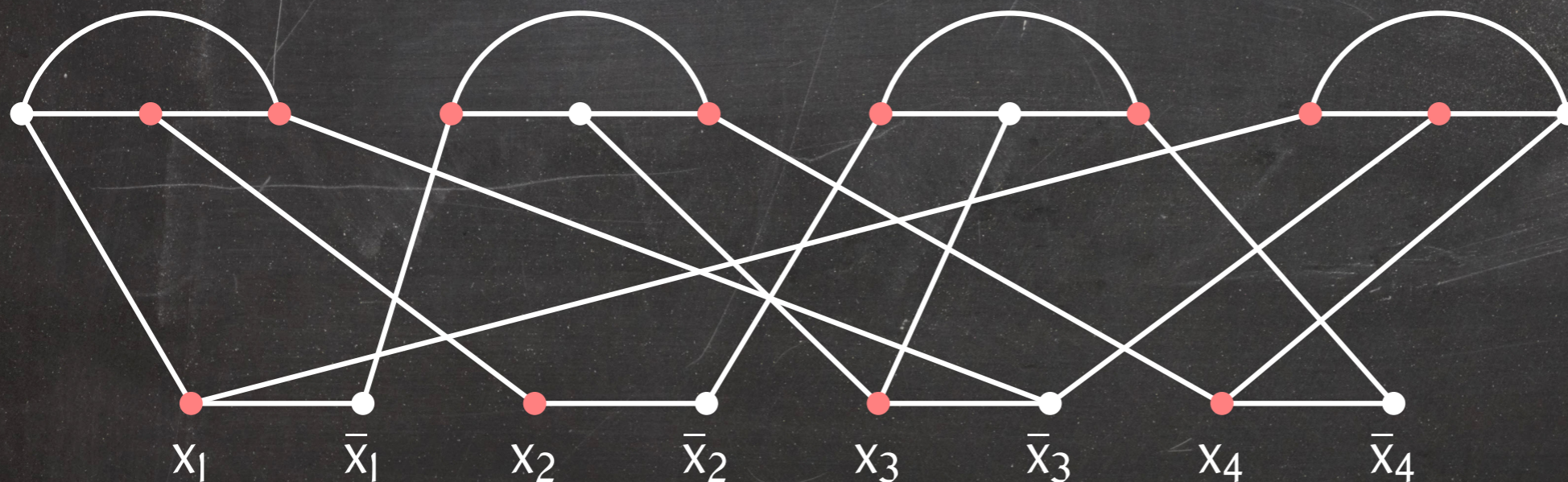
$m$  = number of clauses

**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment

Vertex cover

# Vertex Cover is NP-Hard

$n$  = number of variables

$m$  = number of clauses

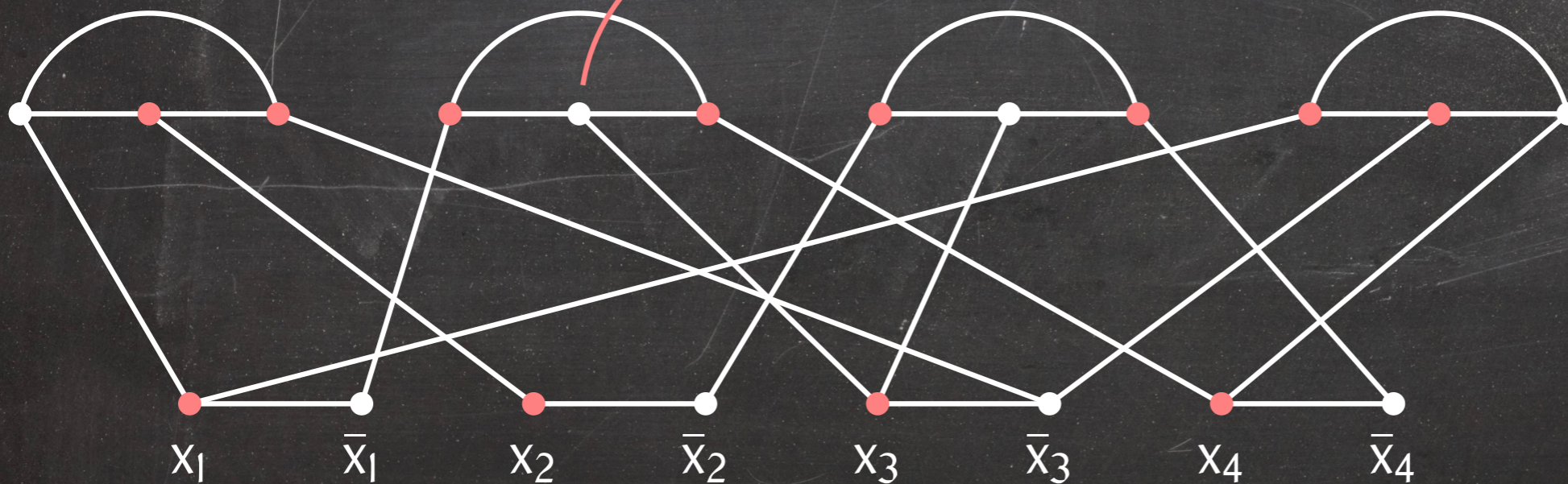
**Observation:** Any vertex cover of  $G_F$  of size  $n + 2m$  contains one vertex per variable widget and two vertices per clause widget.

**Lemma:**  $F$  is satisfiable if and only if  $G_F$  has a vertex cover of size  $n + 2m$ .

*covered this part of the clause*

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$



Truth assignment

Vertex cover

# Vertex Cover is NP-Complete

Since Vertex Cover is NP-hard, we only have to verify that it is in NP:



# Vertex Cover is NP-Complete

Since Vertex Cover is NP-hard, we only have to verify that it is in NP:

Let  $VC = \{(G, k) \mid G \text{ has a vertex cover of size } k\}$ .

To prove that  $VC \in NP$ , we have to prove that there exists a language  $VC' \in P$  such that  $(G, k) \in VC$  if and only if  $(G, k, y) \in VC'$  for some  $y$  with  $|y| \in O(|(G, k)|^c)$ .

# Vertex Cover is NP-Complete

Since Vertex Cover is NP-hard, we only have to verify that it is in NP:

Let  $VC = \{(G, k) \mid G \text{ has a vertex cover of size } k\}$ .

To prove that  $VC \in NP$ , we have to prove that there exists a language  $VC' \in P$  such that  $(G, k) \in VC$  if and only if  $(G, k, y) \in VC'$  for some  $y$  with  $|y| \in O(|(G, k)|^c)$ .

Let  $VC' = \{(G, k, C) \mid C \text{ is a vertex cover of } G \text{ of size } k\}$ .

# Vertex Cover is NP-Complete

Since Vertex Cover is NP-hard, we only have to verify that it is in NP:

Let  $VC = \{(G, k) \mid G \text{ has a vertex cover of size } k\}$ .

To prove that  $VC \in NP$ , we have to prove that there exists a language  $VC' \in P$  such that  $(G, k) \in VC$  if and only if  $(G, k, y) \in VC'$  for some  $y$  with  $|y| \in O(|(G, k)|^c)$ .

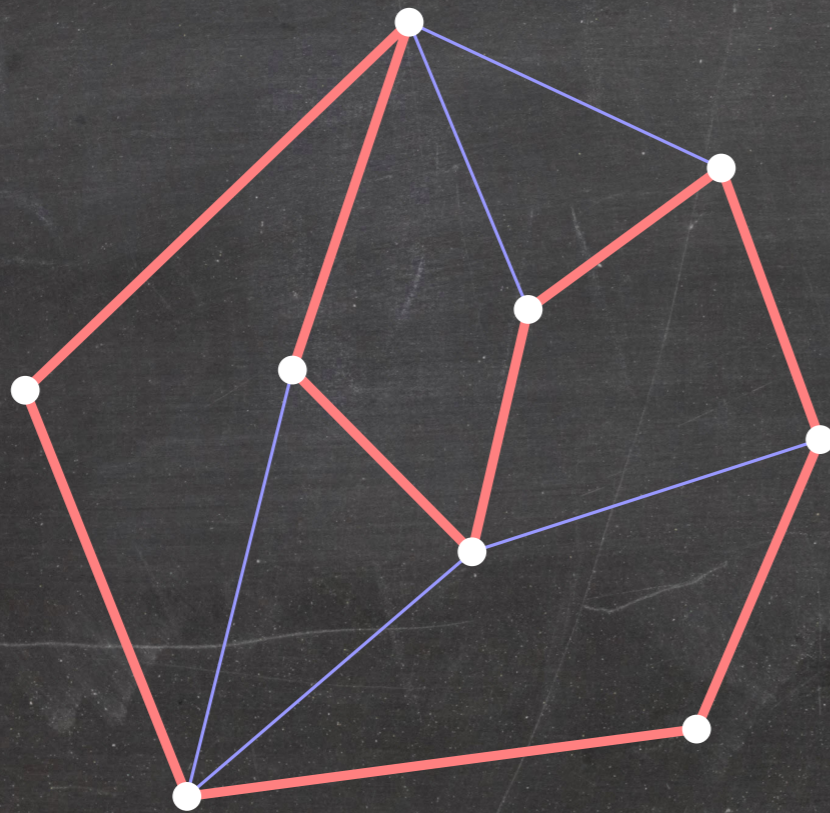
Let  $VC' = \{(G, k, C) \mid C \text{ is a vertex cover of } G \text{ of size } k\}$ .

$VC' \in P$ :

- We can test in polynomial time whether every vertex in  $C$  belongs to  $G$ .
- We can test in polynomial time whether  $|C| = k$ .
- We can test in polynomial time whether every edge of  $G$  has at least one endpoint in  $C$ .

# Hamiltonian Cycle

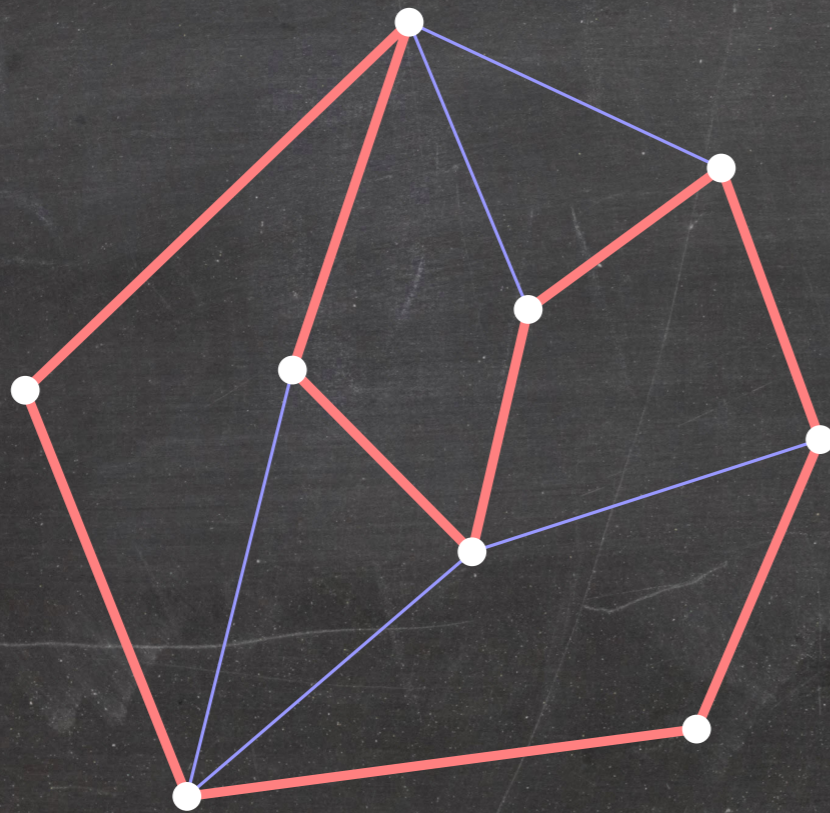
A **Hamiltonian cycle** of a graph  $G$  is a simple cycle that contains all vertices of  $G$  and whose edges are edges of  $G$ .



# Hamiltonian Cycle

A **Hamiltonian cycle** of a graph  $G$  is a simple cycle that contains all vertices of  $G$  and whose edges are edges of  $G$ .

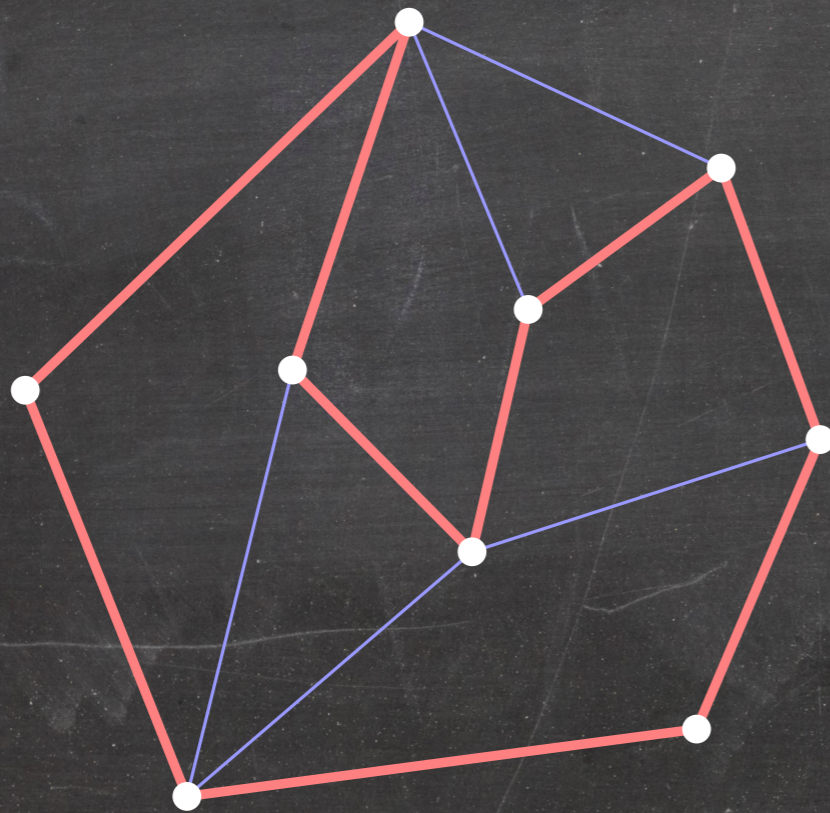
A graph  $G$  is **Hamiltonian** if it has a Hamiltonian cycle.



# Hamiltonian Cycle

A **Hamiltonian cycle** of a graph  $G$  is a simple cycle that contains all vertices of  $G$  and whose edges are edges of  $G$ .

A graph  $G$  is **Hamiltonian** if it has a Hamiltonian cycle.

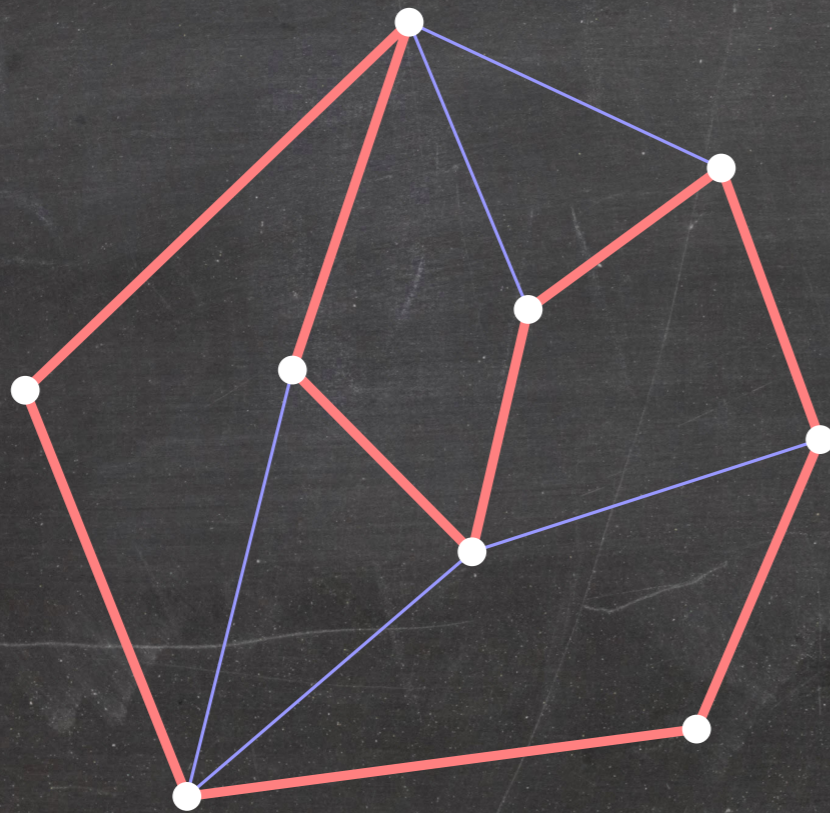


Hamiltonian

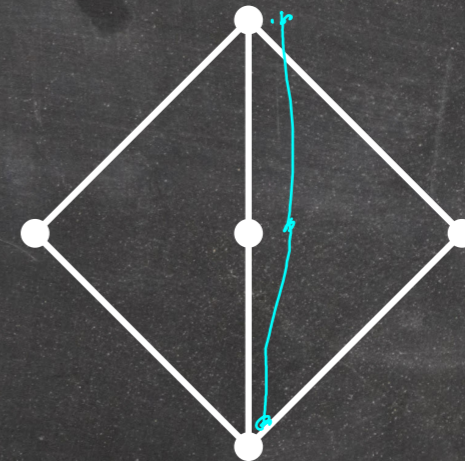
# Hamiltonian Cycle

A **Hamiltonian cycle** of a graph  $G$  is a simple cycle that contains all vertices of  $G$  and whose edges are edges of  $G$ .

A graph  $G$  is **Hamiltonian** if it has a Hamiltonian cycle.



Hamiltonian



not Hamiltonian

# Hamiltonian Cycle is NP-Complete

**Hamiltonian Cycle Problem:** Decide whether a given graph  $G$  is Hamiltonian.



# Hamiltonian Cycle is NP-Complete

**Hamiltonian Cycle Problem:** Decide whether a given graph  $G$  is Hamiltonian.

**Exercise:** Verify that Hamiltonian Cycle is in NP.

# Hamiltonian Cycle is NP-Complete

**Hamiltonian Cycle Problem:** Decide whether a given graph  $G$  is Hamiltonian.

**Exercise:** Verify that Hamiltonian Cycle is in NP.

**To prove:** Hamiltonian Cycle is NP-hard.

# Hamiltonian Cycle is NP-Complete

**Hamiltonian Cycle Problem:** Decide whether a given graph  $G$  is Hamiltonian.

**Exercise:** Verify that Hamiltonian Cycle is in NP.

**To prove:** Hamiltonian Cycle is NP-hard.

**Reduction from Vertex Cover:** Given a vertex cover instance  $(G, k)$ , we build a graph  $G'$  that has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete

**Hamiltonian Cycle Problem:** Decide whether a given graph  $G$  is Hamiltonian.

**Exercise:** Verify that Hamiltonian Cycle is in NP.

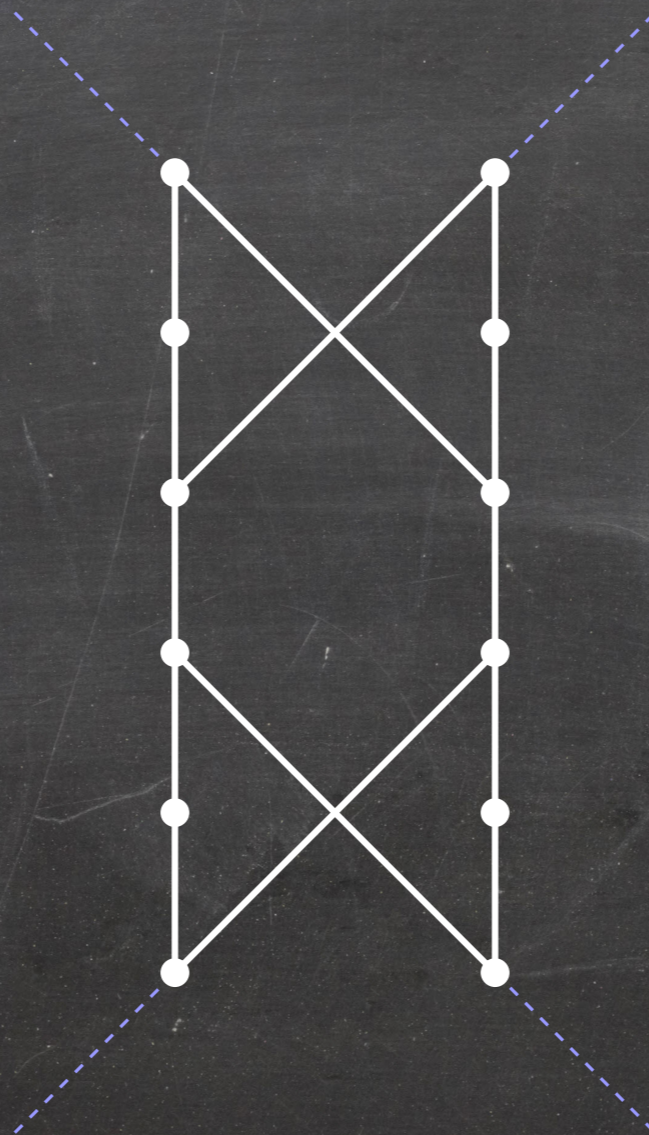
**To prove:** Hamiltonian Cycle is NP-hard.

**Reduction from Vertex Cover:** Given a vertex cover instance  $(G, k)$ , we build a graph  $G'$  that has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

Again, it is trivial to verify that the construction takes polynomial time.

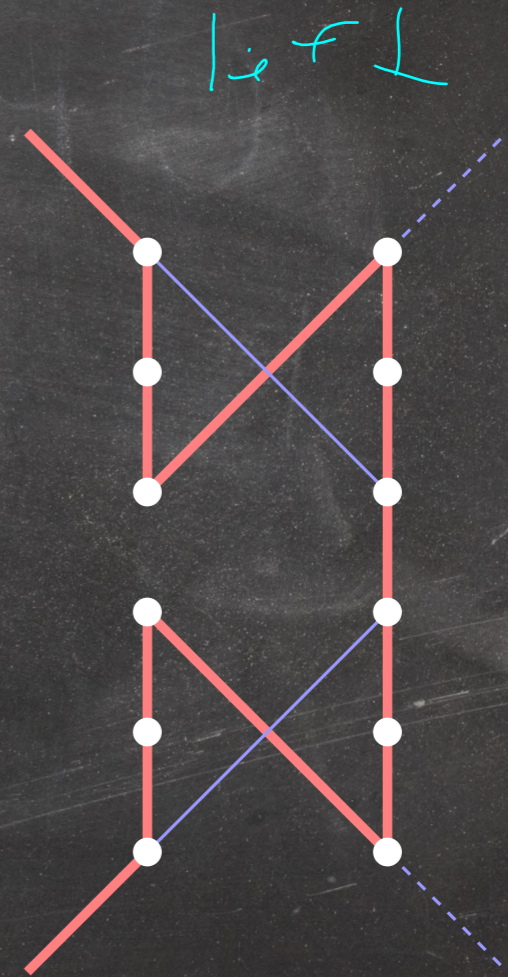
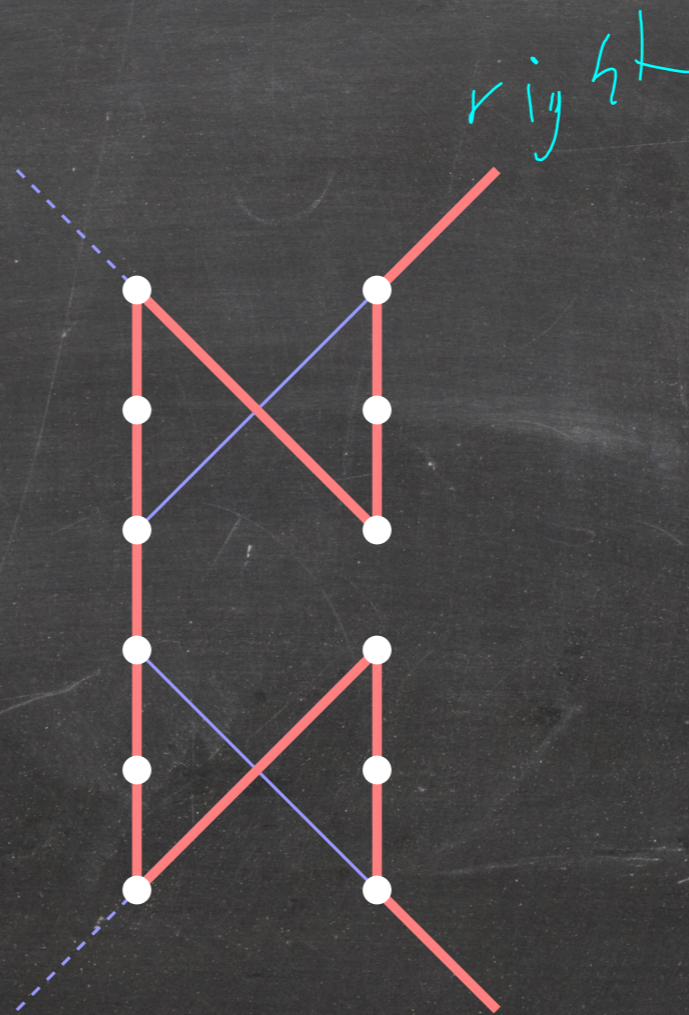
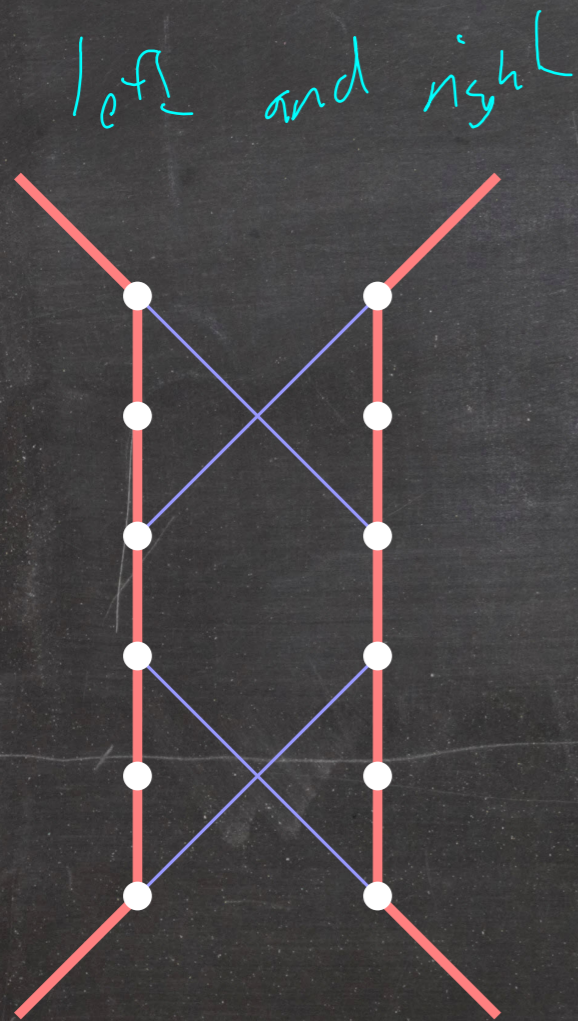
# Edge Widgets

We build  $G'$  from **edge widgets**.



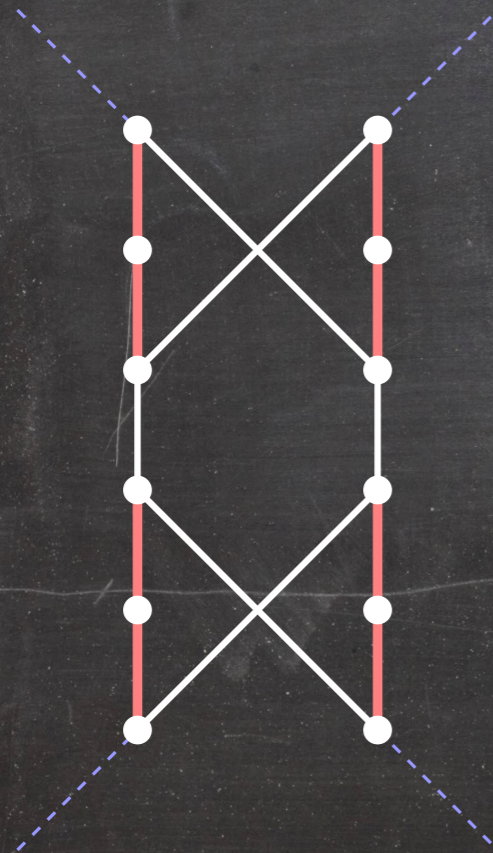
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



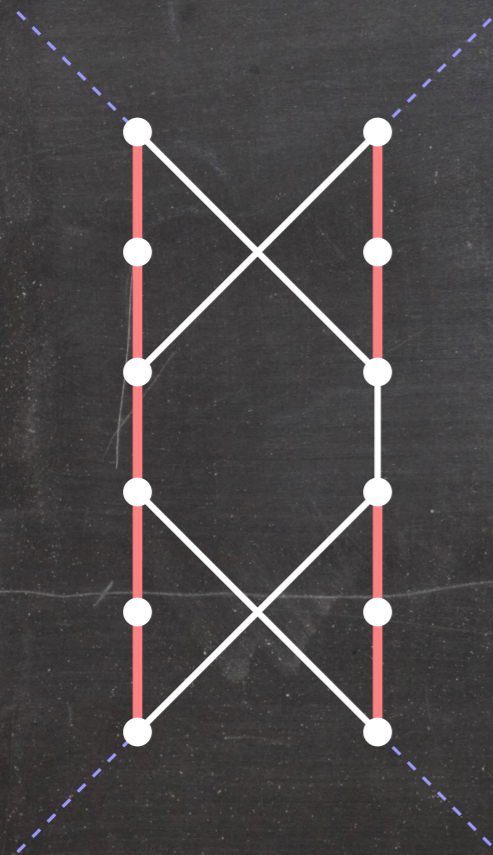
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



# Edge Widgets

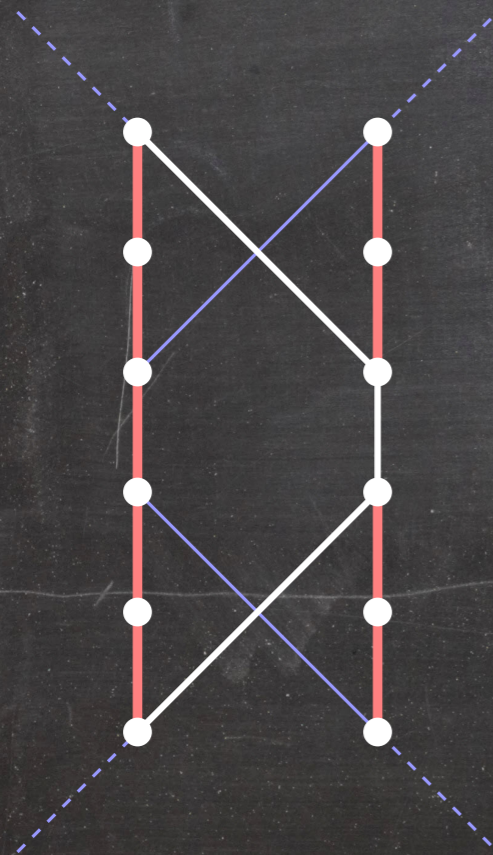
**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.





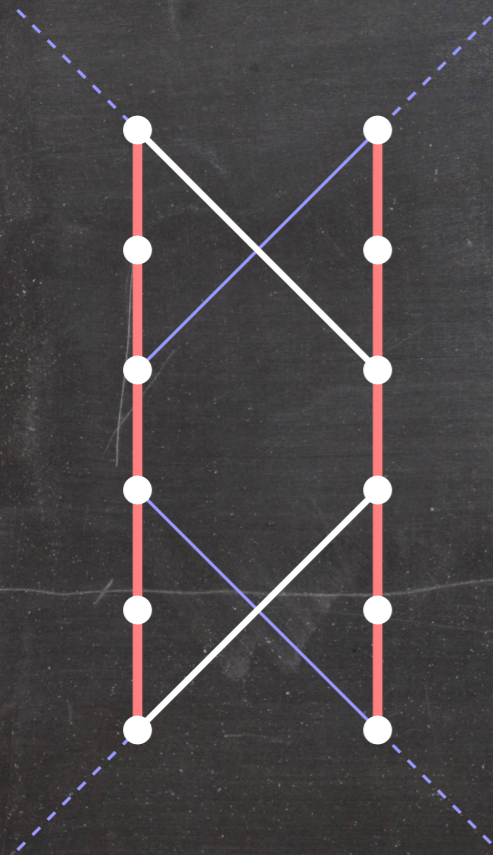
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



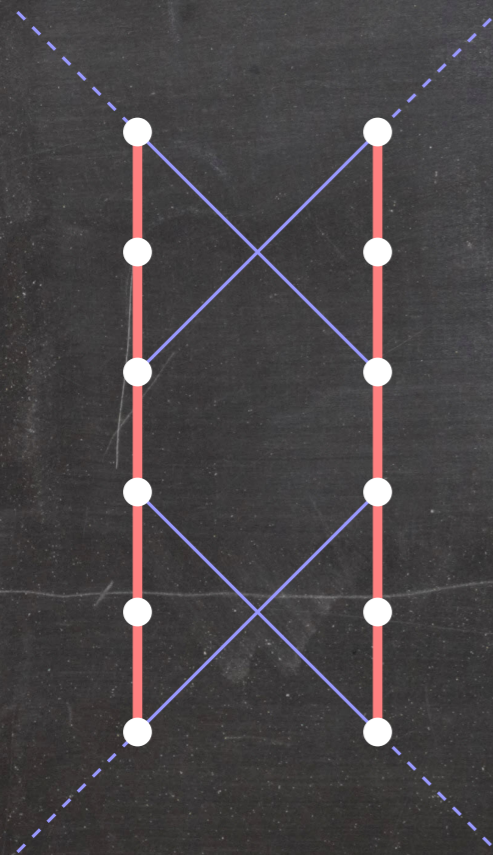
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



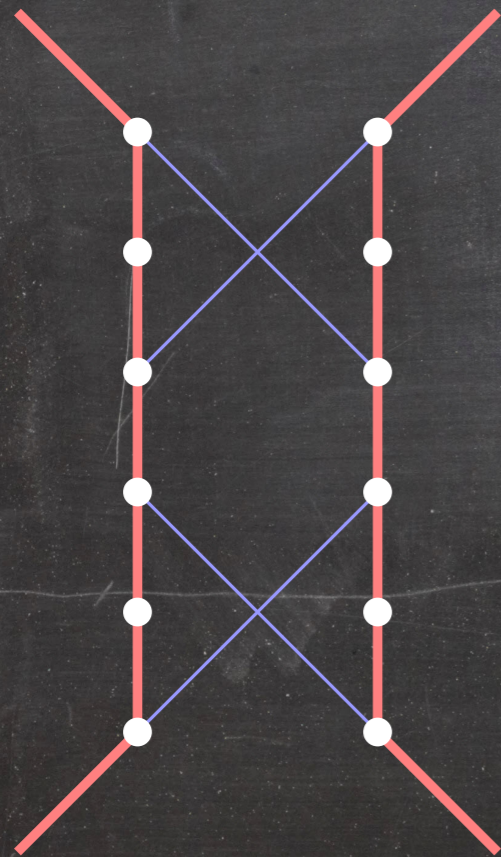
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



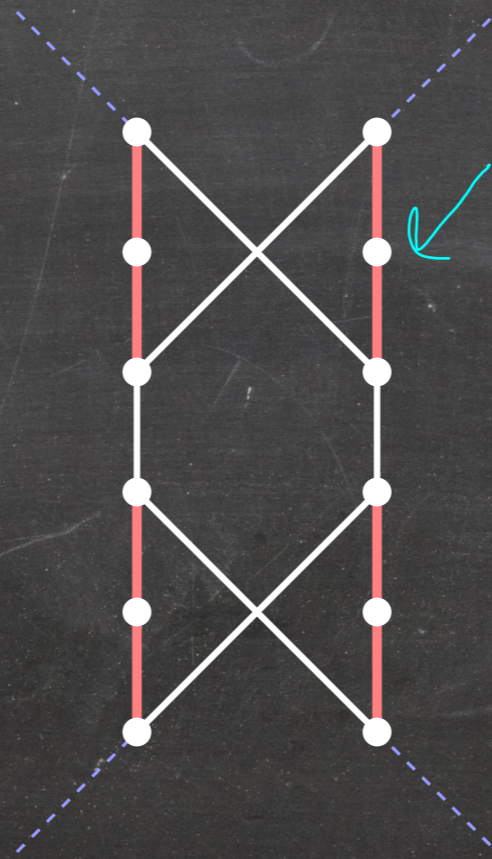
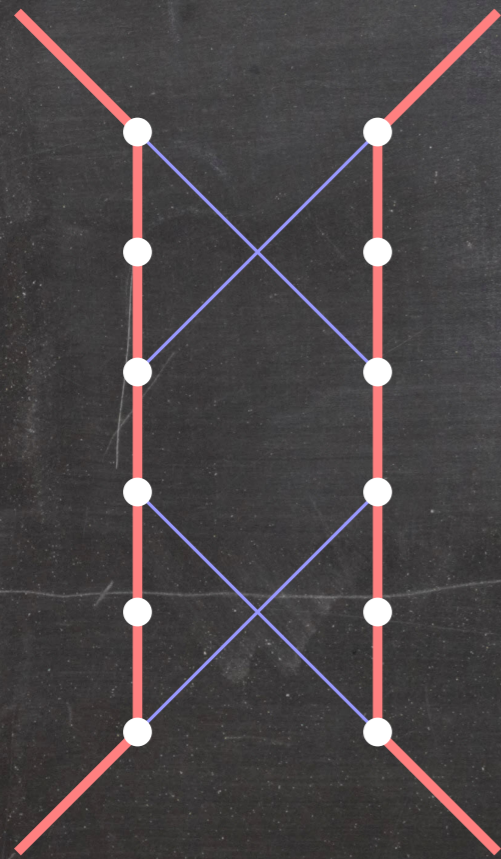
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



# Edge Widgets

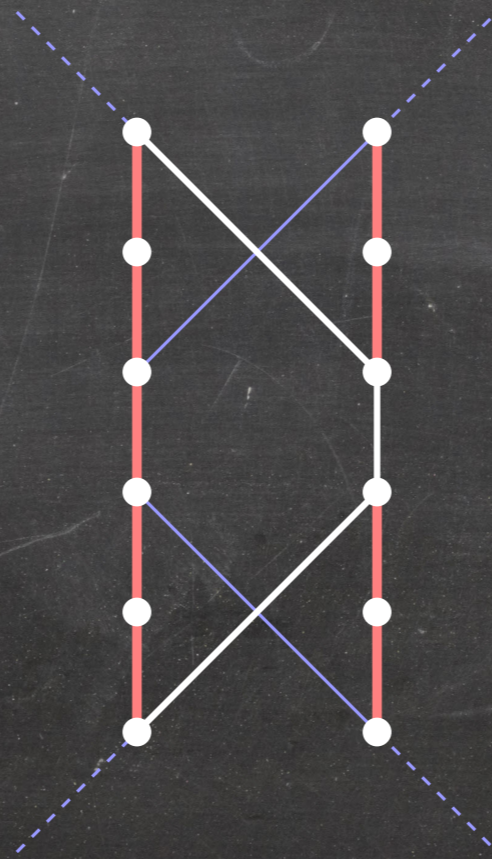
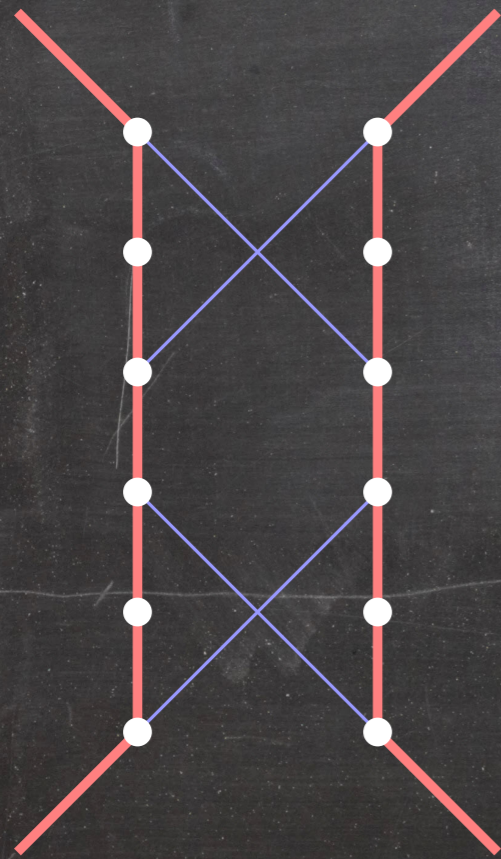
**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



Can't miss this vertex

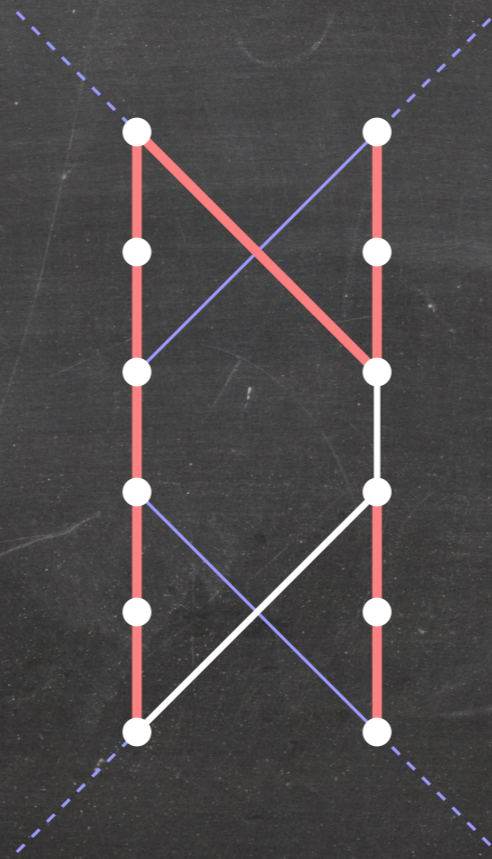
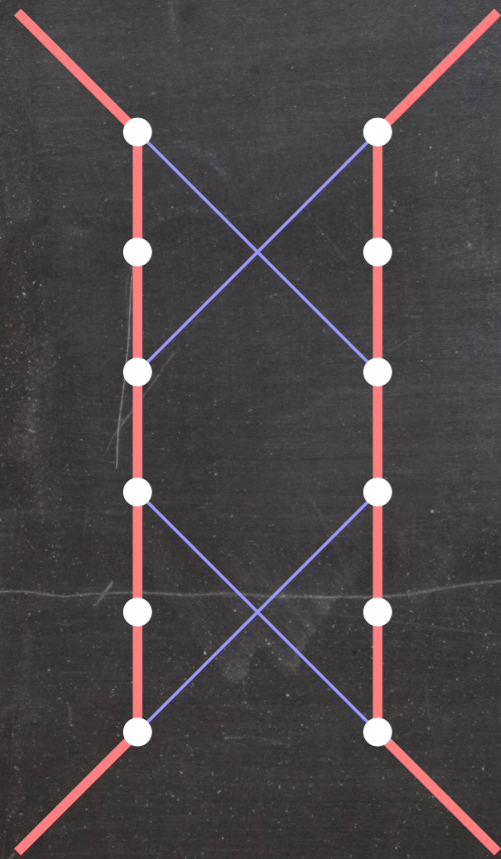
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



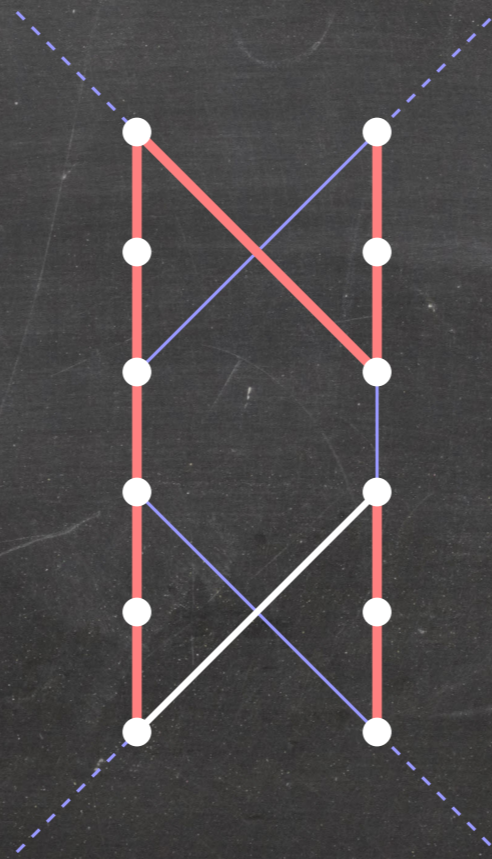
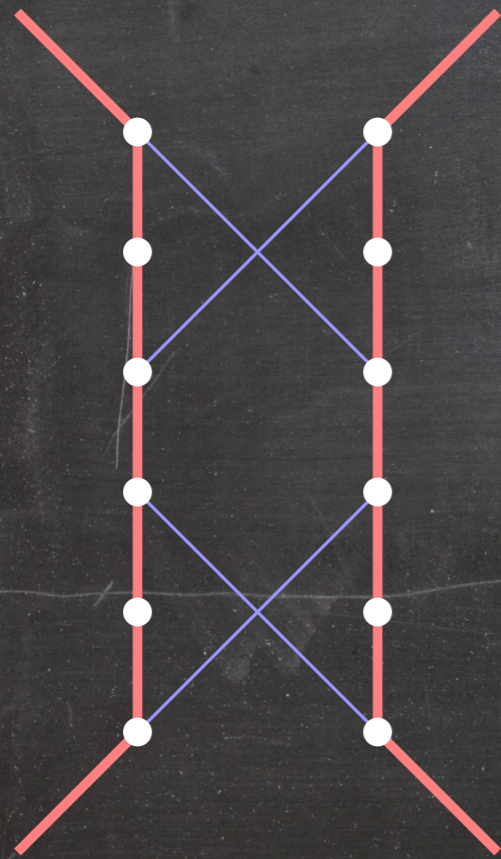
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



# Edge Widgets

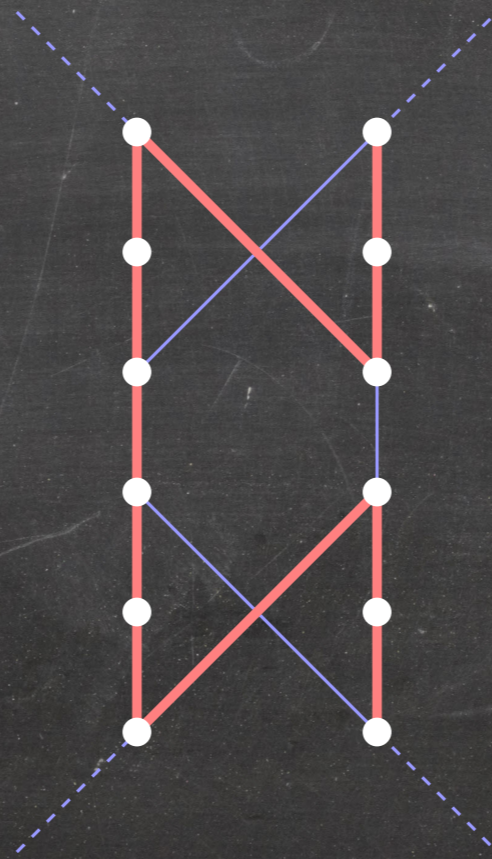
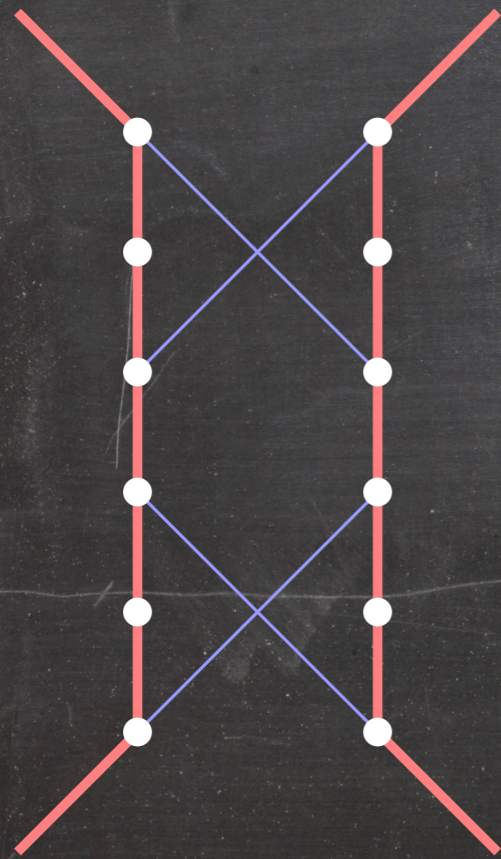
**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.





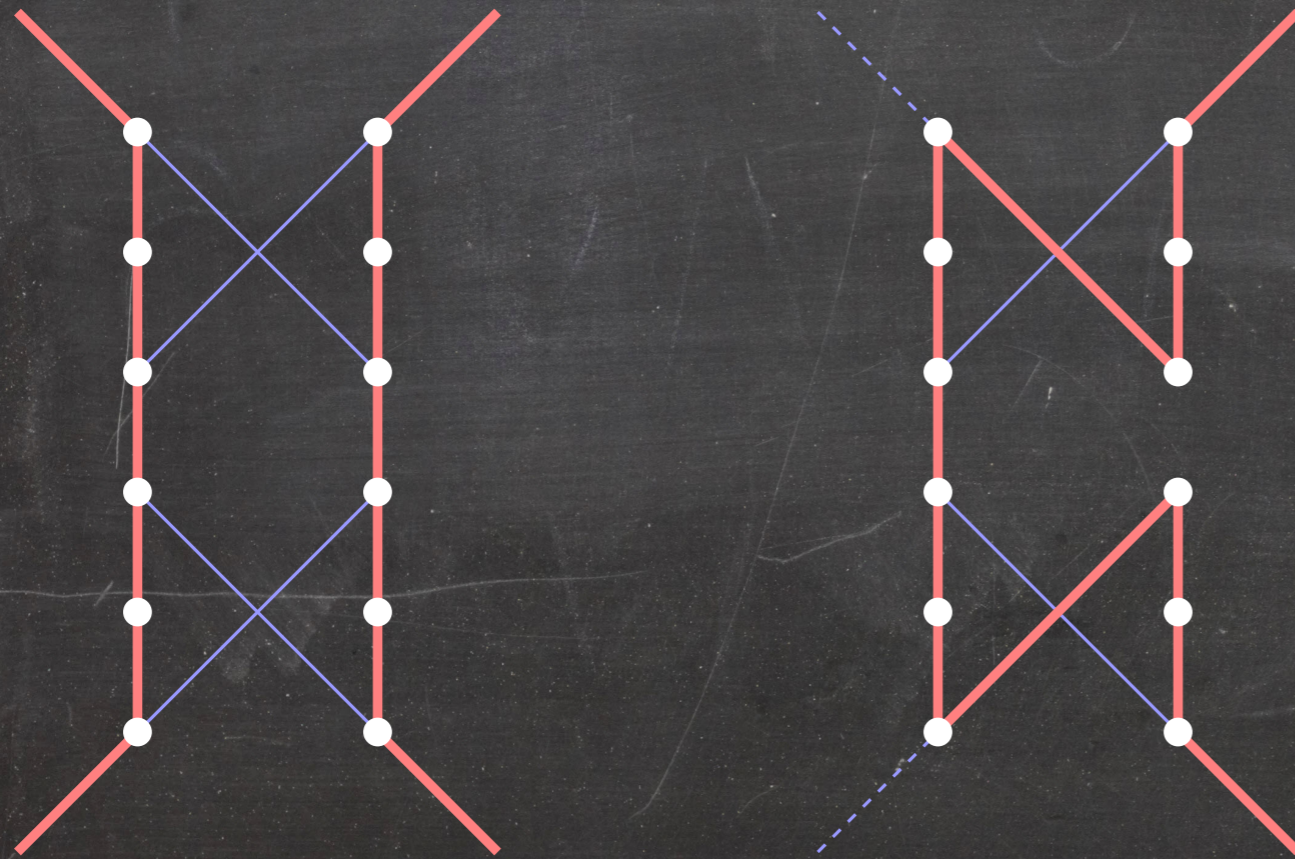
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



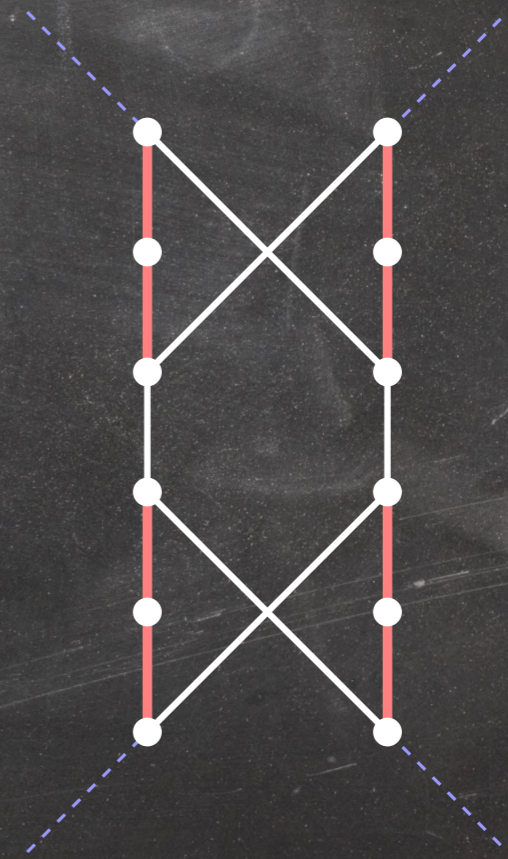
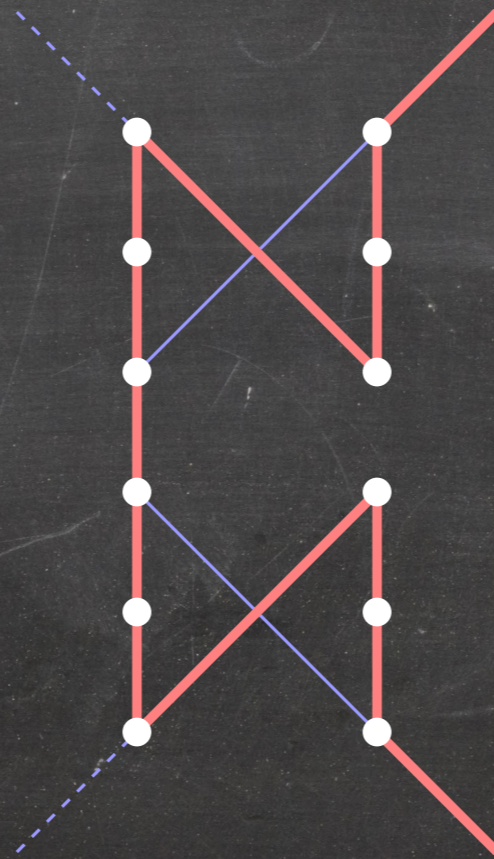
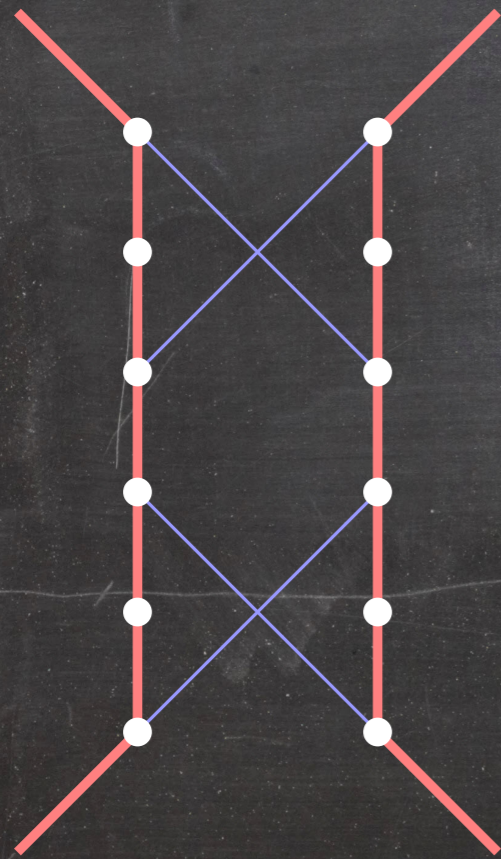
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



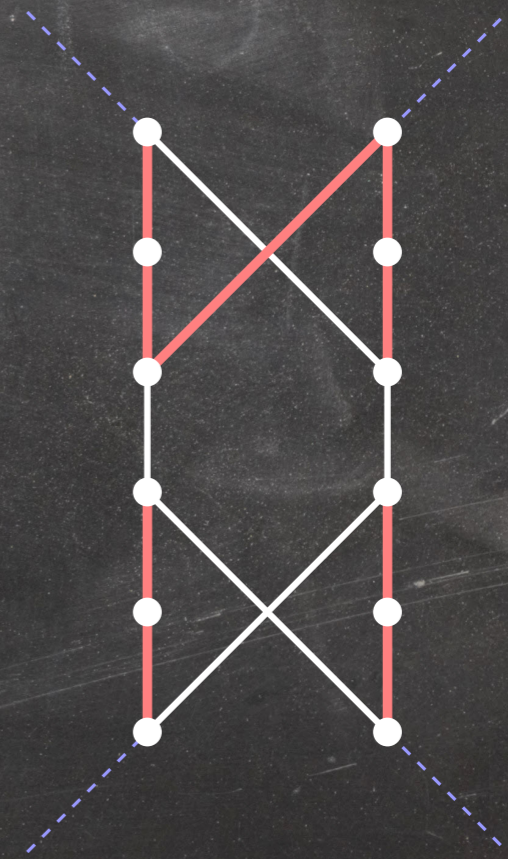
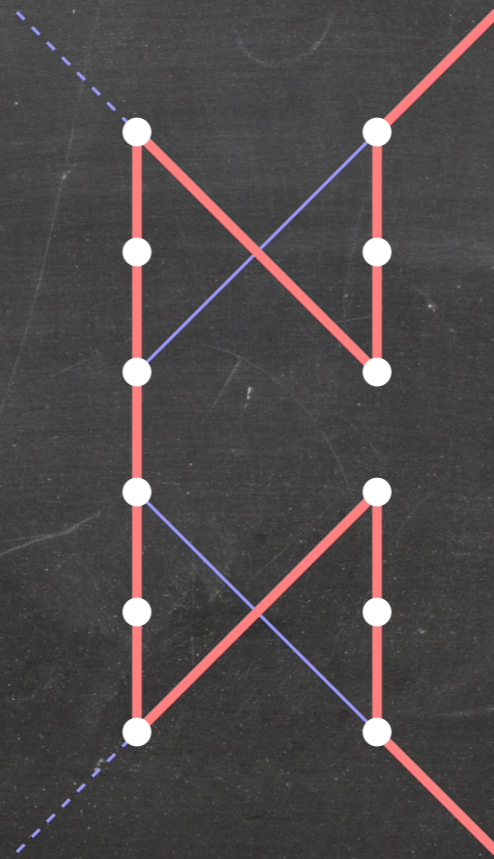
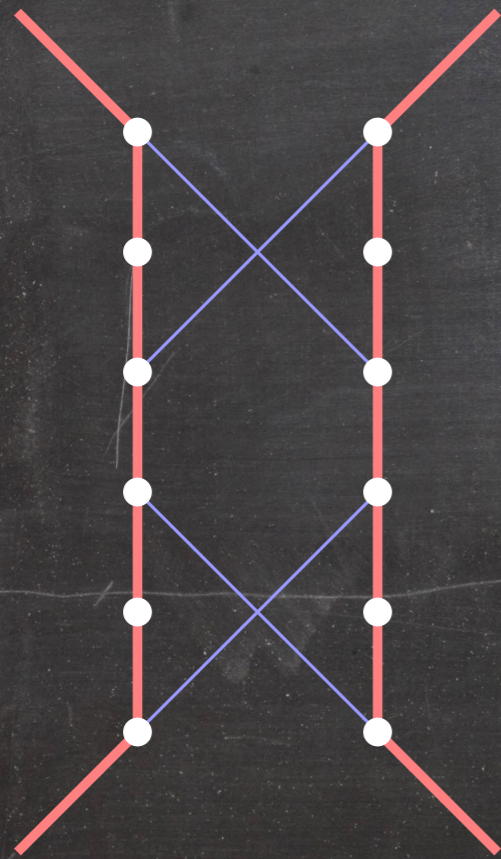
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



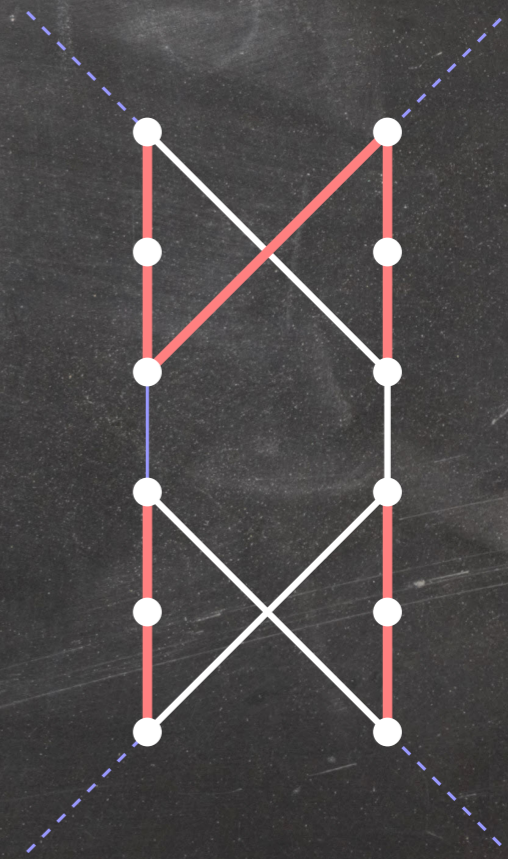
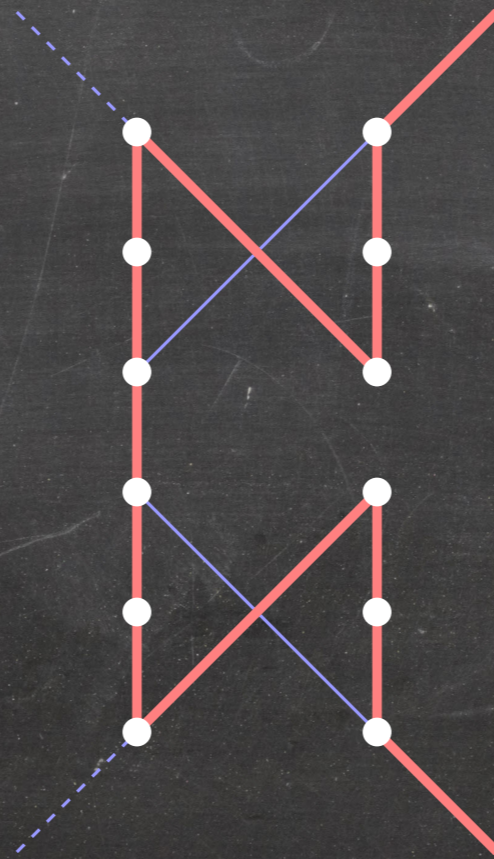
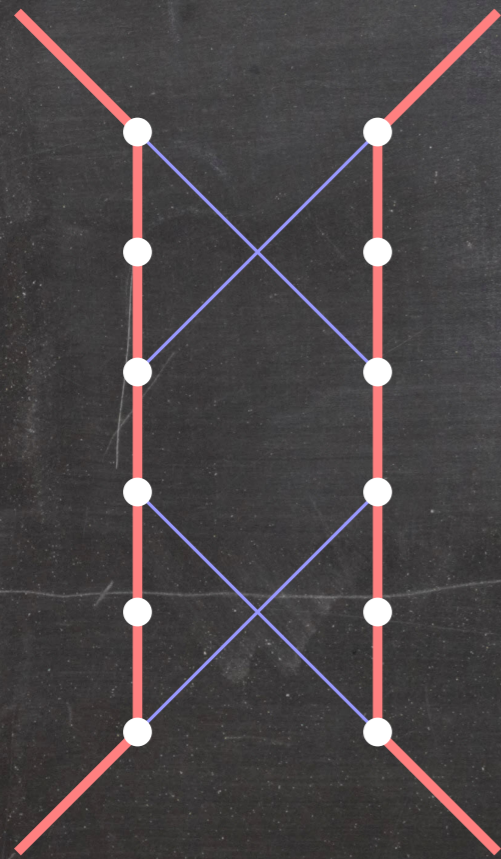
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



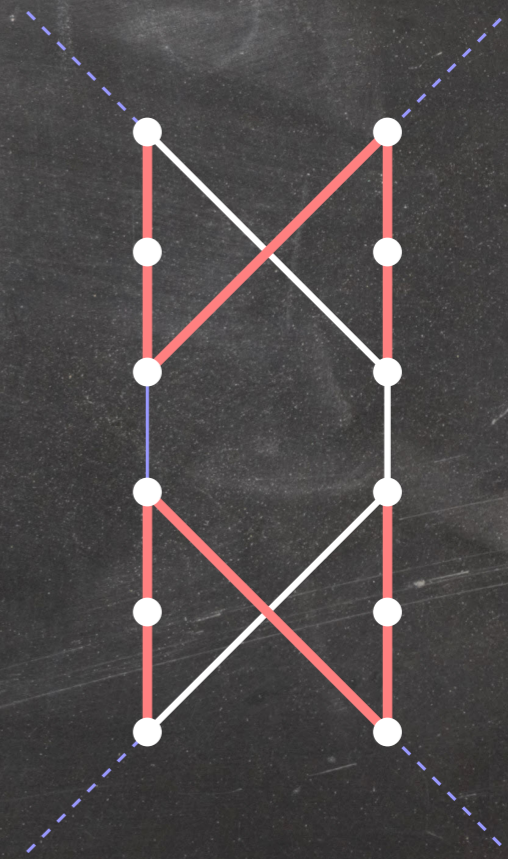
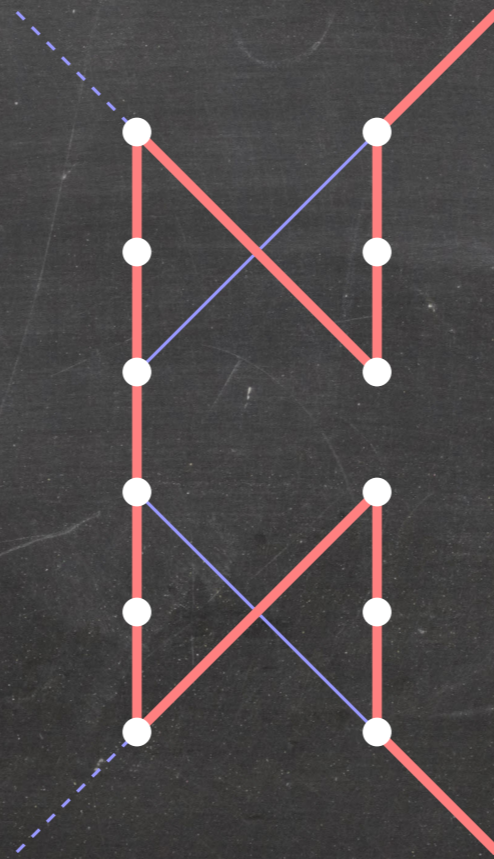
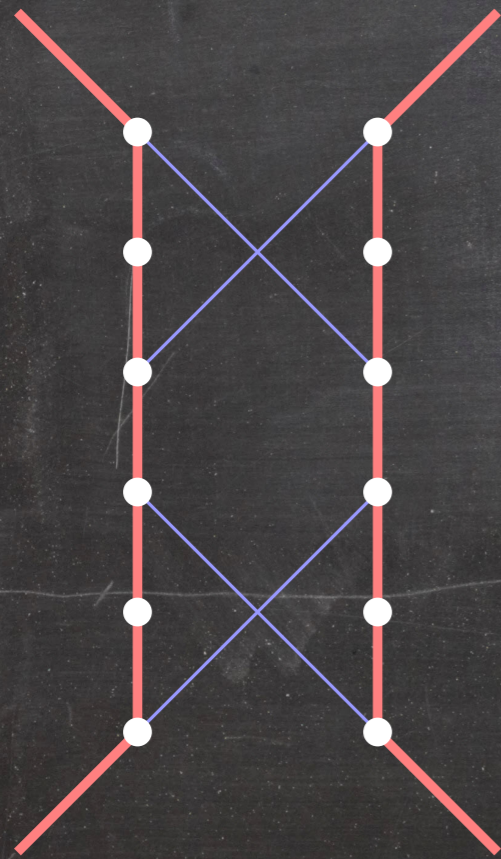
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



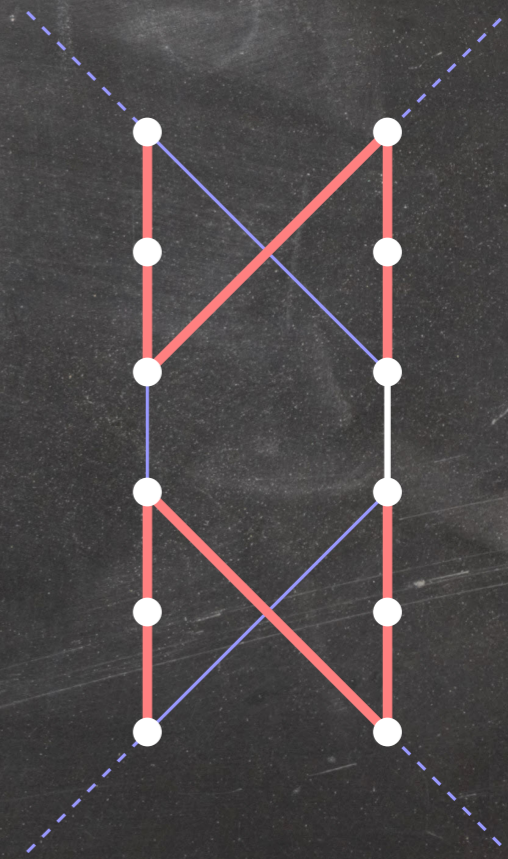
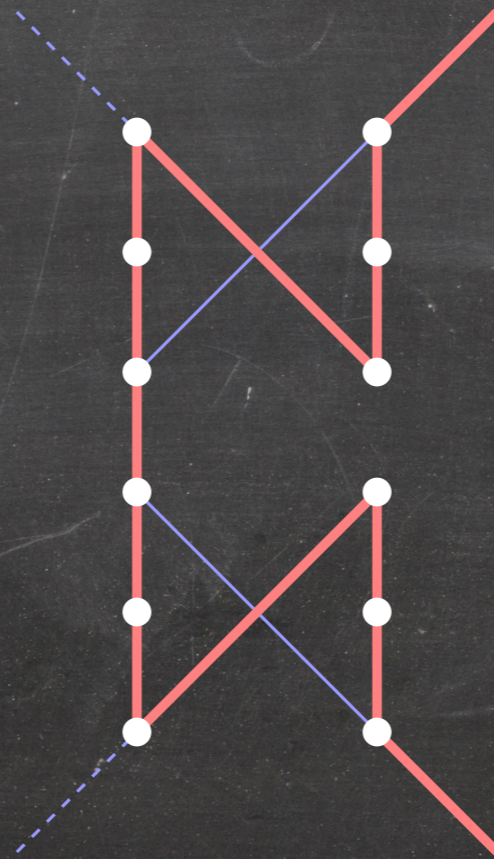
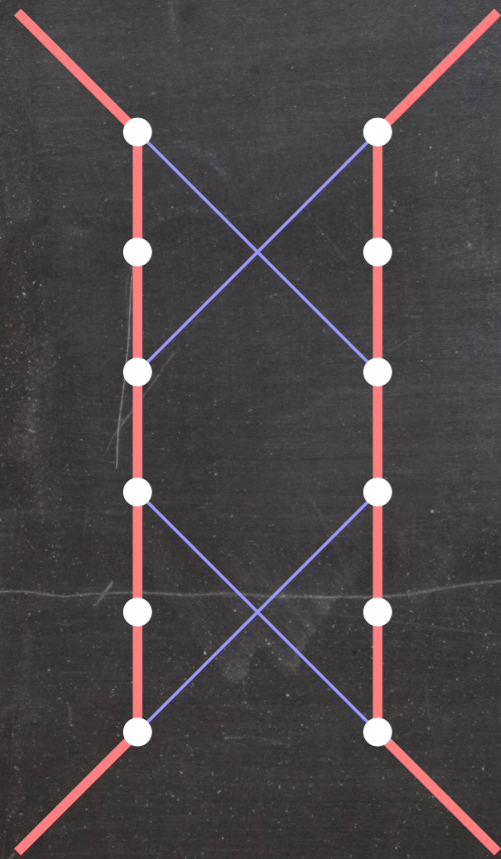
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



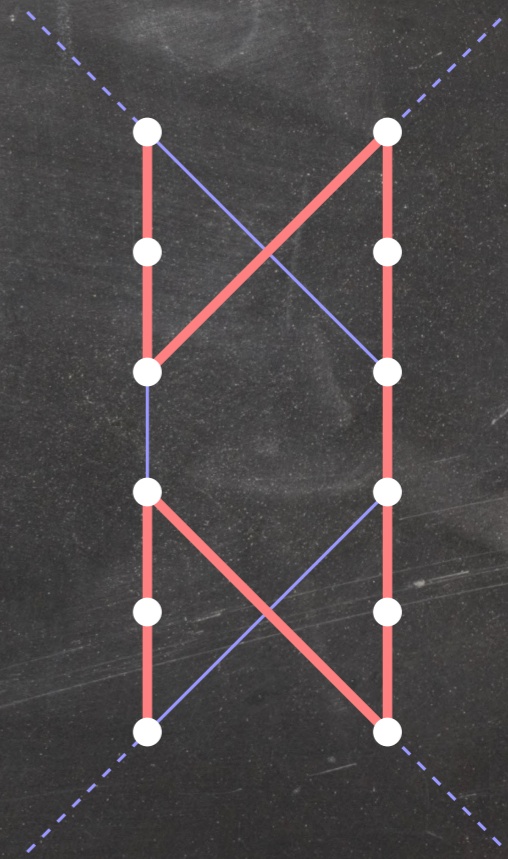
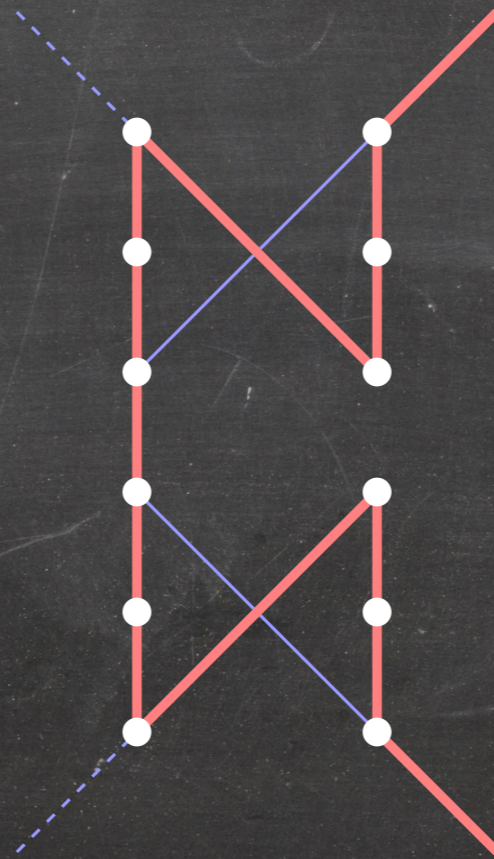
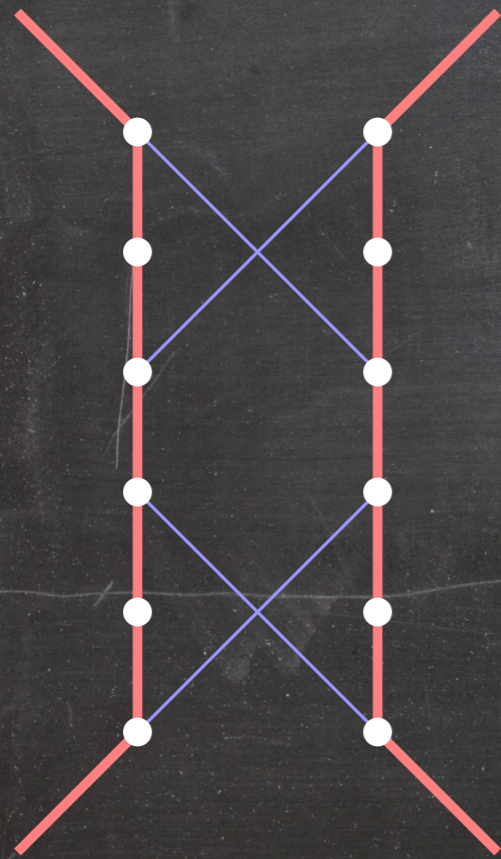
# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



# Edge Widgets

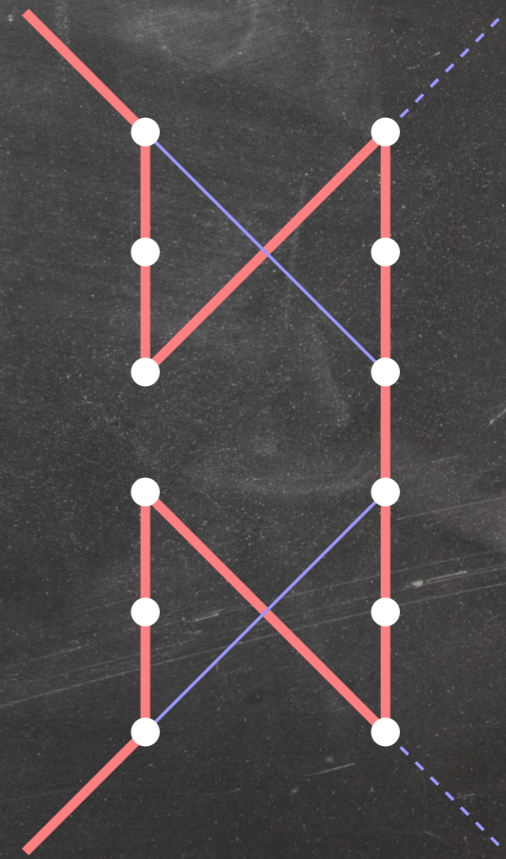
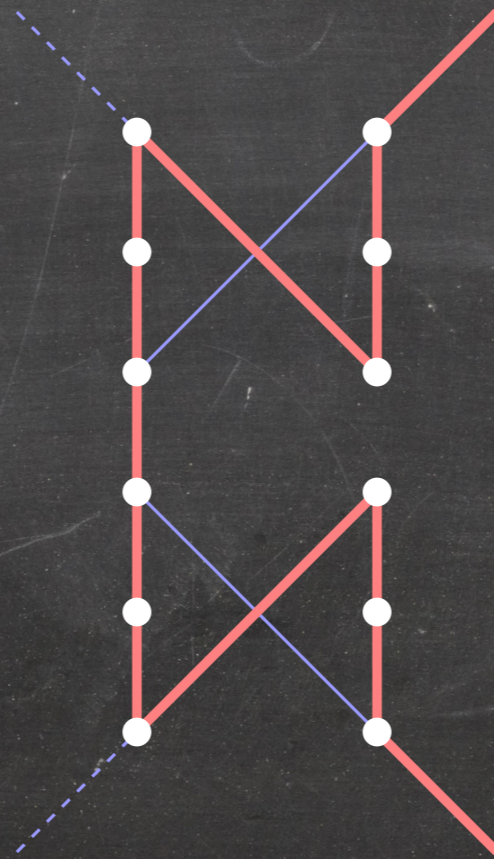
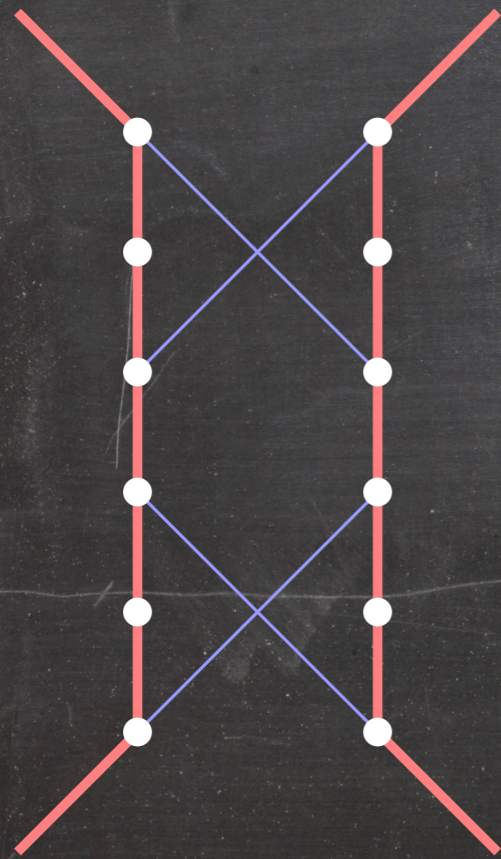
**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



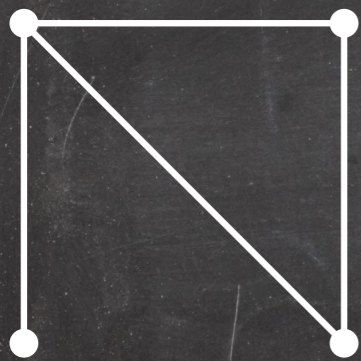


# Edge Widgets

**Observation:** Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.

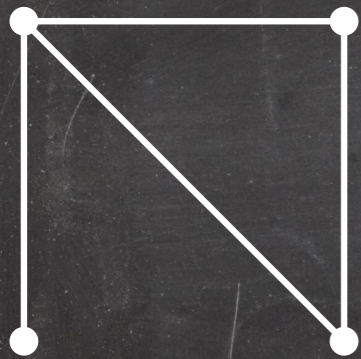


# Hamiltonian Cycle is NP-Complete

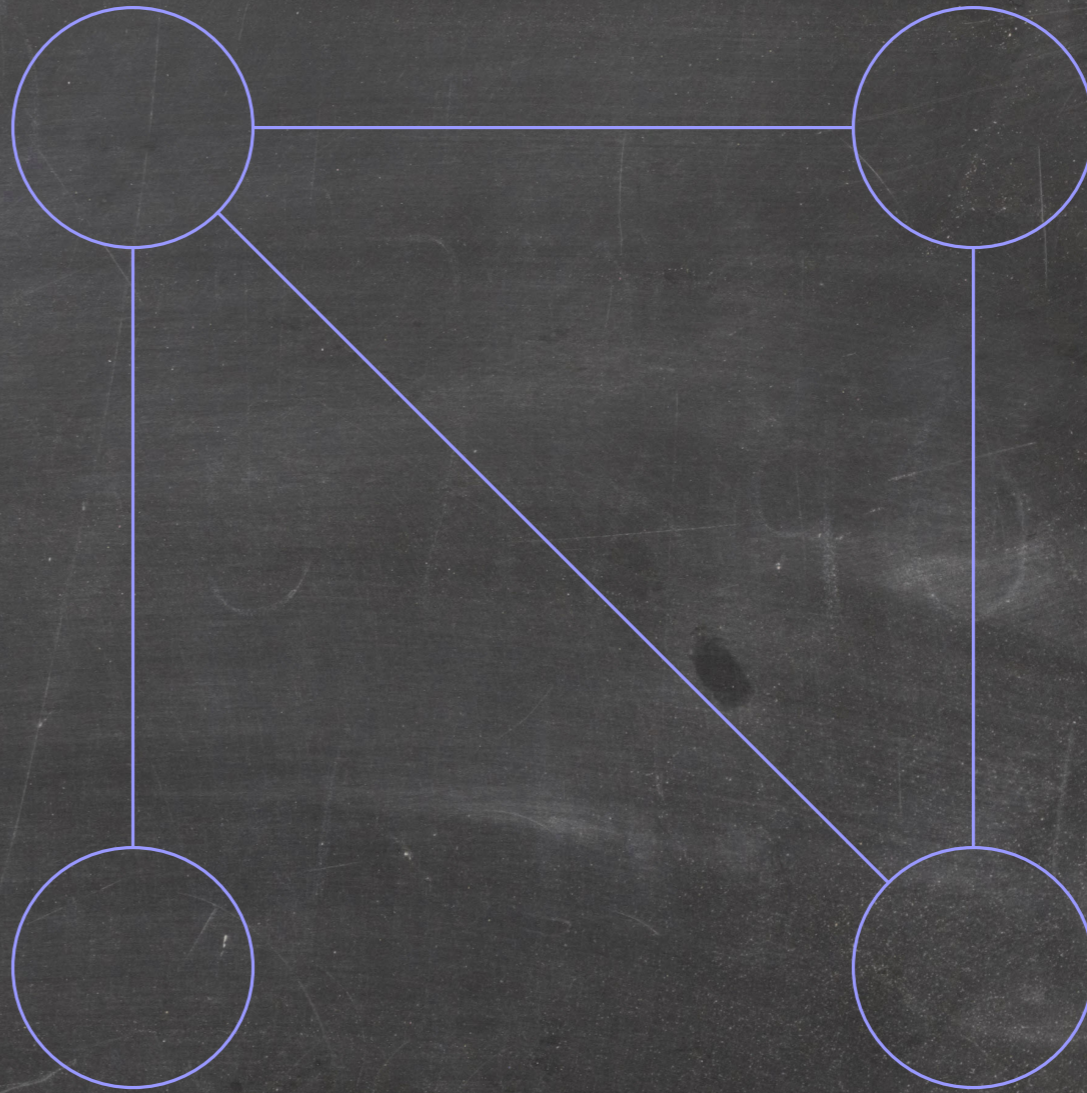


$G$   
 $k = 2$

# Hamiltonian Cycle is NP-Complete

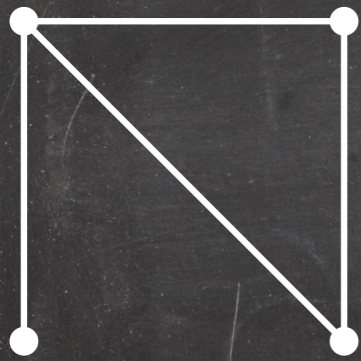


$G$   
 $k = 2$

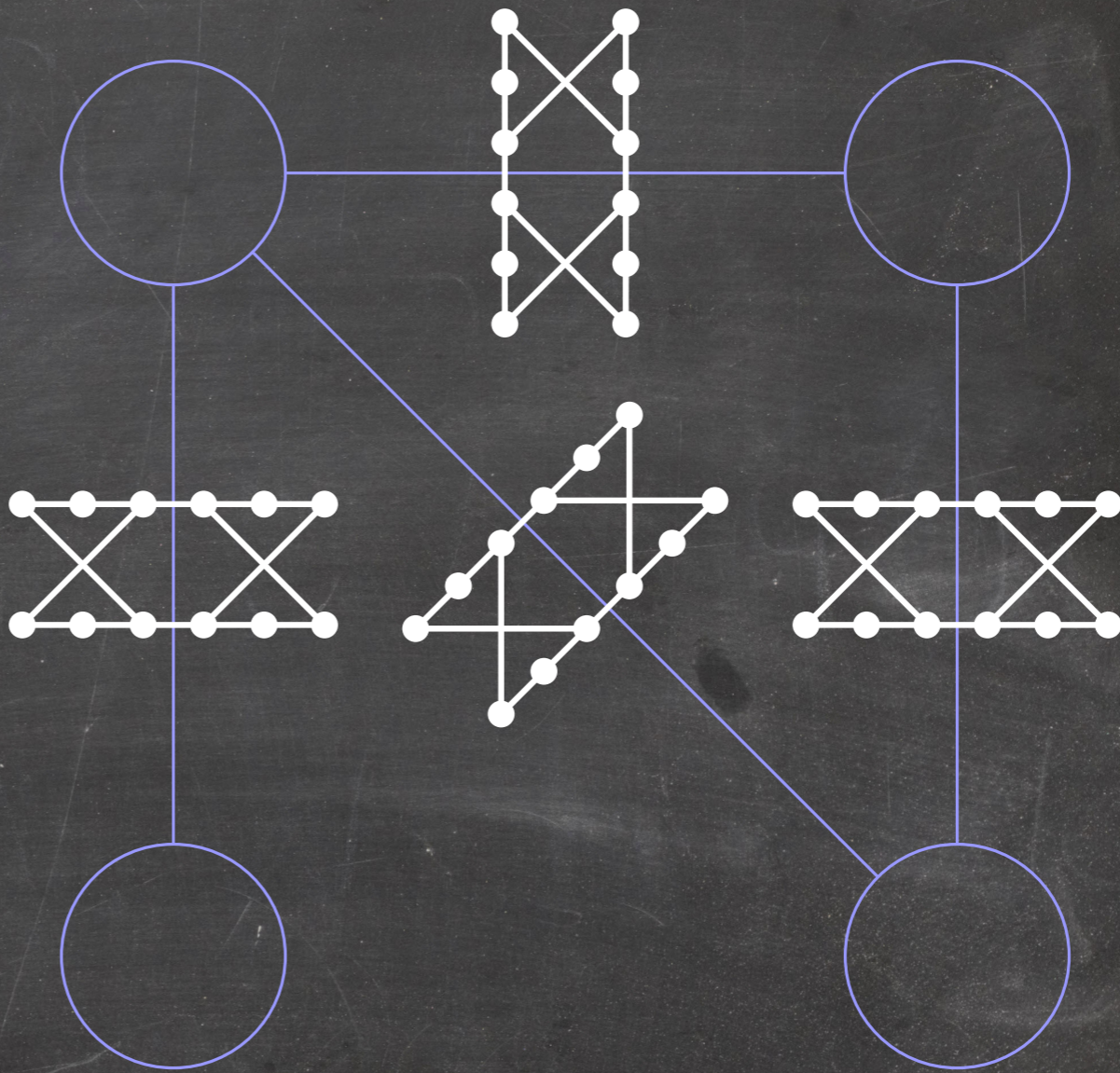


$G'$

# Hamiltonian Cycle is NP-Complete

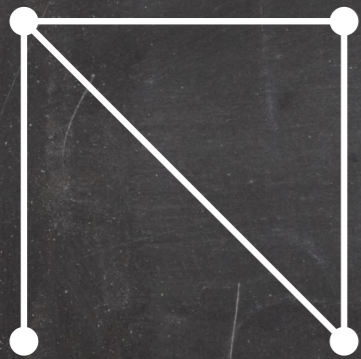


$G$   
 $k = 2$

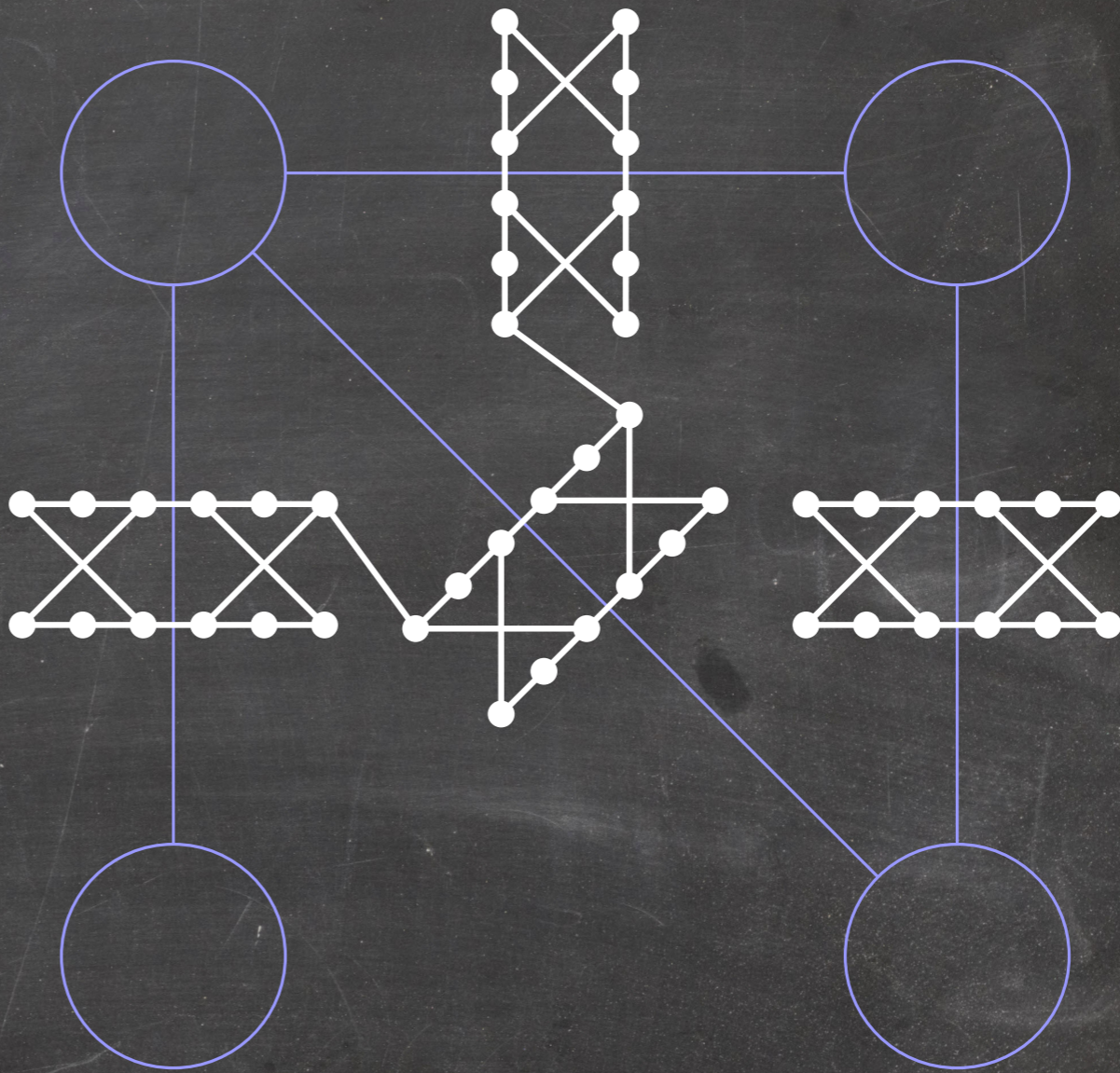


$G'$

# Hamiltonian Cycle is NP-Complete

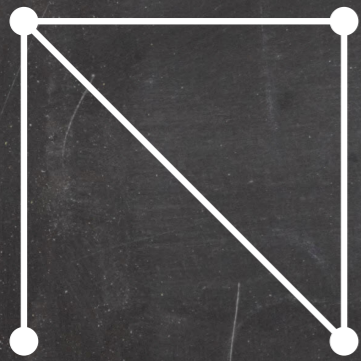


$G$   
 $k = 2$

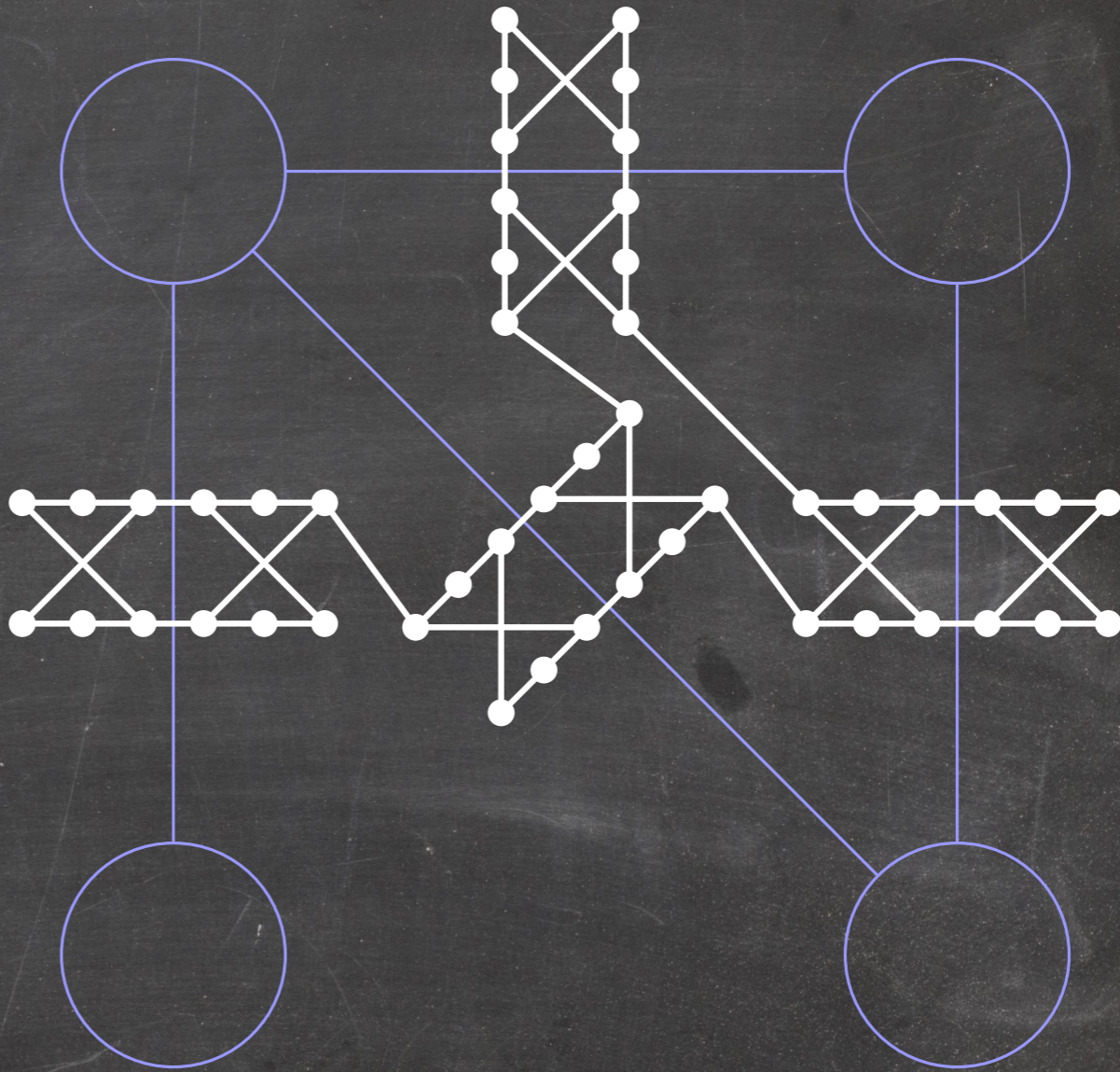


$G'$

# Hamiltonian Cycle is NP-Complete

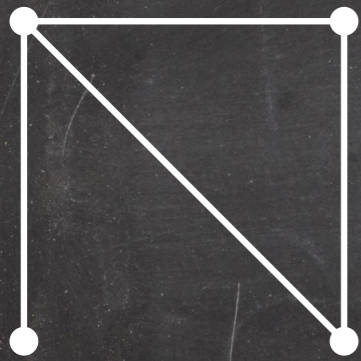


$G$   
 $k = 2$

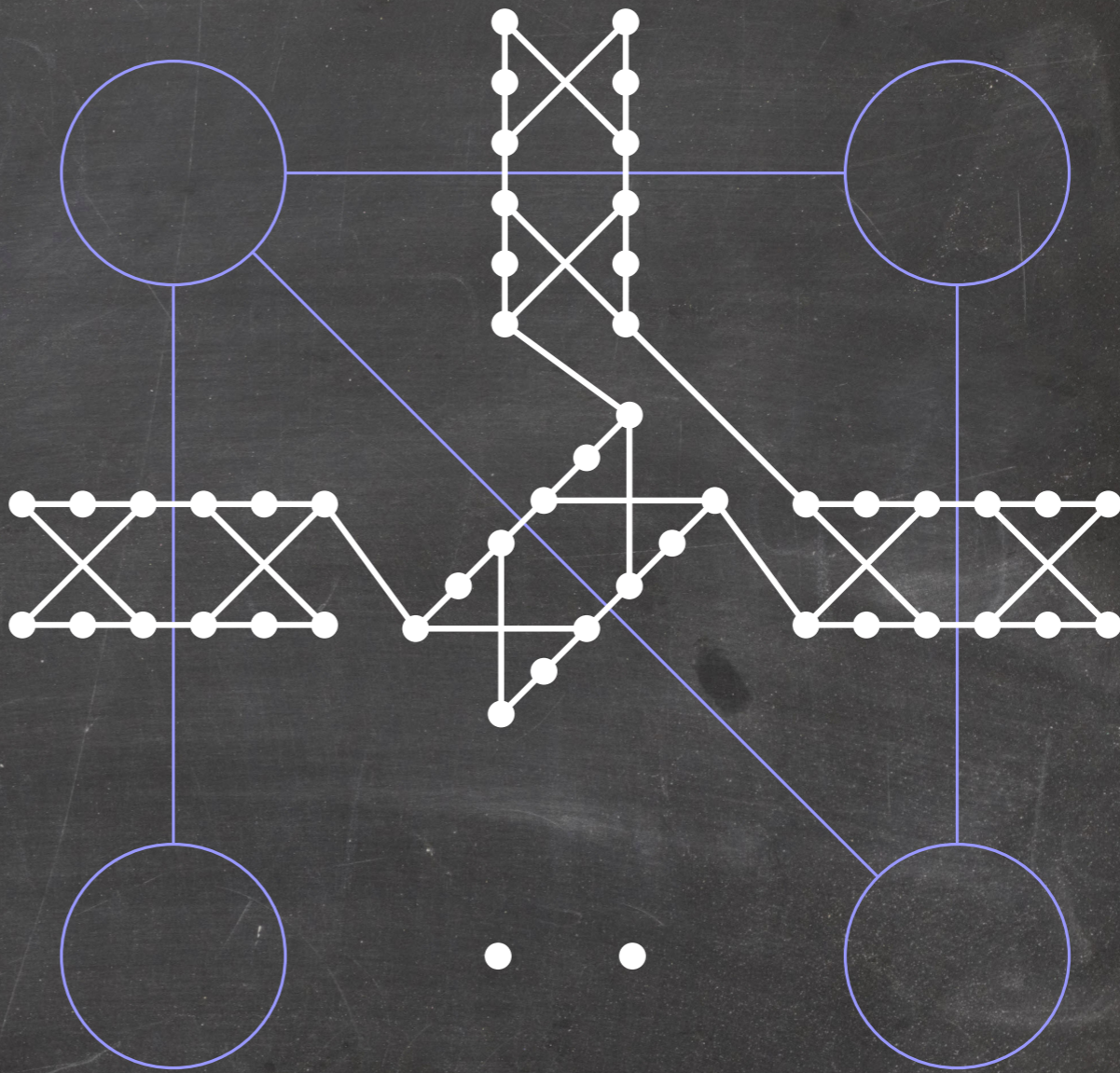


$G'$

# Hamiltonian Cycle is NP-Complete

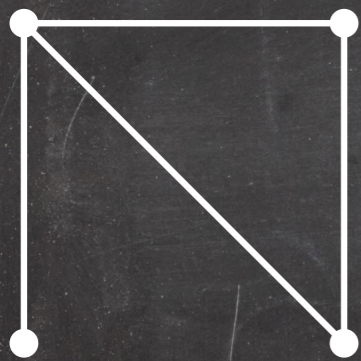


$G$   
 $k = 2$

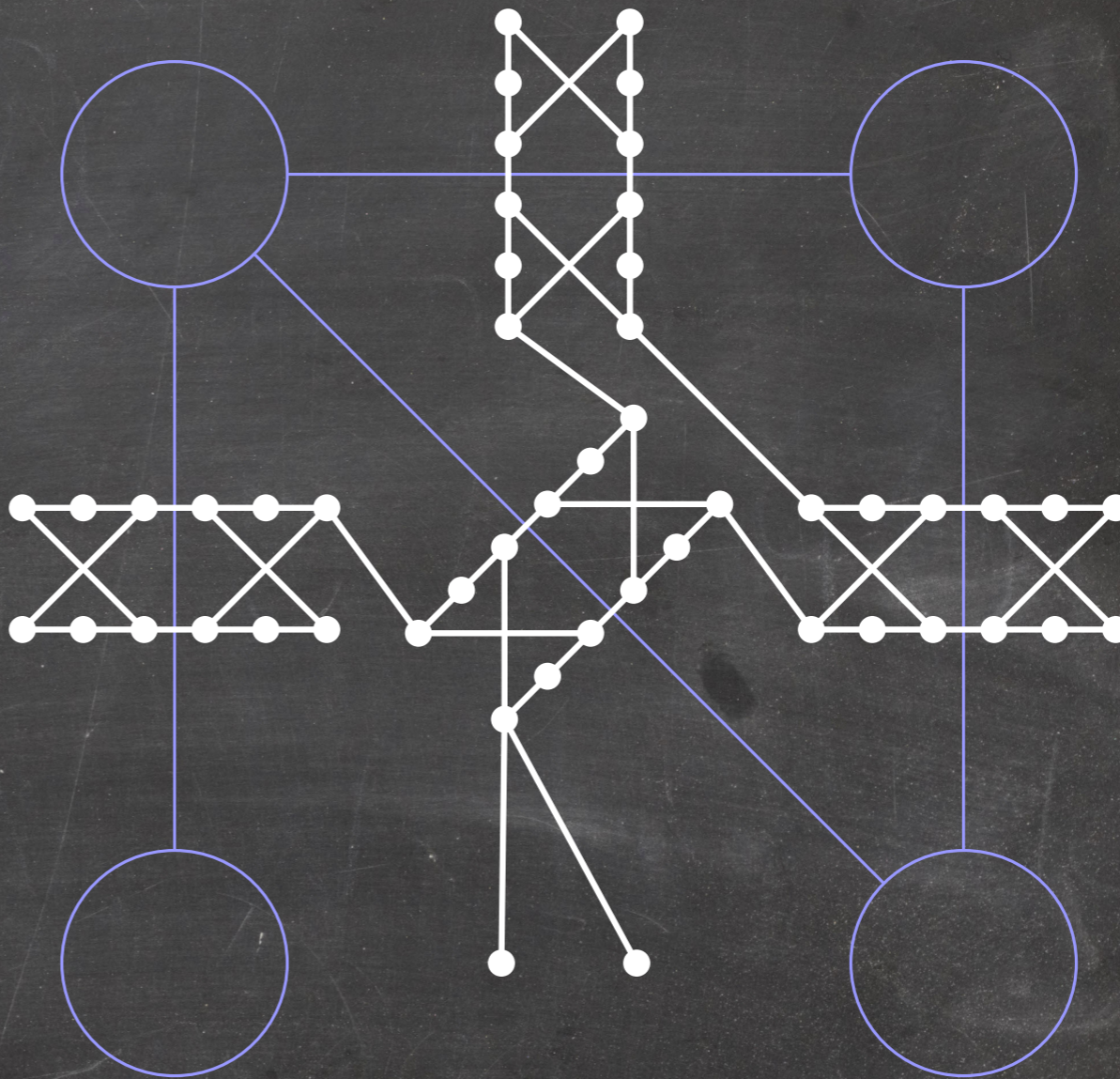


$G'$

# Hamiltonian Cycle is NP-Complete



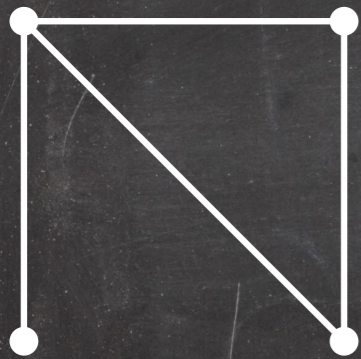
$G$   
 $k = 2$



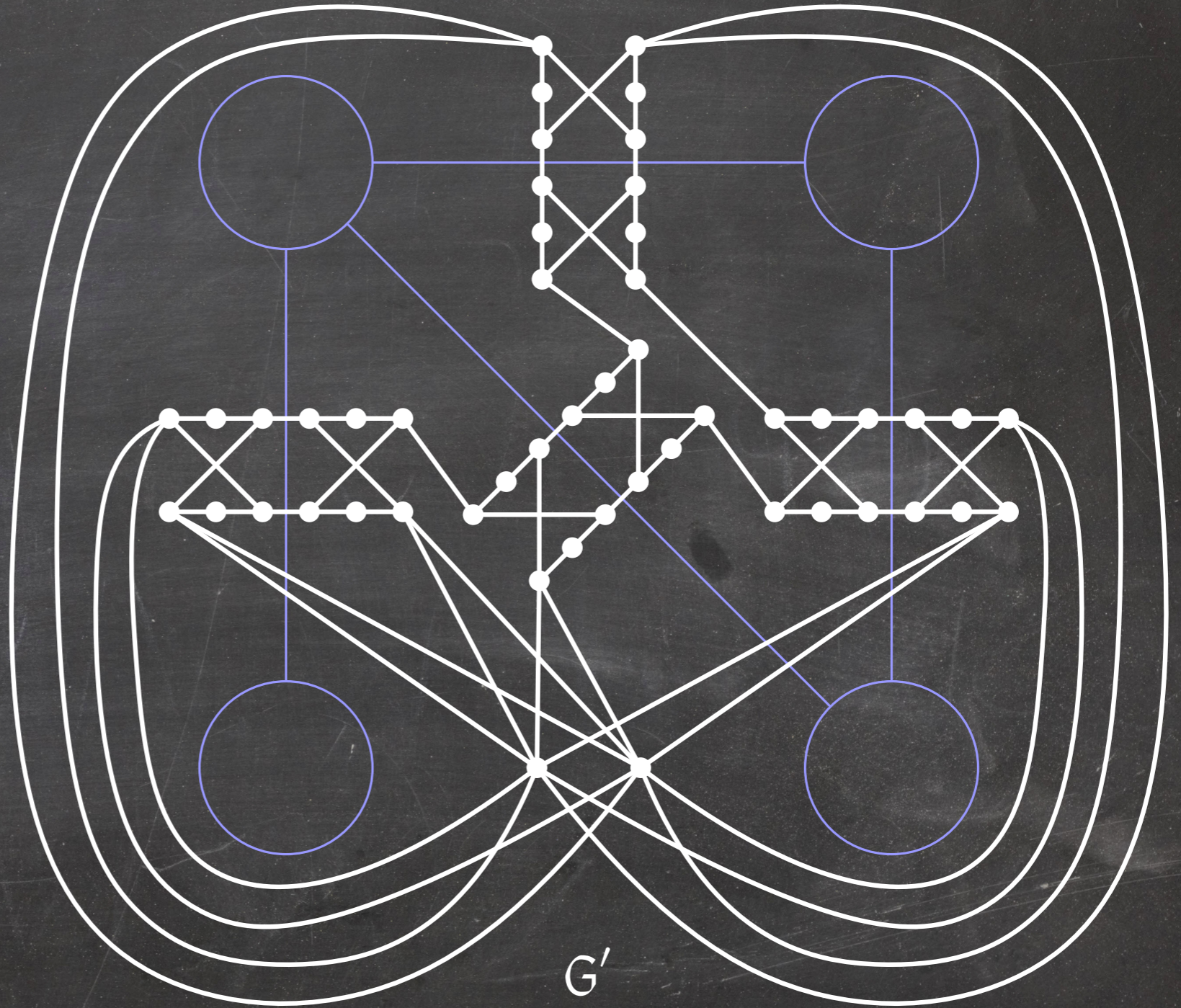
$G'$



# Hamiltonian Cycle is NP-Complete

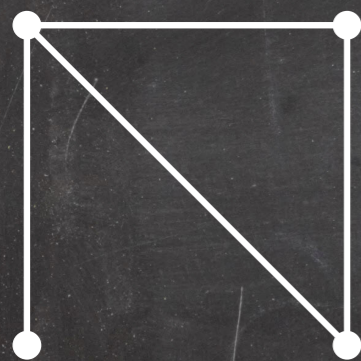


$G$   
 $k = 2$

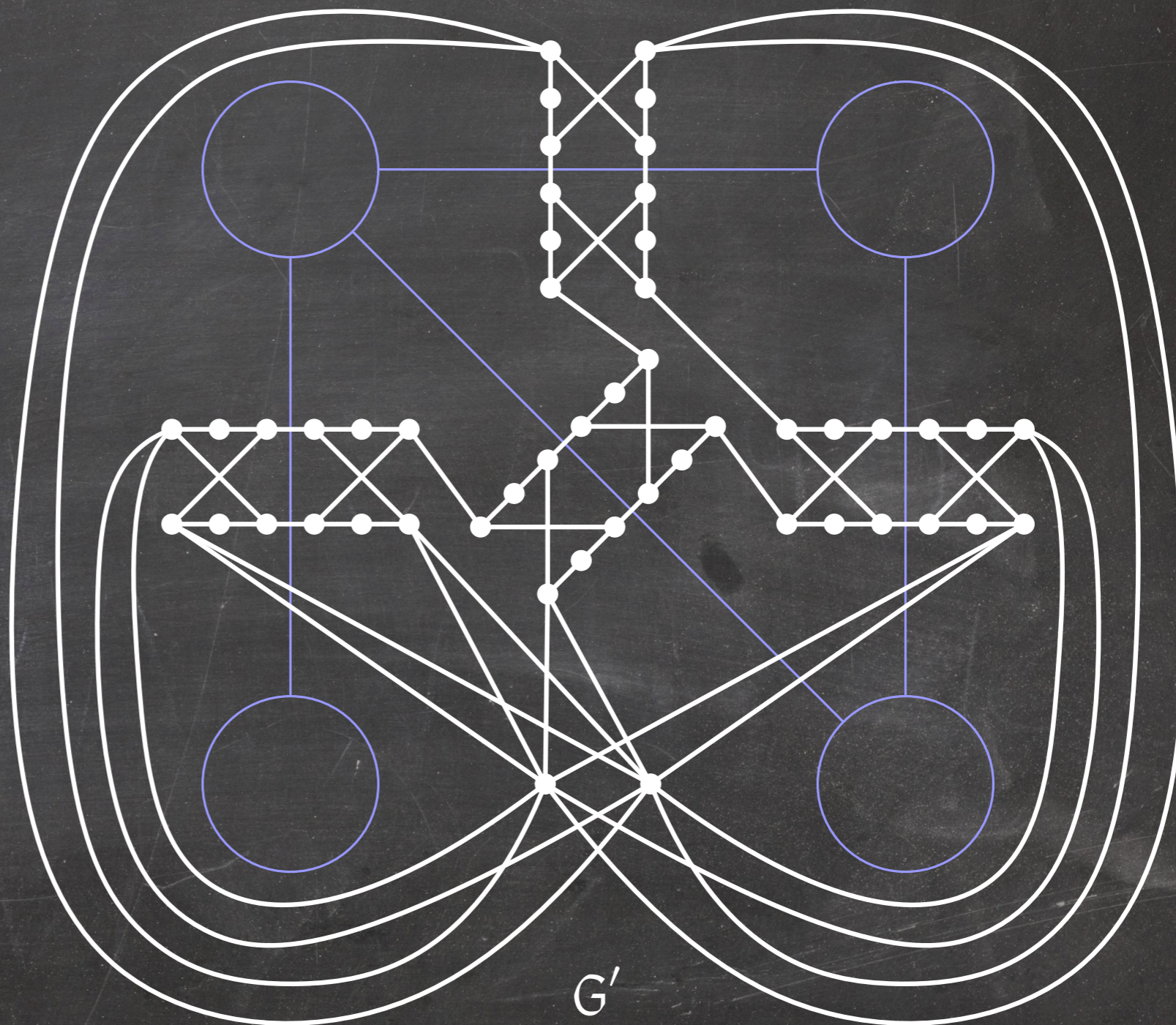


$G'$

# Hamiltonian Cycle is NP-Complete

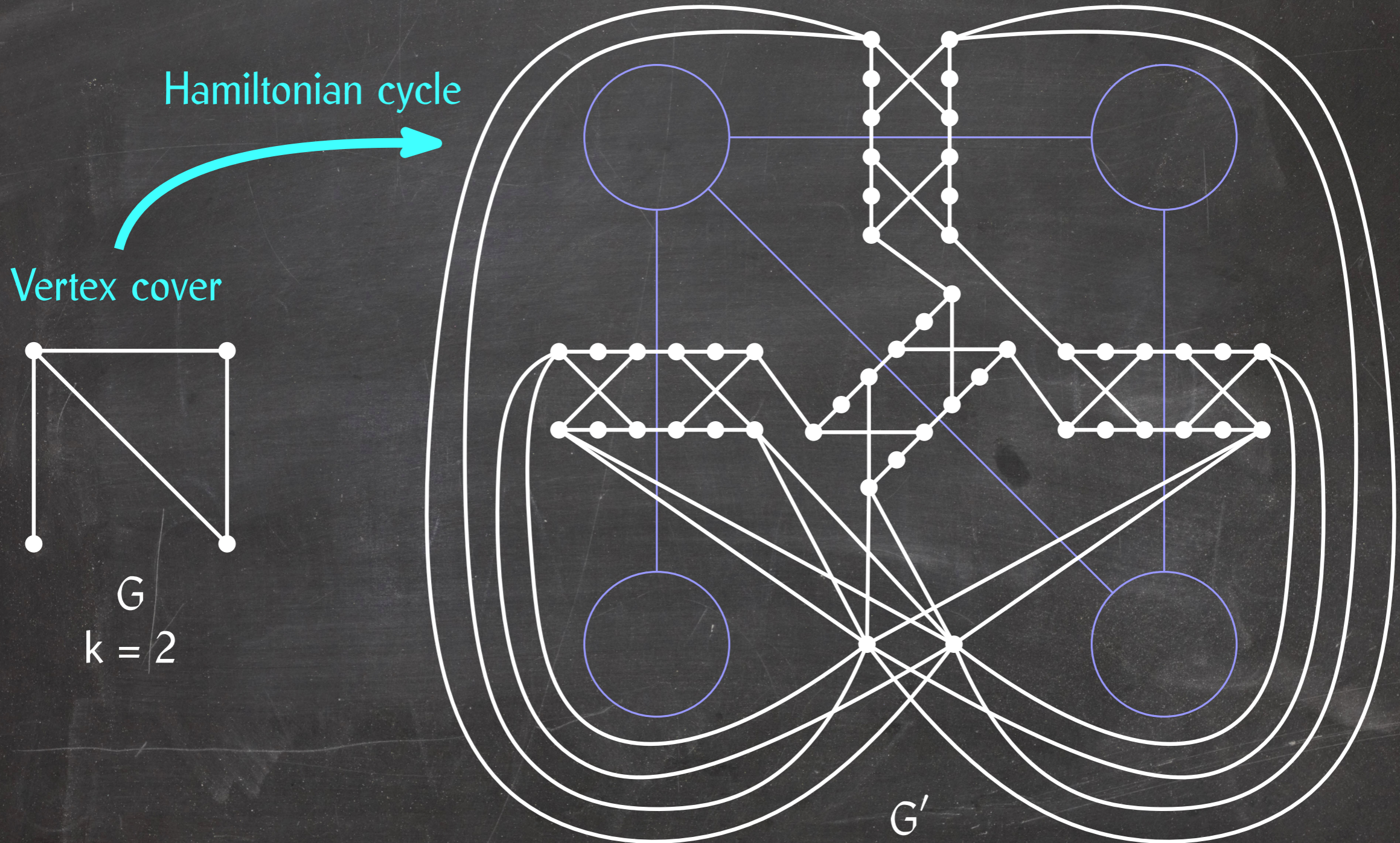


$G$   
 $k = 2$



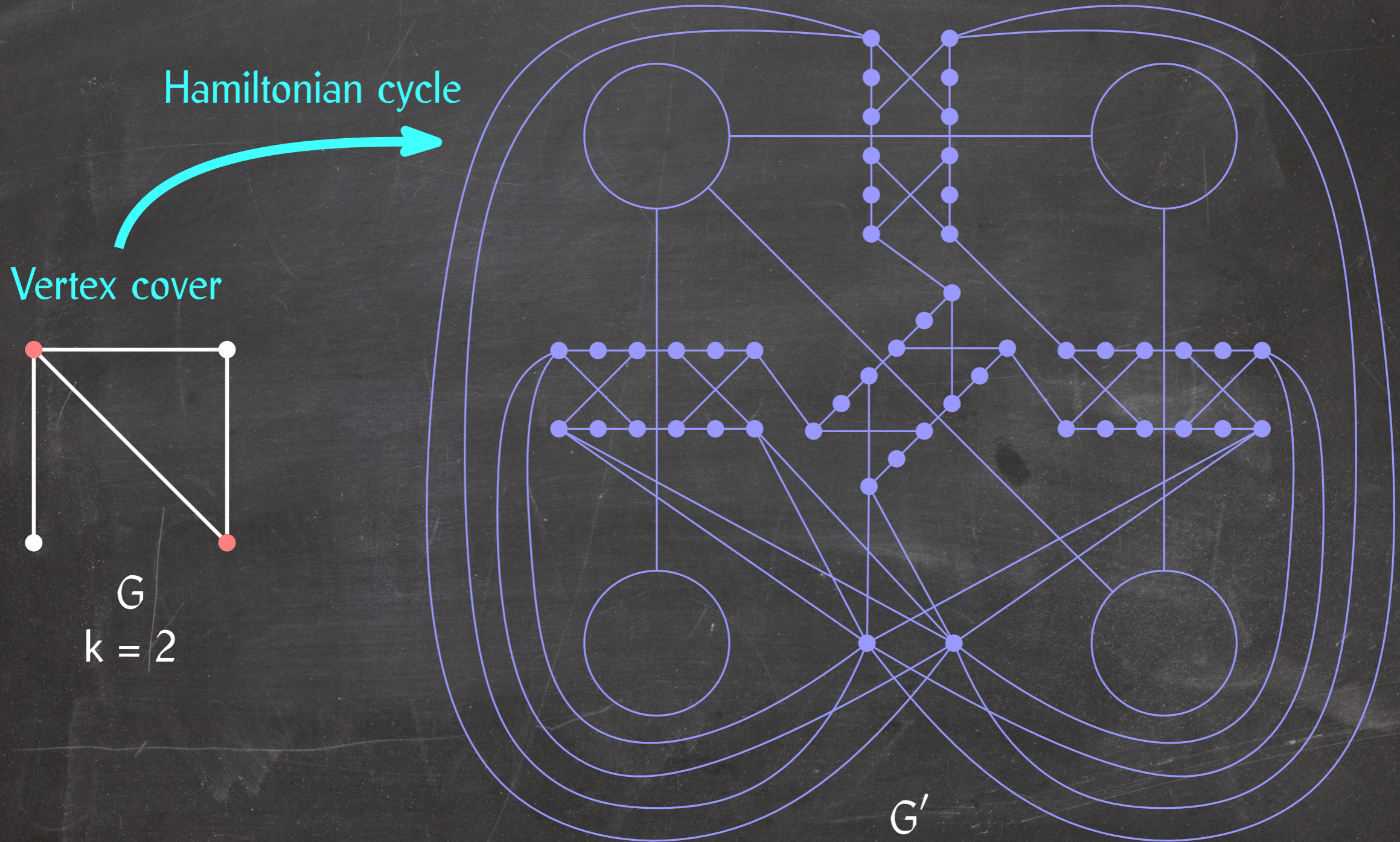
**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete



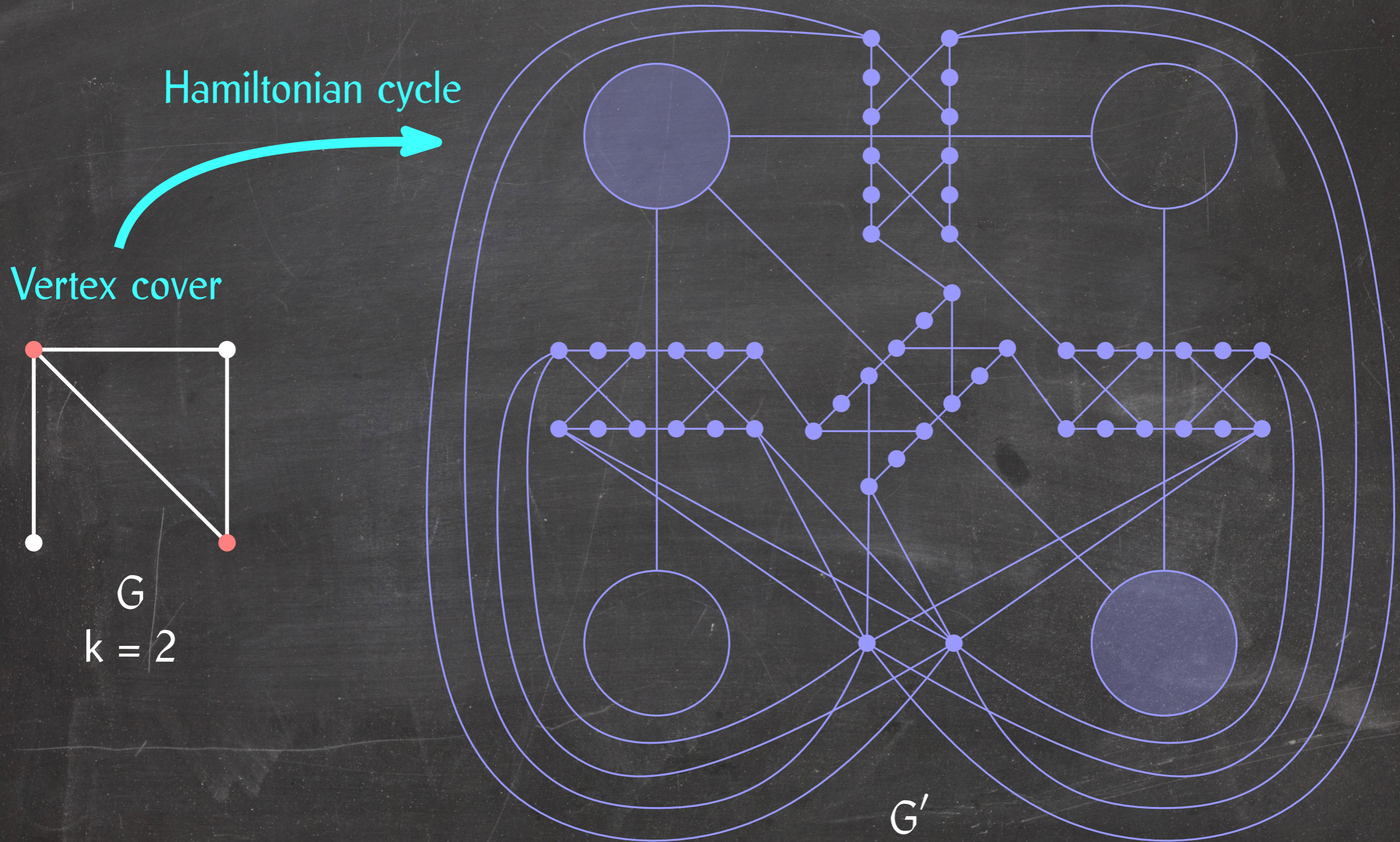
**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete



**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete

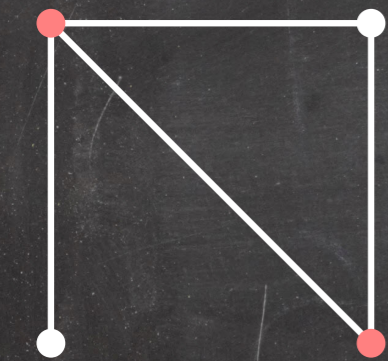


**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

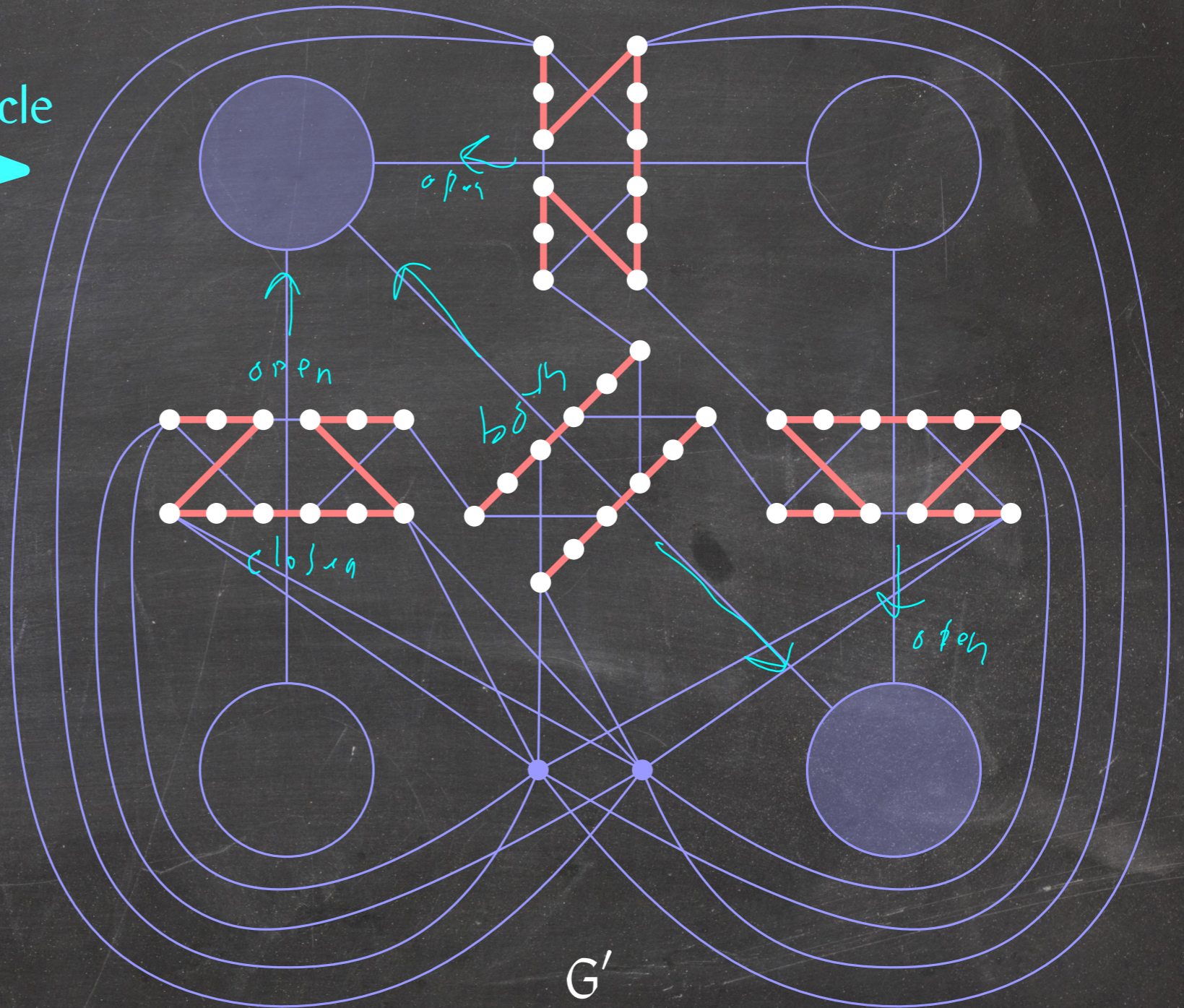
# Hamiltonian Cycle is NP-Complete

Hamiltonian cycle

Vertex cover

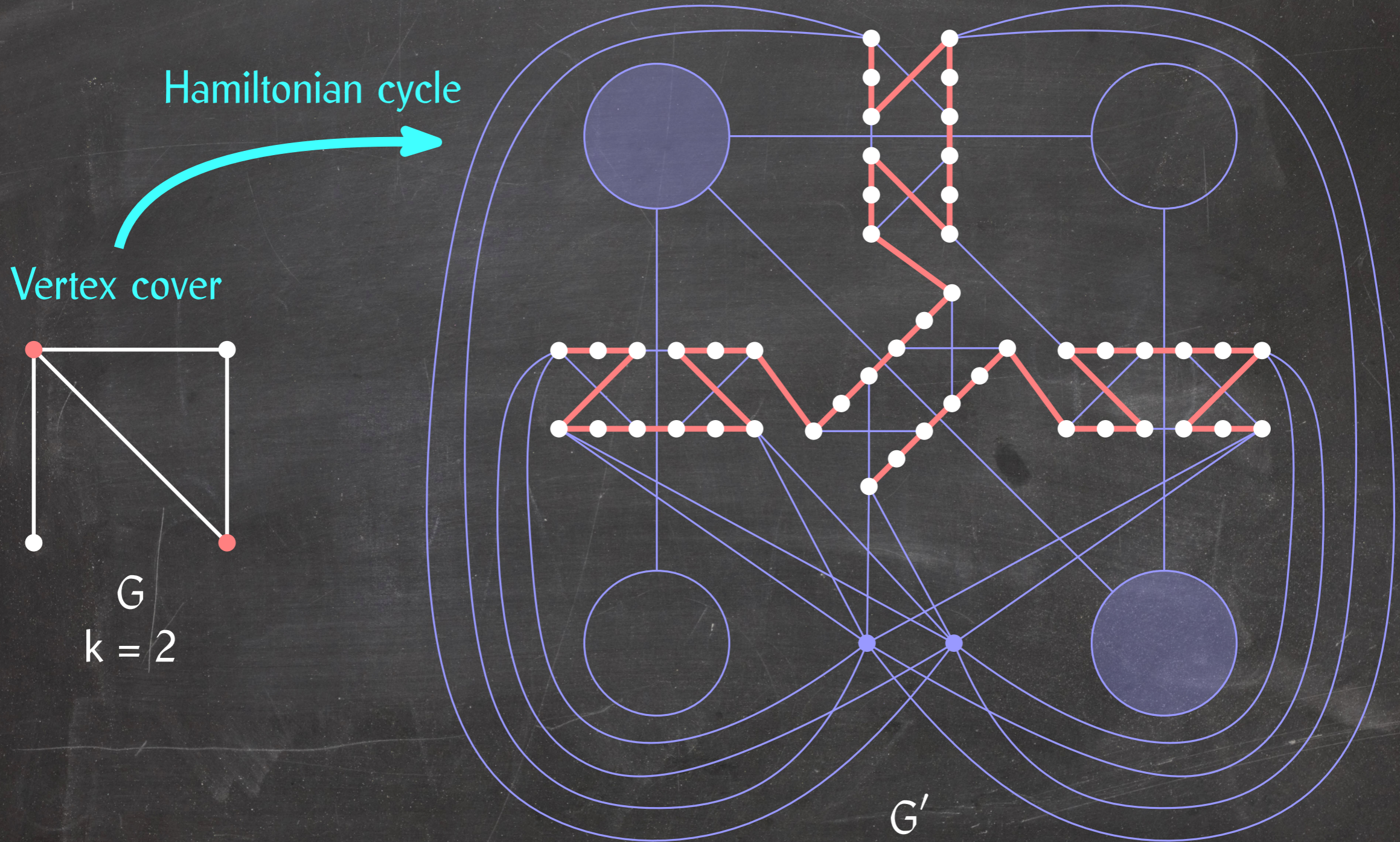


$G$   
 $k = 2$



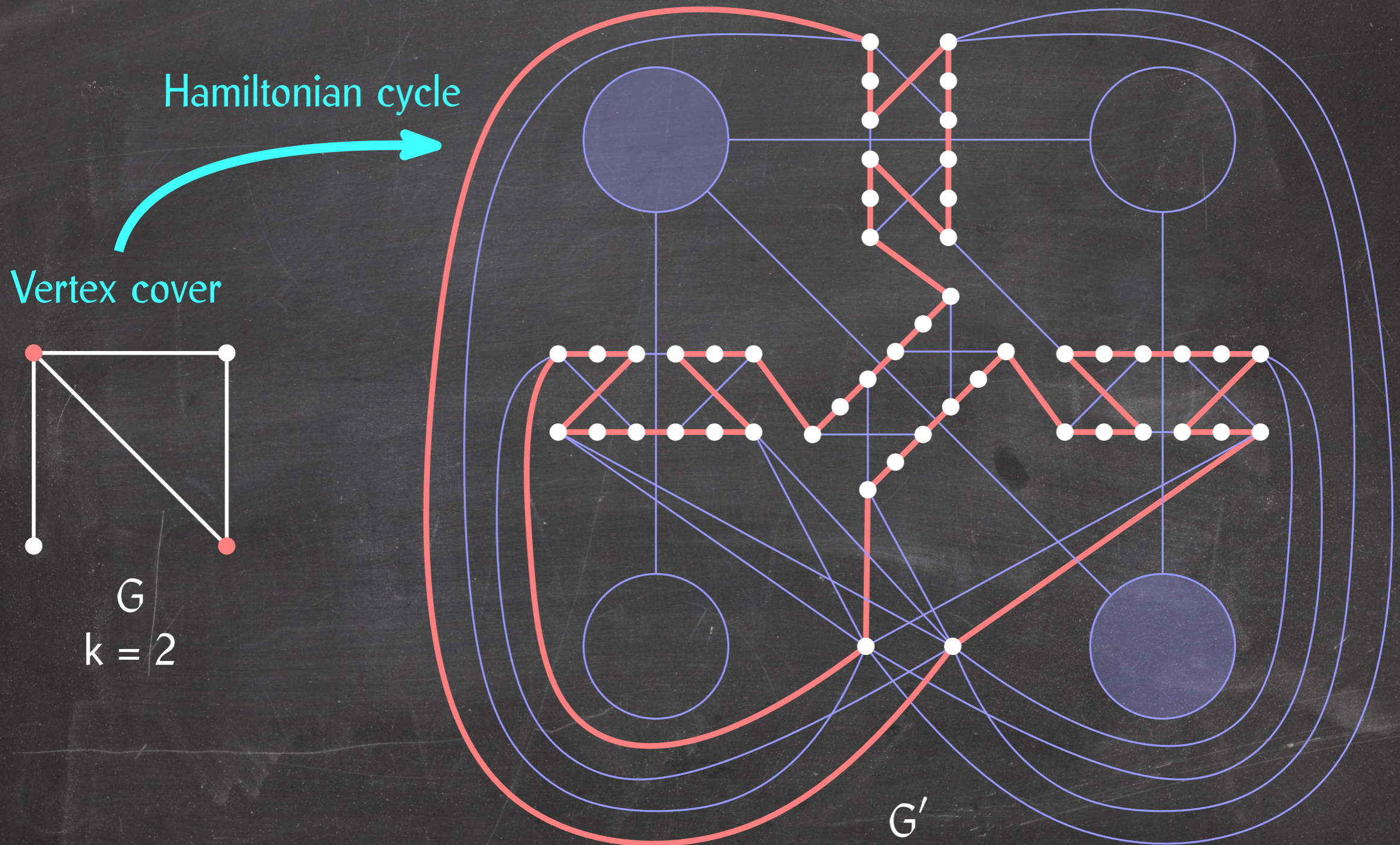
**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete



**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

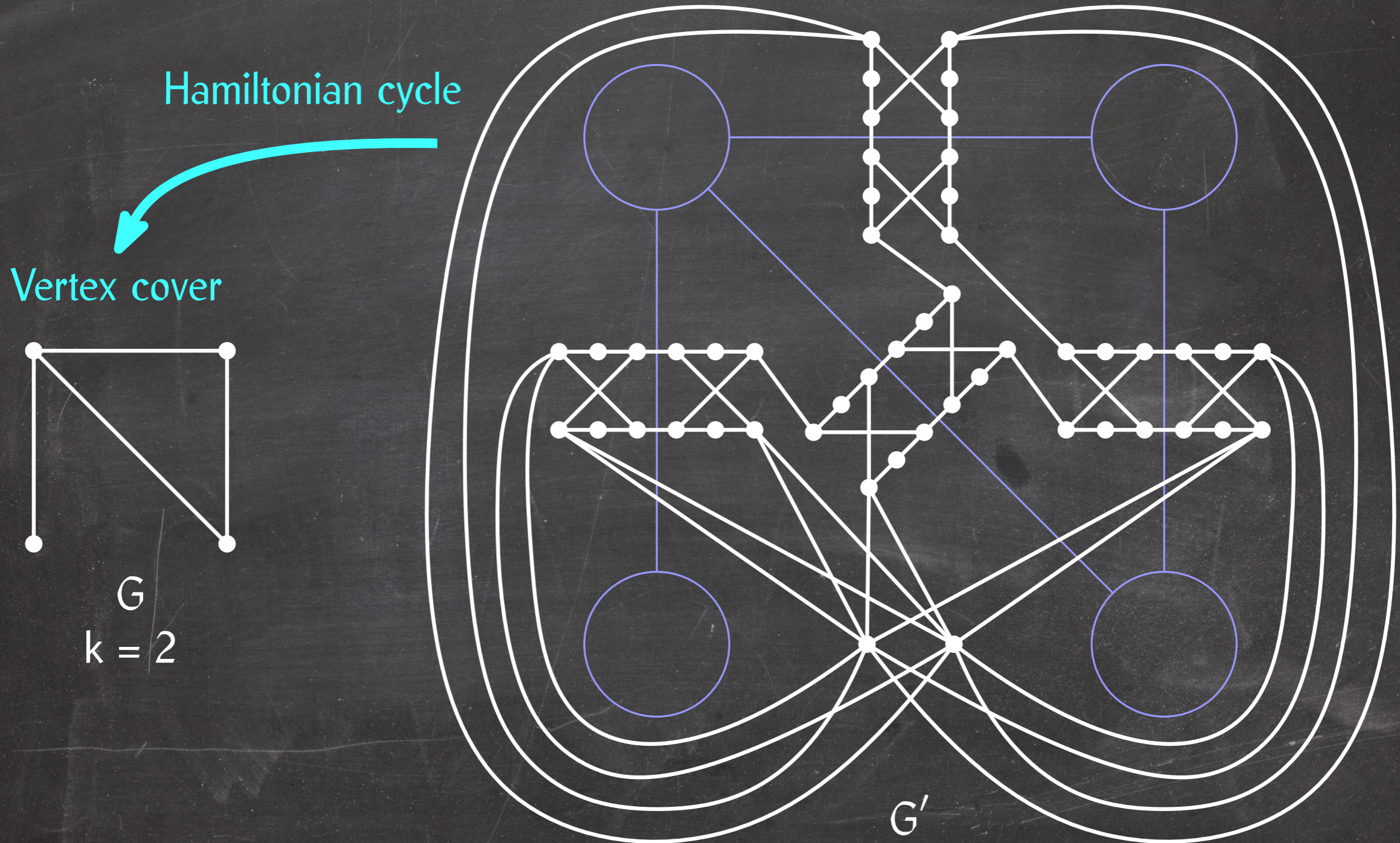
# Hamiltonian Cycle is NP-Complete



**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

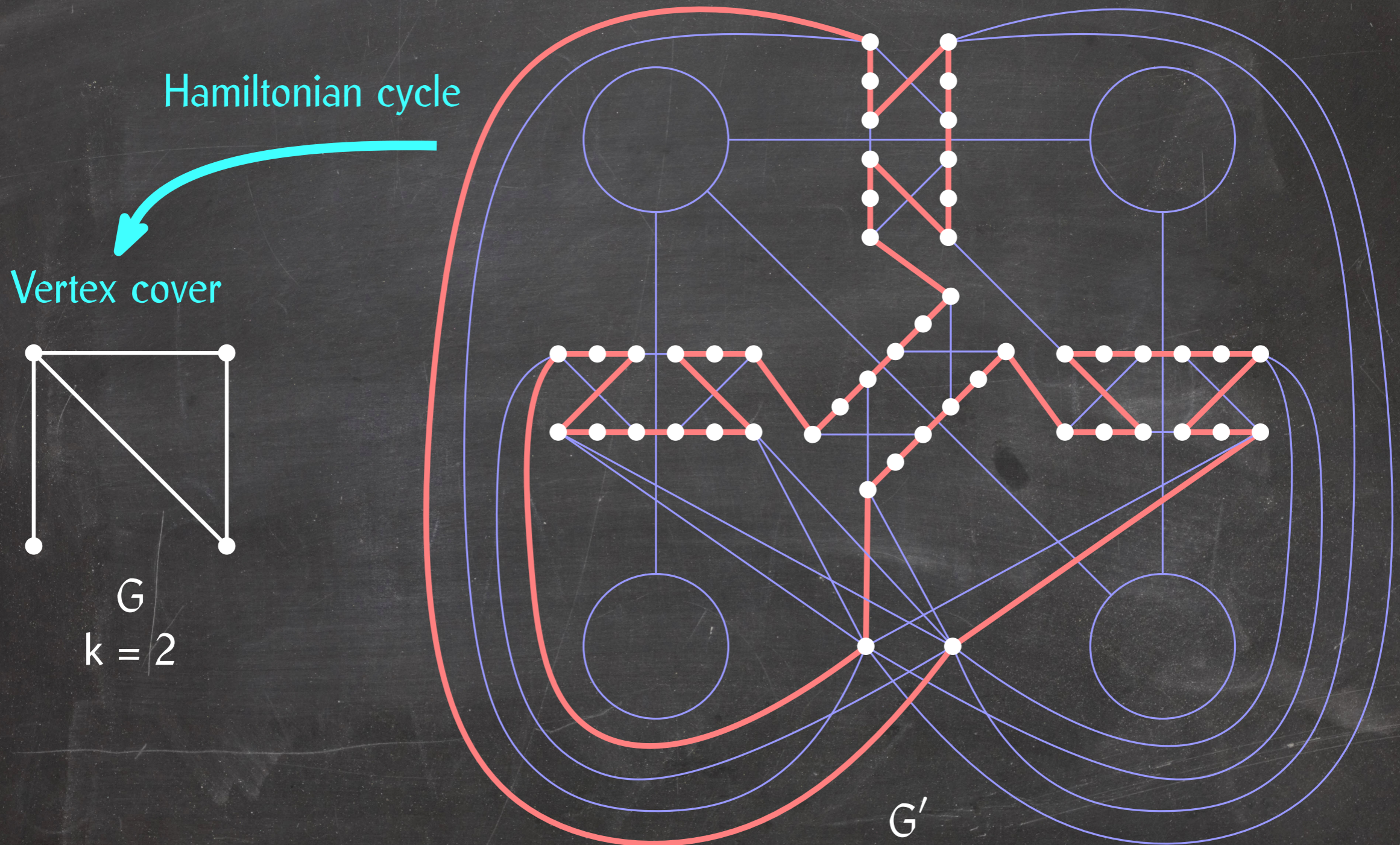


# Hamiltonian Cycle is NP-Complete



**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

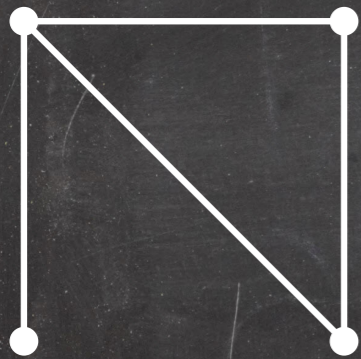
# Hamiltonian Cycle is NP-Complete



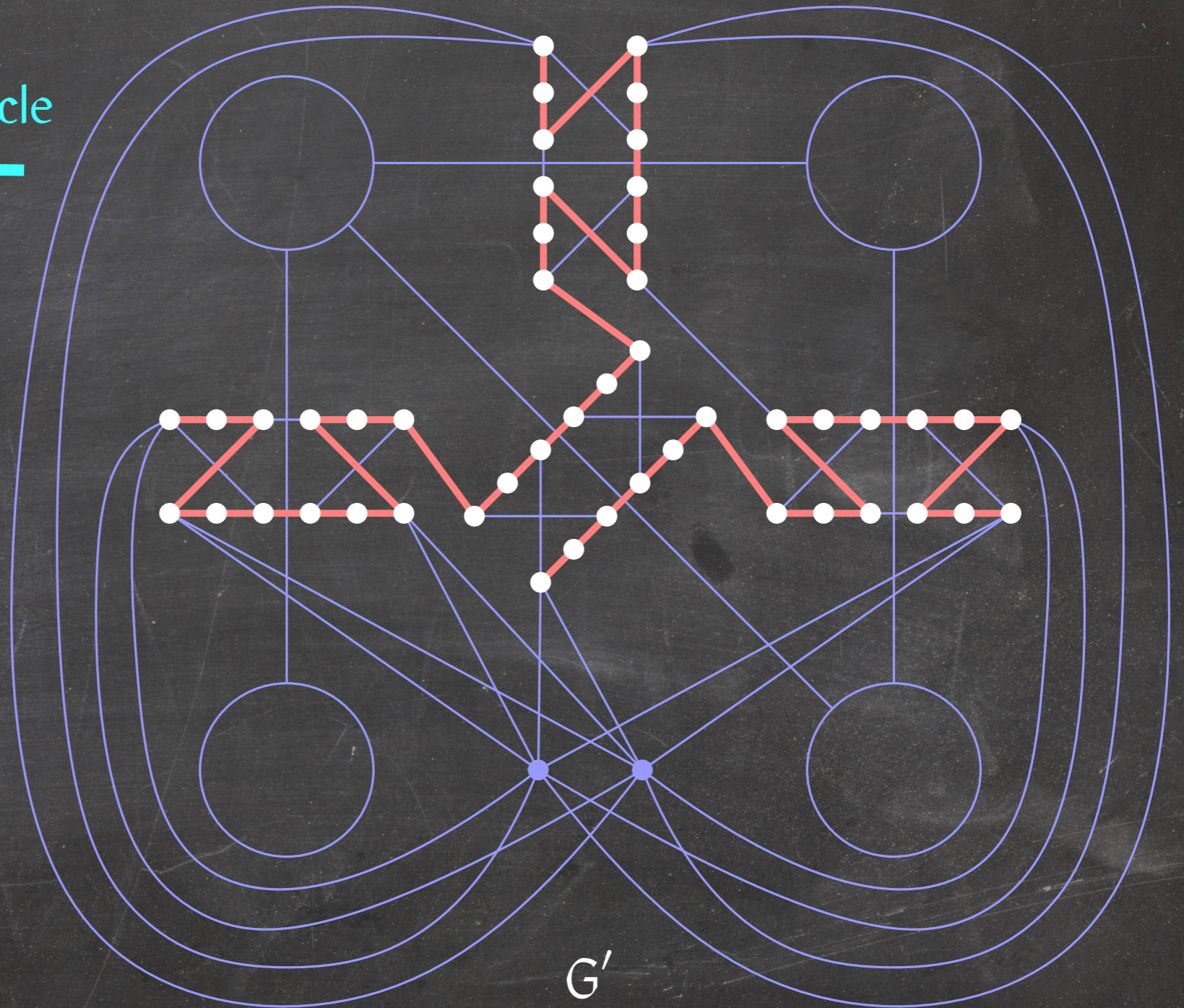
**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete

Hamiltonian cycle  
Vertex cover



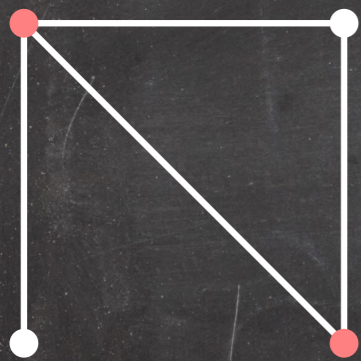
$G$   
 $k = 2$



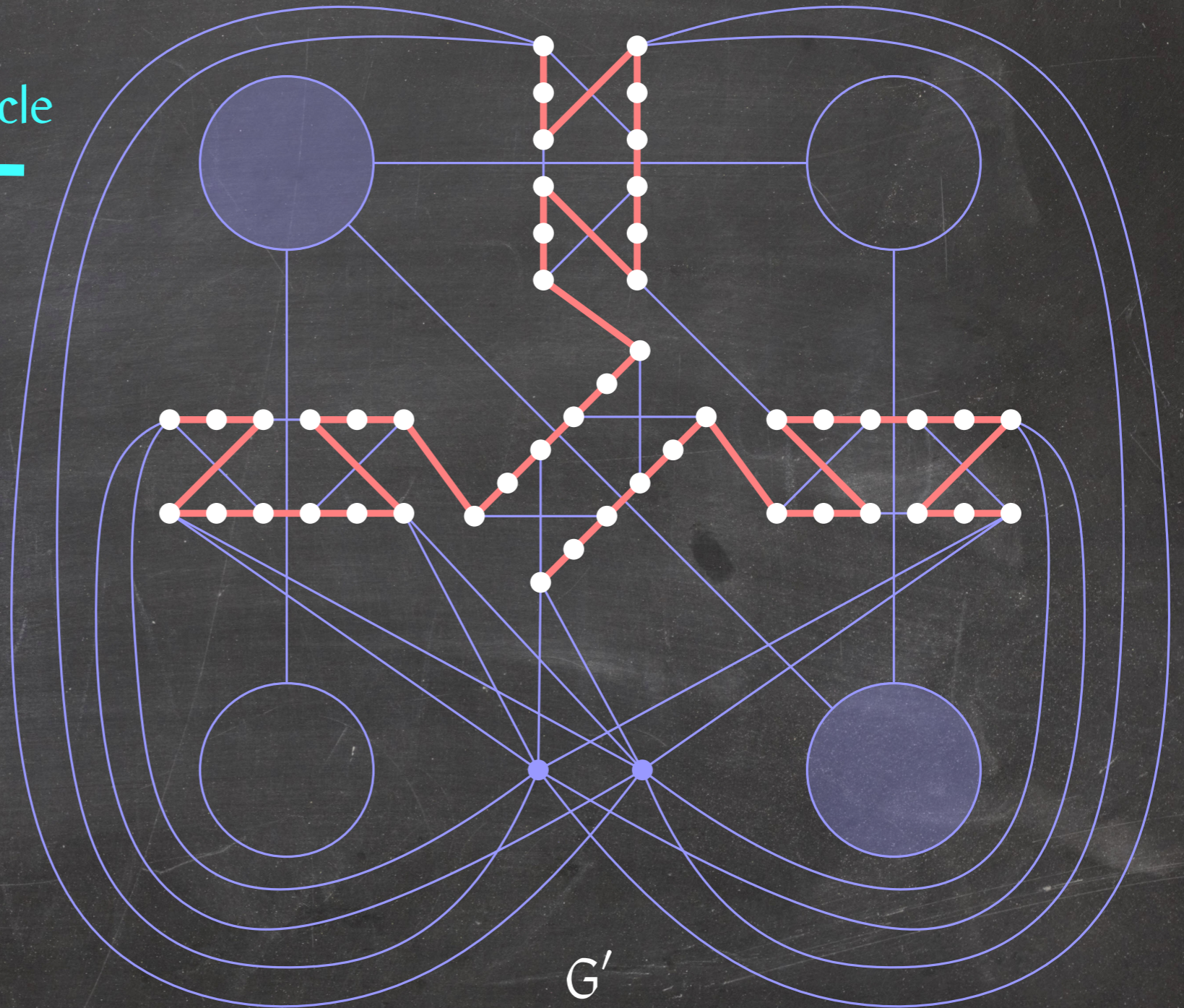
**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Hamiltonian Cycle is NP-Complete

Hamiltonian cycle  
Vertex cover



$G$   
 $k = 2$



**Lemma:** The graph  $G'$  has a Hamiltonian cycle if and only if  $G$  has a vertex cover of size  $k$ .

# Subset Sum

## Given:

- A set  $S = \{x_1, x_2, \dots, x_n\}$  of distinct numbers
- A parameter  $t$

## Question:

Is there a subset  $S' \subseteq S$  such that  $\sum_{x \in S'} x = t$ ?

# Subset Sum

## Given:

- A set  $S = \{x_1, x_2, \dots, x_n\}$  of distinct numbers
- A parameter  $t$

## Question:

Is there a subset  $S' \subseteq S$  such that  $\sum_{x \in S'} x = t$ ?

## Example:

$$S = \{1, 2, 8, 13\}$$

$S$  has a subset  $S'$  whose elements sum to 22, namely  $S' = \{1, 8, 13\}$ , but there is no subset whose elements sum to 12.

# Subset Sum is NP-Complete

**Exercise:** Verify that Subset Sum is in NP.

# Subset Sum is NP-Complete

**Exercise:** Verify that Subset Sum is in NP.

**To prove:** Subset Sum is NP-hard.



# Subset Sum is NP-Complete

**Exercise:** Verify that Subset Sum is in NP.

**To prove:** Subset Sum is NP-hard.

**Reduction from 3-SAT:**

Given a formula  $F$  in 3-CNF, we construct a set  $S_F$  of  $2n + 2m$  numbers with  $n + m$

digits in base-10 notation and a number  $t = \sum_{i=0}^{n-1} 10^{i+m} + \sum_{i=0}^{m-1} 4 \cdot 10^i$ :

$$\underbrace{11 \dots 1}_{n \text{ variable digits}} \quad \underbrace{44 \dots 4}_{m \text{ clause digits}}$$

# Subset Sum is NP-Complete

**Exercise:** Verify that Subset Sum is in NP.

**To prove:** Subset Sum is NP-hard.

## Reduction from 3-SAT:

Given a formula  $F$  in 3-CNF, we construct a set  $S_F$  of  $2n + 2m$  numbers with  $n + m$

digits in base-10 notation and a number  $t = \sum_{i=0}^{n-1} 10^{i+m} + \sum_{i=0}^{m-1} 4 \cdot 10^i$ :

$$\underbrace{11 \dots 1}_{n \text{ variable digits}} \quad \underbrace{44 \dots 4}_{m \text{ clause digits}}$$

There will be a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$  if and only if  $F$  is satisfiable.

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

$x_1$   
 $\bar{x}_1$   
 $x_2$   
 $\bar{x}_2$   
 $x_3$   
 $\bar{x}_3$   
 $x_4$   
 $\bar{x}_4$

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

$x_1$   
 $\bar{x}_1$   
 $x_2$   
 $\bar{x}_2$   
 $x_3$   
 $\bar{x}_3$   
 $x_4$   
 $\bar{x}_4$

Slack numbers

$s_1$   
 $s'_1$   
 $s_2$   
 $s'_2$   
 $s_3$   
 $s'_3$   
 $s_4$   
 $s'_4$

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

$x_1$   
 $\bar{x}_1$   
 $x_2$   
 $\bar{x}_2$   
 $x_3$   
 $\bar{x}_3$   
 $x_4$   
 $\bar{x}_4$

Slack numbers

$s_1$   
 $s'_1$   
 $s_2$   
 $s'_2$   
 $s_3$   
 $s'_3$   
 $s_4$   
 $s'_4$

---

t    1 1 1 1    4 4 4 4

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers	$x_1$	1							
	$\bar{x}_1$	1							
	$x_2$								
	$\bar{x}_2$								
	$x_3$								
	$\bar{x}_3$								
	$x_4$								
	$\bar{x}_4$								
Slack numbers	$s_1$								
	$s'_1$								
	$s_2$								
	$s'_2$								
	$s_3$								
	$s'_3$								
	$s_4$								
	$s'_4$								
	$t$	1	1	1	1	4	4	4	4

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

Slack numbers

$x_1$	1				
$\bar{x}_1$	1				
$x_2$		1			
$\bar{x}_2$		1			
$x_3$			1		
$\bar{x}_3$			1		
$x_4$				1	
$\bar{x}_4$				1	
$s_1$					
$s'_1$					
$s_2$					
$s'_2$					
$s_3$					
$s'_3$					
$s_4$					
$s'_4$					
t	1	1	1	1	4 4 4 4

*x<sub>1</sub> or  $\bar{x}_1$  chosen*



# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

Slack numbers

$x_1$	1				1				
$\bar{x}_1$	1								
$x_2$		1							
$\bar{x}_2$		1							
$x_3$			1						
$\bar{x}_3$			1						
$x_4$				1					
$\bar{x}_4$				1					
$s_1$									
$s'_1$									
$s_2$									
$s'_2$									
$s_3$									
$s'_3$									
$s_4$									
$s'_4$									
	t	1	1	1	1	4	4	4	4

$x_1$  in clause  $s_1, s_4$

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

Slack numbers

$x_1$	1			1	1	
$\bar{x}_1$	1					1
$x_2$		1				
$\bar{x}_2$		1				
$x_3$			1			
$\bar{x}_3$			1			
$x_4$				1		
$\bar{x}_4$				1		
$s_1$						
$s'_1$						
$s_2$						
$s'_2$						
$s_3$						
$s'_3$						
$s_4$						
$s'_4$						
	t	1	1	1	1	4
						4
						4
						4

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers	$x_1$	1			1		1	
	$\bar{x}_1$	1					1	
	$x_2$		1		1			
	$\bar{x}_2$		1					1
	$x_3$			1		1	1	
	$\bar{x}_3$			1		1		1
	$x_4$				1		1	1
	$\bar{x}_4$				1			1
Slack numbers	$s_1$							
	$s'_1$							
	$s_2$							
	$s'_2$							
	$s_3$							
	$s'_3$							
	$s_4$							
	$s'_4$							
	t	1	1	1	1	4	4	4



# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers	$x_1$	1		1		1			
	$\bar{x}_1$	1			1				
	$x_2$		1		1				
	$\bar{x}_2$		1				1		
	$x_3$			1		1	1		
	$\bar{x}_3$			1		1		1	
	$x_4$				1		1	1	
	$\bar{x}_4$				1			1	
Slack numbers	$s_1$				1				
	$s'_1$				2				
	$s_2$				⋮	1			
	$s'_2$				⋮	2			
	$s_3$				⋮	⋮	1		
	$s'_3$				⋮	⋮	2		
	$s_4$				⋮	⋮	⋮	1	
	$s'_4$				⋮	⋮	⋮	2	
	t	1	1	1	1	4	4	4	4

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers	$x_1$	1	0	0	0	1	0	0	1
	$\bar{x}_1$	1	0	0	0	0	1	0	0
	$x_2$	0	1	0	0	1	0	0	0
	$\bar{x}_2$	0	1	0	0	0	0	1	0
	$x_3$	0	0	1	0	0	1	1	0
	$\bar{x}_3$	0	0	1	0	1	0	0	1
	$x_4$	0	0	0	1	0	1	0	1
	$\bar{x}_4$	0	0	0	1	0	0	1	0
Slack numbers	$s_1$	0	0	0	0	1	0	0	0
	$s'_1$	0	0	0	0	2	0	0	0
	$s_2$	0	0	0	0	0	1	0	0
	$s'_2$	0	0	0	0	0	2	0	0
	$s_3$	0	0	0	0	0	0	1	0
	$s'_3$	0	0	0	0	0	0	2	0
	$s_4$	0	0	0	0	0	0	0	1
	$s'_4$	0	0	0	0	0	0	0	2
t	1	1	1	1	4	4	4	4	

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers

$x_1$	1	0	0	0	1	0	0	1
$\bar{x}_1$	1	0	0	0	0	1	0	0
$x_2$	0	1	0	0	1	0	0	0
$\bar{x}_2$	0	1	0	0	0	0	1	0
$x_3$	0	0	1	0	0	1	1	0
$\bar{x}_3$	0	0	1	0	1	0	0	1
$x_4$	0	0	0	1	0	1	0	1
$\bar{x}_4$	0	0	0	1	0	0	1	0
$s_1$	0	0	0	0	1	0	0	0
$s'_1$	0	0	0	0	2	0	0	0
$s_2$	0	0	0	0	0	1	0	0
$s'_2$	0	0	0	0	0	2	0	0
$s_3$	0	0	0	0	0	0	1	0
$s'_3$	0	0	0	0	0	0	2	0
$s_4$	0	0	0	0	0	0	0	1
$s'_4$	0	0	0	0	0	0	0	2

Slack numbers

**Lemma:**  $F$  is satisfiable if and only if there

is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .  $\frac{11 \dots 1}{n} = \frac{44 \dots 4}{m}$

t 1 1 1 1 4 4 4 4

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1	0	0	0	1	0	0	1
$\bar{x}_1$	1	0	0	0	0	1	0	0
$x_2$	0	1	0	0	1	0	0	0
$\bar{x}_2$	0	1	0	0	0	0	1	0
$x_3$	0	0	1	0	0	1	1	0
$\bar{x}_3$	0	0	1	0	1	0	0	1
$x_4$	0	0	0	1	0	1	0	1
$\bar{x}_4$	0	0	0	1	0	0	1	0
$s_1$	0	0	0	0	1	0	0	0
$s'_1$	0	0	0	0	2	0	0	0
$s_2$	0	0	0	0	0	1	0	0
$s'_2$	0	0	0	0	0	2	0	0
$s_3$	0	0	0	0	0	0	1	0
$s'_3$	0	0	0	0	0	0	2	0
$s_4$	0	0	0	0	0	0	0	1
$s'_4$	0	0	0	0	0	0	0	2
t	1	1	1	1	4	4	4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .



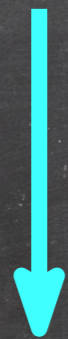


# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1					1			1
$x_2$		1					1		
$\bar{x}_2$		1							1
$x_3$			1					1	1
$\bar{x}_3$			1				1		1
$x_4$				1				1	1
$\bar{x}_4$				1					1
	$t$	1	1	1	1	4	4	4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1				1		1
$x_2$		1			1		
$x_3$			1			1	1
$x_4$				1		1	1
	$t$	1	1	1	1	4	4
						4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1				1		1
$x_2$		1			1		
$x_3$			1			1	1
$x_4$				1		1	1
	$t$	1	1	1	1	2	2
						1	2

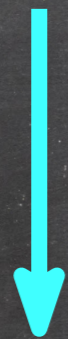
**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1			1		1			
$x_2$		1		1					
$x_3$			1		1	1			
$x_4$				1	1	1			
$s'_1$				2					
	$t$	1	1	1	1	4	2	1	2

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1				1		1		
$x_2$		1			1				
$x_3$			1		1	1	1		
$x_4$				1	1	1	1	1	
$s'_1$					2				
$s'_2$						2			
$s_3$							1		
$s'_3$								2	
$s'_4$								2	
	$t$	1	1	1	1	4	4	4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

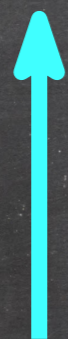
$x_1$	1				1		1		
$\bar{x}_1$	1						1		
$x_2$		1			1				
$\bar{x}_2$		1						1	
$x_3$			1			1	1		
$\bar{x}_3$			1		1			1	
$x_4$				1		1		1	
$\bar{x}_4$				1			1		
$s_1$					1				
$s'_1$					2				
$s_2$						1			
$s'_2$						2			
$s_3$							1		
$s'_3$							2		
$s_4$								1	
$s'_4$								2	
$t$		1	1	1	1	4	4	4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1			1		1	
$x_2$		1			1		
$x_3$			1			1	1
$x_4$				1		1	1
$s'_1$					2		
$s'_2$						2	
$s_3$							1
$s'_3$							2
$s'_4$							2
$t$	1	1	1	1	4	4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .



# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1			1		1	
$x_2$		1			1		
$x_3$			1			1	1
$x_4$				1		1	1
$s'_1$					2		
$s'_2$						2	
$s_3$							1
$s'_3$							2
$s'_4$							2
$t$	1	1	1	1	4	4	4

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1				1		1
$x_2$		1				1	
$x_3$			1			1	1
$x_4$				1			1
	$t$	1	1	1	1	2	2
						1	2

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$x_1 = x_2 = x_3 = x_4 = \text{true}$$

Truth assignment



Subset  $S'$

Literal numbers

Slack numbers

$x_1$	1				1		1
$x_2$		1				1	
$x_3$			1			1	1
$x_4$				1			1
	$t$	1	1	1	1	2	2
						1	2

**Lemma:**  $F$  is satisfiable if and only if there is a subset  $S' \subseteq S_F$  such that  $\sum_{x \in S'} x = t$ .

# Summary

Many important problems are NP-hard or NP-complete.

## Examples:

- Satisfiability
- Vertex cover
- Subset sum
- Hamiltonian cycle
- Clique
- Independent set
- ...

$P =$  solve in polynomial time  
 $NP =$  verify in polynomial time

$NP\text{-hard} =$  if in  $P$   
then  $NP = P$

$NP\text{-complete} = NP$  and  $NP\text{-hard}$

These problems are unlikely to be solvable in polynomial time.

## Techniques to cope with NP-hardness:

- Parameterized algorithms
- Approximation algorithms
- Heuristics