Greedy Algorithms

Textbook Reading

Chapters 16, 17, 21, 23 & 24

Overview

Design principle:

Make progress towards a globally optimal solution by making locally optimal choices, hence the name.

Problems:

- Interval scheduling
- Minimum spanning tree
- Shortest paths
- Minimum-length codes

Proof techniques:

- Induction
- The greedy algorithm “stays ahead”
- Exchange argument

Data structures:

- Priority queue
- Union-find data structure

Interval Scheduling

Given:

A set of activities competing for time intervals on a certain resource
(E.g., classes to be scheduled competing for a classroom)

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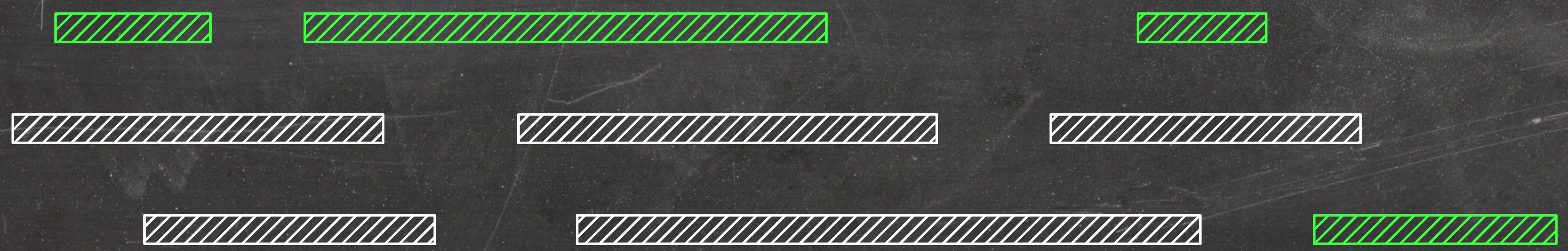

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A Greedy Framework for Interval Scheduling

FindSchedule(S)

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while S is not empty
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Main questions:

- Can we choose an arbitrary interval I in each iteration?
- How do we choose interval I in each iteration?

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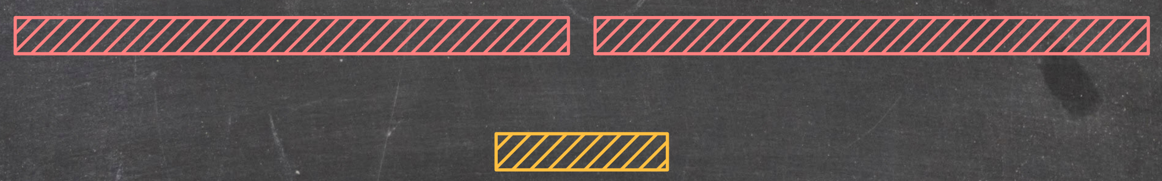

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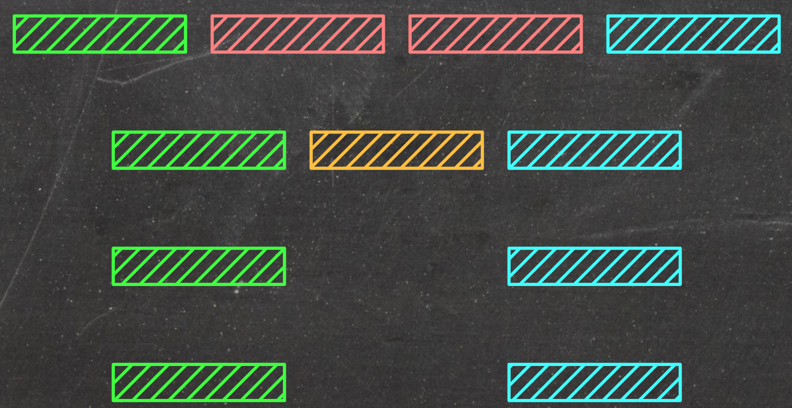

The Strategy That Works

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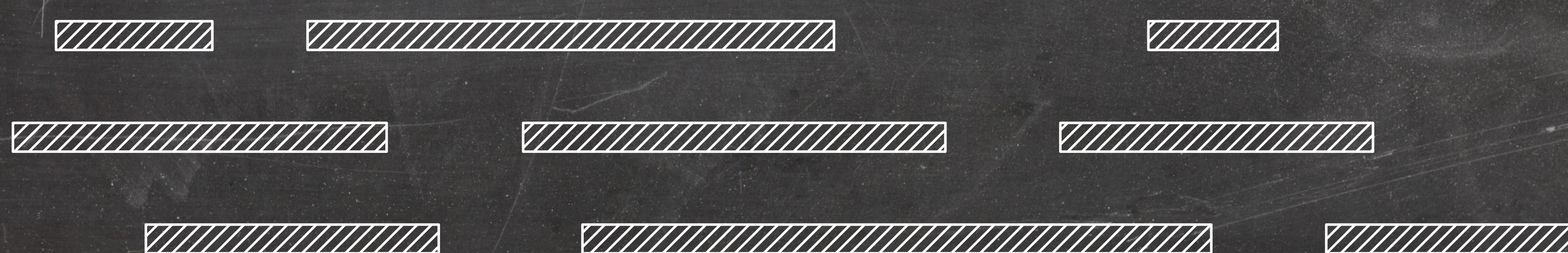

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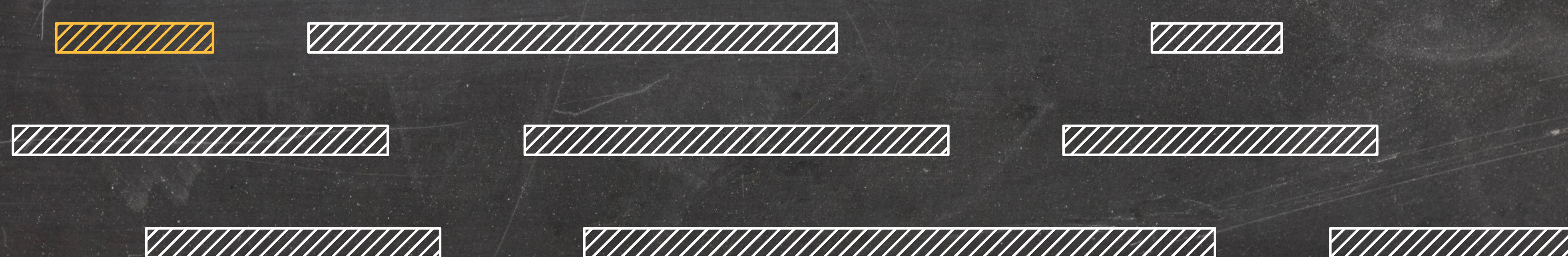

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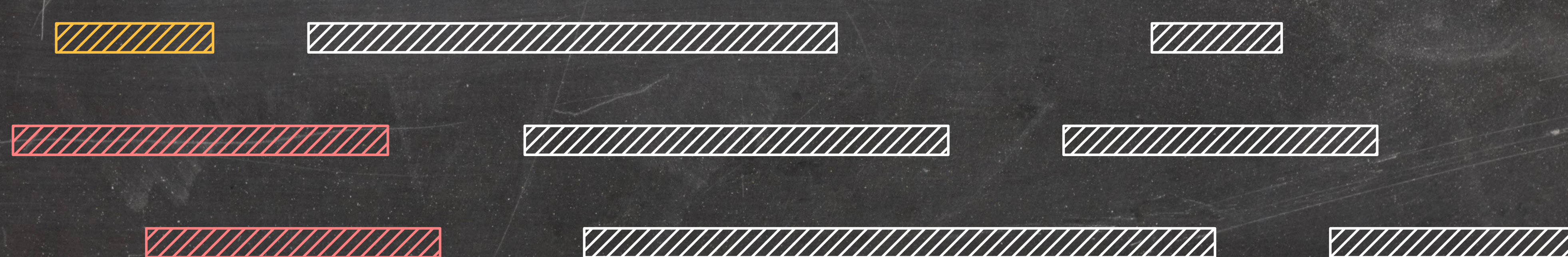

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Let $O_1 \prec O_2 \prec \cdots \prec O_m$ be an optimal schedule.

Prove by induction on j that I_j ends no later than O_j.

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$\Rightarrow$ Since O_{j+1} starts after O_j ends, it also starts after I_j ends.

$\Rightarrow$ If $k < m$, FindSchedule inspects O_{k+1} after I_k and thus would have added it to its output, a contradiction.

The Greedy Algorithm Stays Ahead

Lemma: FindSchedule finds a maximum-cardinality set of conflict-free intervals.

Proof by induction:

Base case(s): Verify that the claim holds for a set of initial instances.

Inductive step: Prove that, if the claim holds for the first k instances, it holds for the (k + 1)st instance.

The Greedy Algorithm Stays Ahead

Lemma: FindSchedule finds a maximum-cardinality set of conflict-free intervals.

Base case: I_1 ends no later than O_1 because both I_1 and O_1 are chosen from S and I_1 is the interval in S that ends first.

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$\Rightarrow$ O_{k+1} does not conflict with $I_1, I_2, \ldots, I_k$.

$\Rightarrow$ I_{k+1} ends no later than O_{k+1} because it is the interval that ends first among all intervals that do not conflict with $I_1, I_2, \ldots, I_k$.

Implementing The Algorithm

FindSchedule(S)

```
S′ = [ ]
sort the intervals in S by increasing finish times
S′.append(S[1])
f = S[1].f
for i = 2 to |S|
    do if S[i].s > f
            then S′.append(S[i])
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Lemma: A maximum-cardinality set of non-conflicting intervals can be found in $O(n \lg n)$ time.

Minimum Spanning Tree

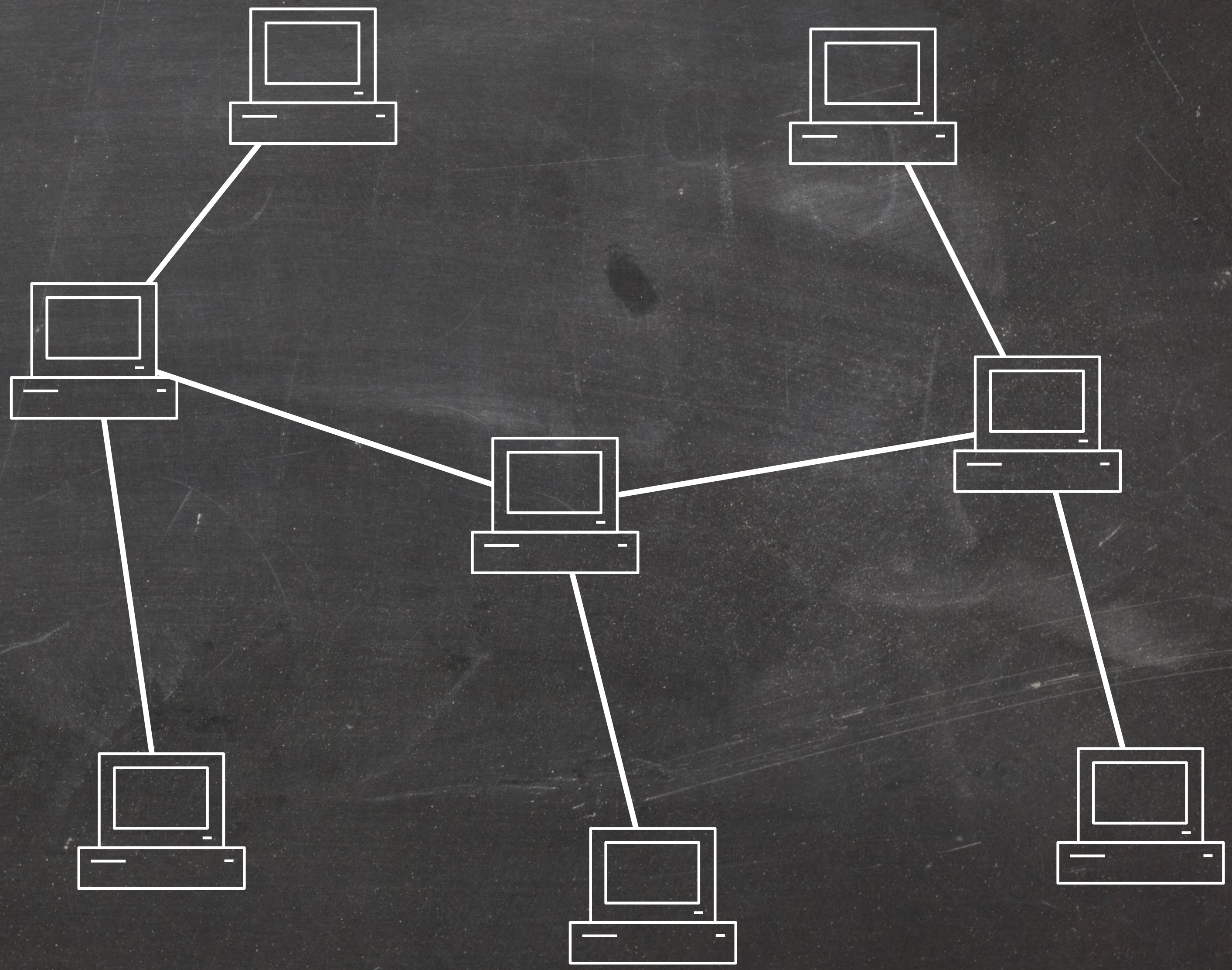
Given: n computers

Goal: Connect them so that every computer can communicate with every other computer.

We don't care whether the connection between any pair of computers is short.

We don't care about fault tolerance.

Every foot of cable costs us $1.

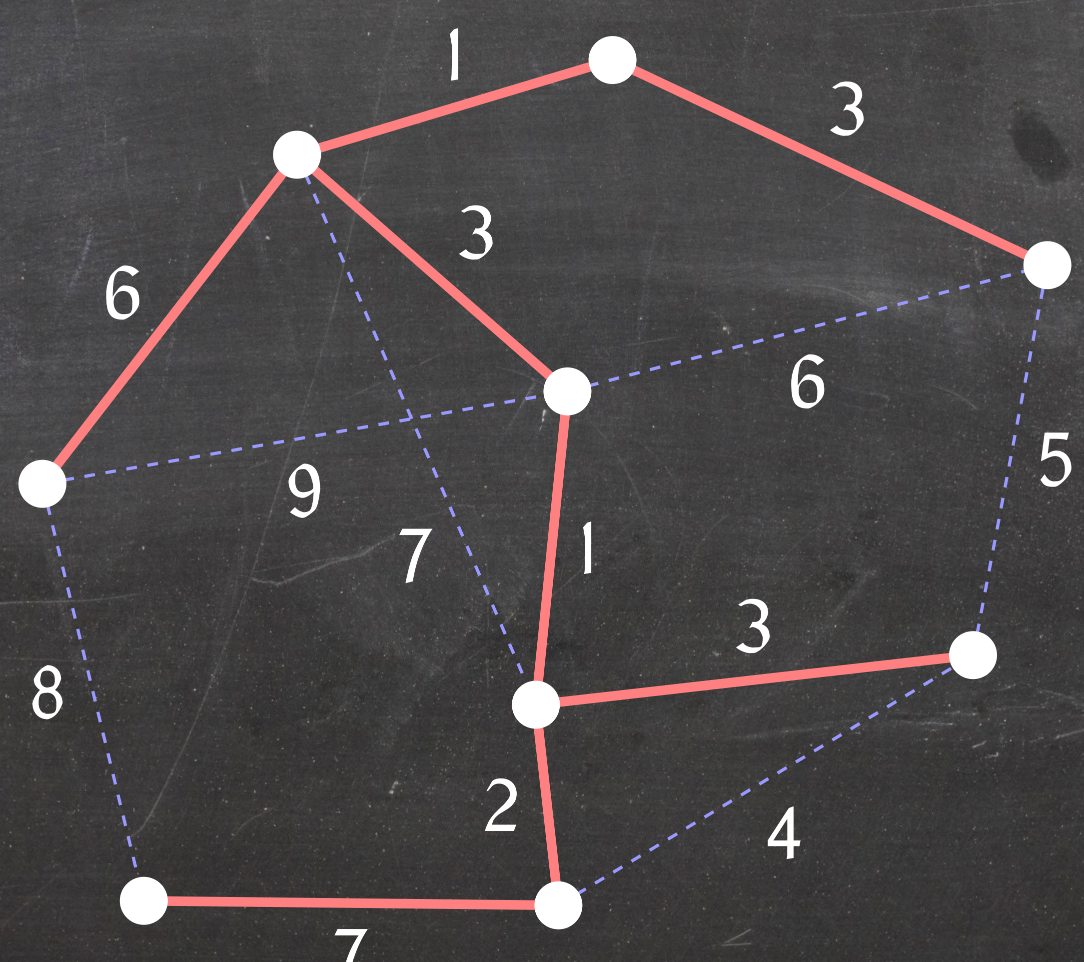


$\Rightarrow$ We want the cheapest possible network.

Minimum Spanning Tree

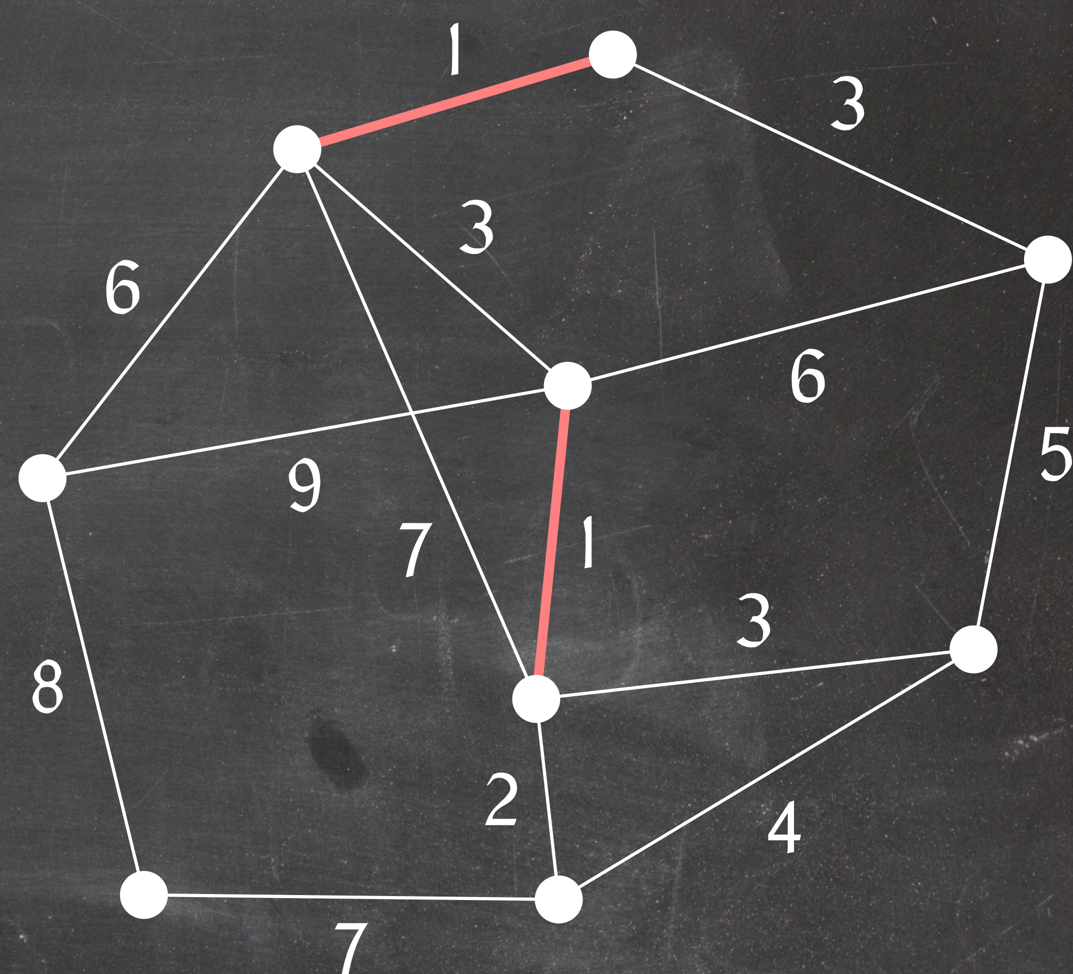
Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a minimum spanning tree (MST) T of G is a spanning tree with minimum total weight

$$w(T) = \sum_{e \in T} w(e).$$



Kruskal's Algorithm

Greedy choice: Pick the shortest edge



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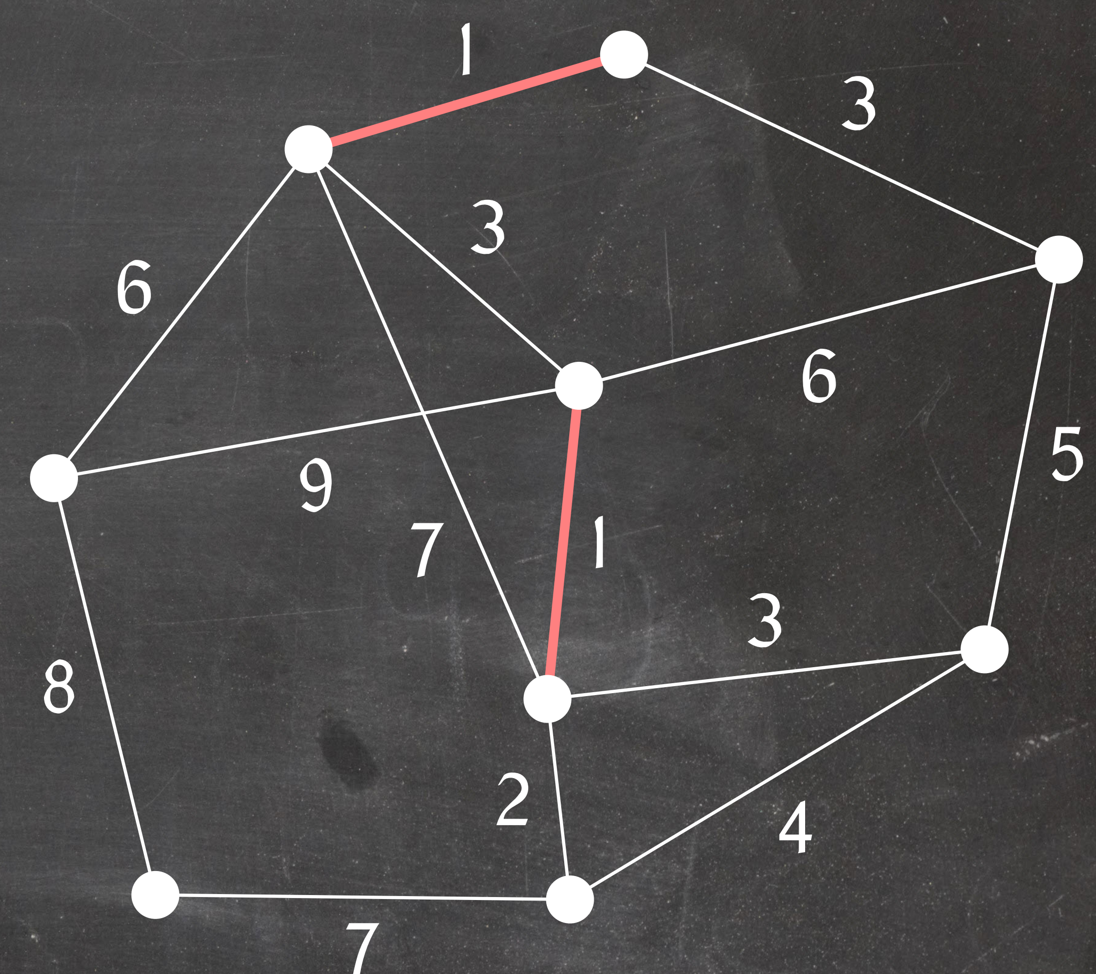

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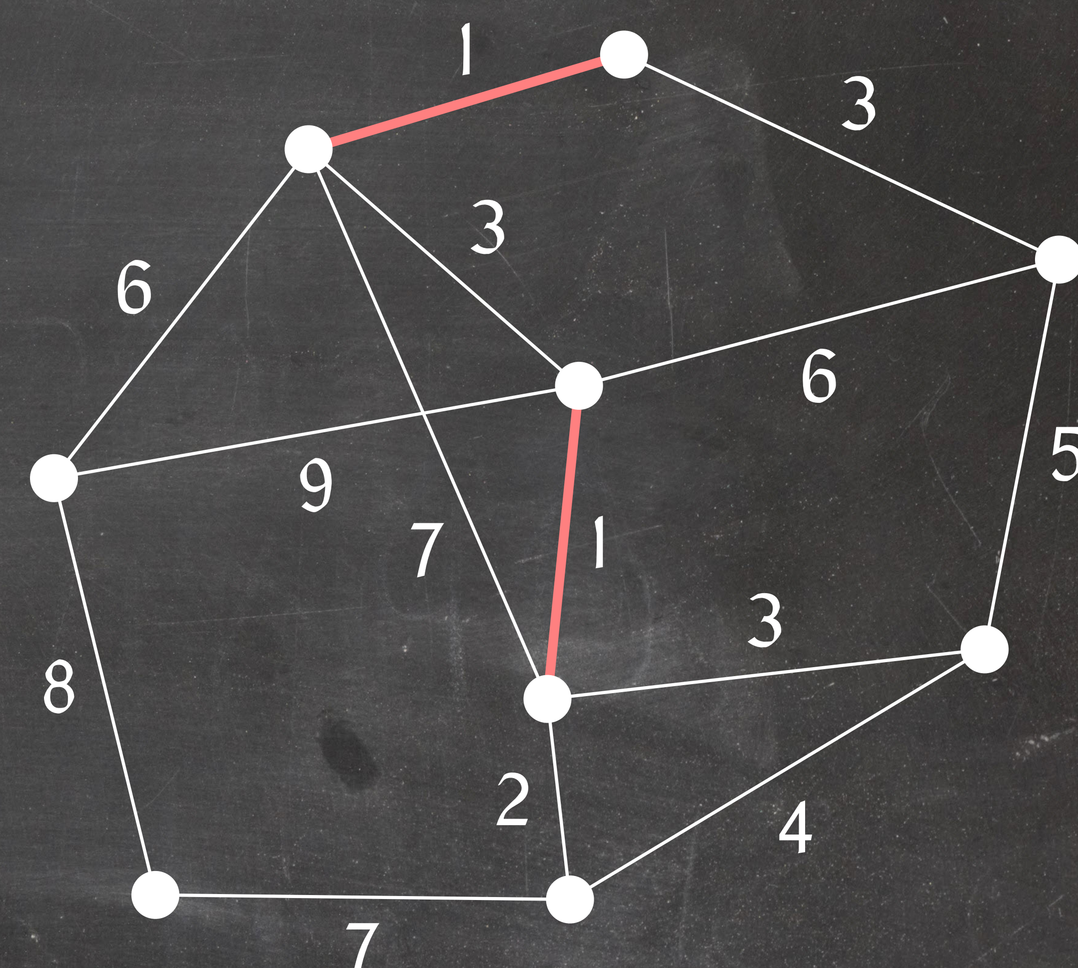

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T = (V, ∅)
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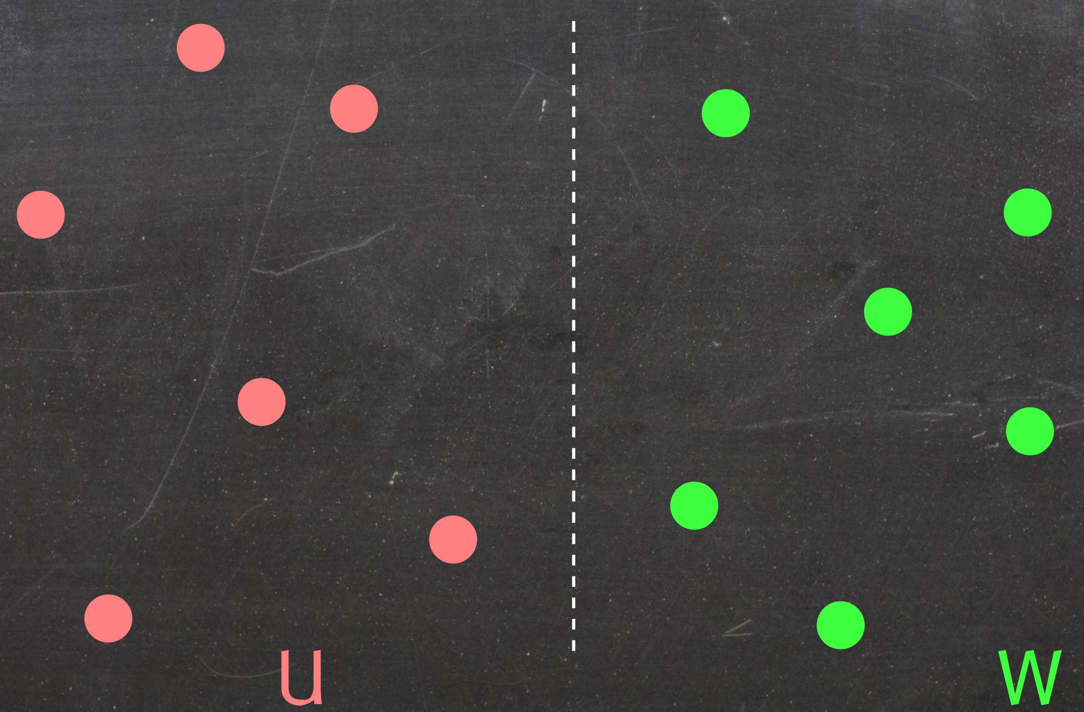
A Cut Theorem

A cut is a partition (U, W) of V into two non-empty subsets: $\emptyset \subset U \subset V$ and $W = V \setminus U$.



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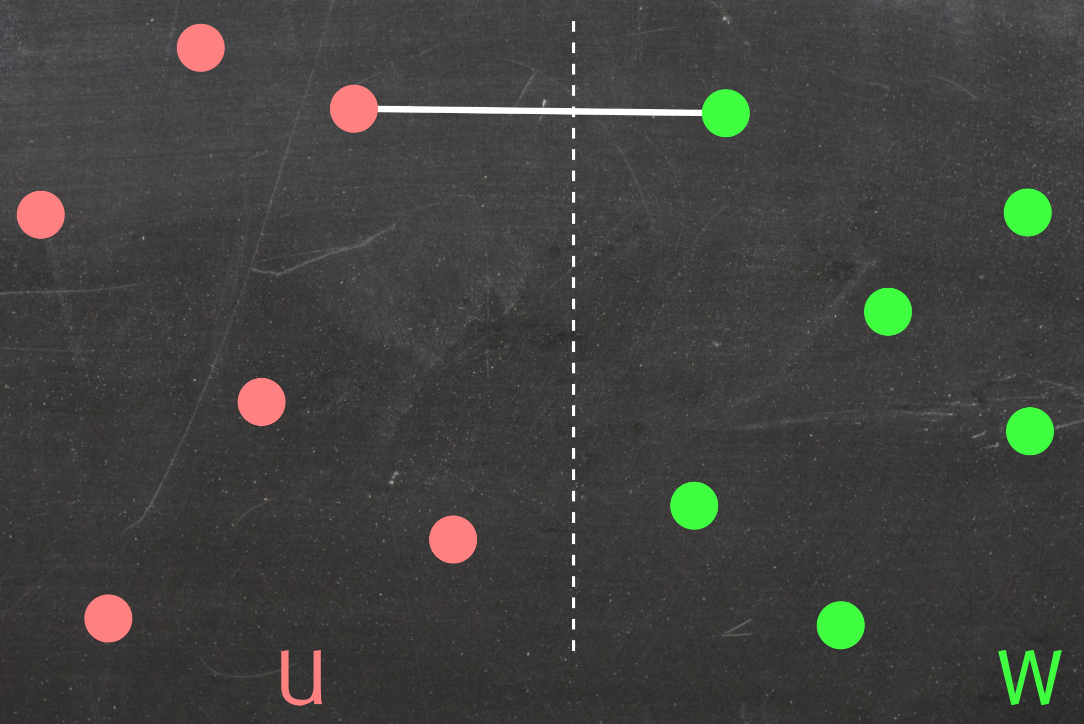

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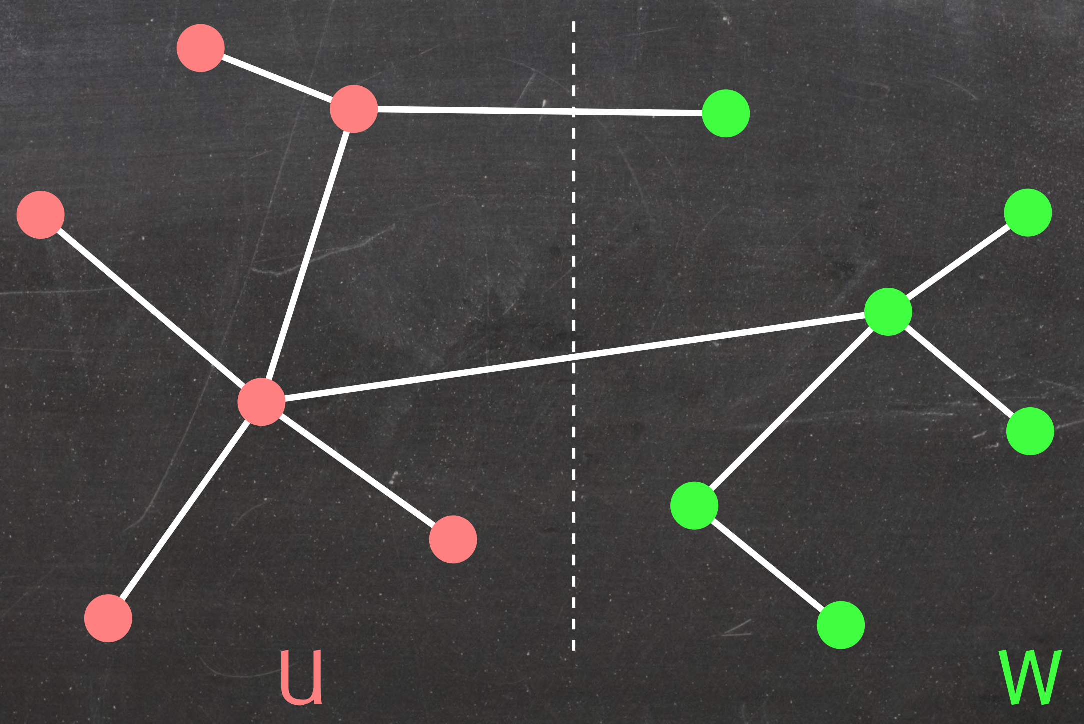
Theorem: Let T be a minimum spanning tree, let (U, W) be an arbitrary cut, and let e be the cheapest edge crossing the cut. Then there exists a minimum spanning tree that contains e and all edges of T that do not cross the cut.



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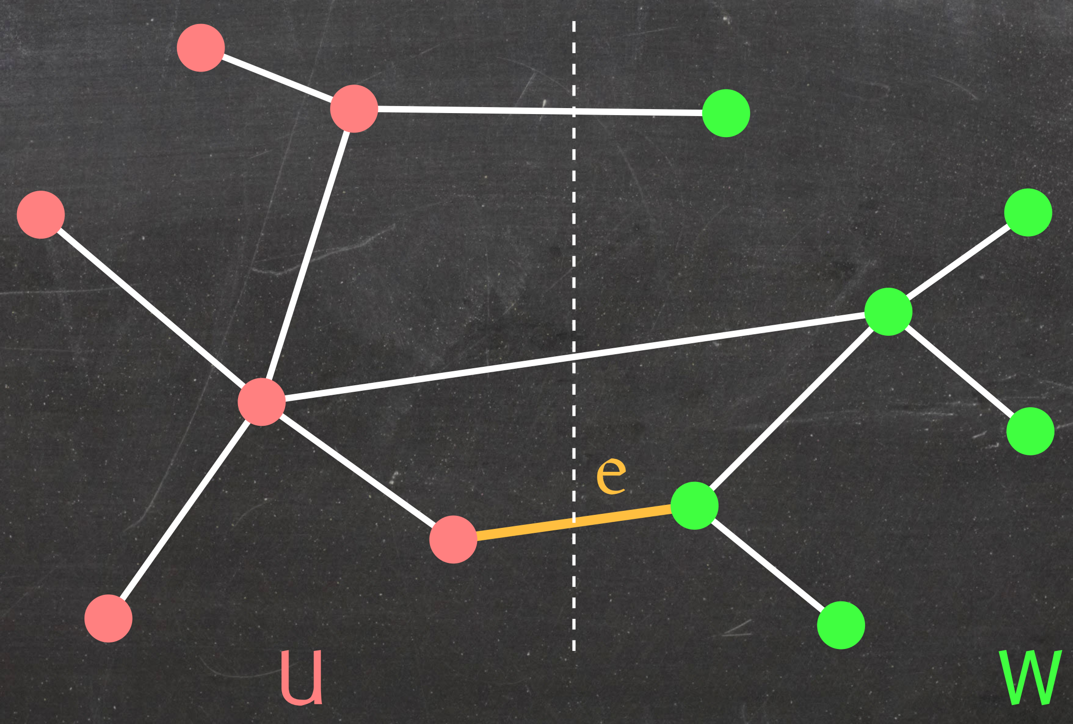

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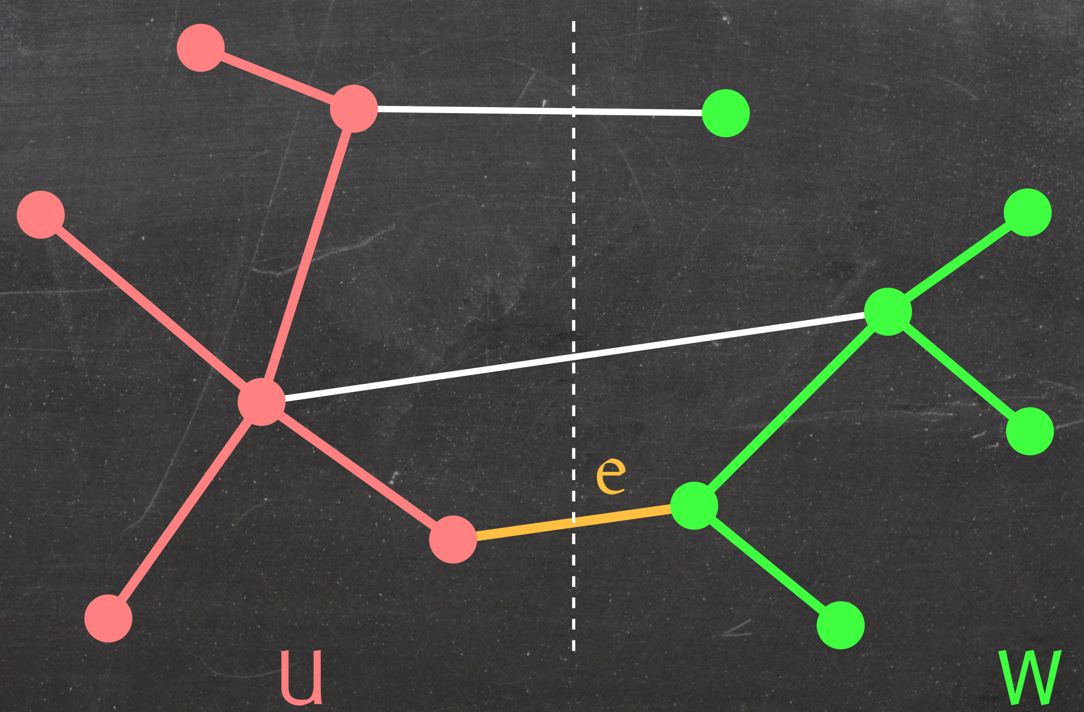

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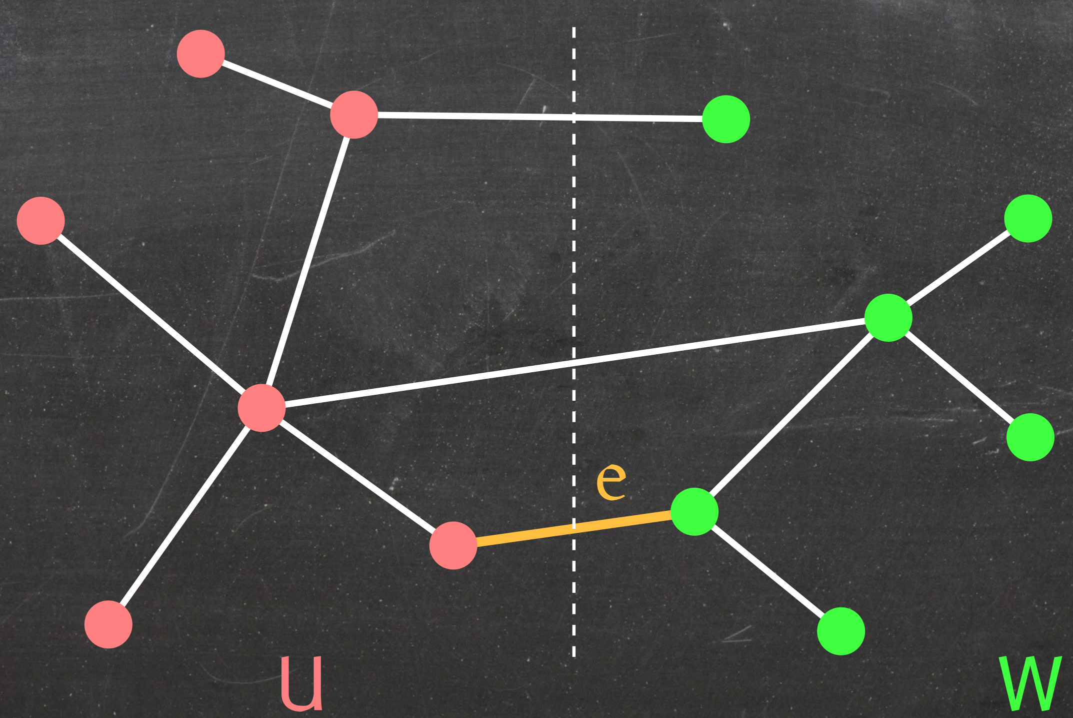

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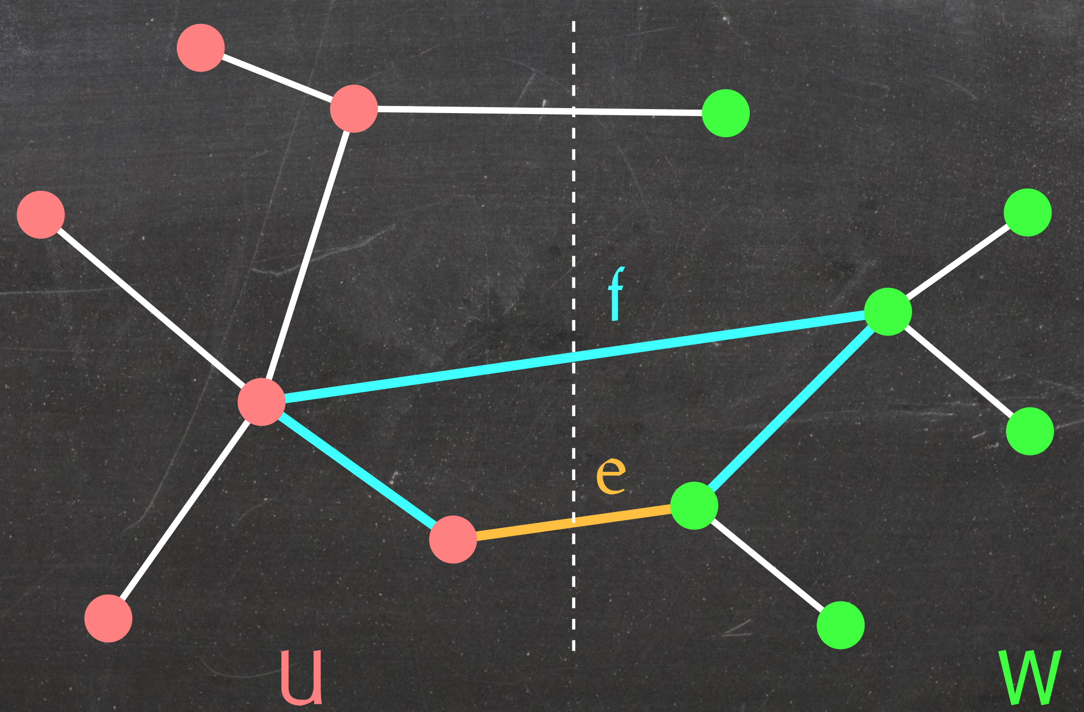

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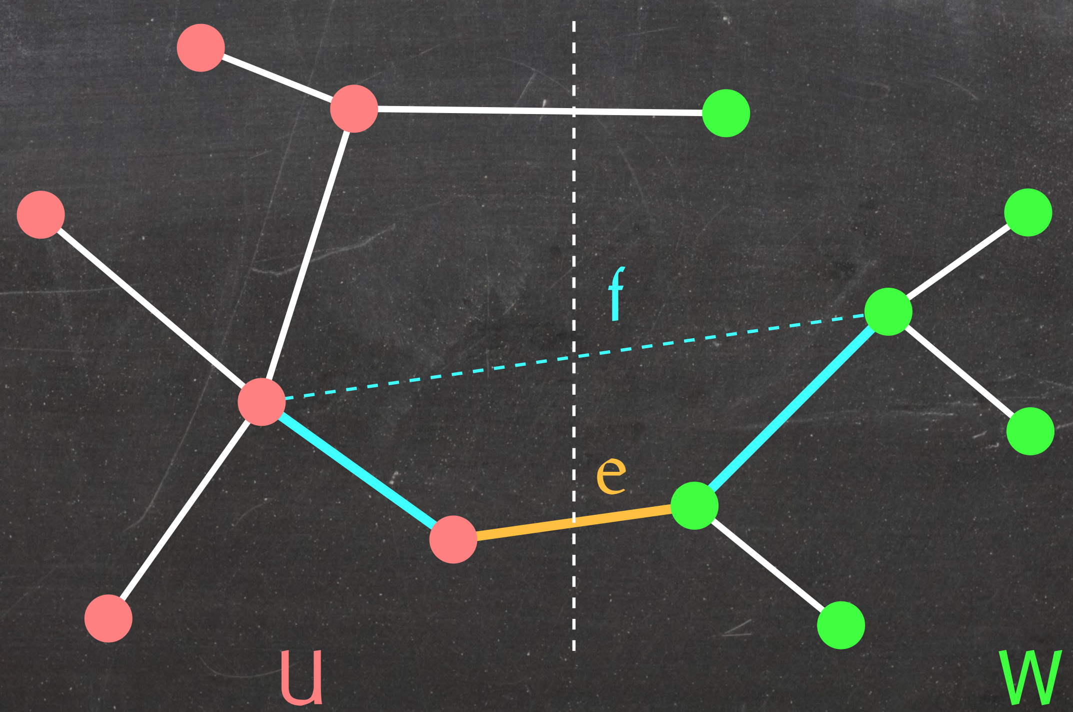

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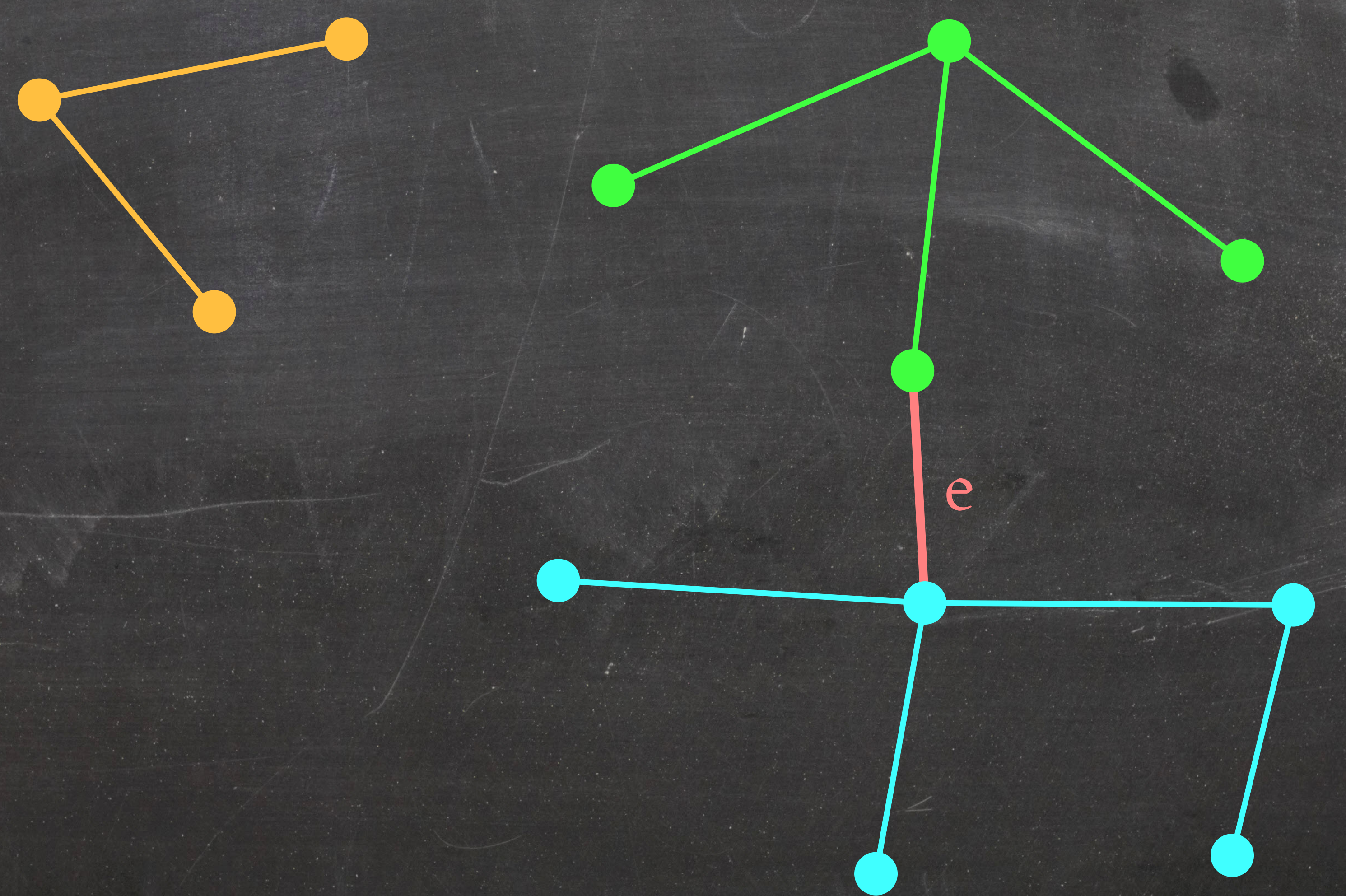

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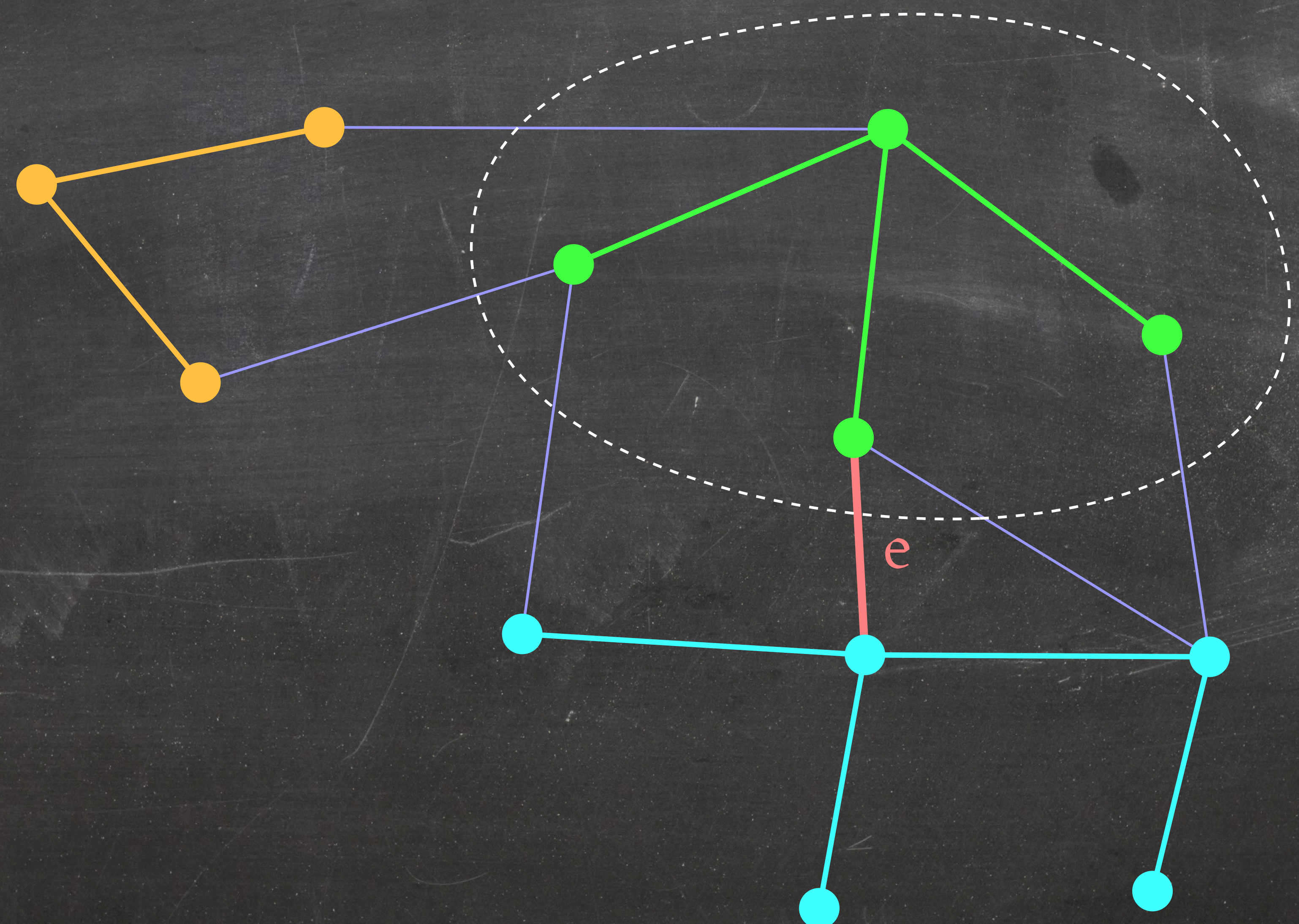

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$$\Downarrow$$

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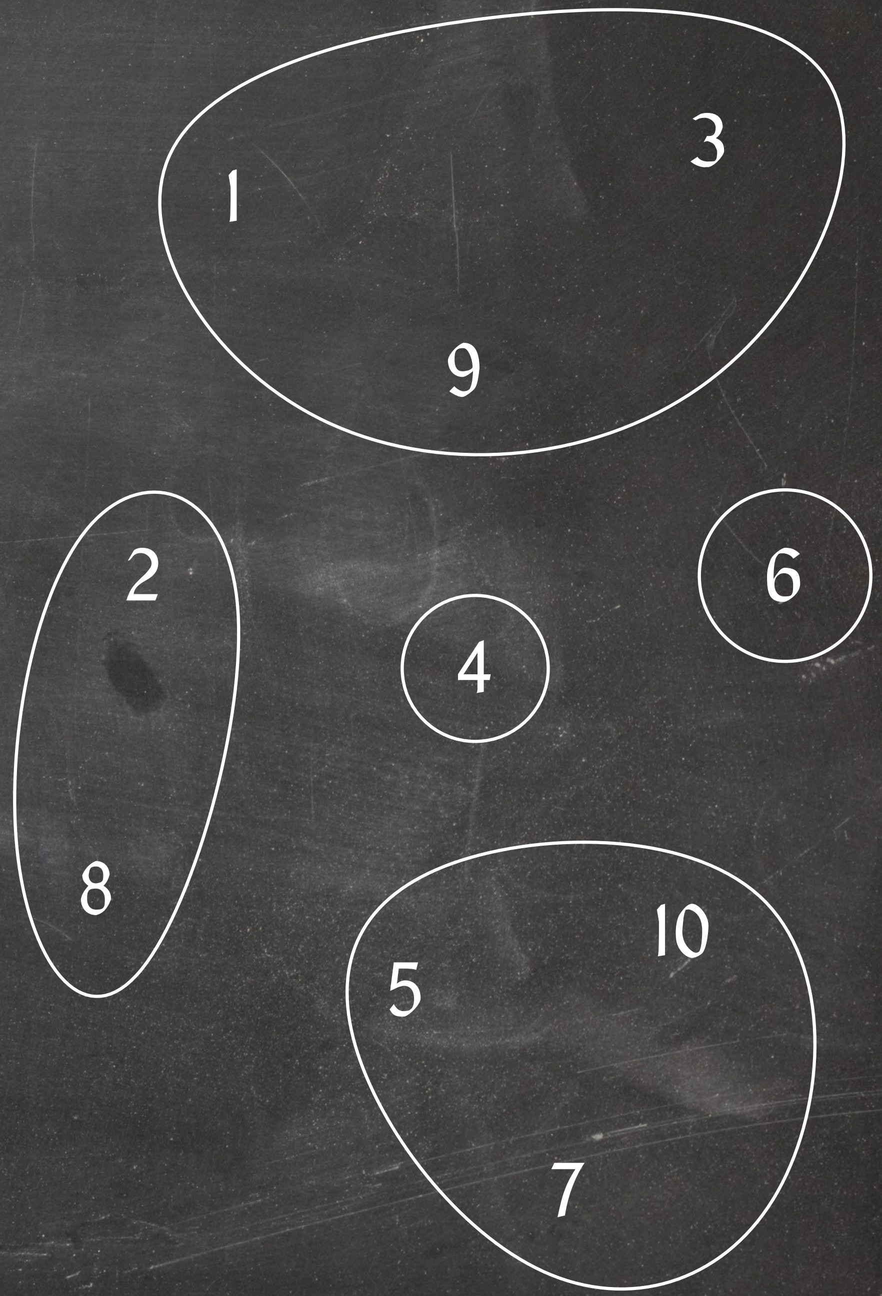
A Union-Find Data Structure

Given a set S of elements, maintain a partition of S into subsets $S_1, S_2, \ldots, S_k$.



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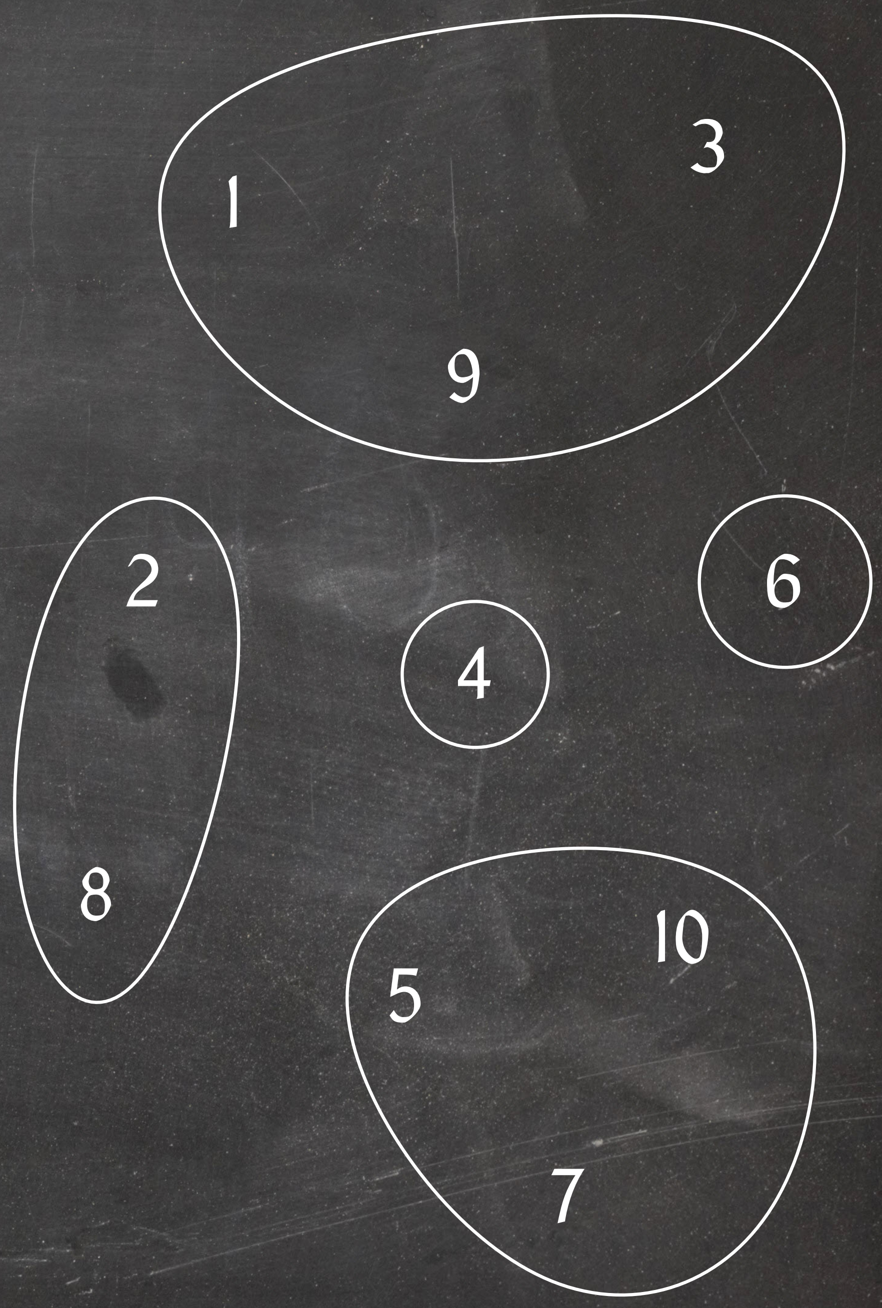

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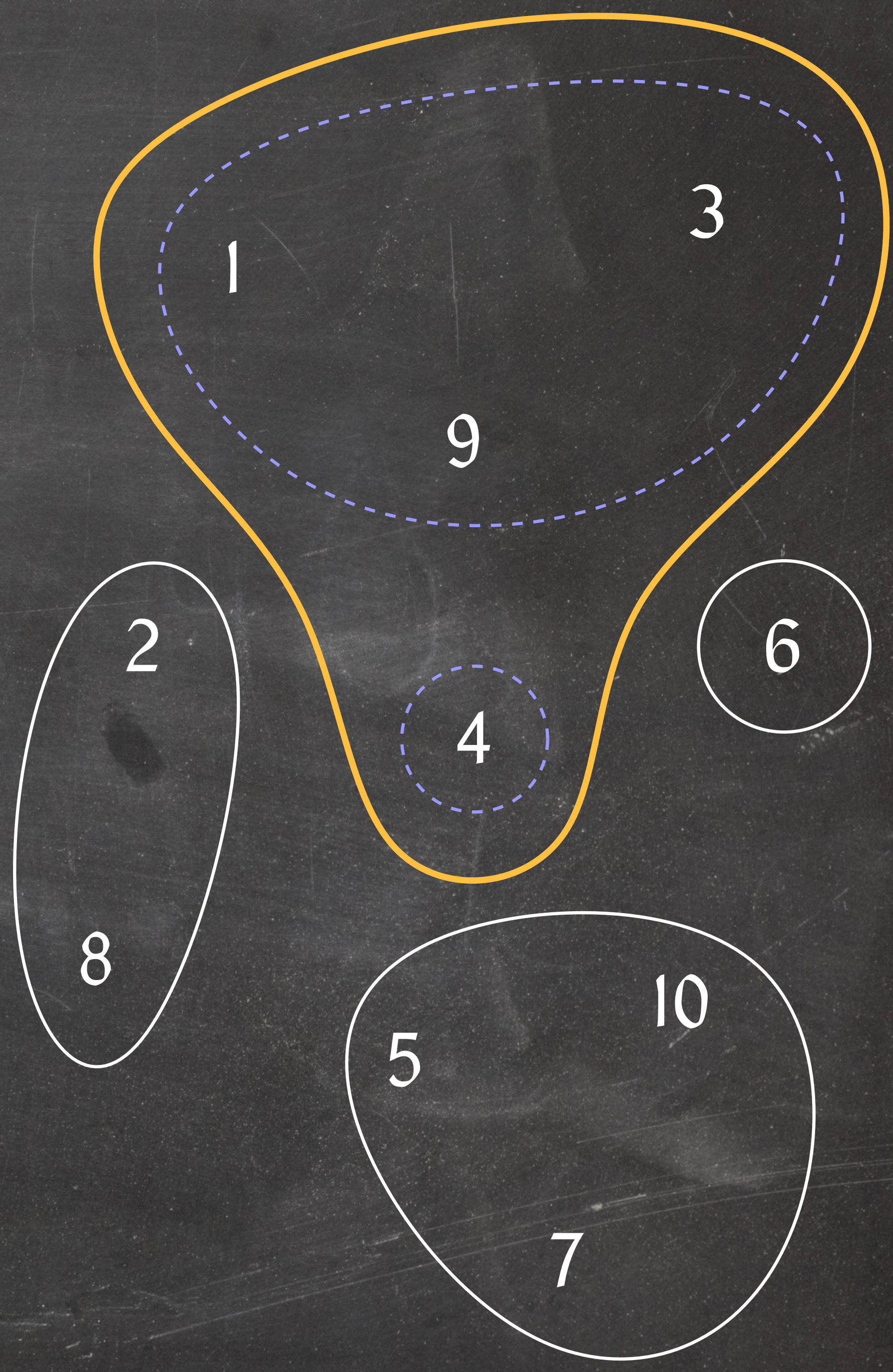

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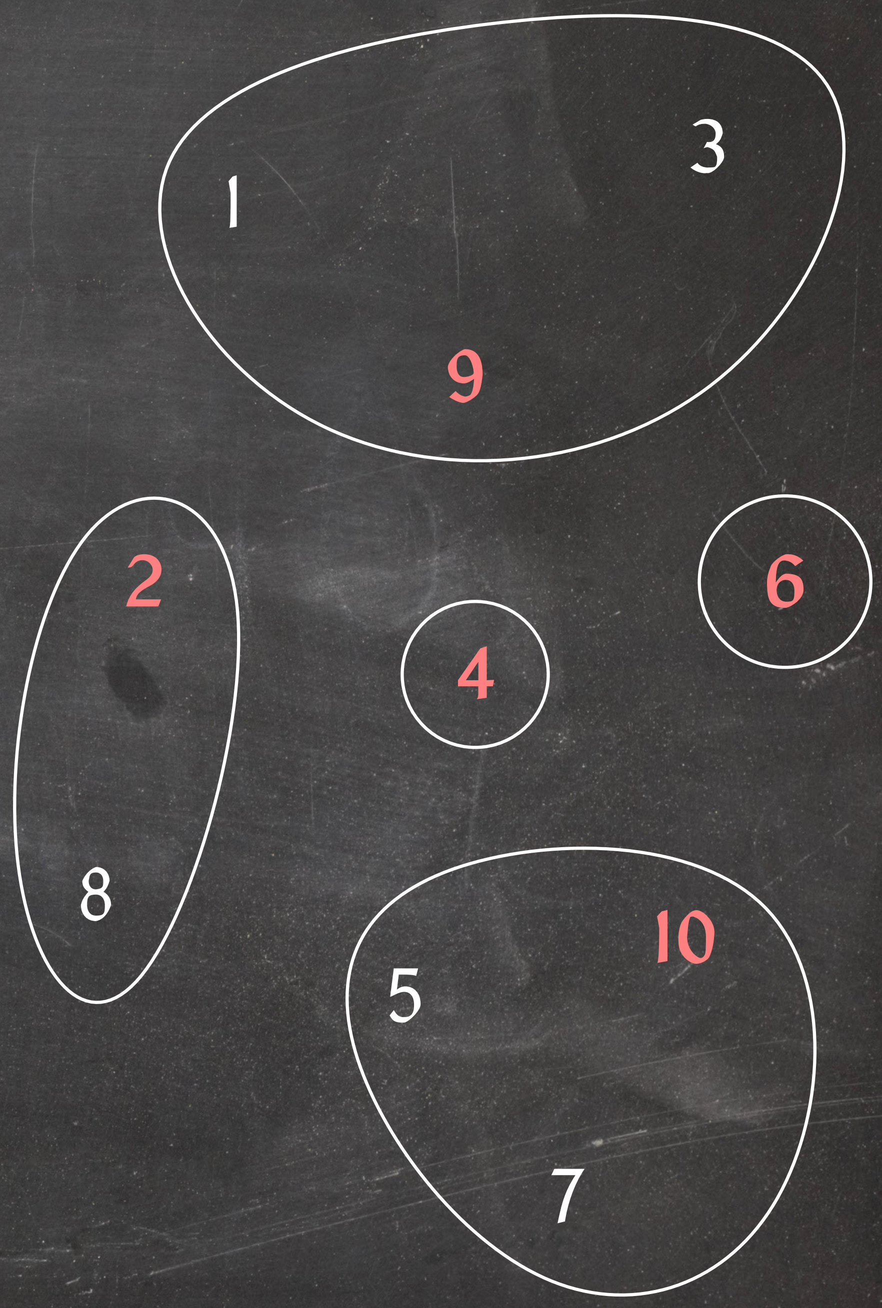
Find(x): Return a representative $r(S_i) \in S_i$ of the set S_i that contains x.



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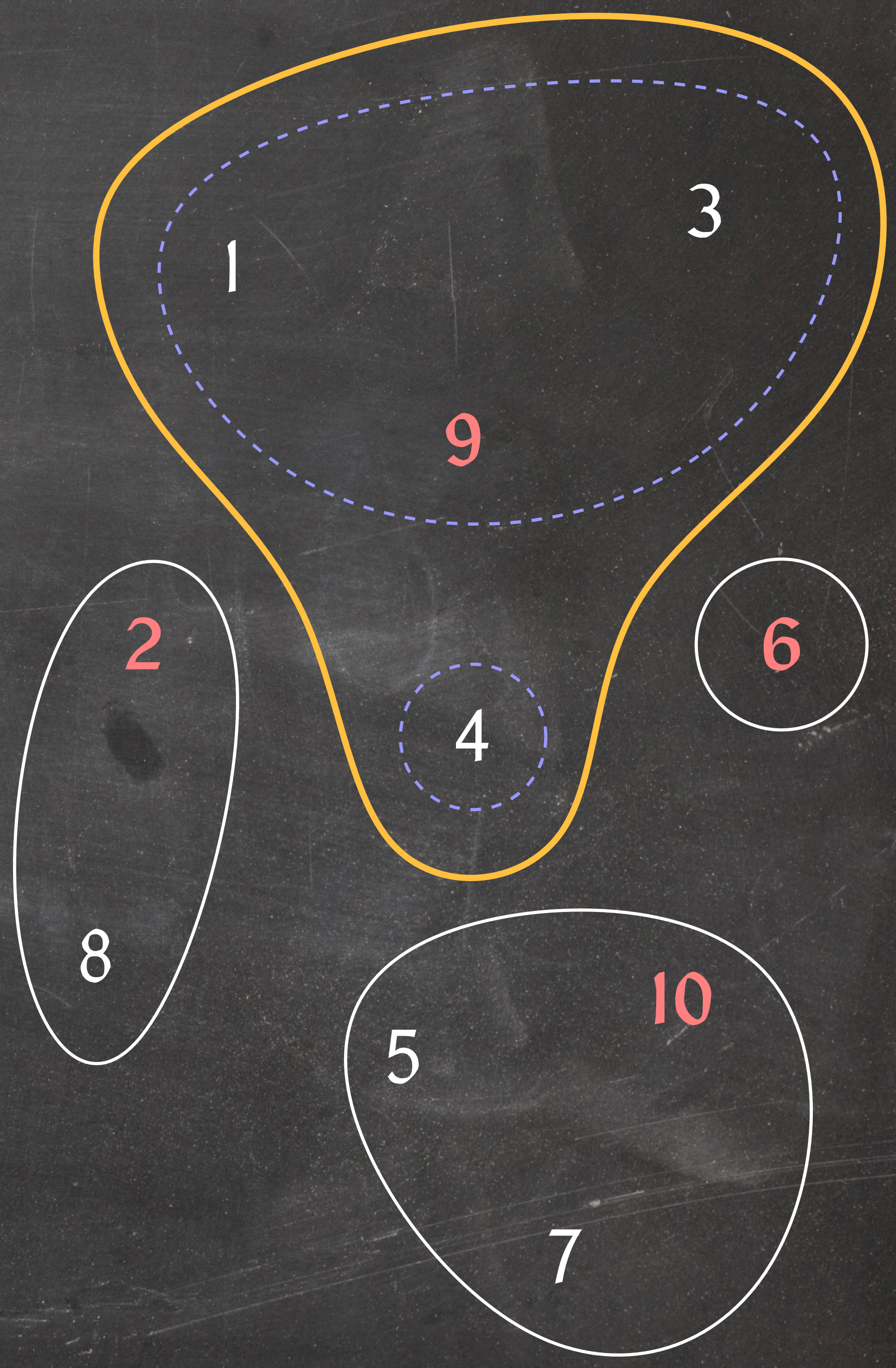

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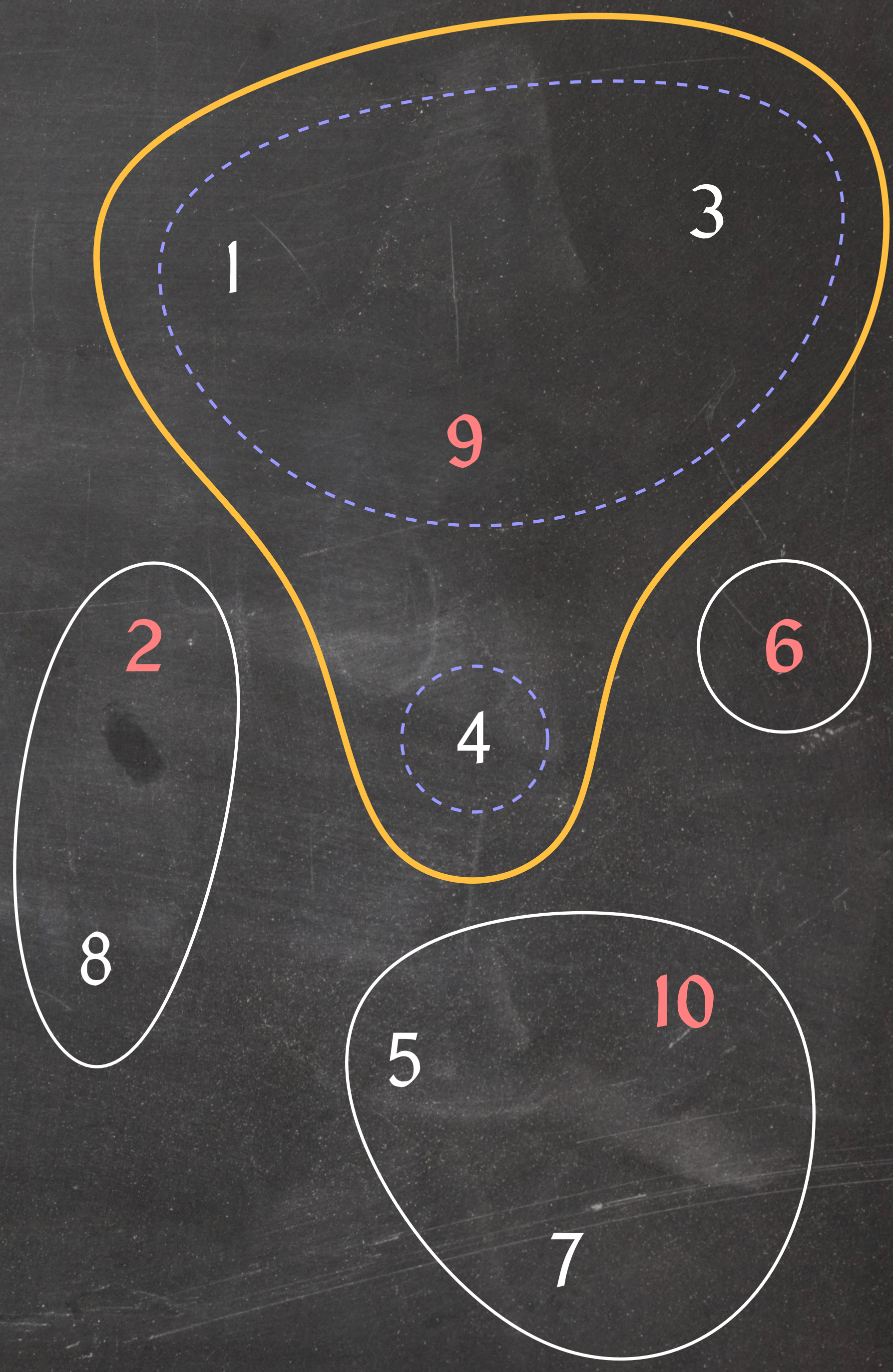
In particular, Find(x) = Find(y) if and only if x and y belong to the same set.



Kruskal's Algorithm Using Union-Find

Idea: Maintain a partition of V into the vertex sets of the connected components of T.

Kruskal(G)

```
T = (V, ∅)
initialize a union-find structure D for V with every vertex v ∈ V in its own set
sort the edges in G by increasing weight
for every edge (v, w) of G, in sorted order
    do if D.find(v) ≠ D.find(w)
          then add (v, w) to T
               D.union(v, w)
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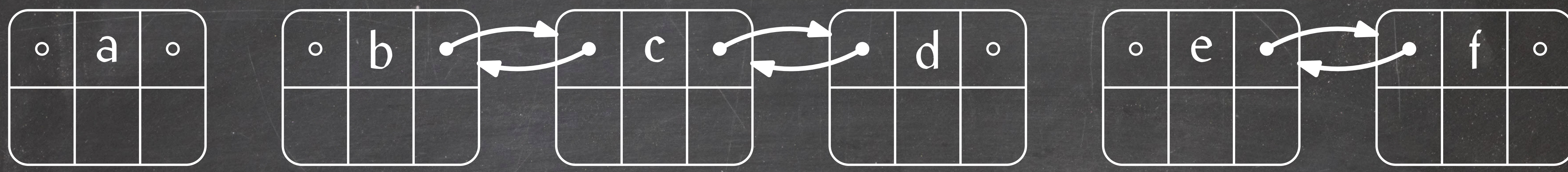
Lemma: Kruskal's algorithm takes $O(m \lg m)$ time plus the cost of $2m$ Find and $n - 1$ Union operations.

A Simple Union-Find Structure



List node:

- A set element
- Pointers to predecessor and successor
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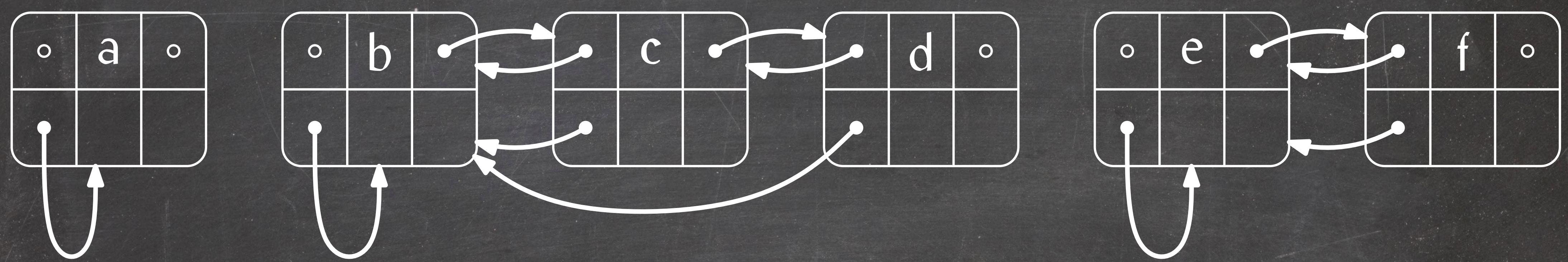

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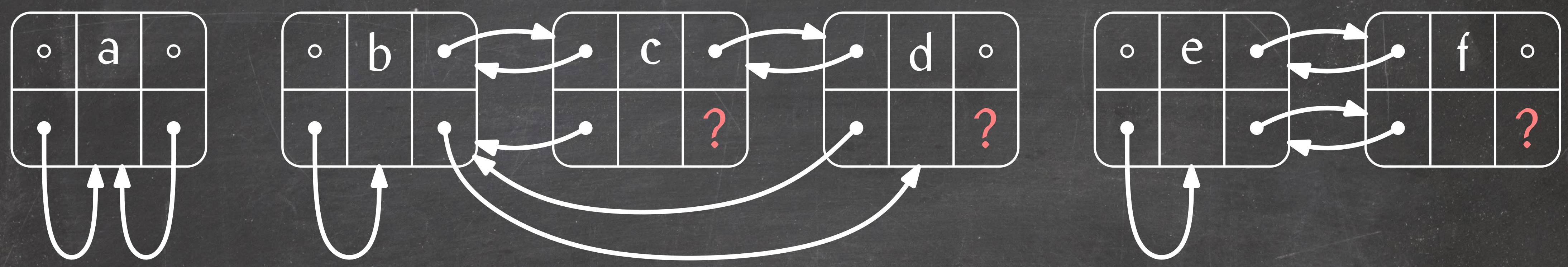

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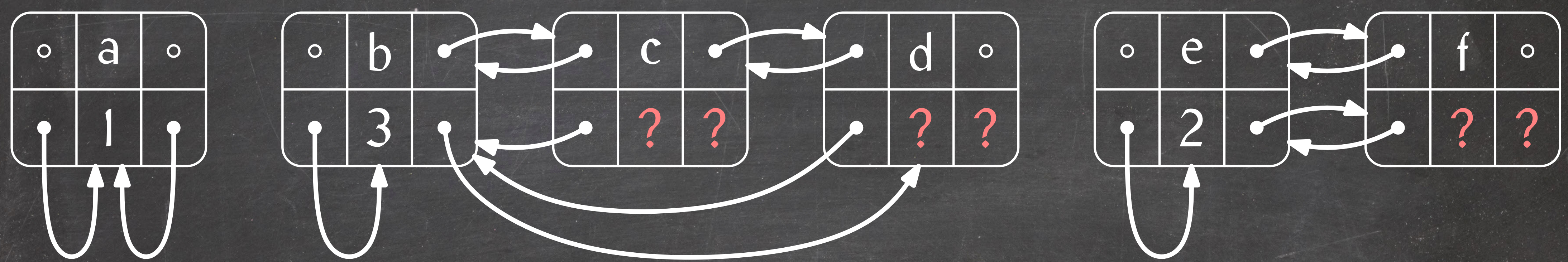

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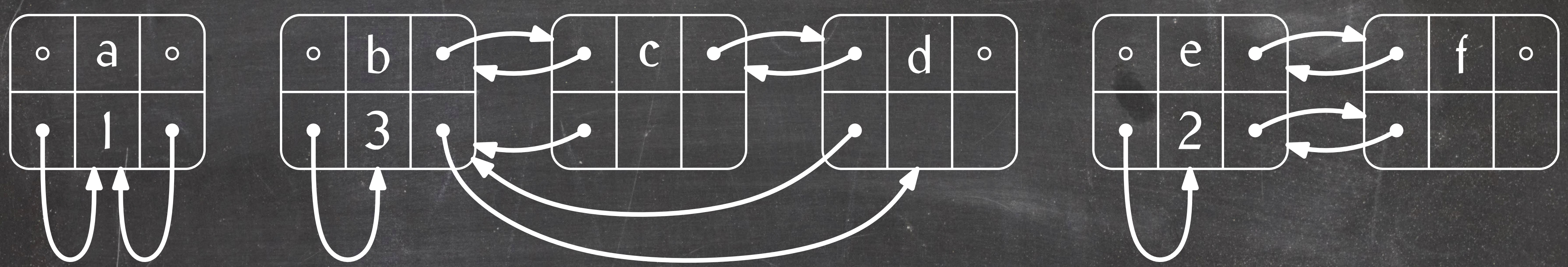
Find

D.find(x)

```
return x.head.key
```



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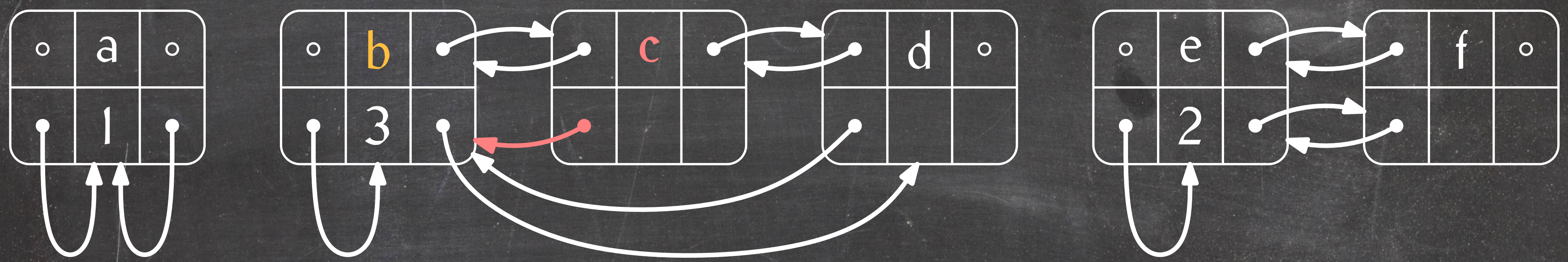

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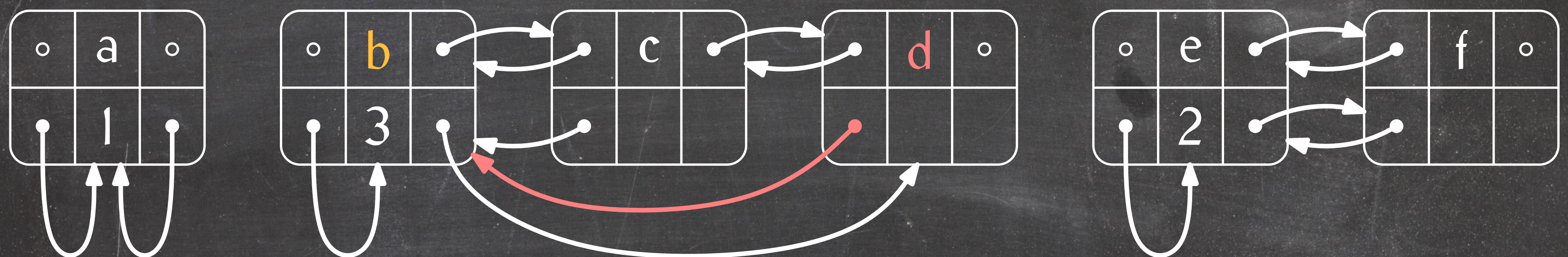

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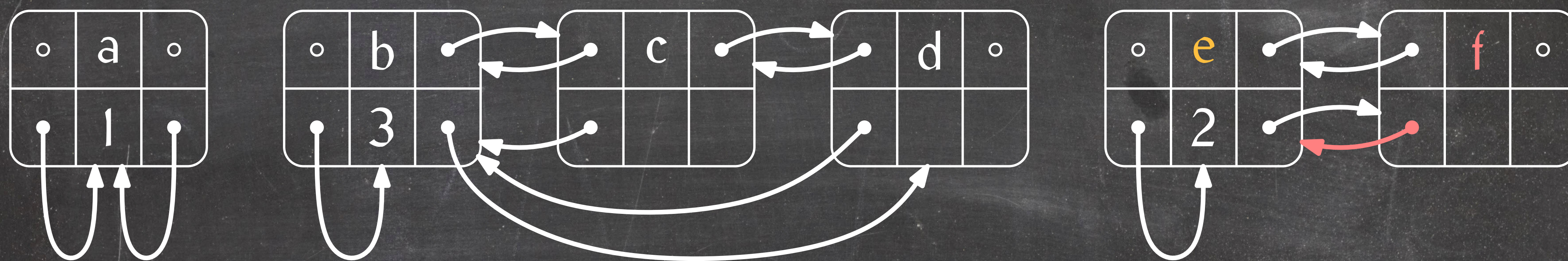

D.find(c) = b

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D.find(f) = e

Union

D.union(x, y)

```
if x.head.listSize < y.head.listSize
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x.head.tail.succ = y.head
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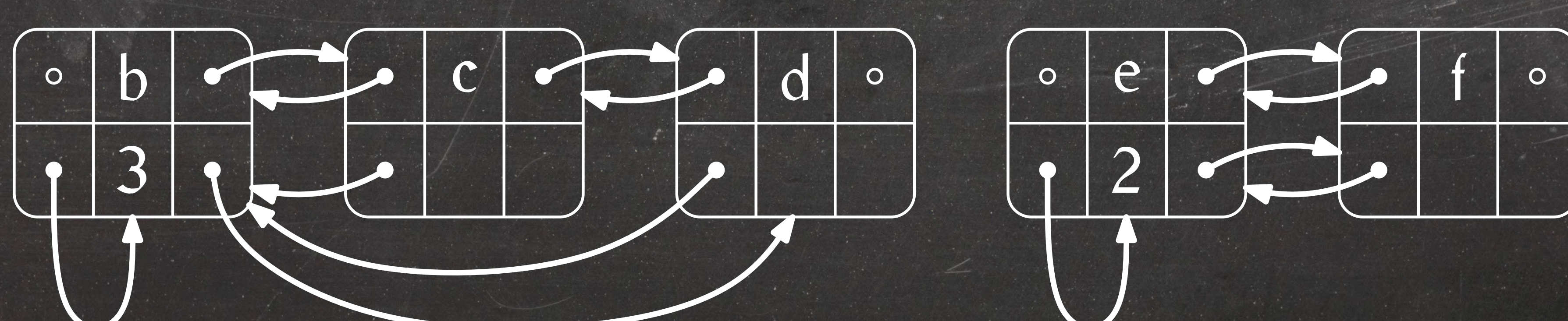

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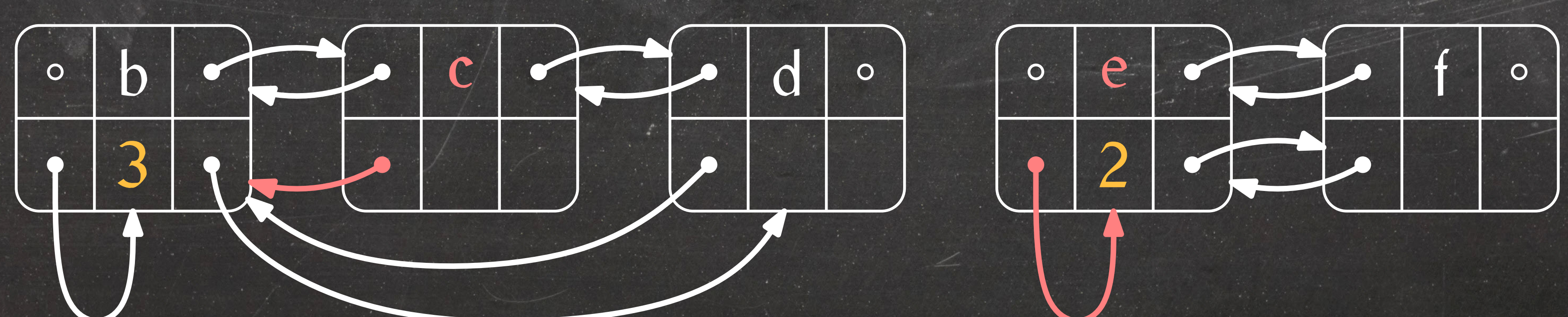

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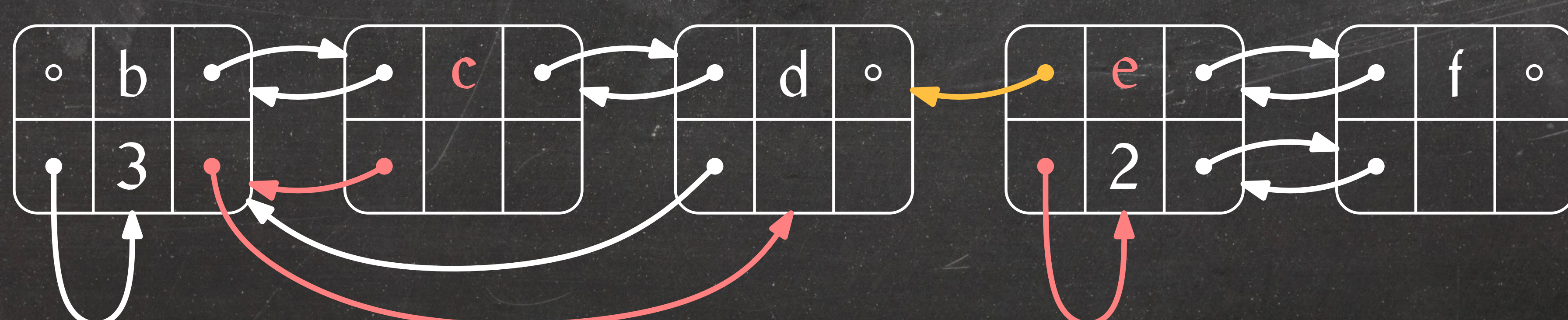

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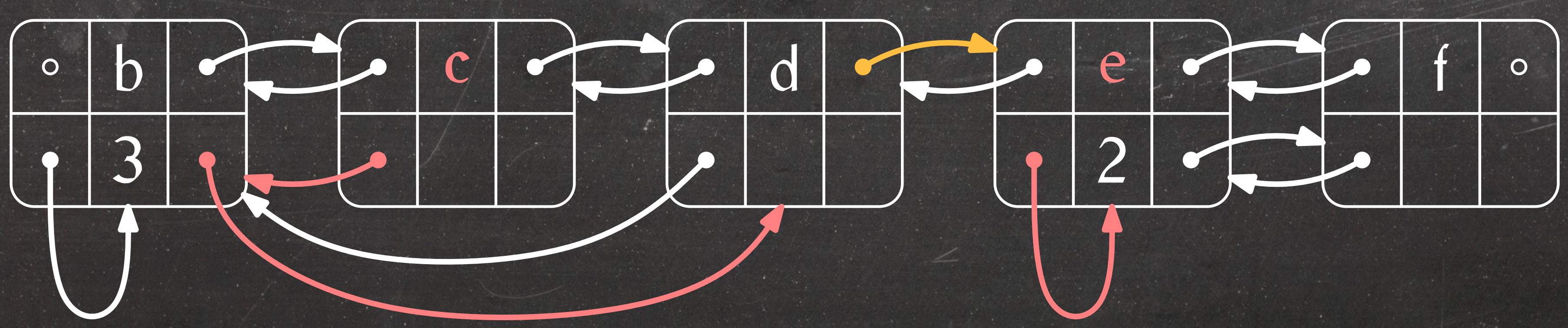

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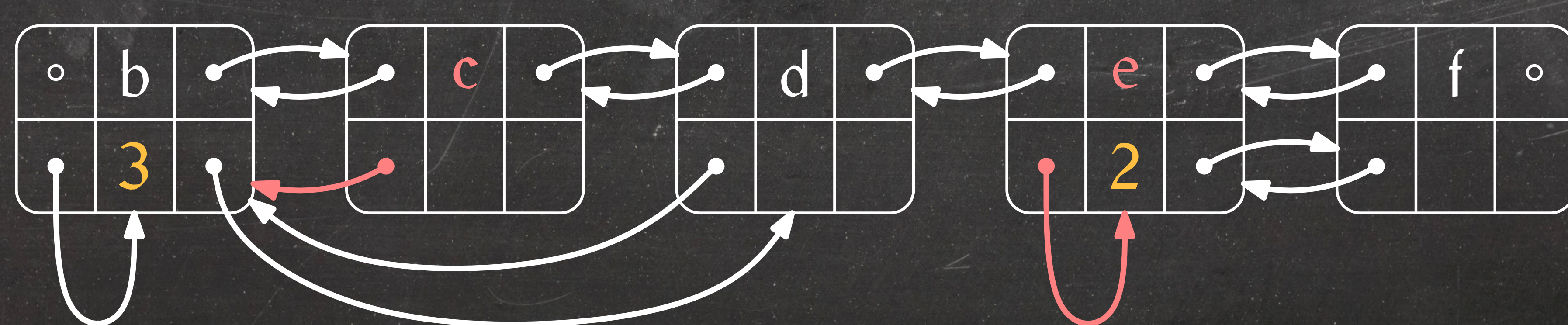

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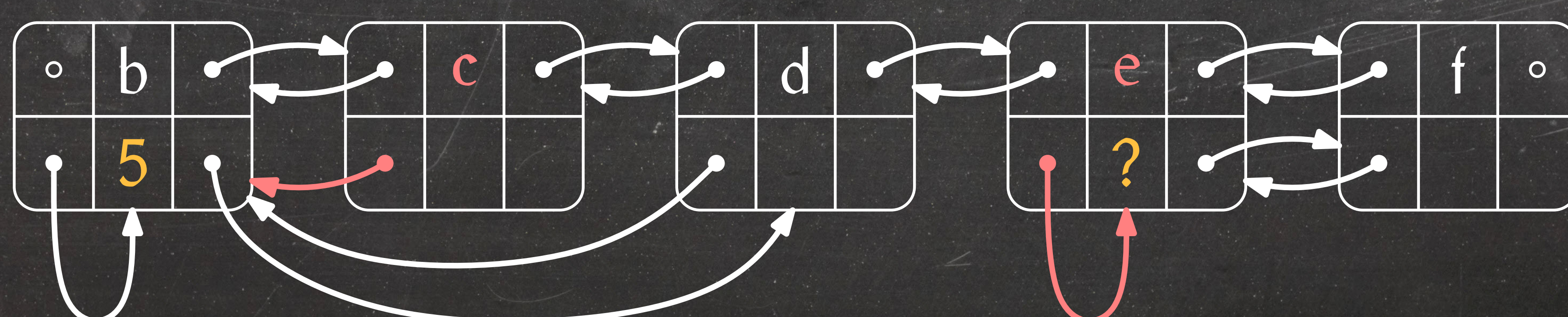

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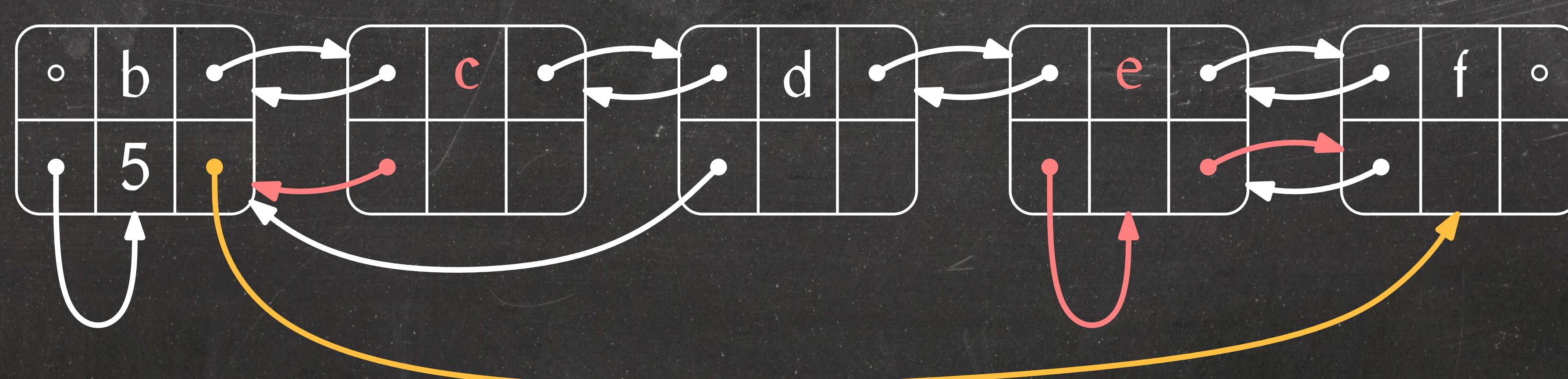

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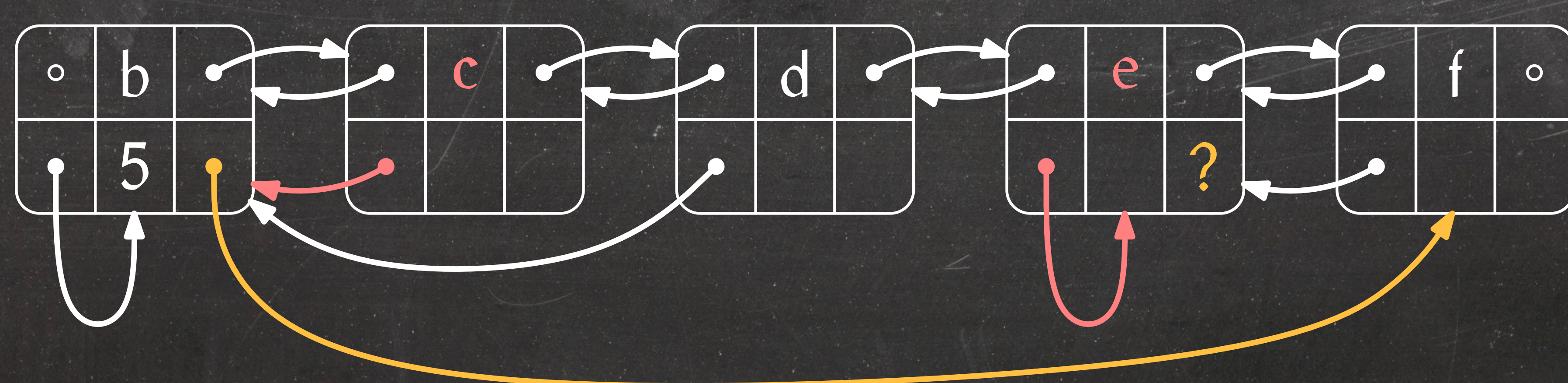

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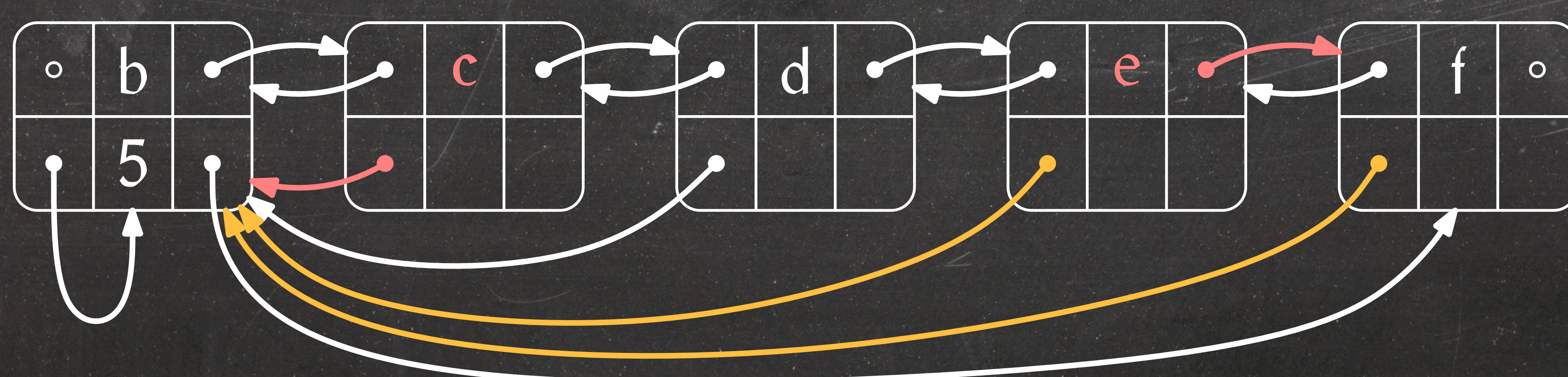

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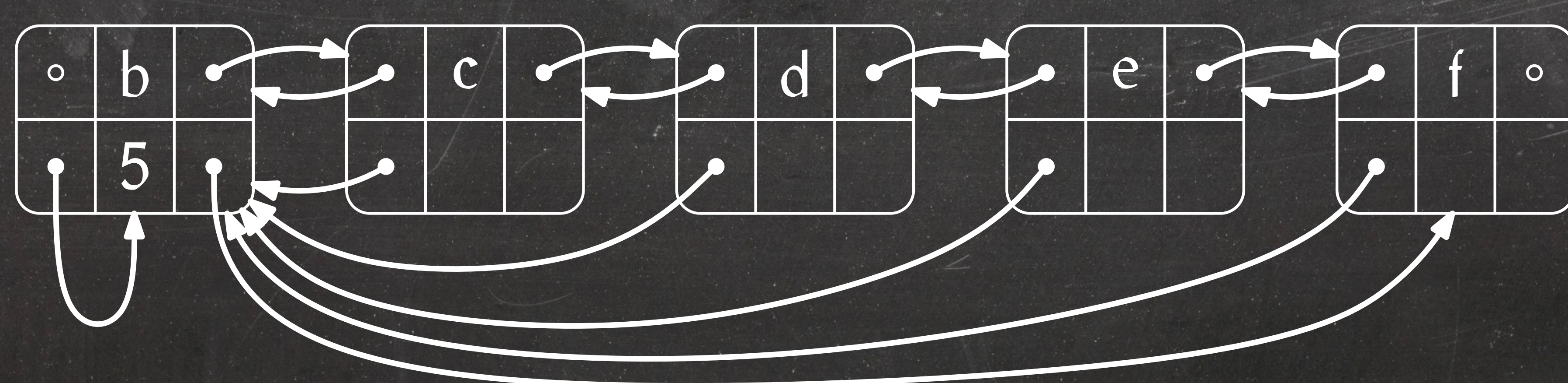

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Inductive step: $i > 0$.

- Consider the ith Union operation where x is in the smaller list.
- Let S_1 and S_2 be the two unioned lists and assume $x \in S_2$.
- Then $|S_1| \geq |S_2| \geq 2^{i-1}$.
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Corollary: $c(x) \leq \lg n$ for all $x \in S$.

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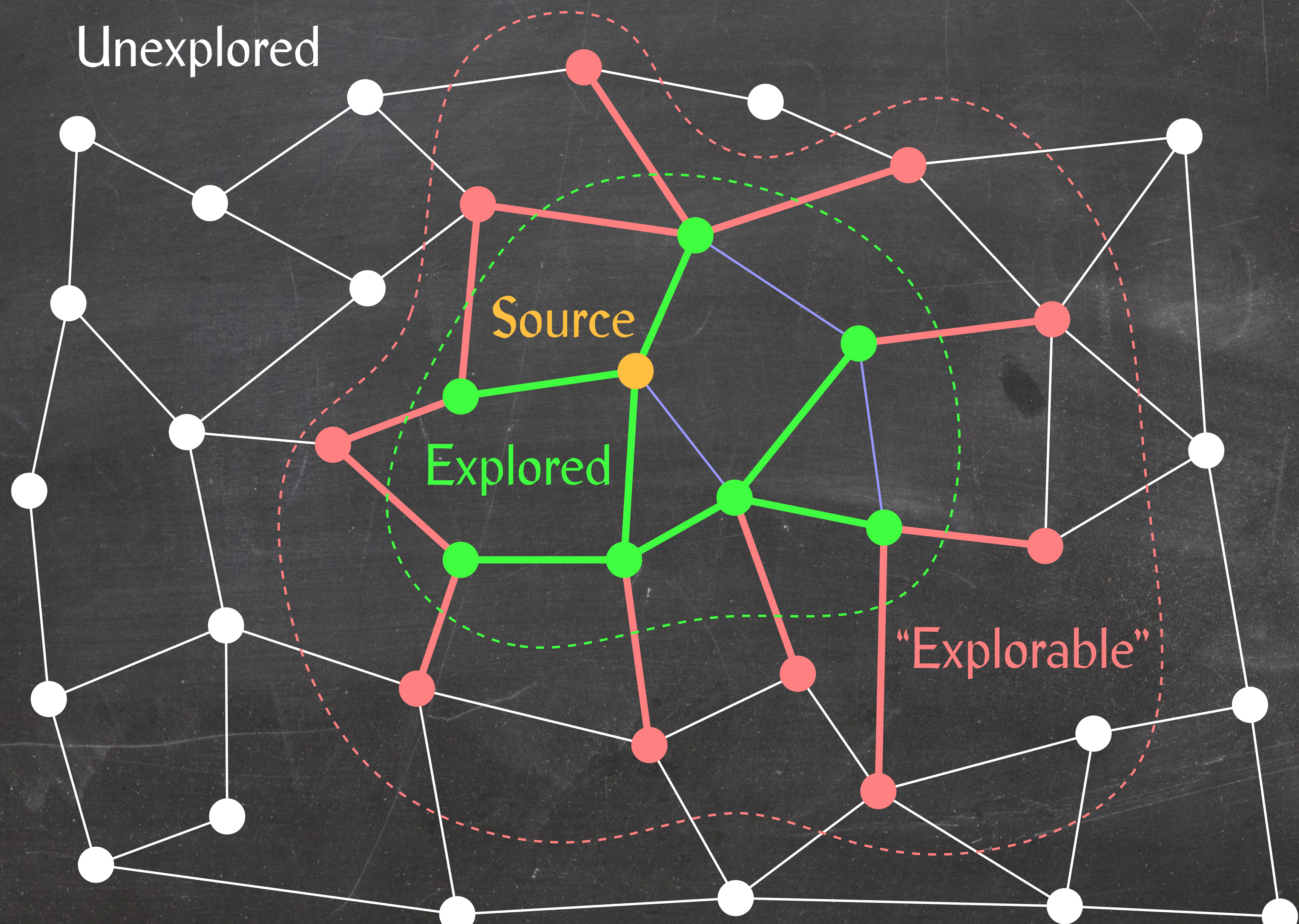
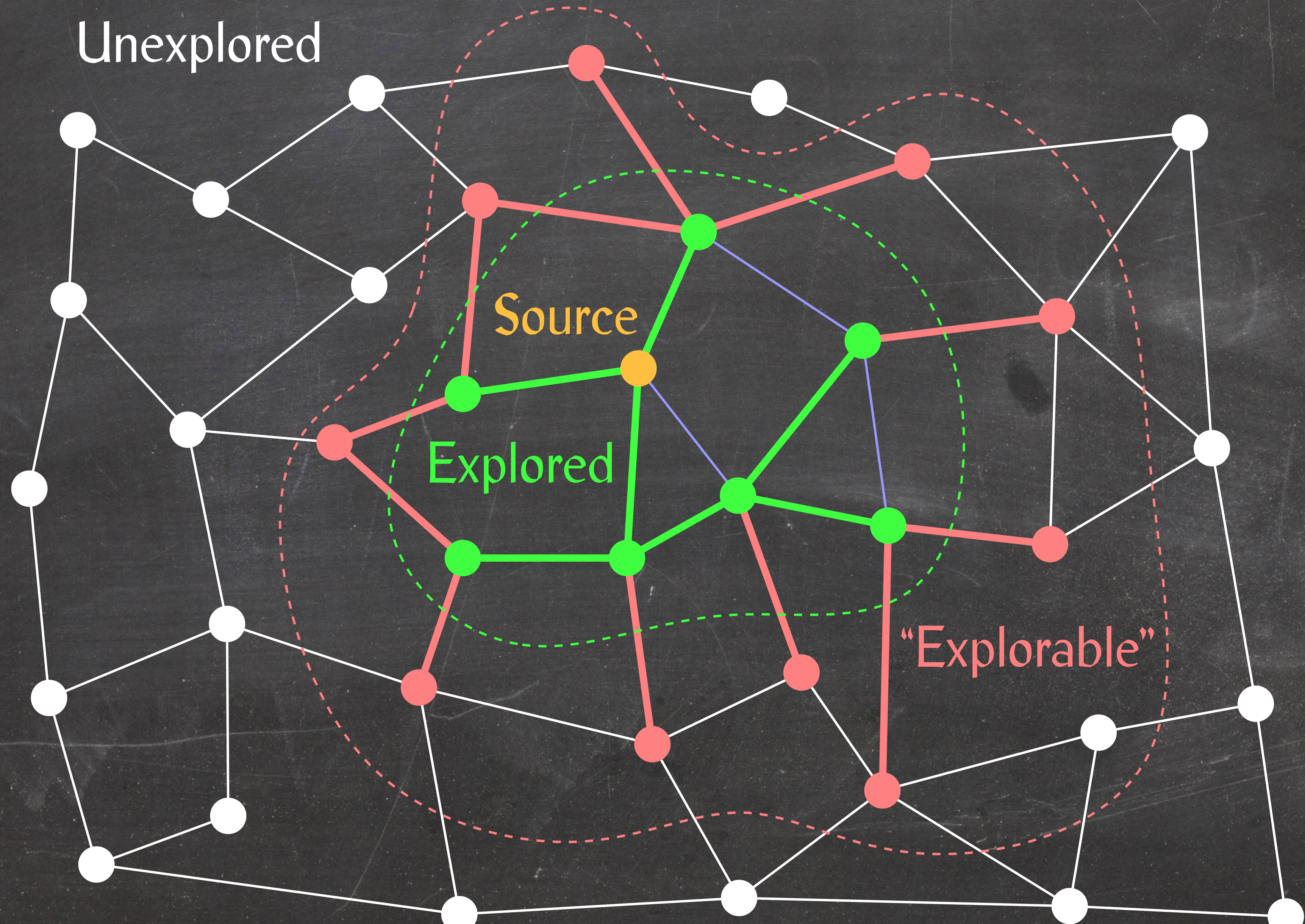
If the graph is connected, then $m \geq n - 1$, so the running time simplifies to $O(m \lg m)$.

The Cut Theorem And Graph Traversal



The Cut Theorem And Graph Traversal

If there exists an MST containing all green edges, then there exists an MST containing all green edges and the cheapest red edge.



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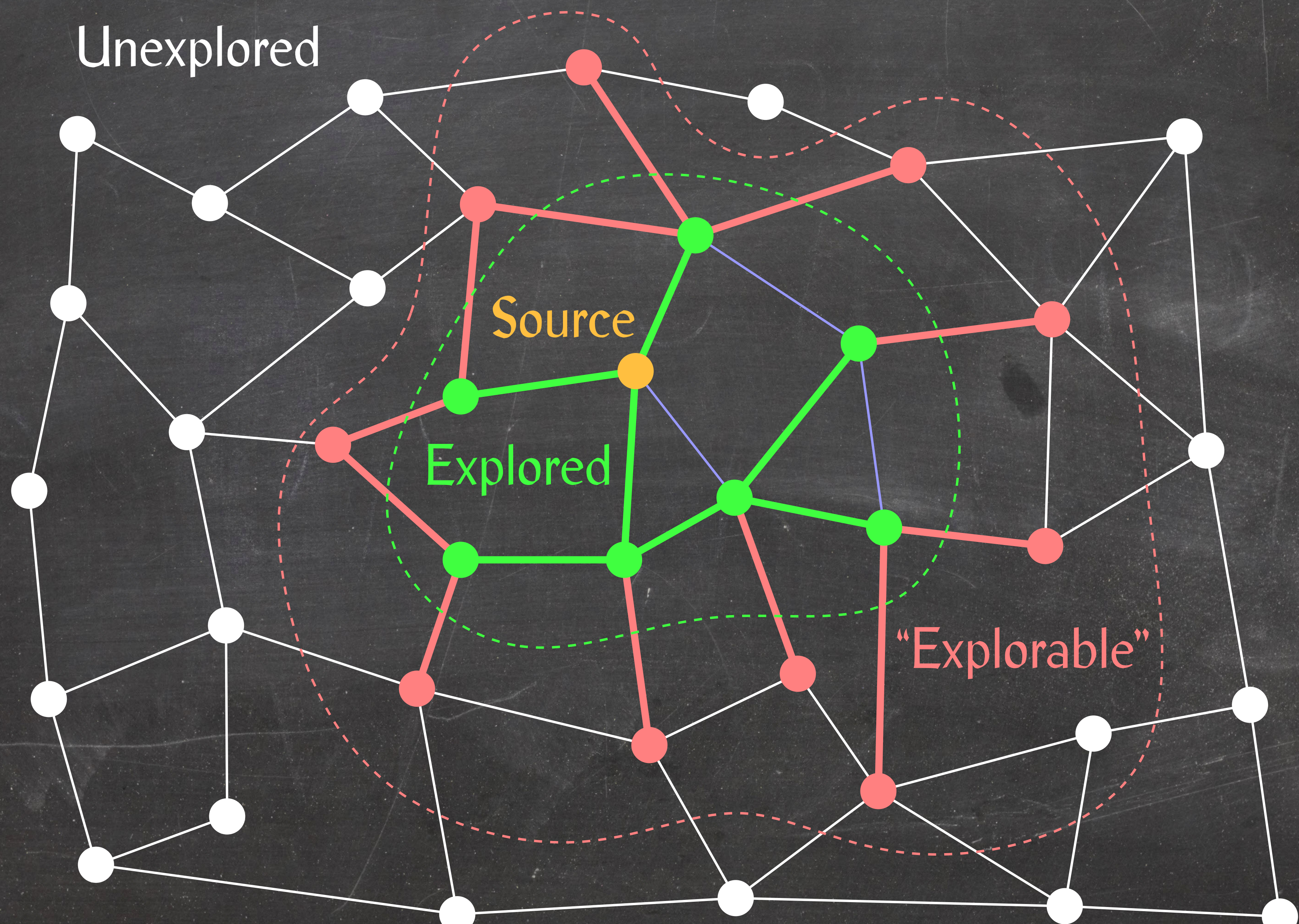

Cut: U = explored vertices, W = V \ U

Prim's Algorithm

Prim(G)

```
T = (V, ∅)
mark all vertices of G as unexplored
mark an arbitrary vertex s as explored
while not all vertices are explored
   do pick the cheapest edge e with exactly one unexplored endpoint v
      mark v as explored
      add e to T
return T
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Once T is connected, we have $T^* = T$.

The Abstract Data Type Priority Queue

Operations:

Q.insert(x, p):	Insert element x with priority p
Q.delete(x):	Delete element x
Q.findMin():	Find and return the element with minimum priority
Q.deleteMin():	Delete the element with minimum priority and return it
Q.decreaseKey(x, p):	Change the priority p_x of x to $\min(p, p_x)$

Delete and DecreaseKey assume they're given a pointer to the place in Q where x is stored.

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Example: A binary heap is a priority queue supporting all operations in $O(\lg |Q|)$ time.

Prim's Algorithm Using A Priority Queue

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Q = an empty priority queue
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Prim's Algorithm Using A Priority Queue

Invariant: Q contains all edges with exactly one unexplored endpoint.

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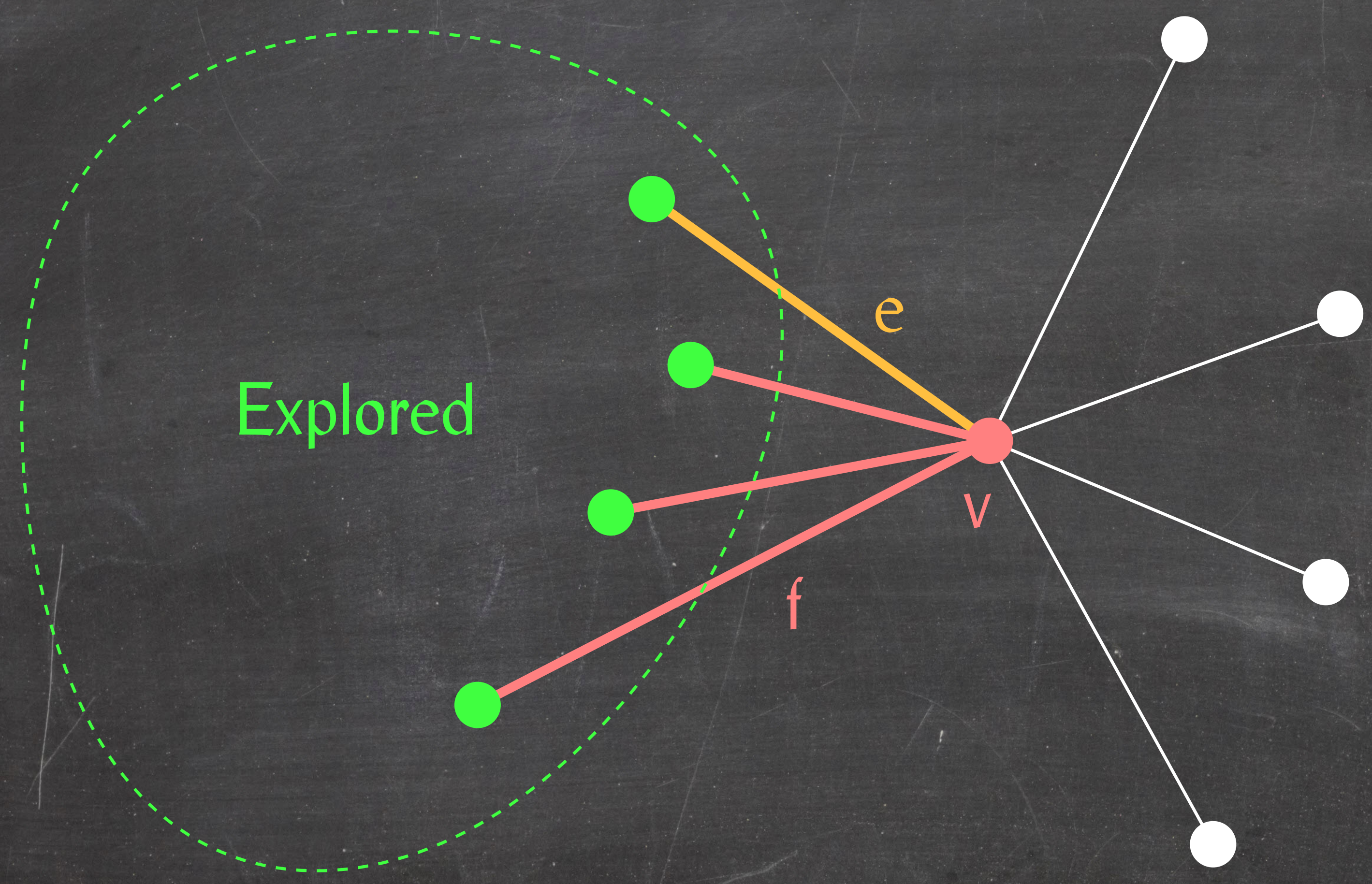
$\Rightarrow$ Every edge is removed from Q once.

$\Rightarrow$ 2m priority queue operations.

Most Edges In Q Are Useless

Observation: Of all the edges connecting an unexplored vertex to explored vertices only the cheapest has a chance of being added to the MST.



$w(e) < w(f)$

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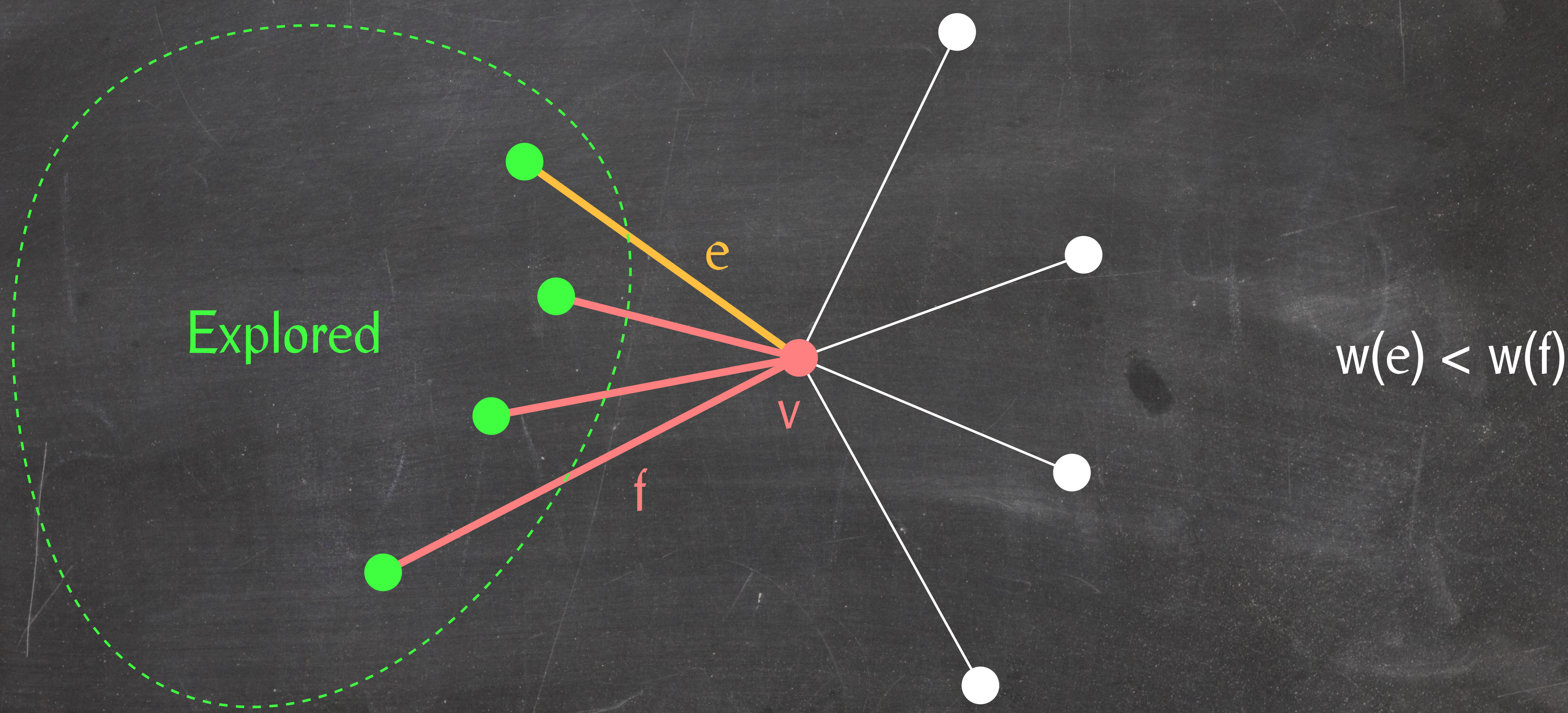

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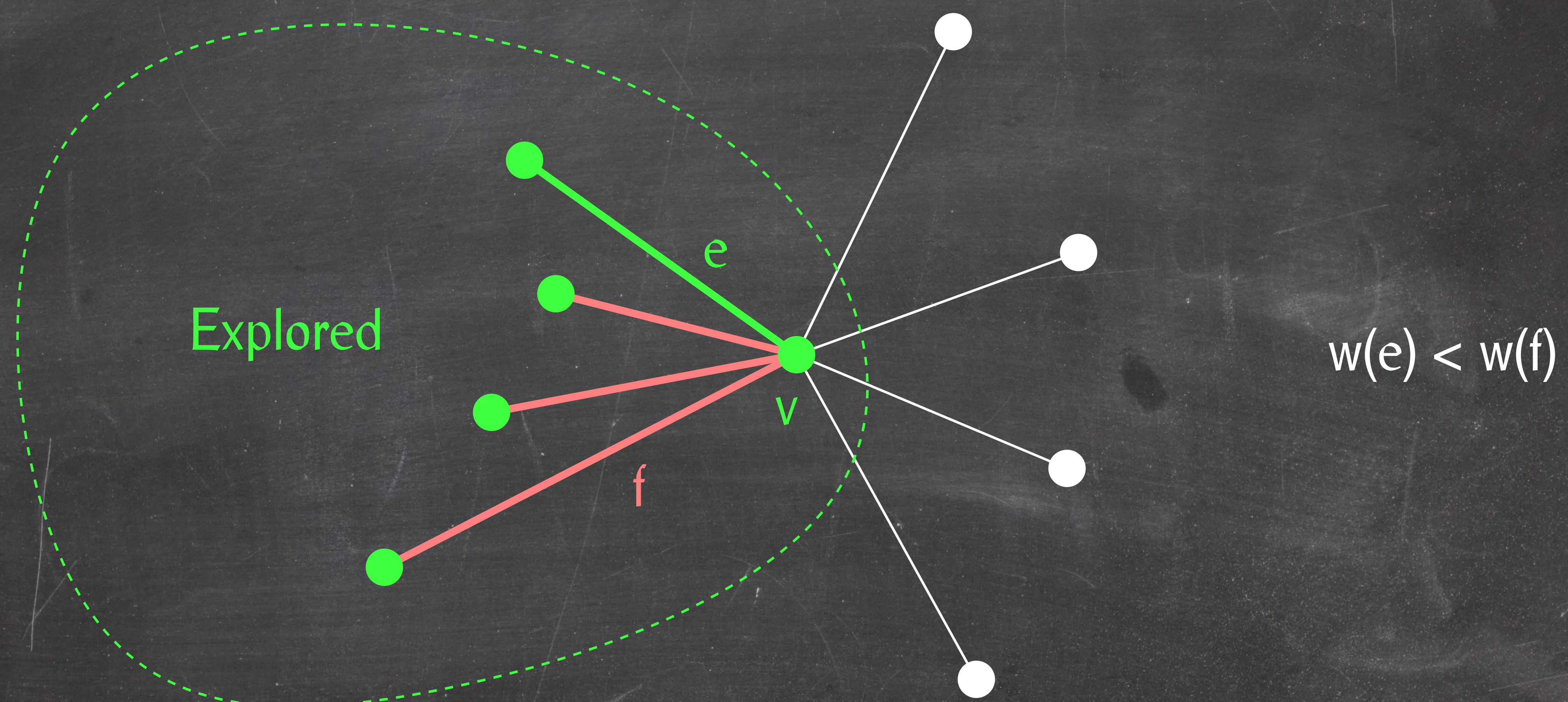

While v is unexplored, all red and orange edges are in Q, so none of the red edges can be the first edge to be removed from Q.

After marking v as explored, both endpoints of red edges are explored, so they cannot be added to T either.

A Faster Version Of Prim's Algorithm

Prim(G)

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T = (V, ∅)
mark every vertex of G as unexplored
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for every edge (s, v) incident to s
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The Thin Heap is a priority queue which supports

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Prim's algorithm performs $n + m$ priority queue operations, n of which are DeleteMin operations.

Lemma: Prim's algorithm takes $O(n \lg n + m)$ time.

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A Thin Heap is built from Thin Trees. Thin Trees are defined inductively.

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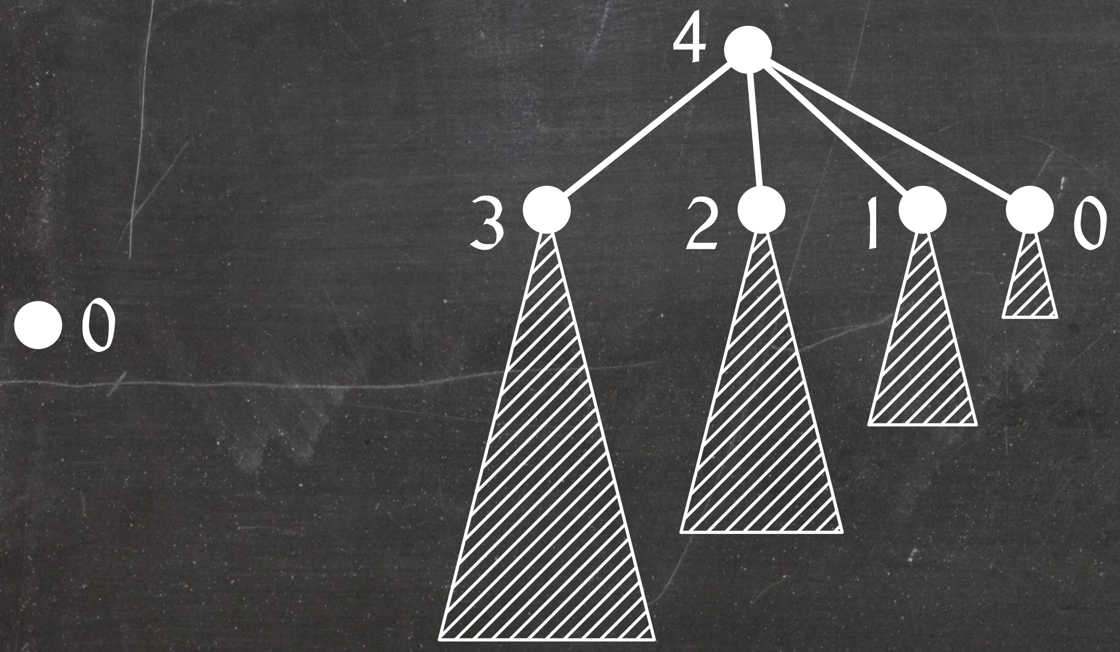

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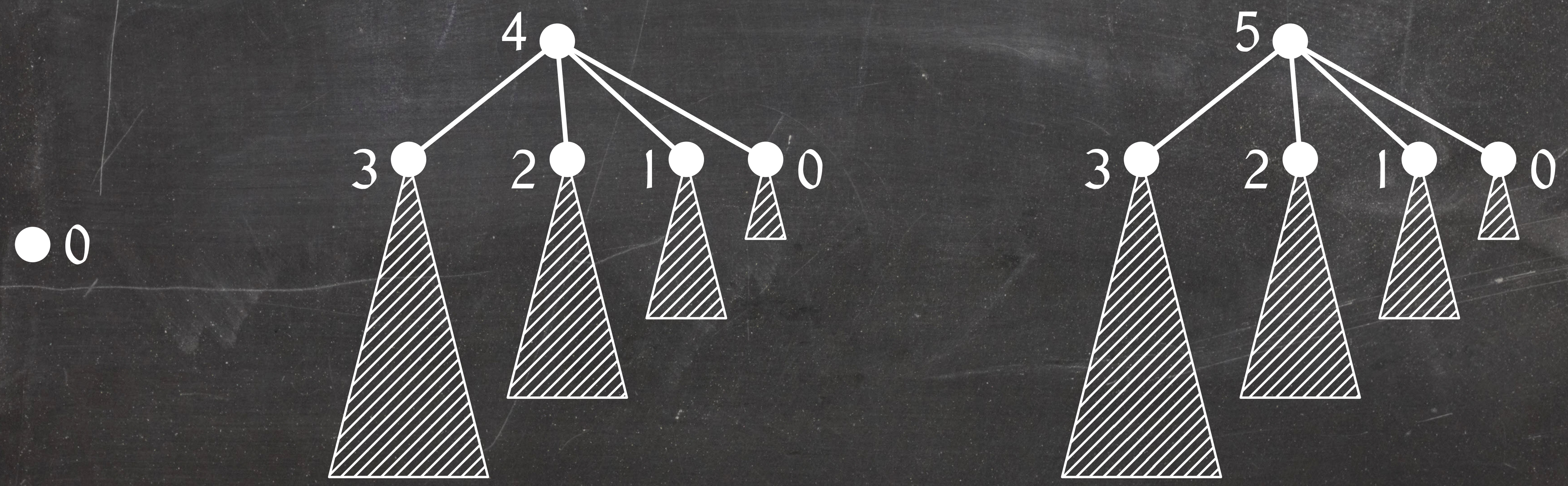

Rank 0 Rank 4, thick Rank 5, thin

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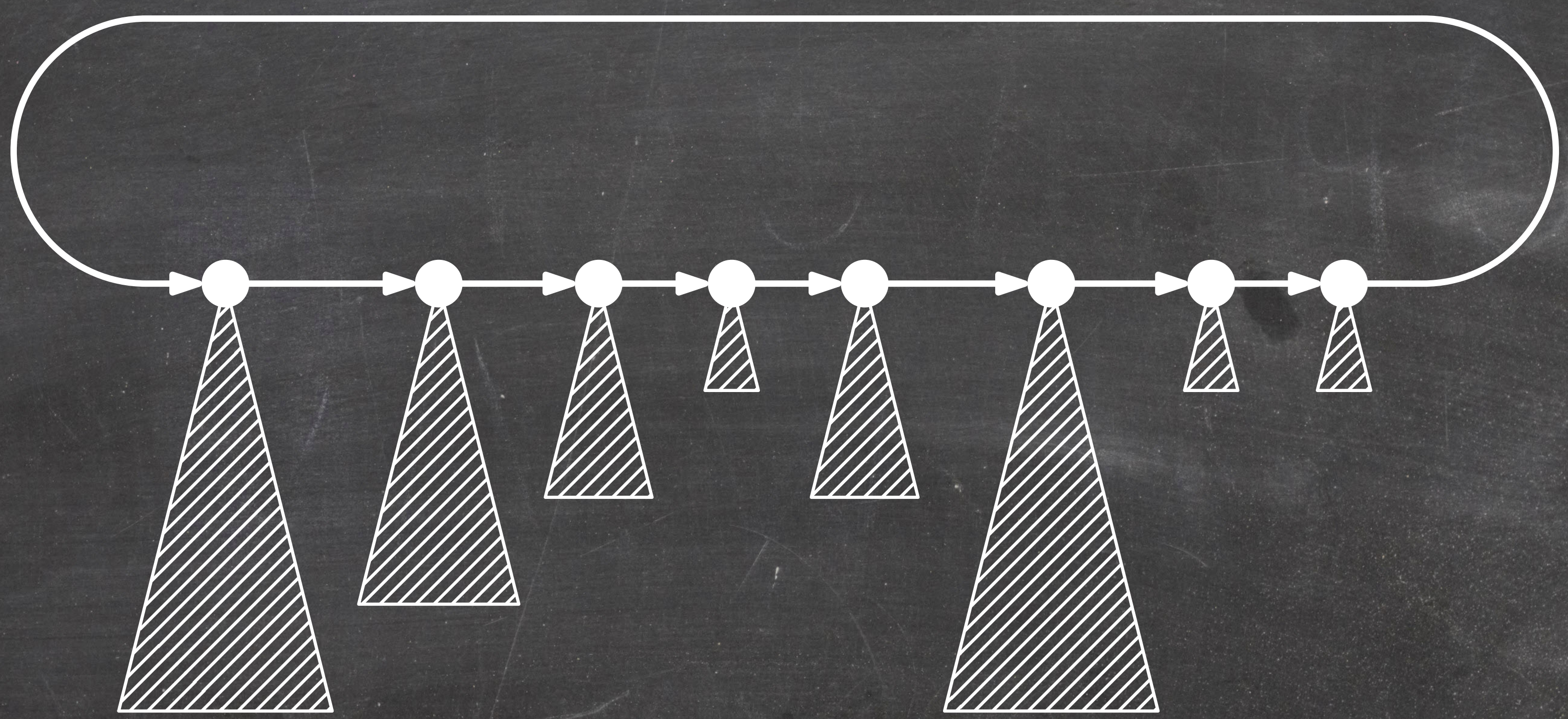
A Thin Heap is a circular list of heap-ordered Thin Trees.



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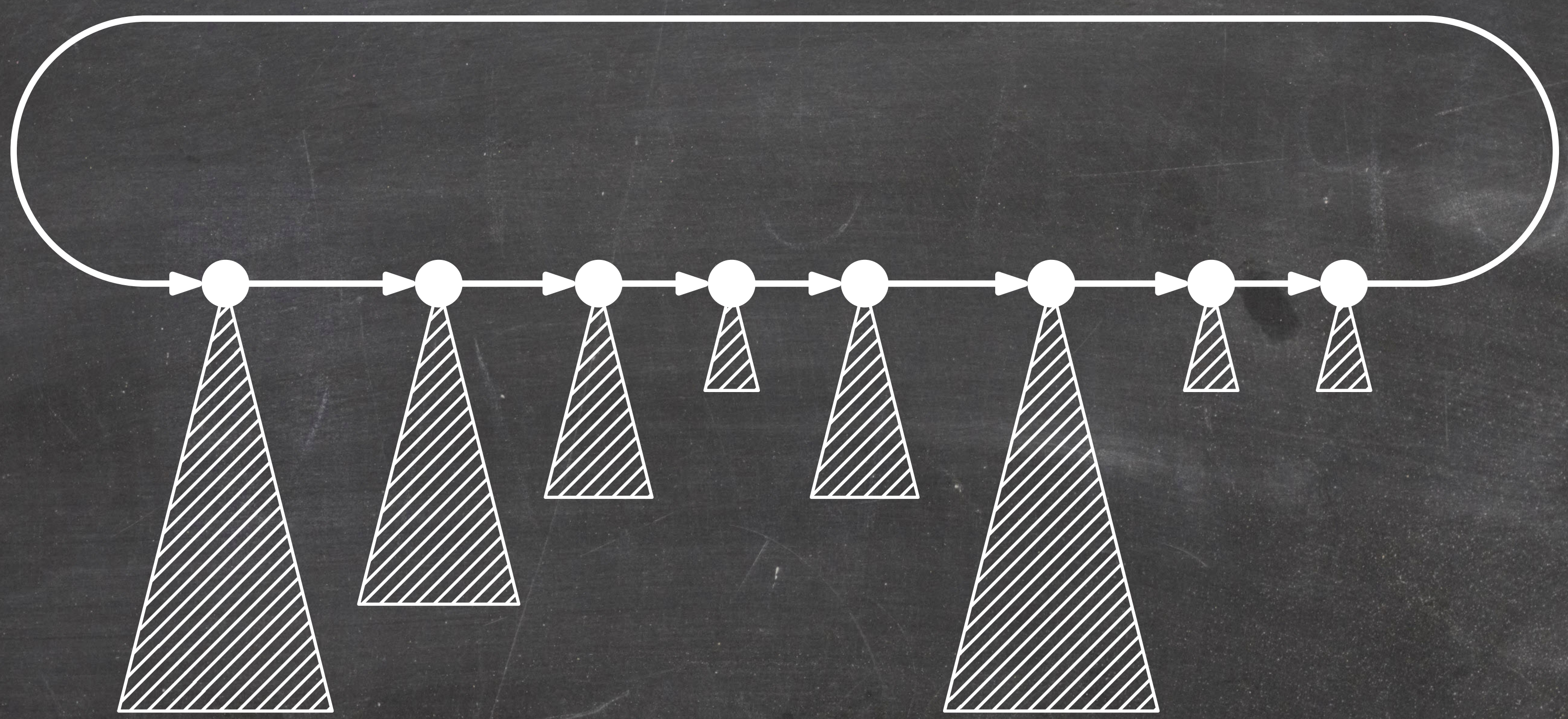
Heap-ordered: Every node stores an element no less than the element stored at its parent.



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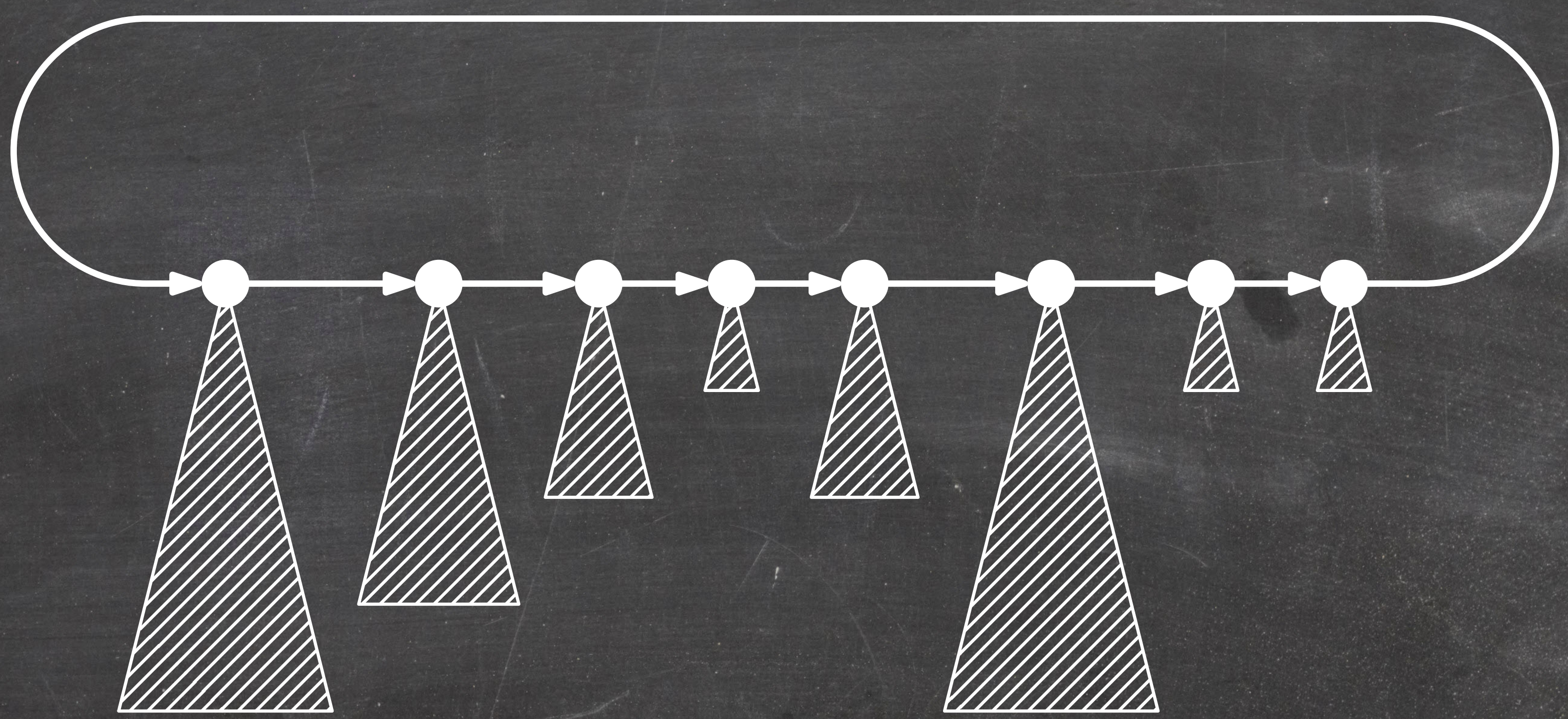

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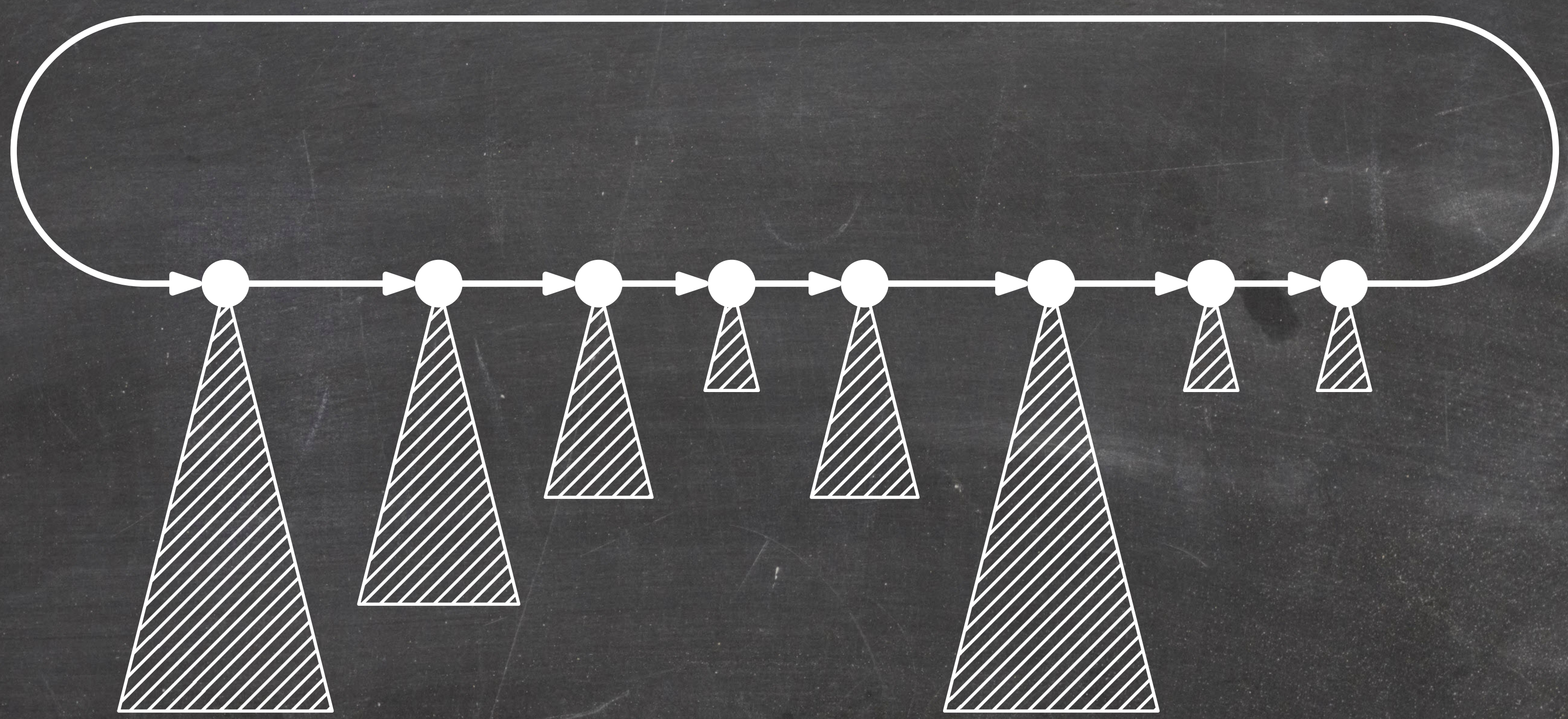

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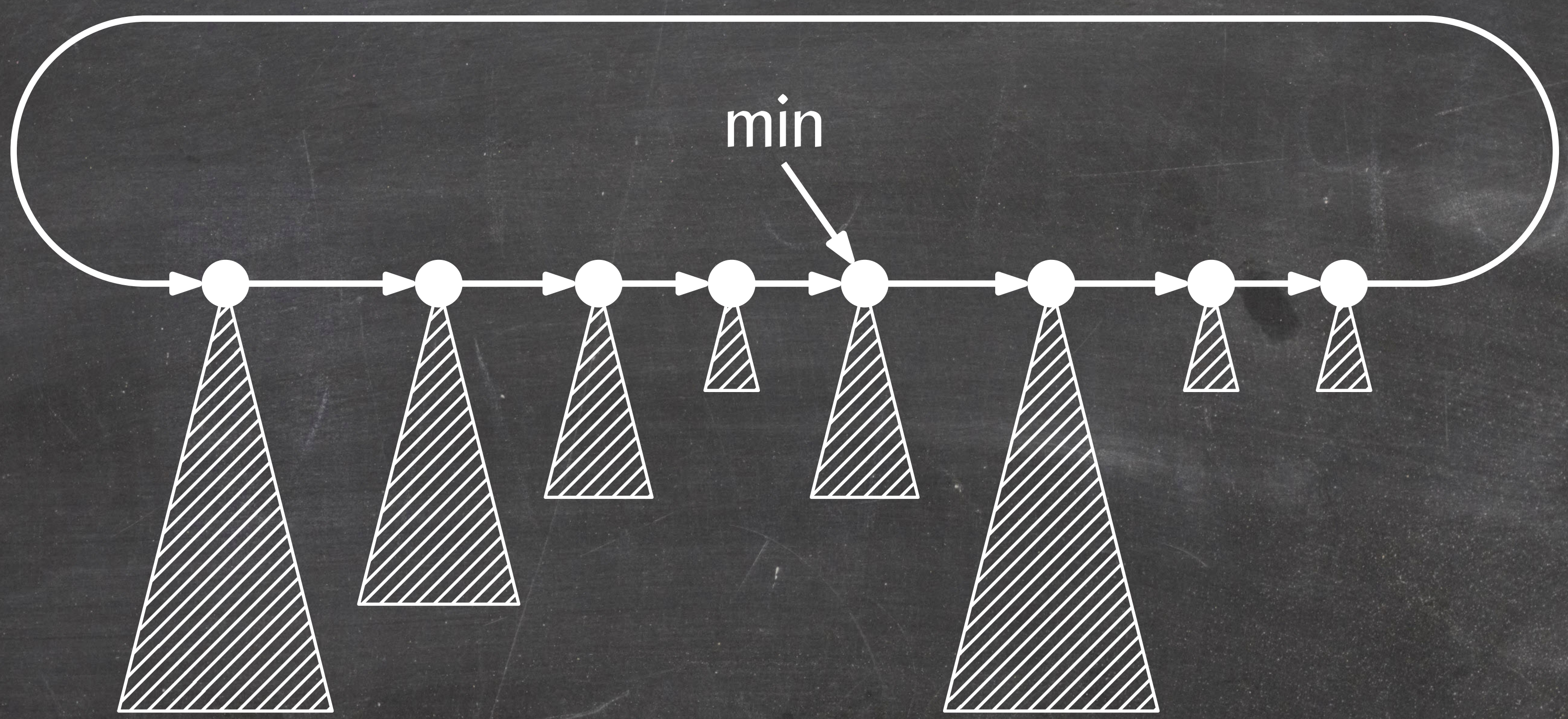


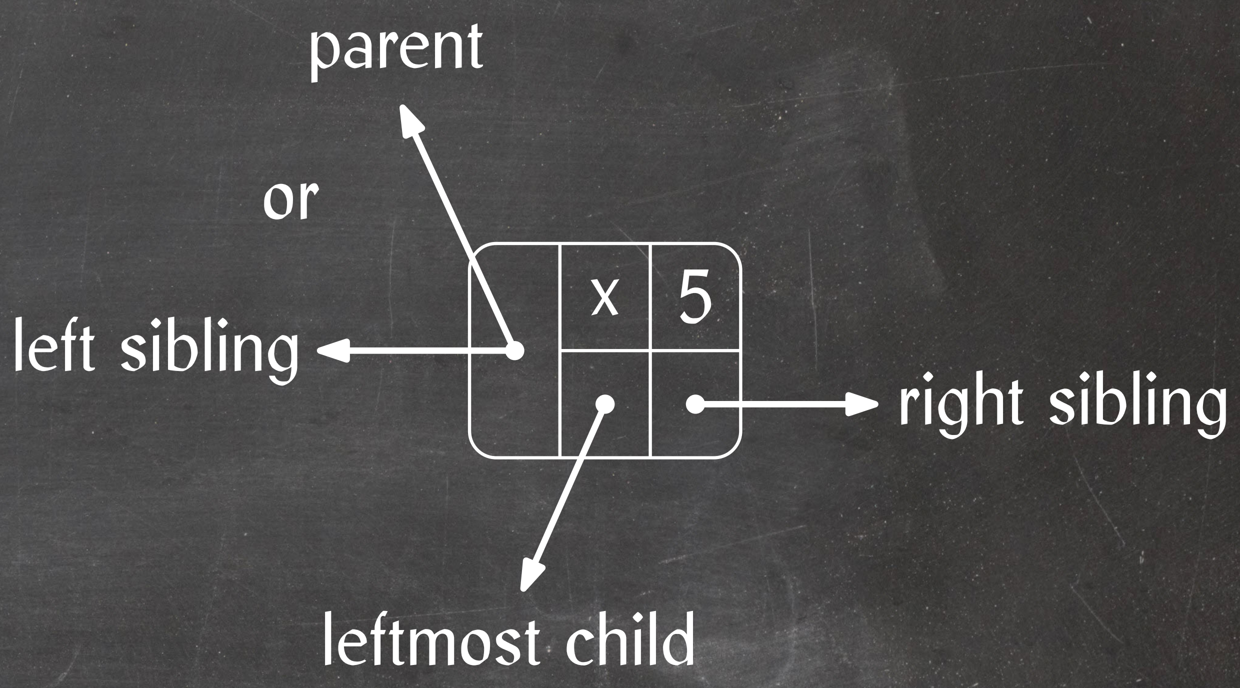
All roots are thick.

The minimum element is stored at one of the roots.

We store a pointer to this root.

Node Representation

- Element stored at the node
- Rank
- Pointer to leftmost child
- Pointer to right sibling
- Pointer to left sibling or parent



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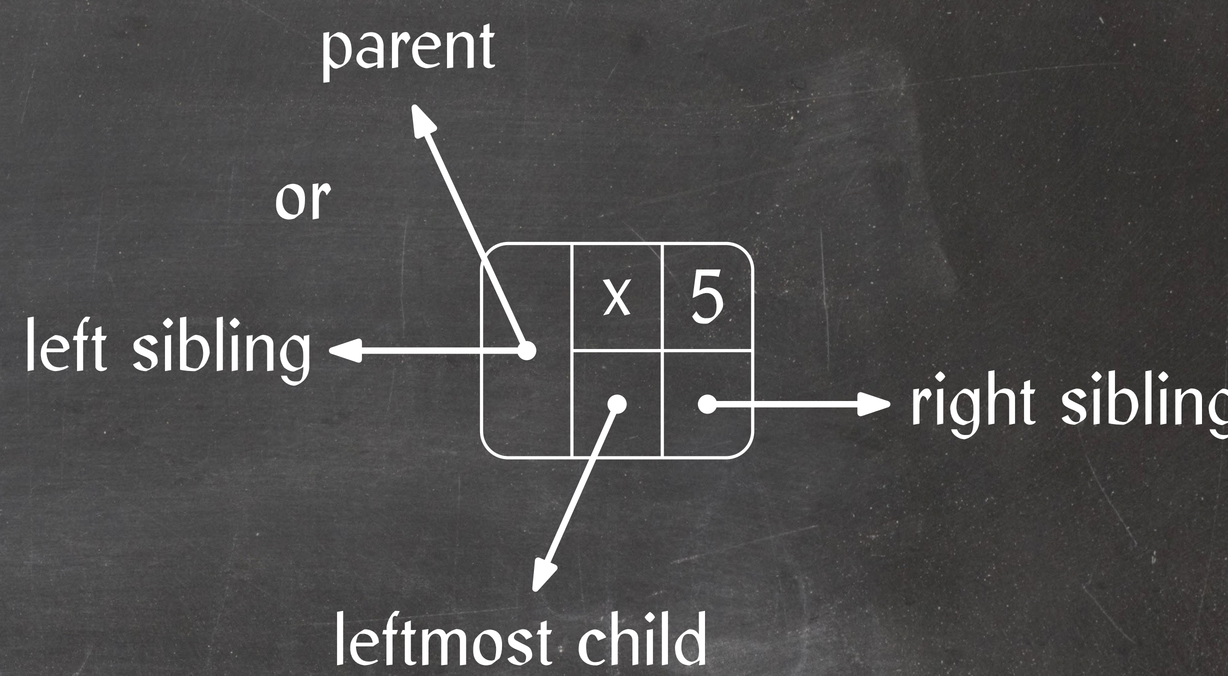
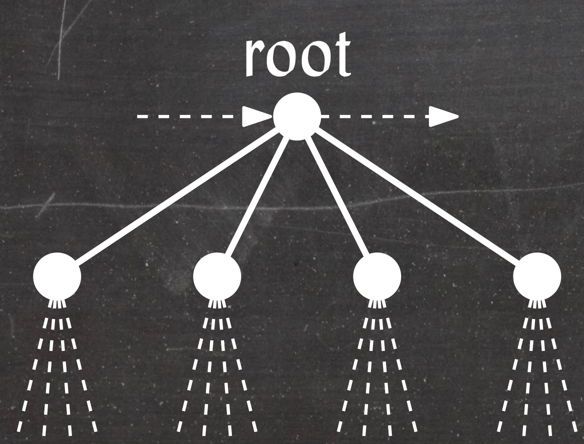
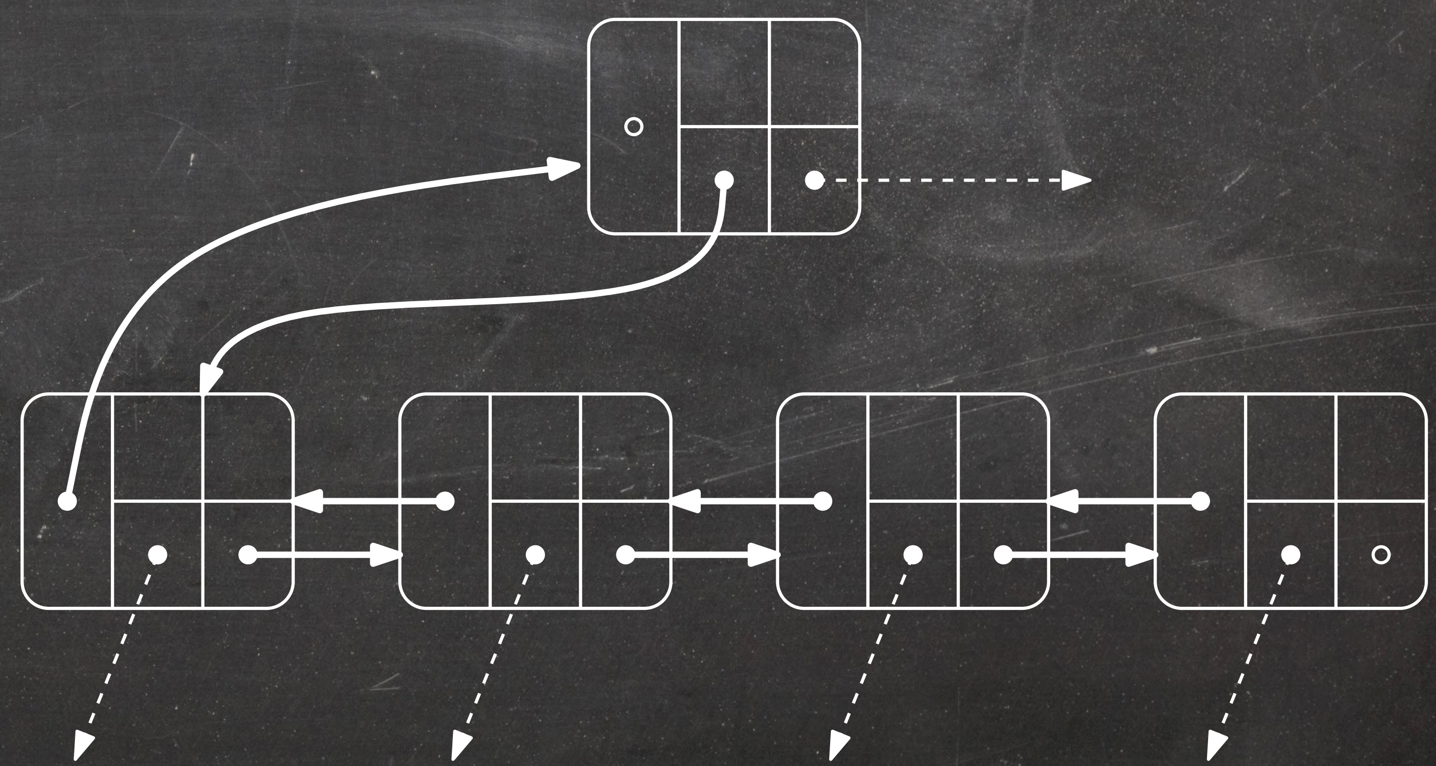
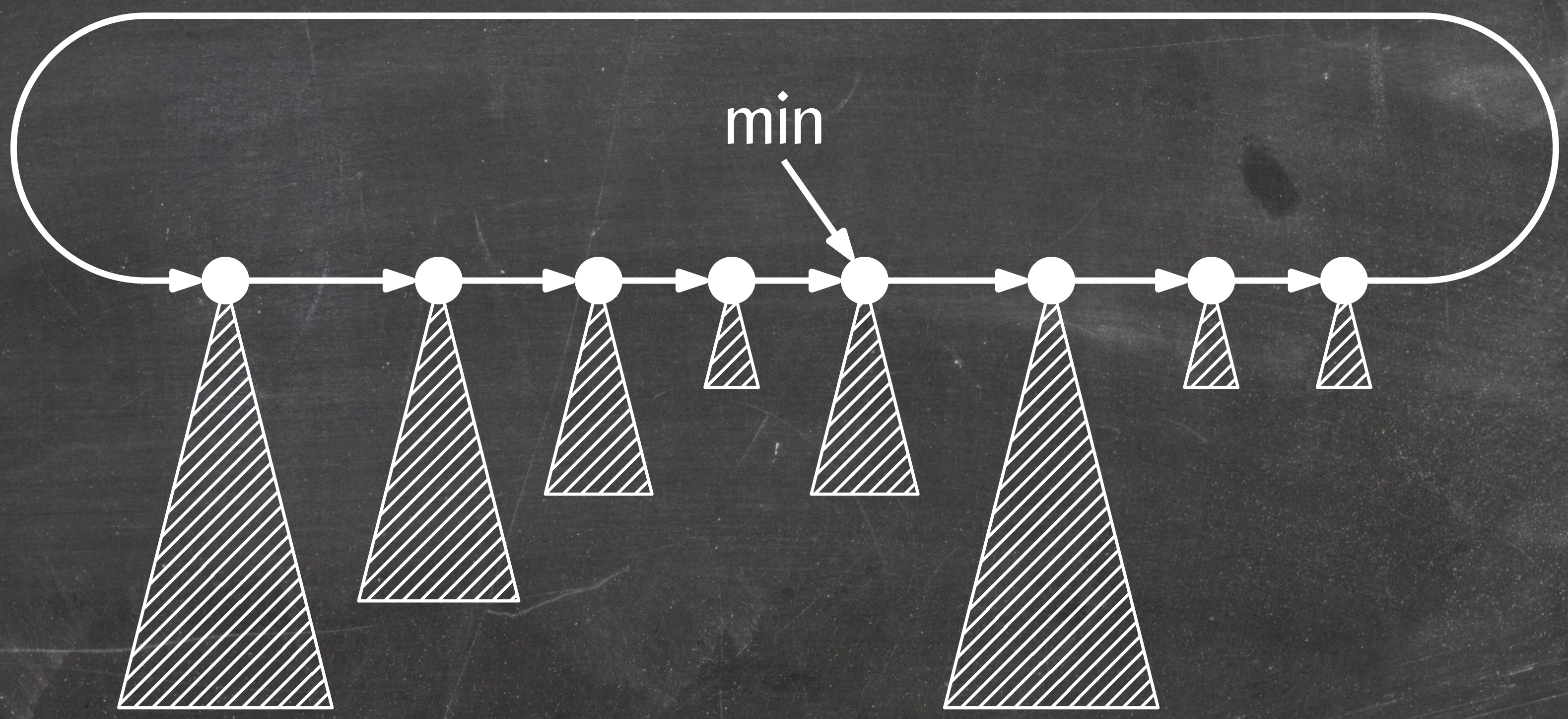






FindMin

... is easy:



Delete

... can be implemented using DecreaseKey and DeleteMin:

Q.delete(x)

Q.decreaseKey(x, $-\infty$)
Q.deleteMin()

Insert

Insert

If Q is empty:

Insert

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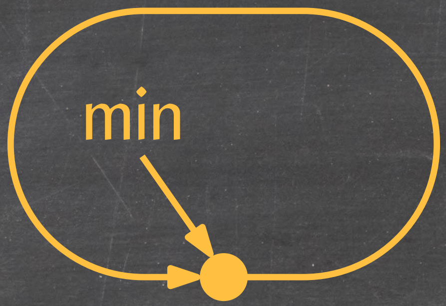

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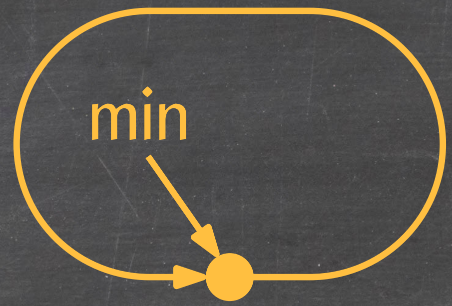

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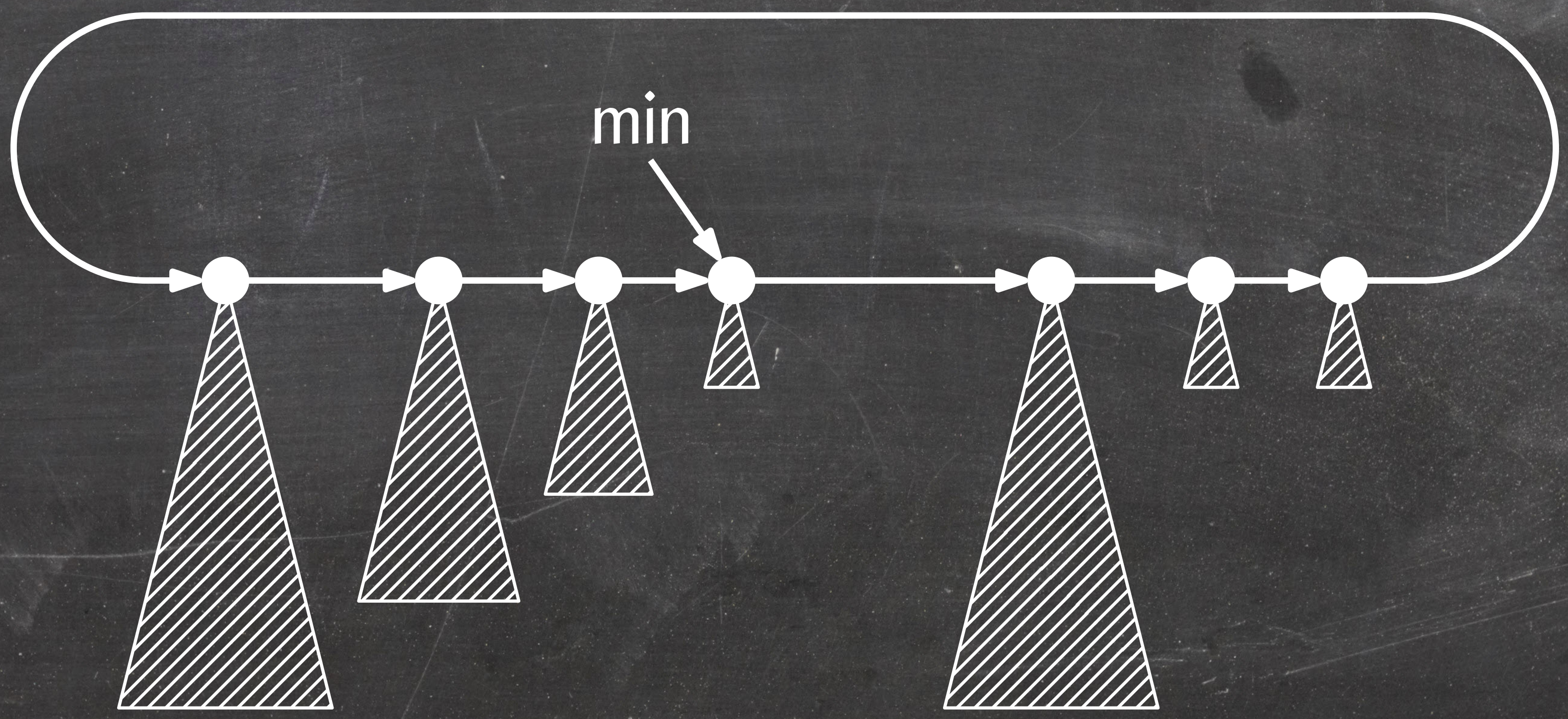

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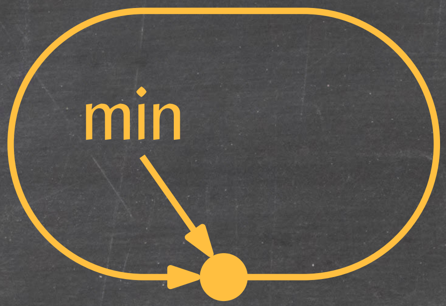

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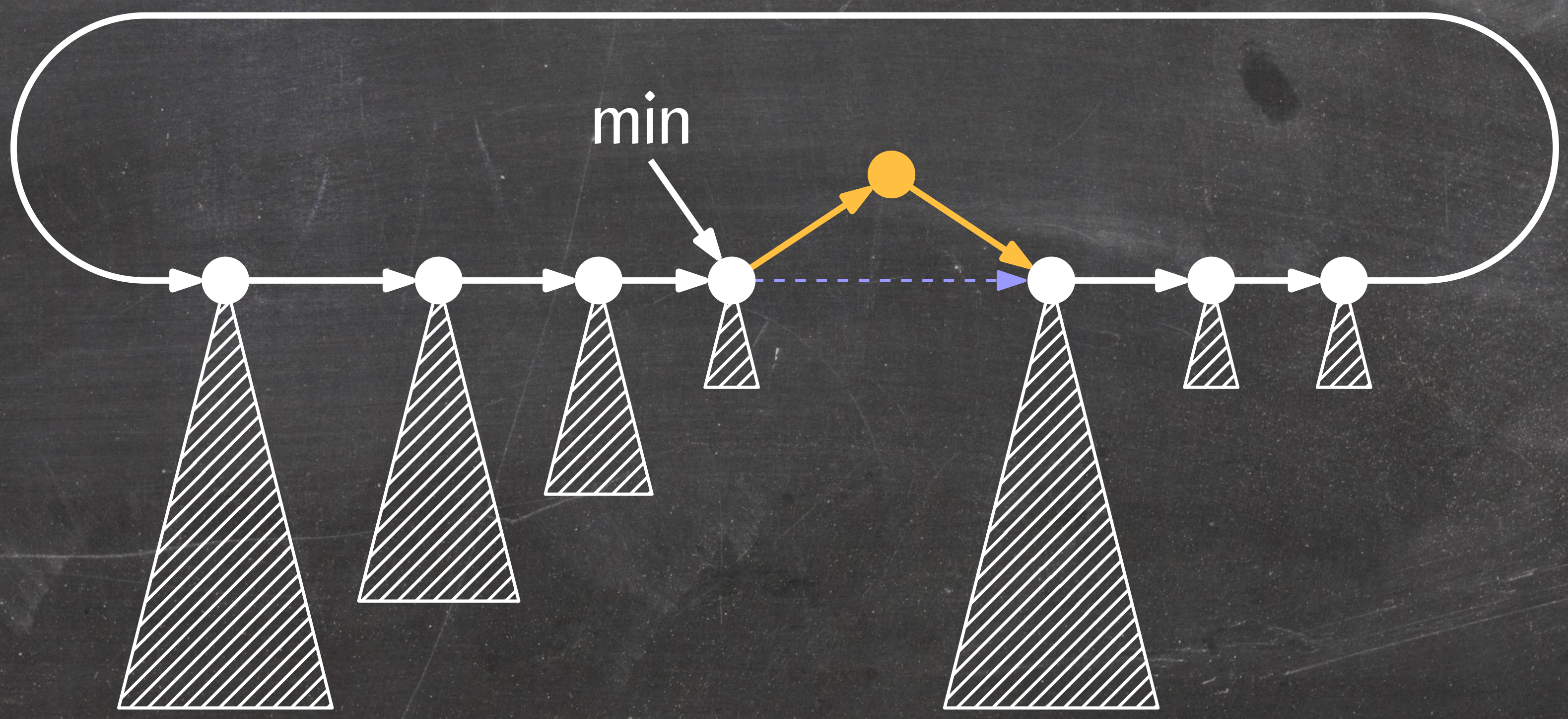

- Insert new element between min and its successor.

Insert

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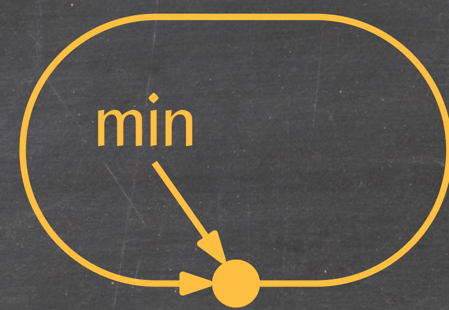

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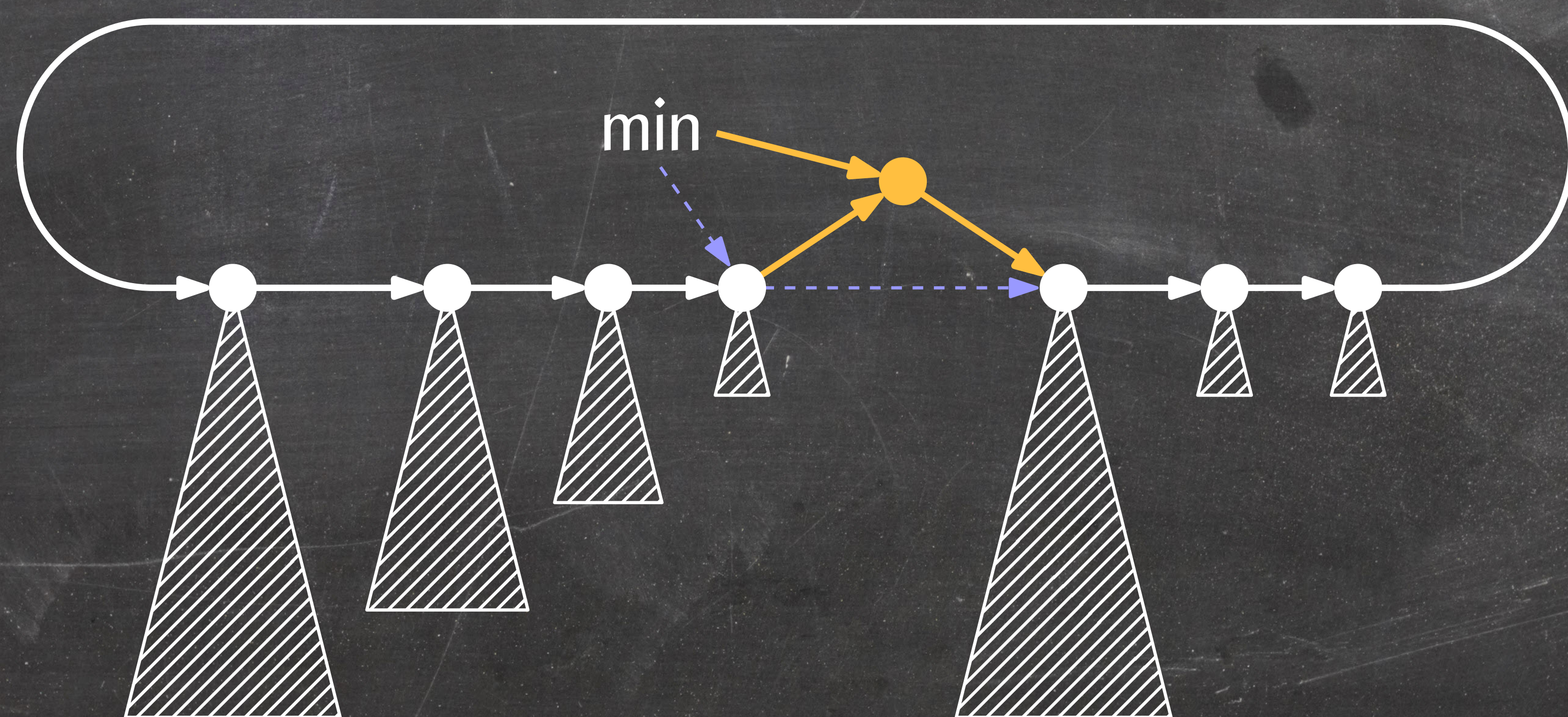
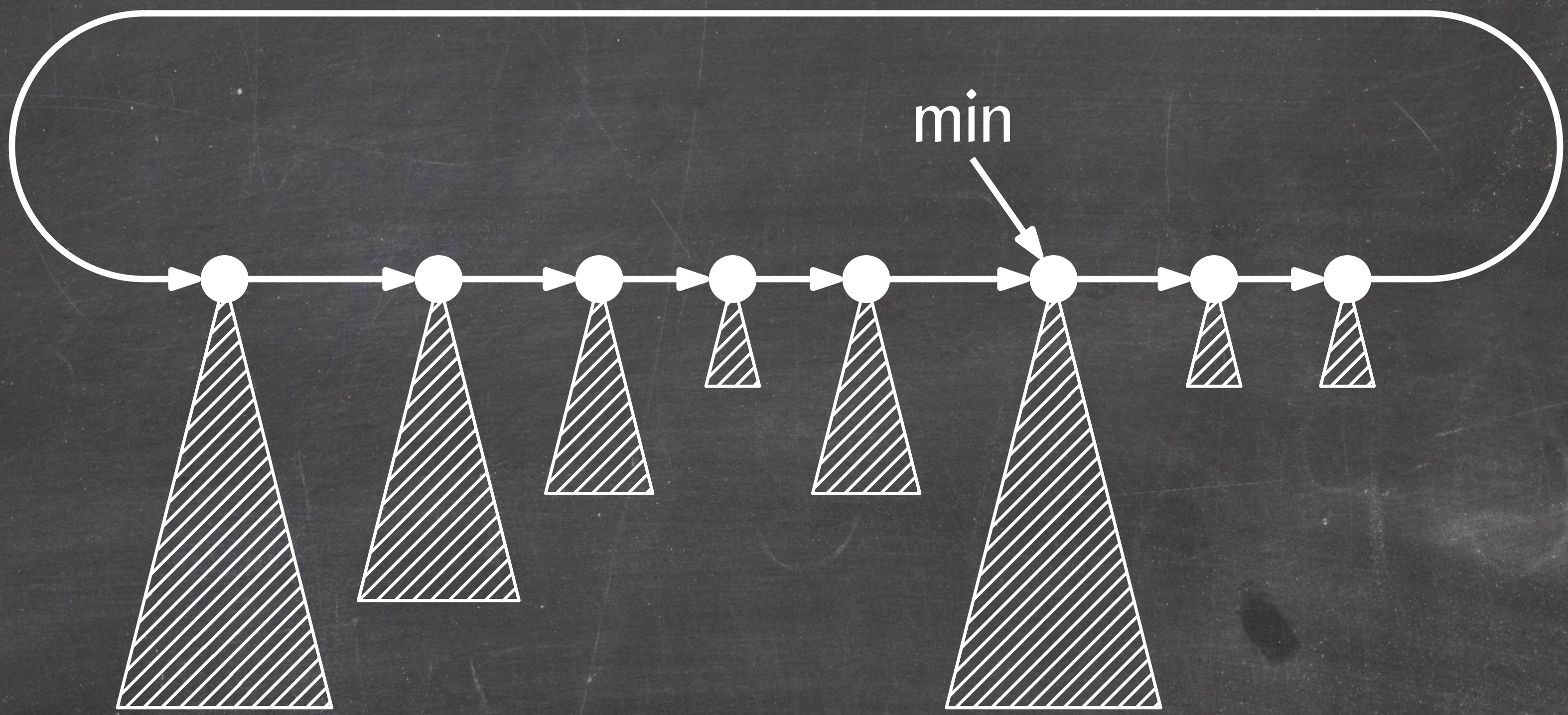


- Insert new element between min and its successor.
- Update min if the new element is the new smallest element.

DeleteMin



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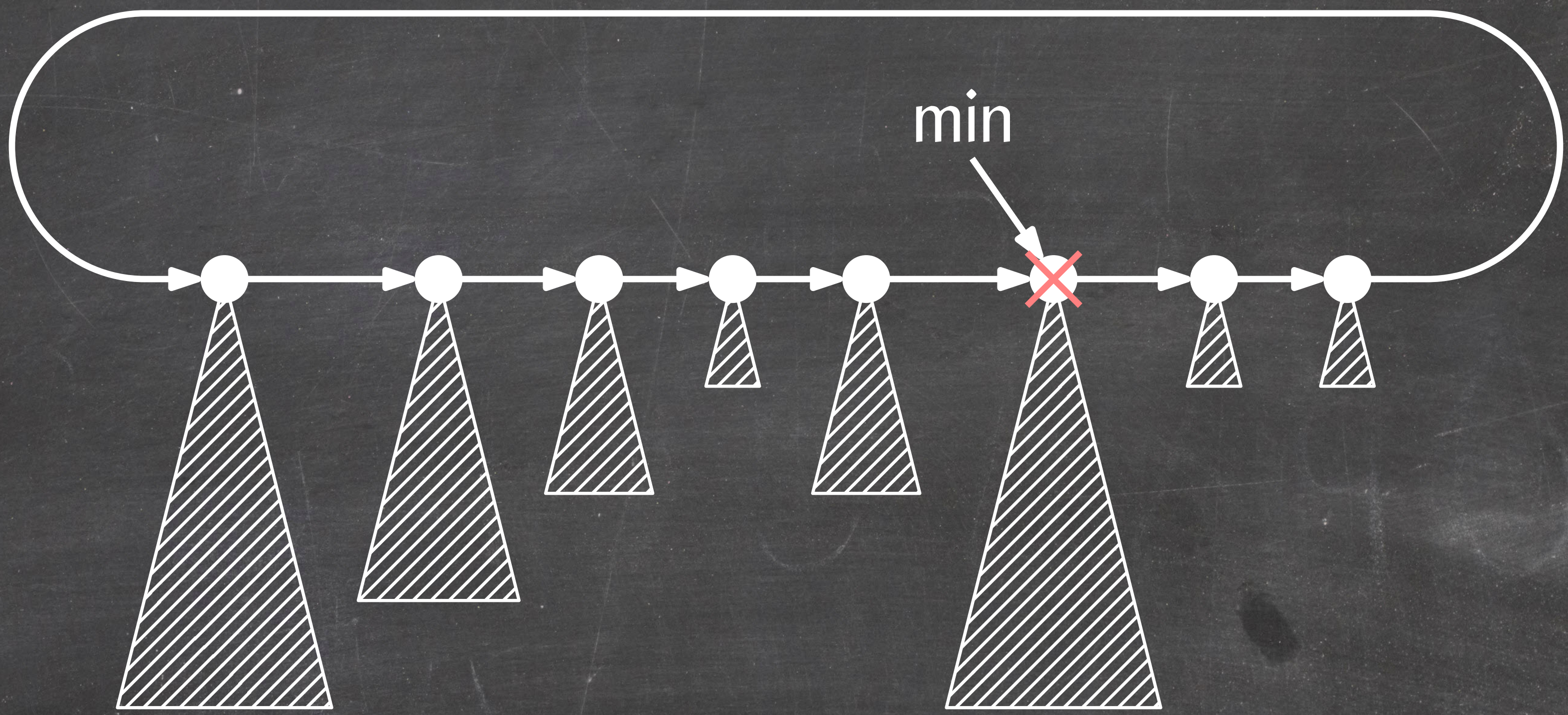

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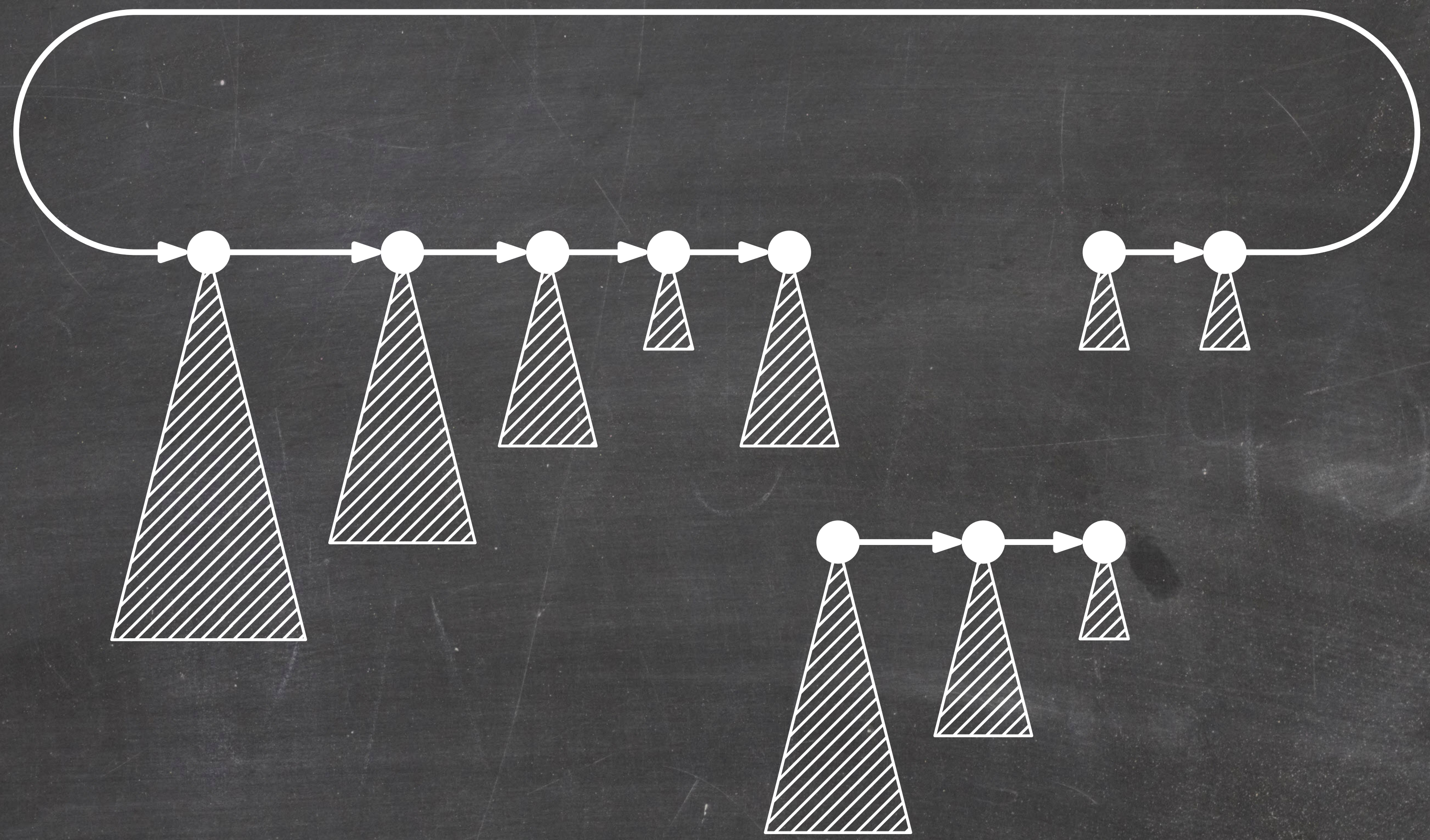

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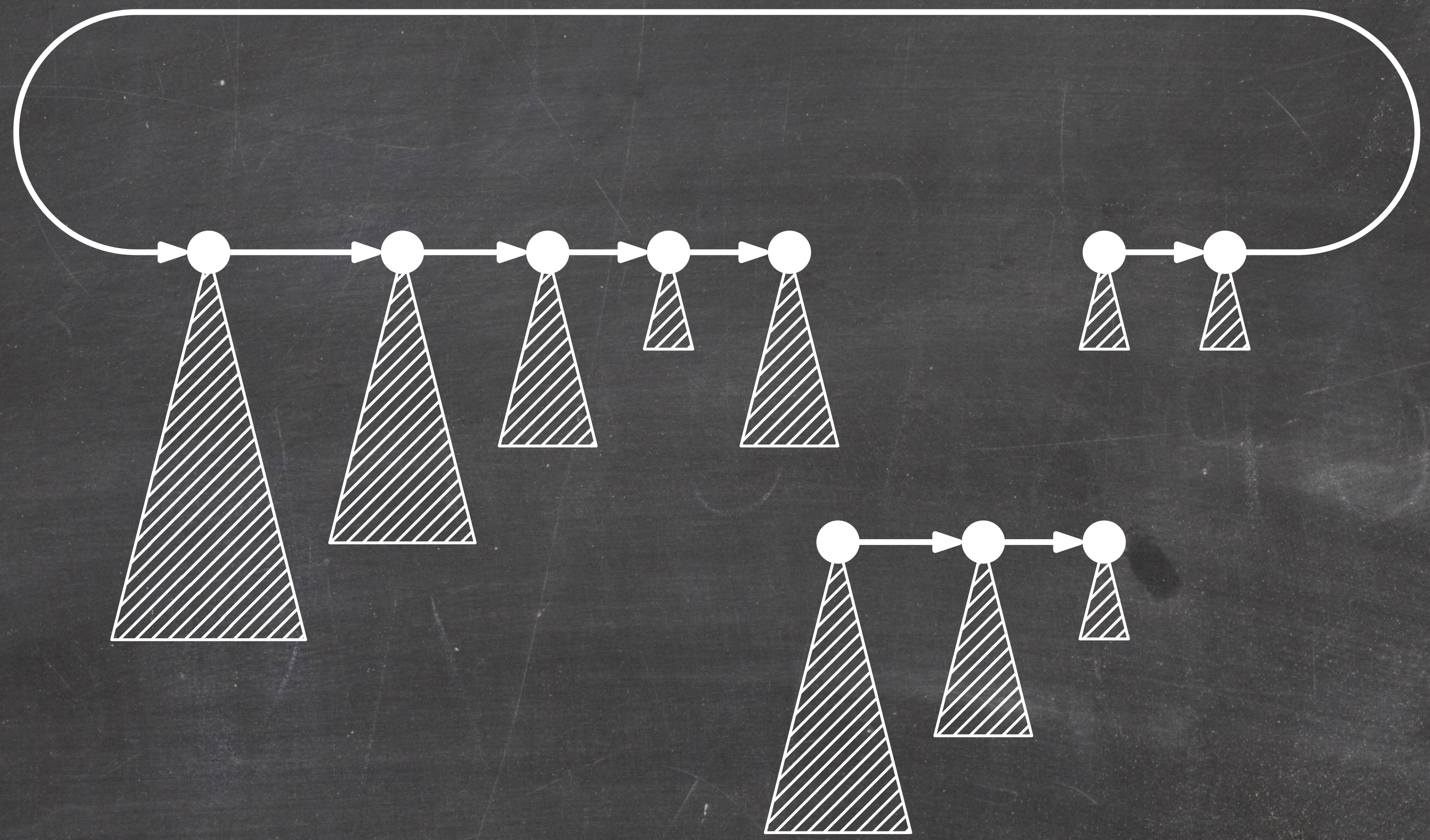

What do we do with the children?

How do we find the new minimum?

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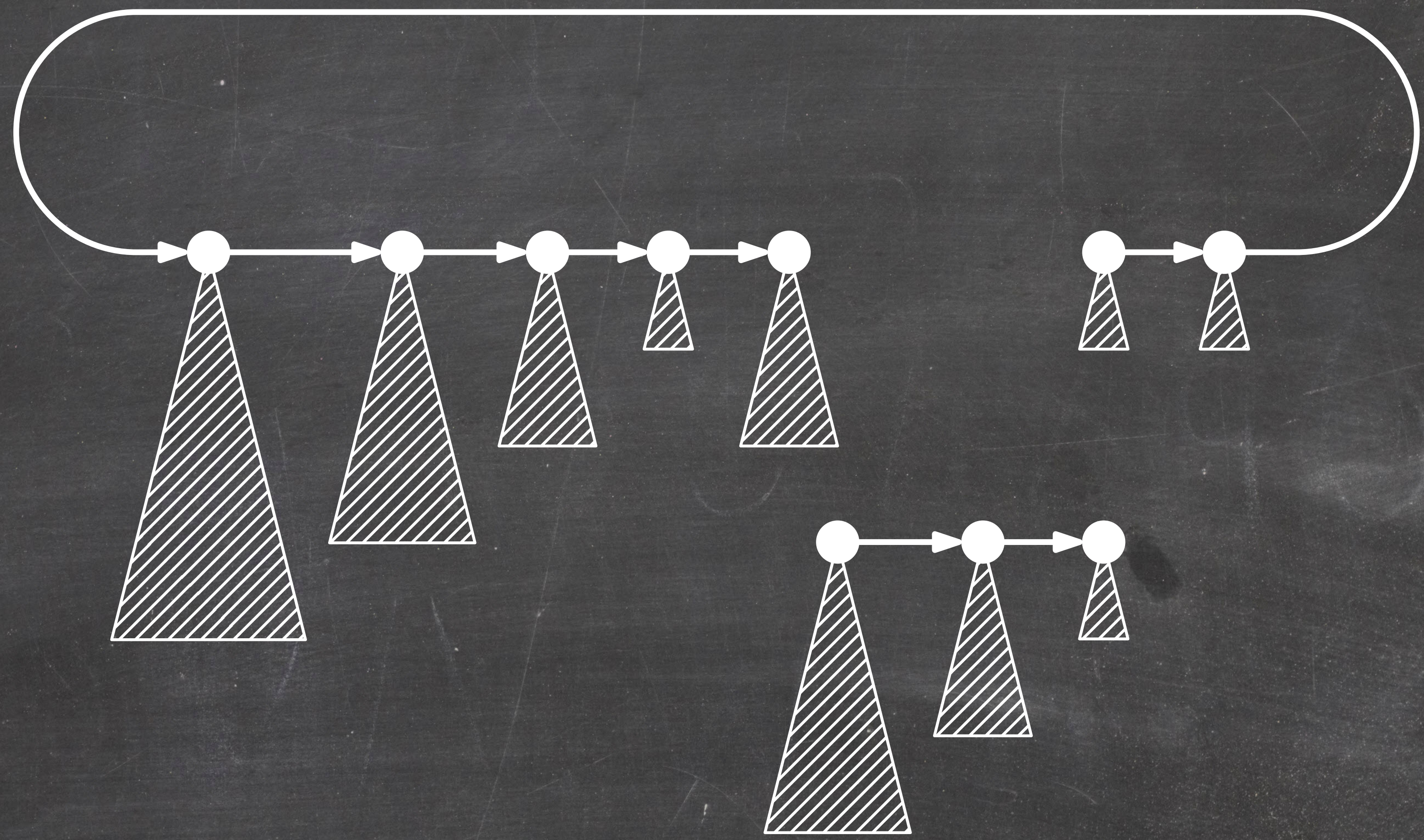

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- Could be one of the children.
- Could be one of the other roots.

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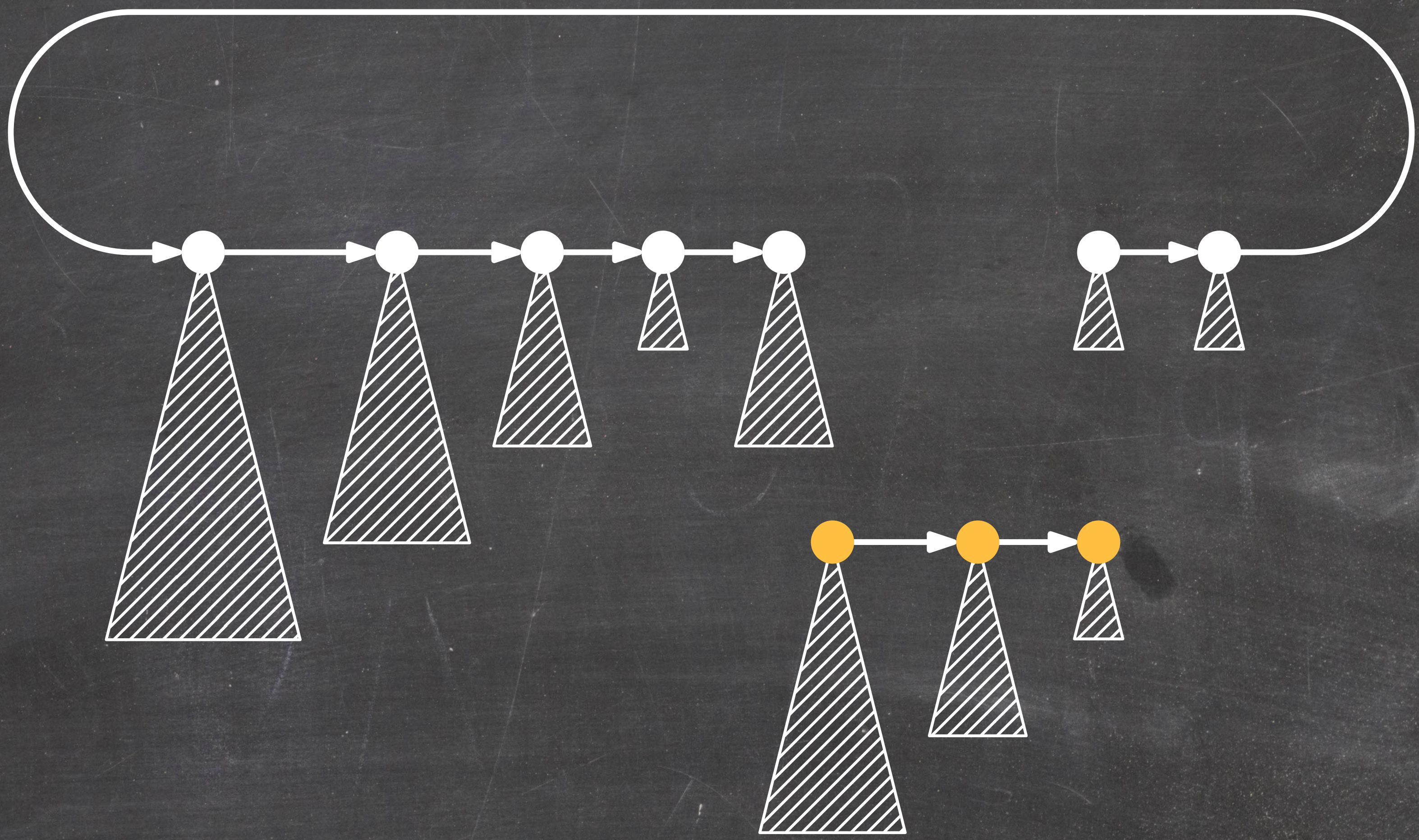

- Ensure all former children of min are thick. How?

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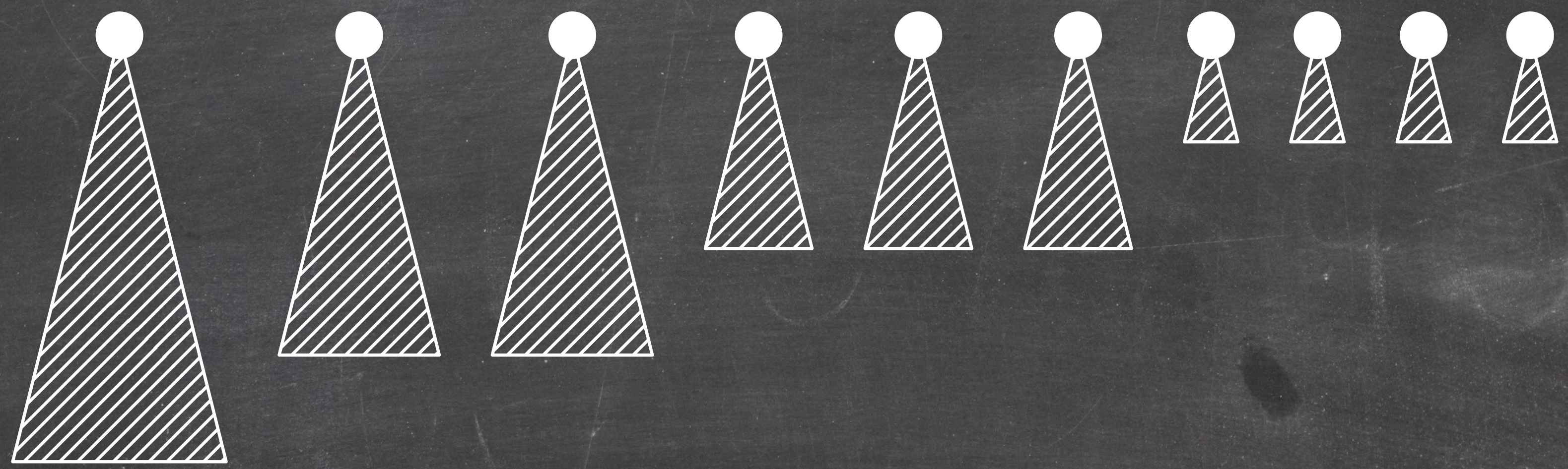

- Ensure all former children of min are thick. How?
- Collect all roots and former children of min.

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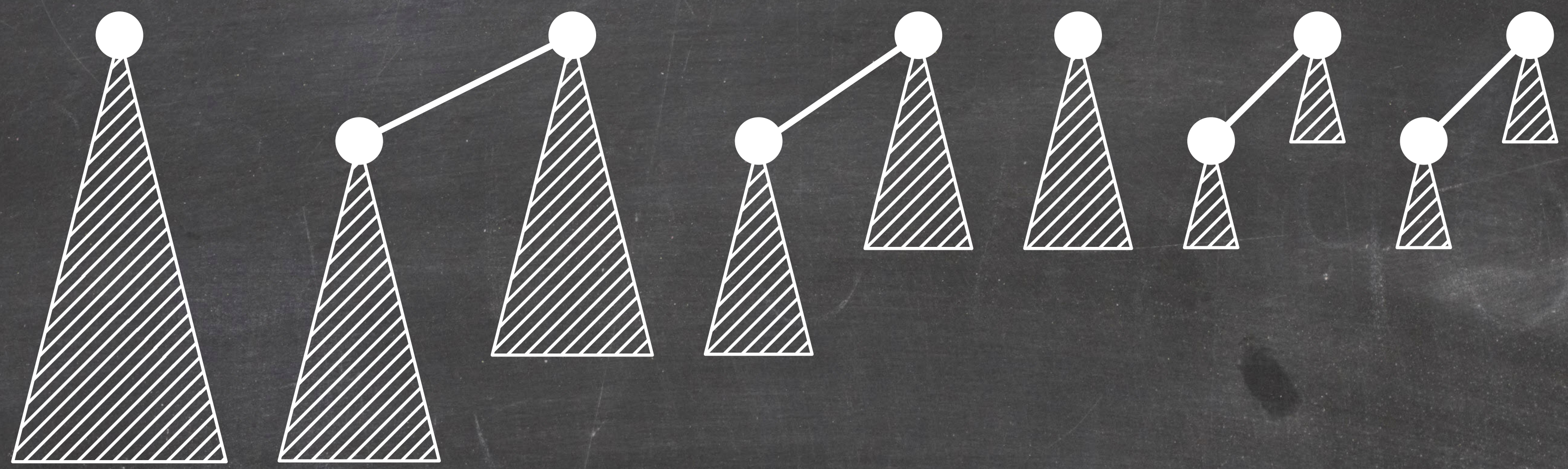

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- Collect all roots and former children of min.
- Link trees of the same rank until at most one tree of each rank remains.

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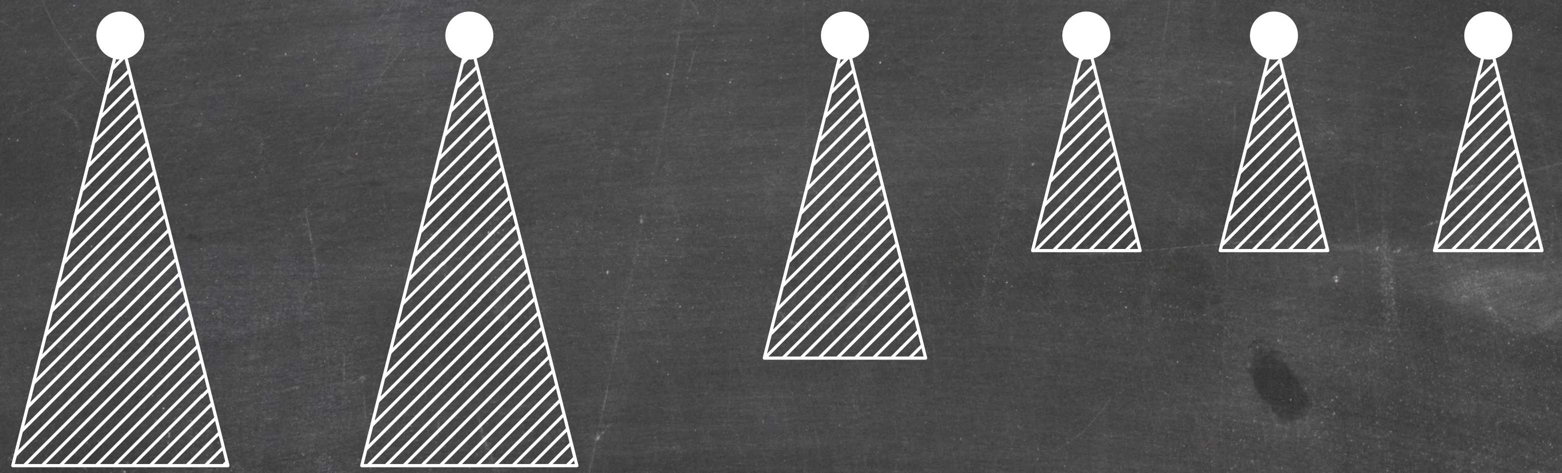

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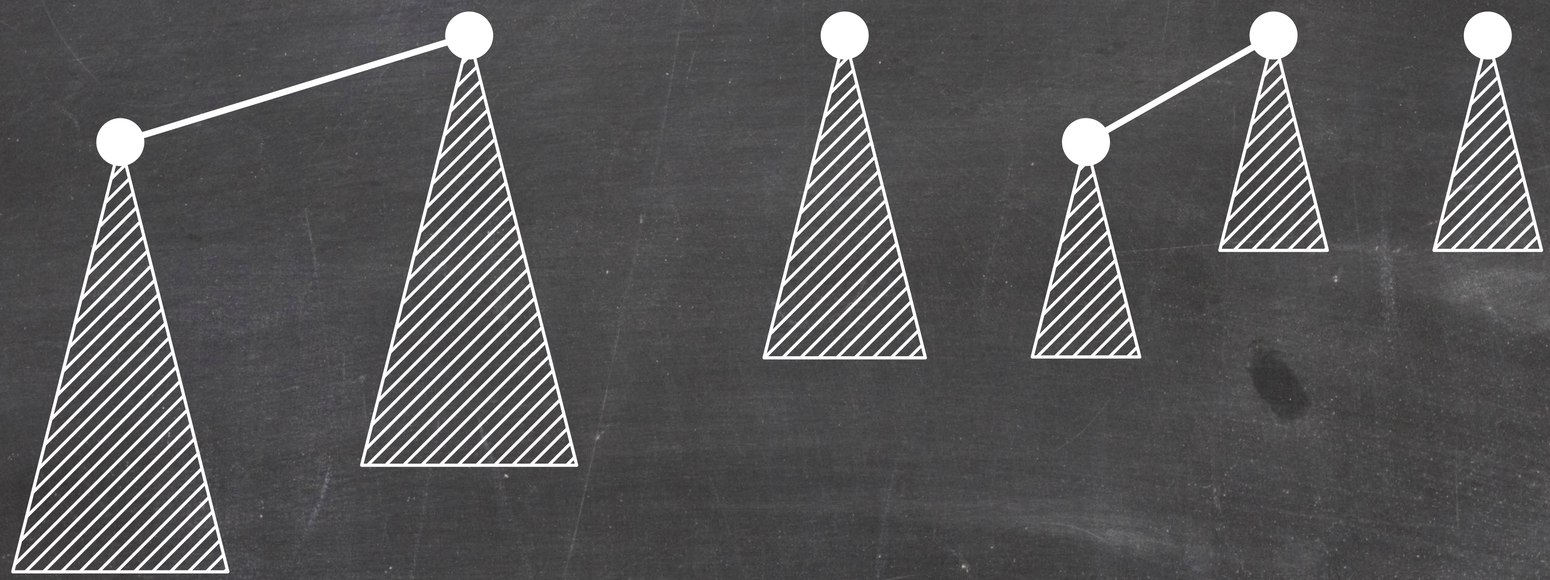

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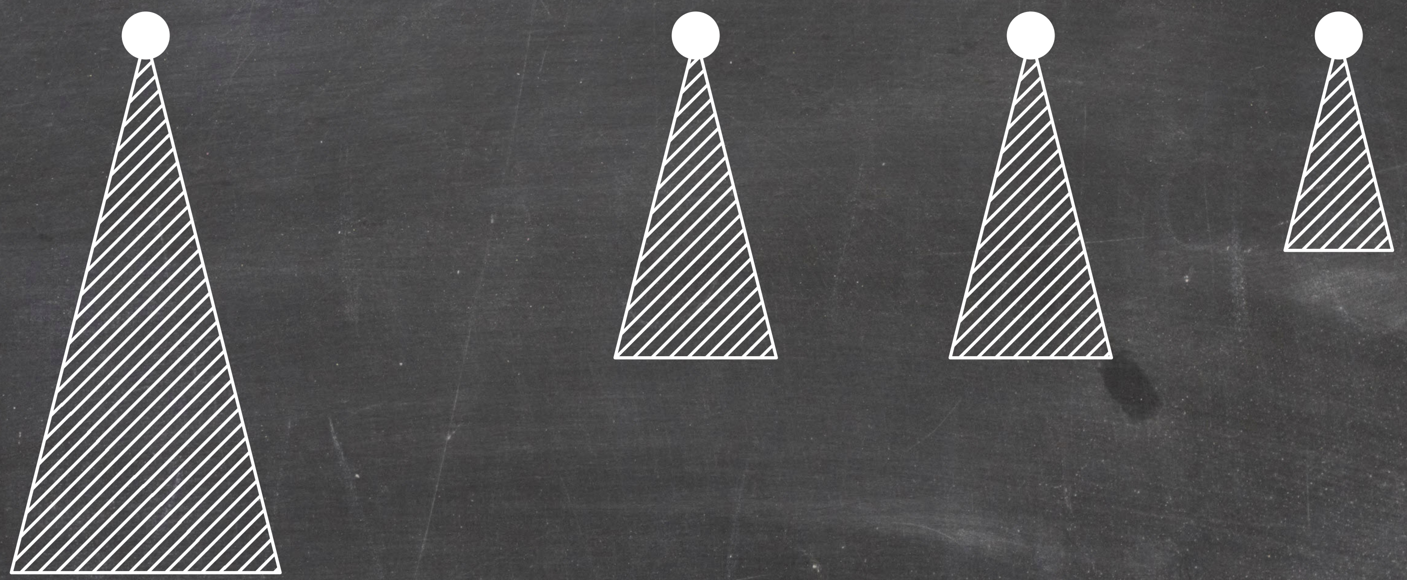

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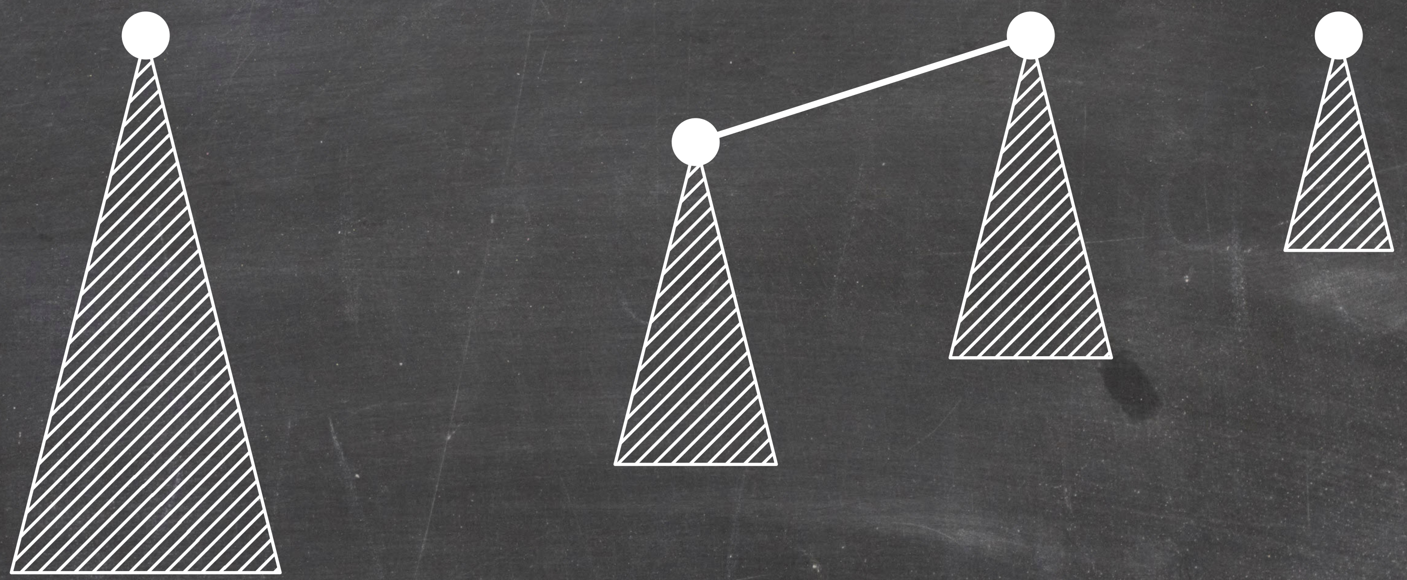

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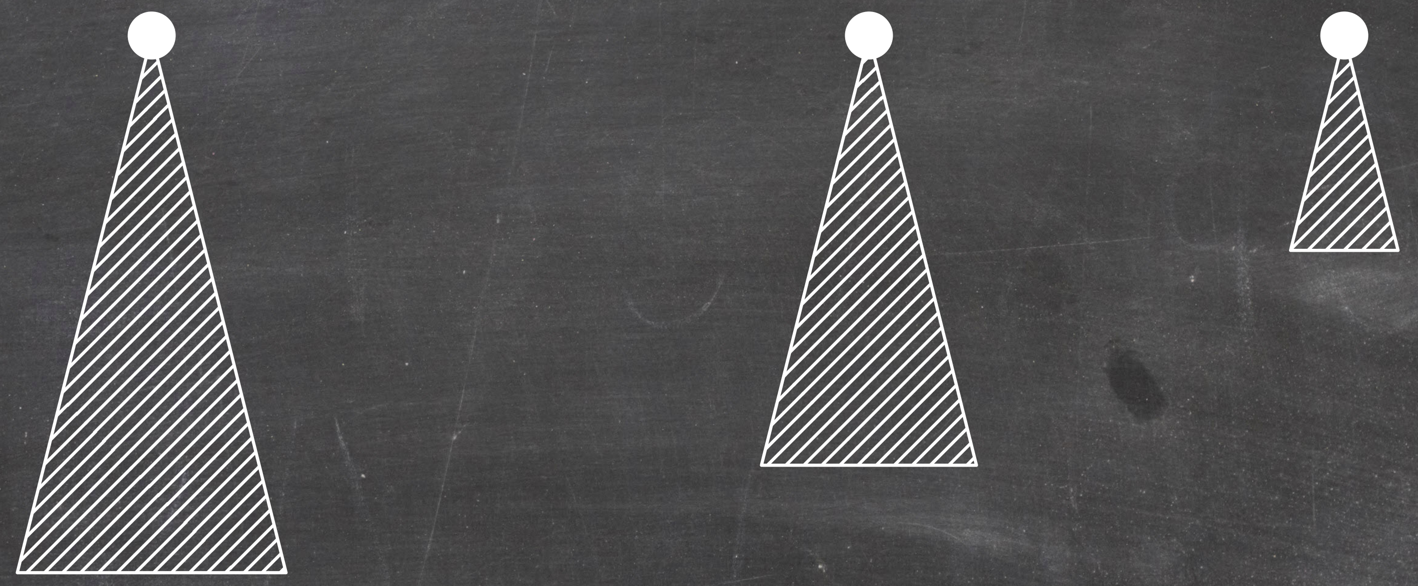

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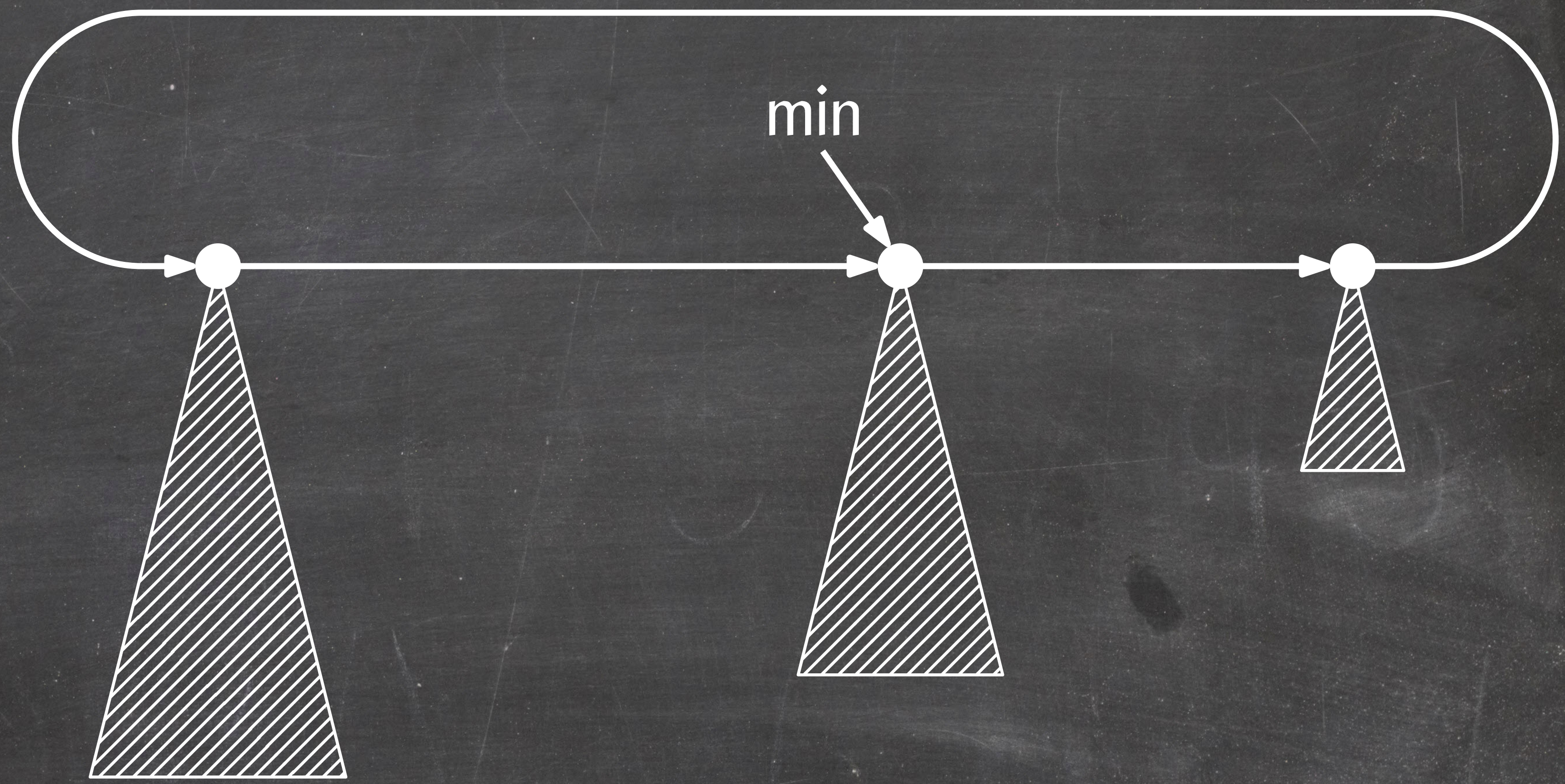
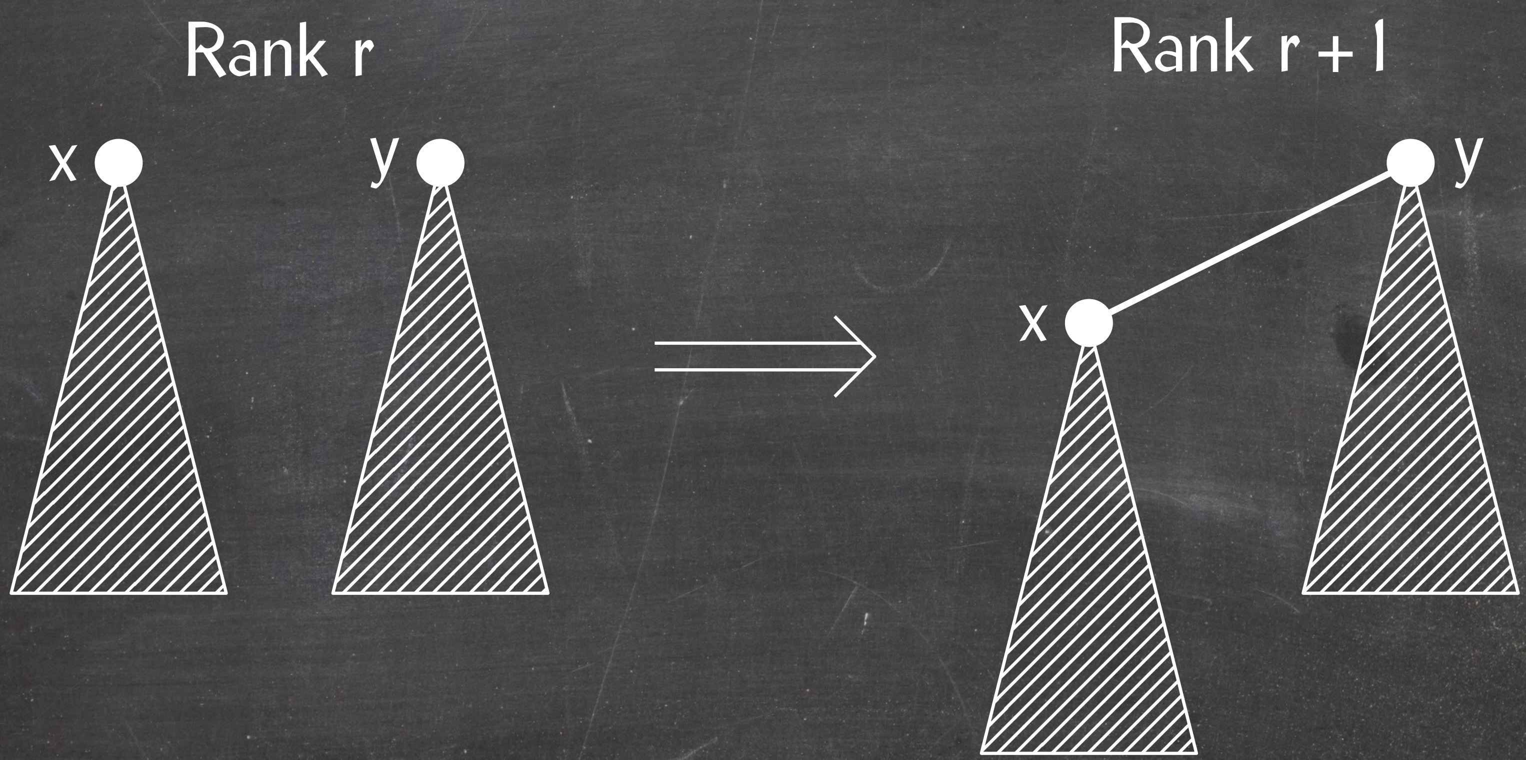


- Ensure all former children of min are thick. How?
- Collect all roots and former children of min.
- Link trees of the same rank until at most one tree of each rank remains.
- Relink roots into circular list and make min point to the minimum root.

Linking

Important: Both nodes need to be thick and of the same rank.

Assume $y < x$ (swap the two trees otherwise).



This produces a valid thin tree:

y had r children of ranks $r-1, r-2, \ldots, 0$ before.

$\Rightarrow$ y has $r+1$ children of ranks $r, r-1, \ldots, 0$ after.

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Lemma: A tree whose root has rank r has at least F_r nodes, where F_r is the rth Fibonacci number.

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Fibonacci numbers:

$$F_k = \begin{cases} 1 & k = 0 \text{ or } k = 1 \\ F_{k-1} + F_{k-2} & \text{otherwise} \end{cases}$$

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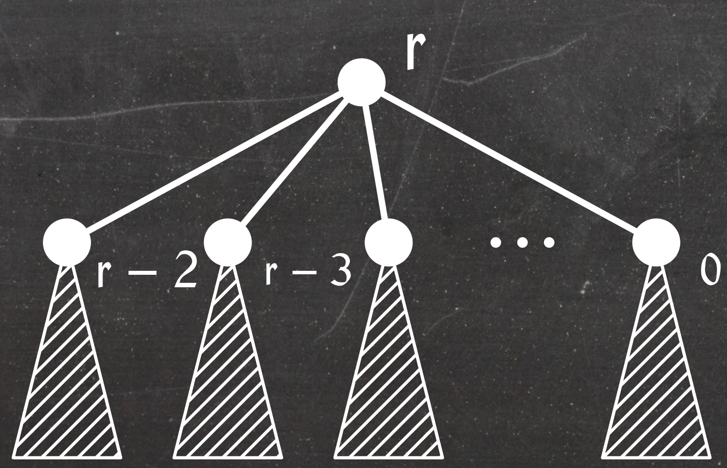

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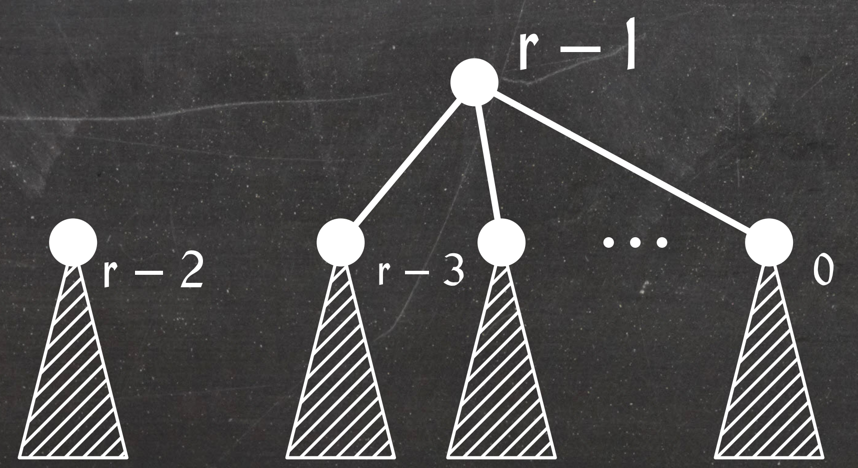

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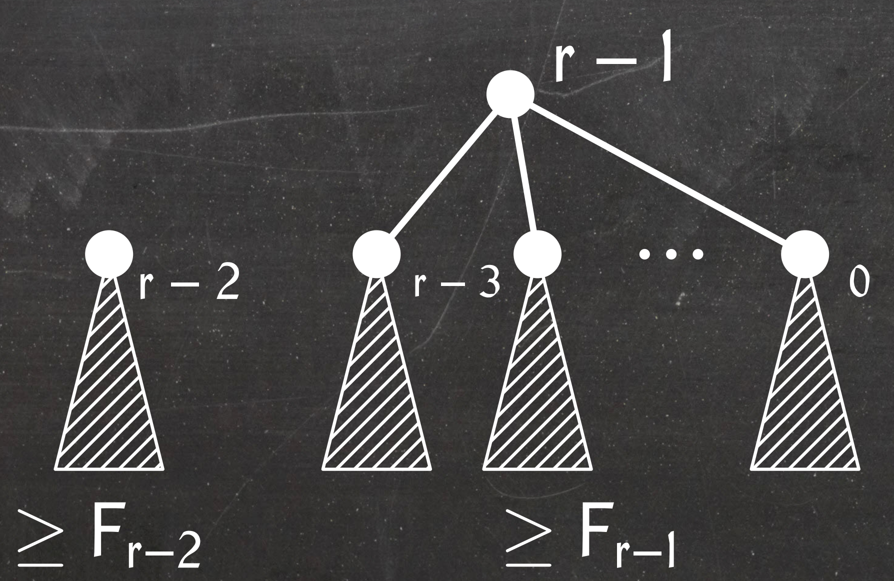

$$F_{r-1} + F_{r-2} = F_r$$

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Corollary: The maximum rank in a Thin Heap storing n elements is $\log_\phi n < 2 \lg n$.

Implementation of DeleteMin

Q.deleteMin()

```
x = Q.min
R = array of size 2 lg n with all its entries initially null.
for every root r other than Q.min
    do LinkTrees(R, r)
for every child c of Q.min
    do decrease c's rank if necessary to make it thick
       LinkTrees(R, c)
Q.min = null
for i = 0 to 2 lg n
    do if R[i] ≠ null
          then R[i].leftSibOrParent = null
               if Q.min = null
                  then Q.min = R[i]
                       Q.min.rightSib = Q.min
                  else R[i].rightSib = Q.min.rightSib
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                       if R[i].val < Q.min.val
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return x.val
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LinkTrees(R, x)

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r = x.rank
while R[r] ≠ null
    do x = Link(x, R[r])
       R[r] = null
       r = r + 1
R[r] = x
```

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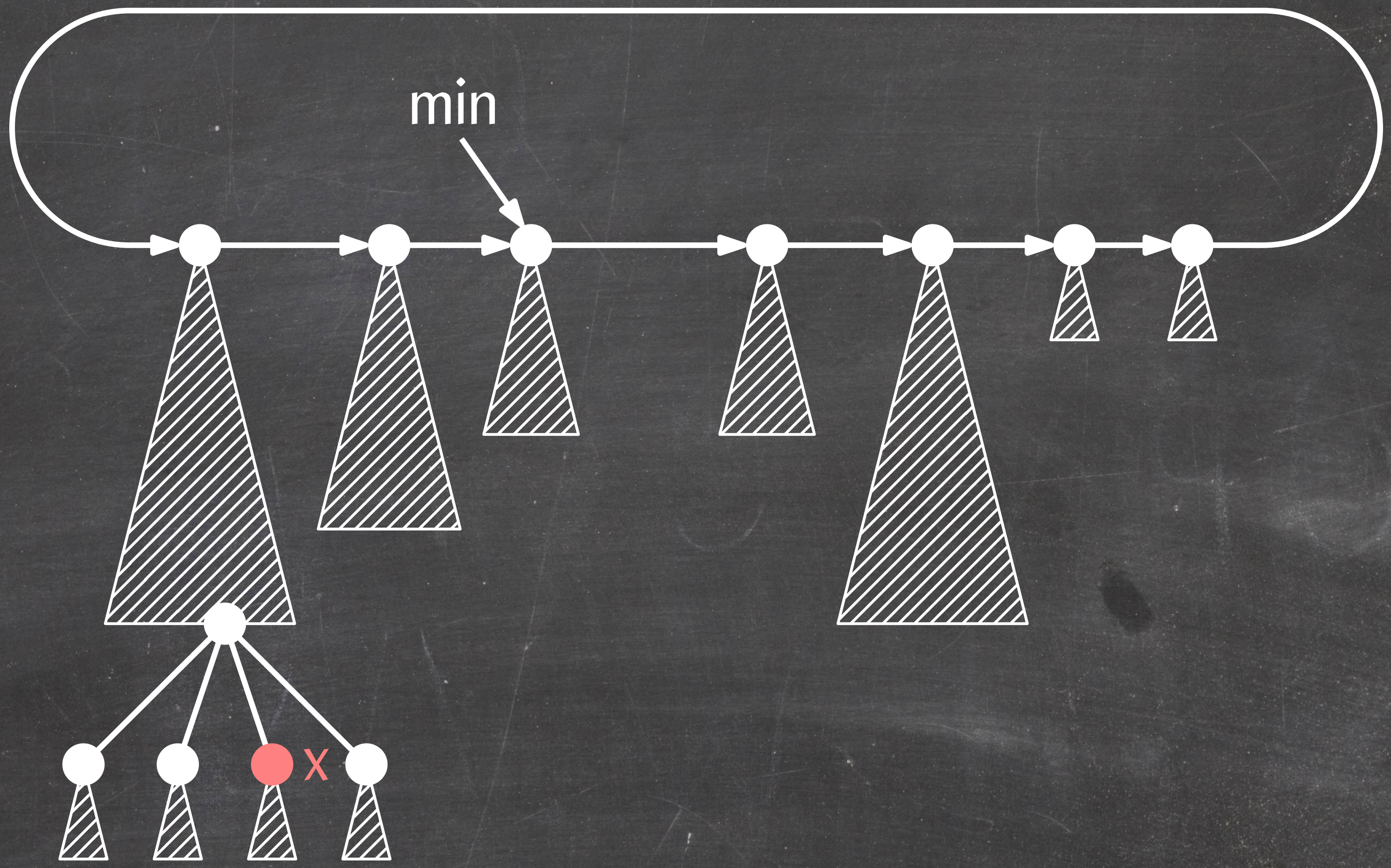
```

Collect remaining trees and form circular list.

DecreaseKey



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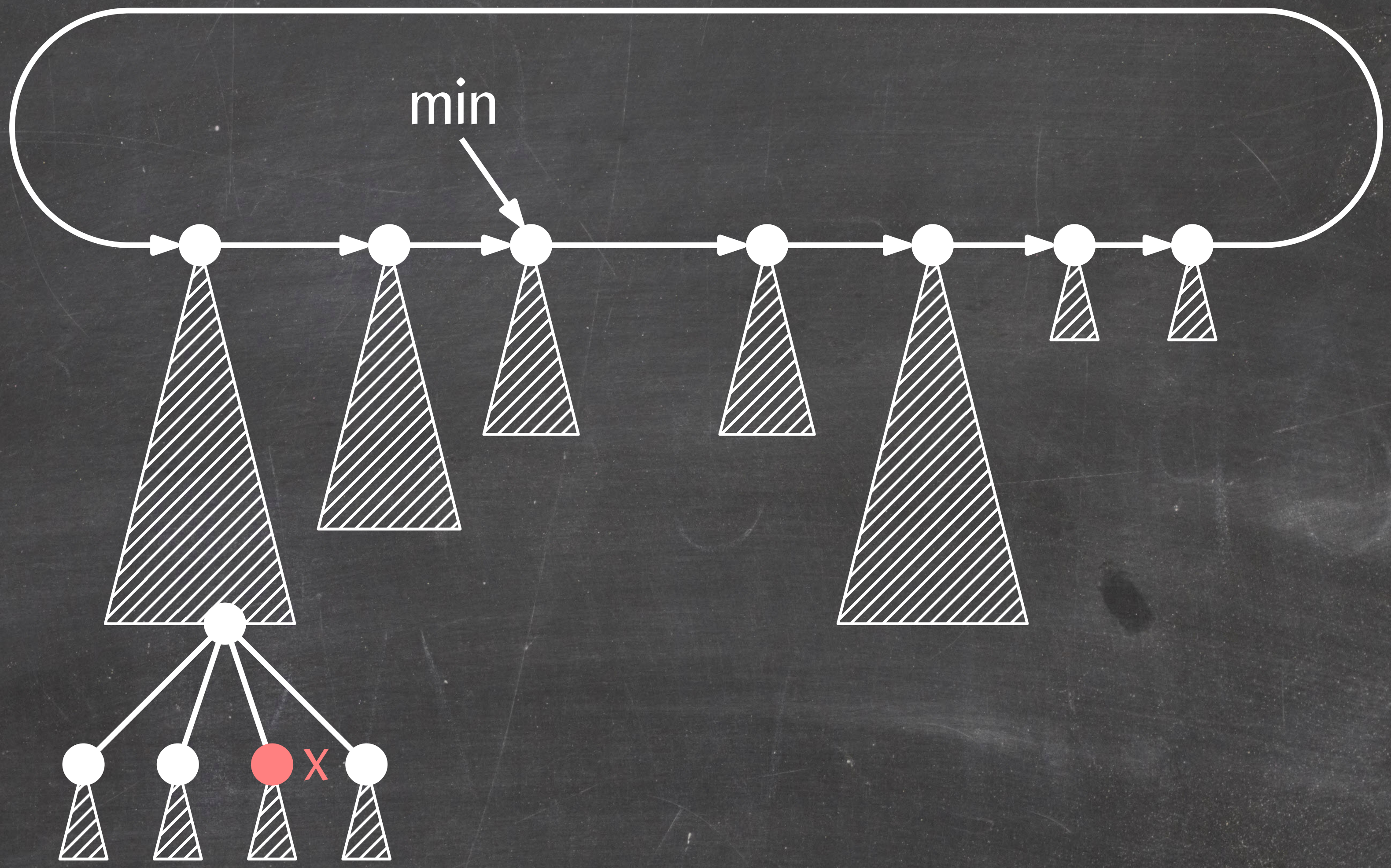

- Update x's priority

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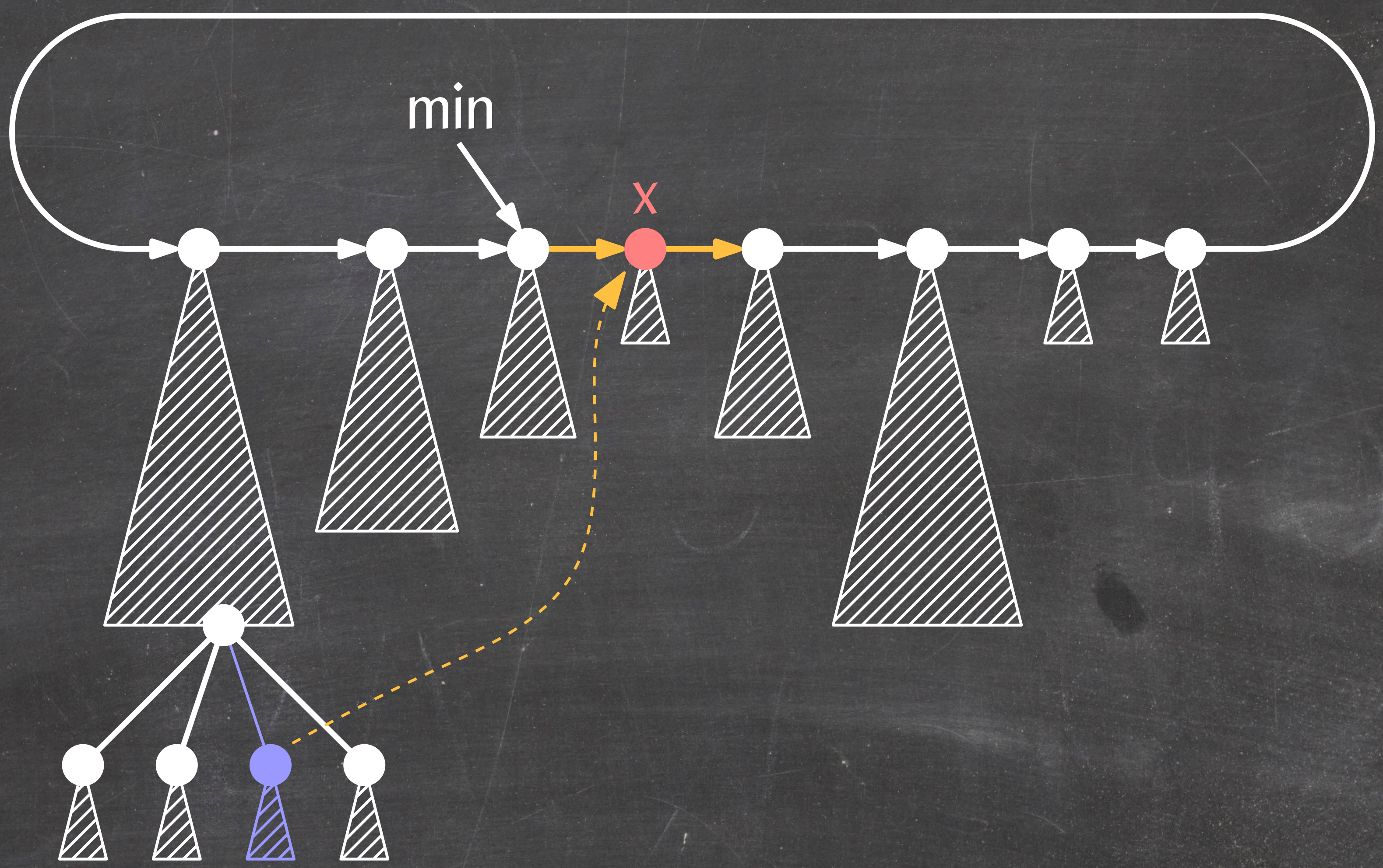

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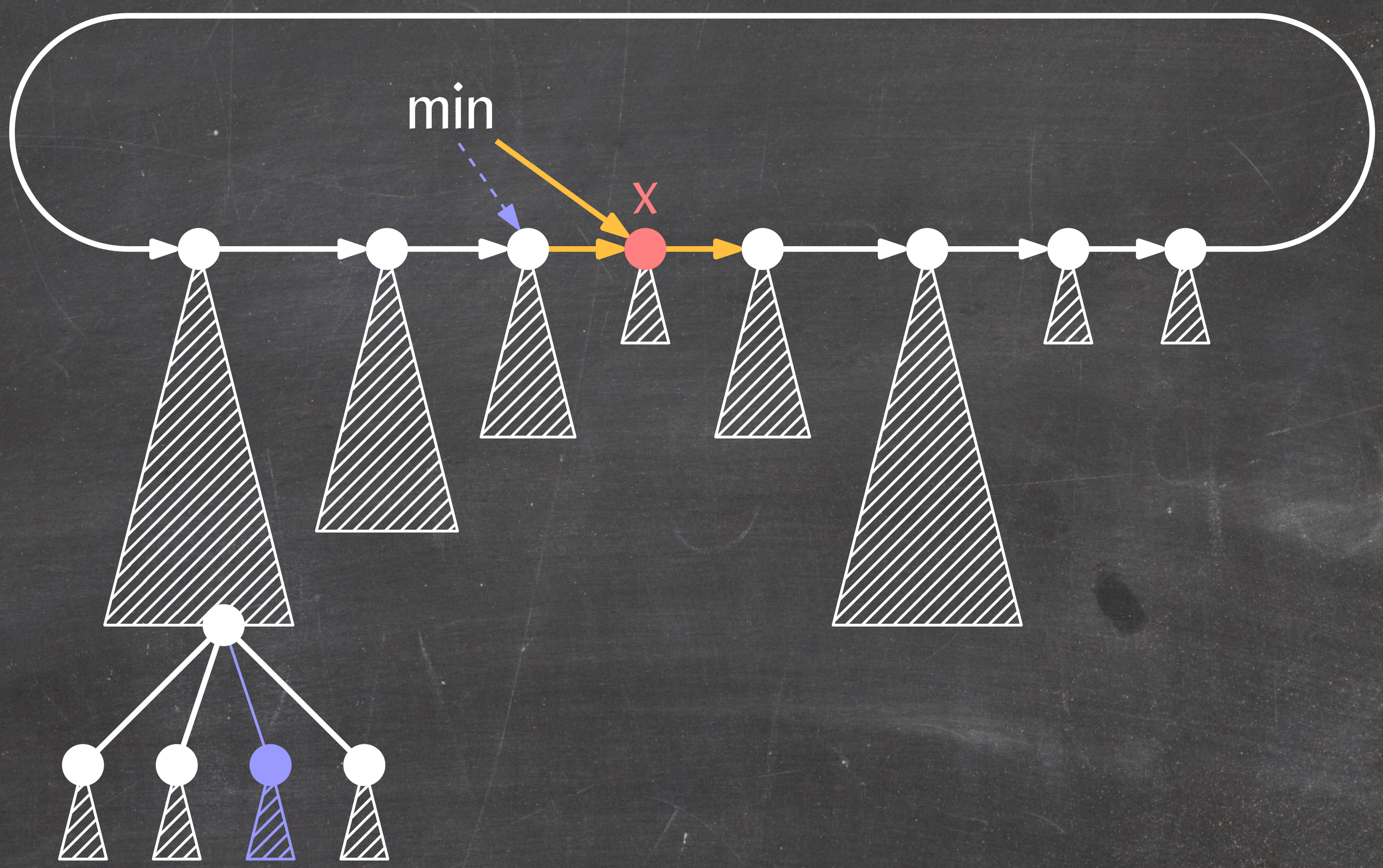

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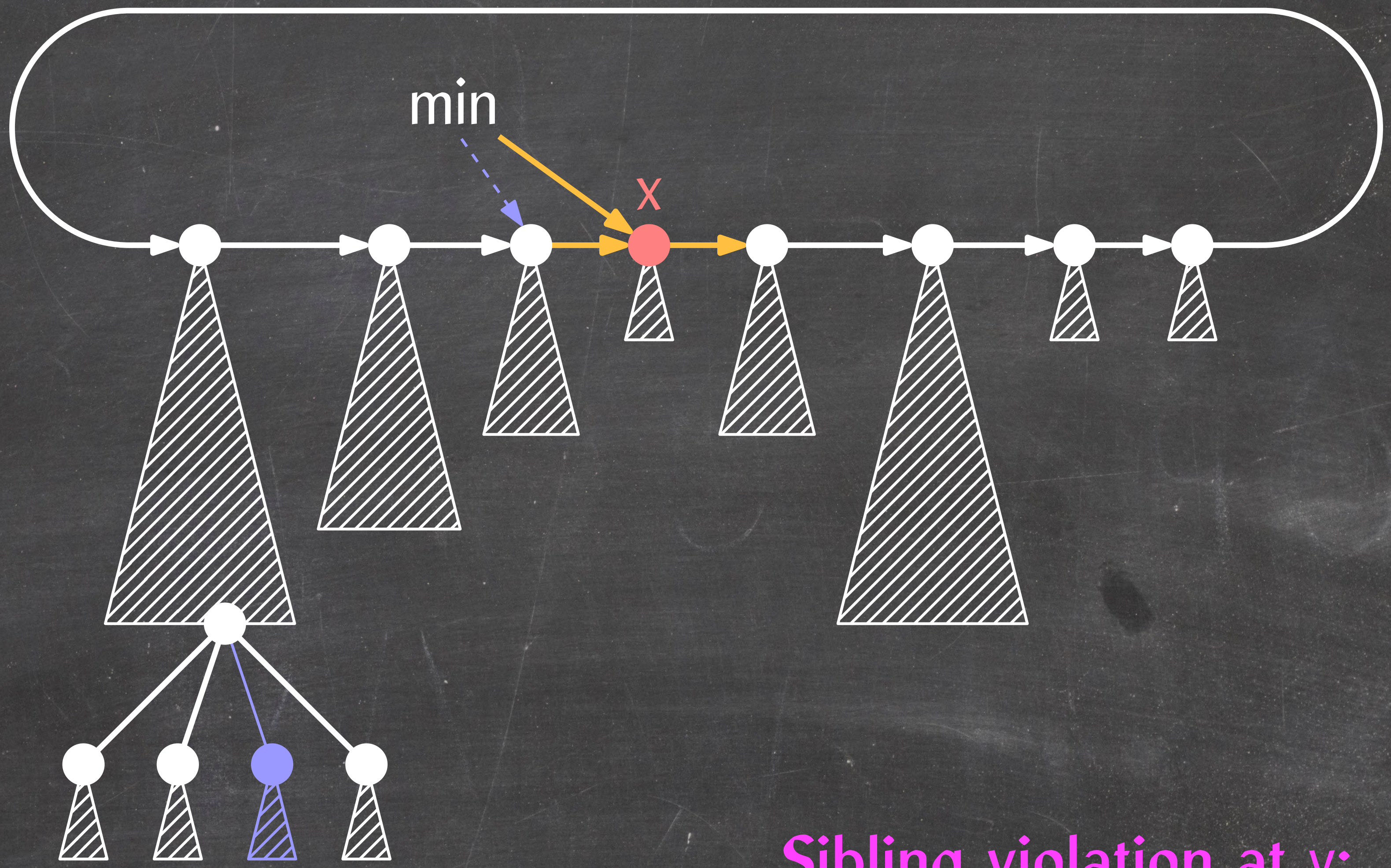

Sibling violation at y:

y.rank > 0 and y has no right sibling or y.rightSib.rank < y.rank − 1.

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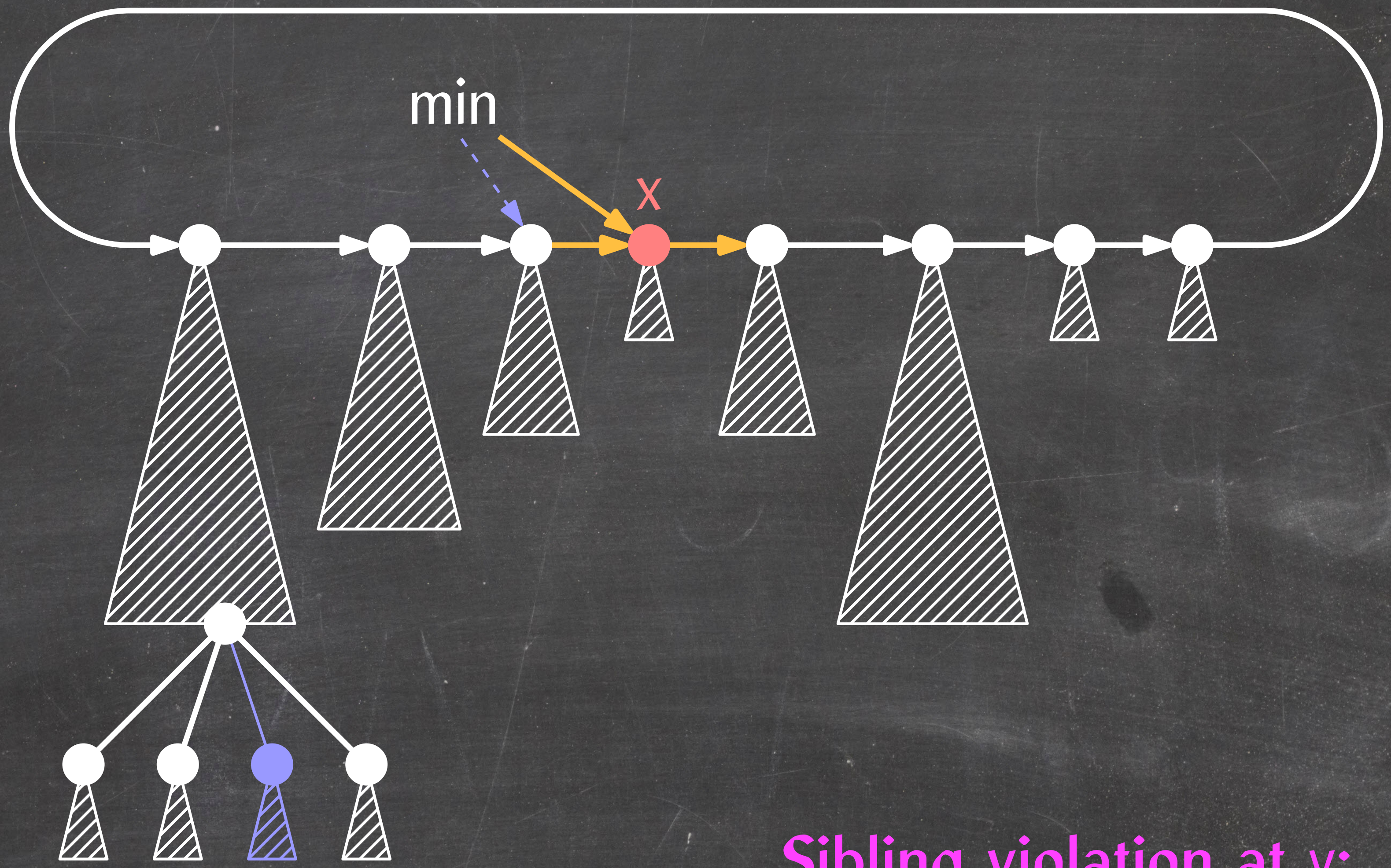

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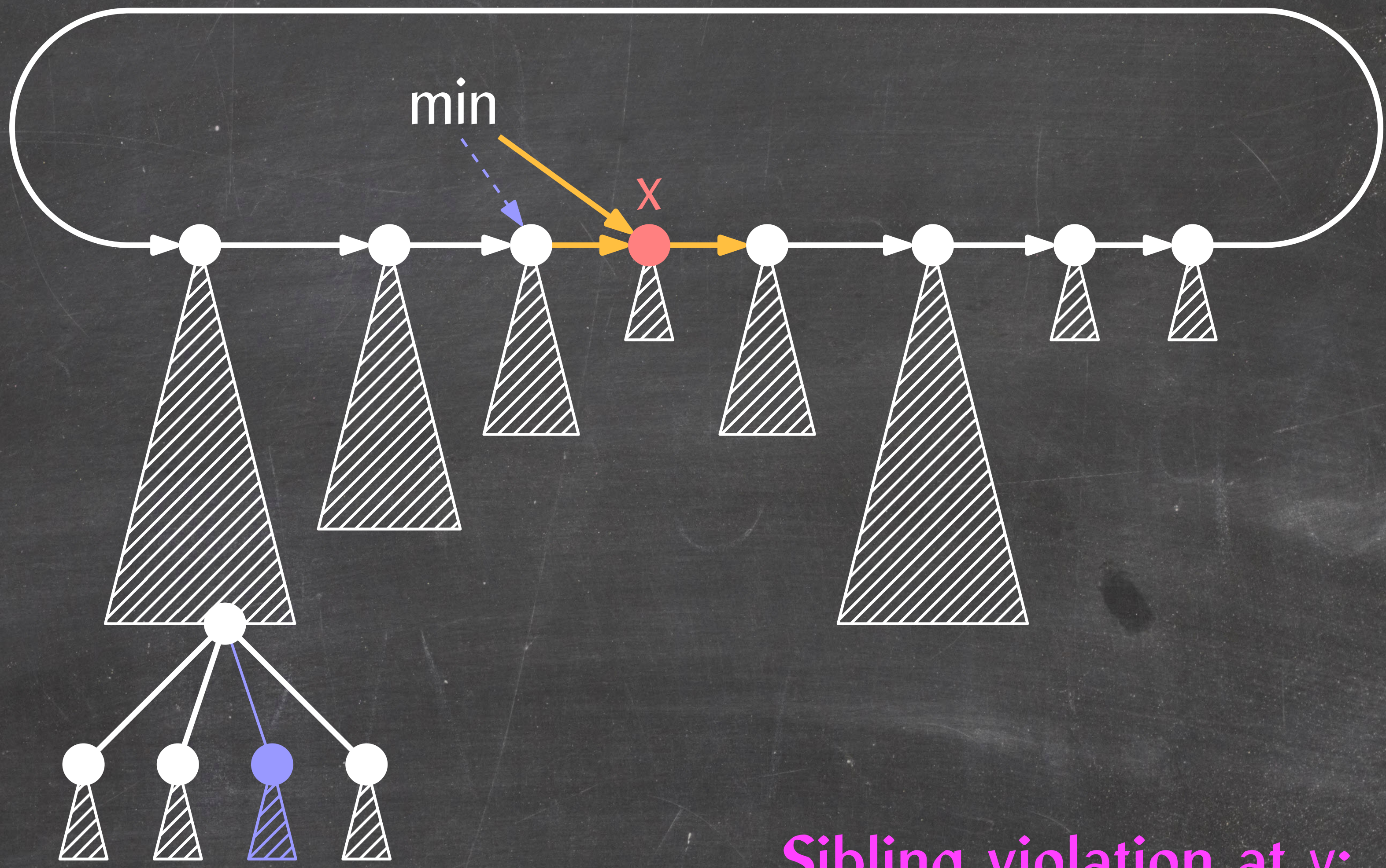

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- Update x's priority
- Make x a root
- Fix parent/sibling violations

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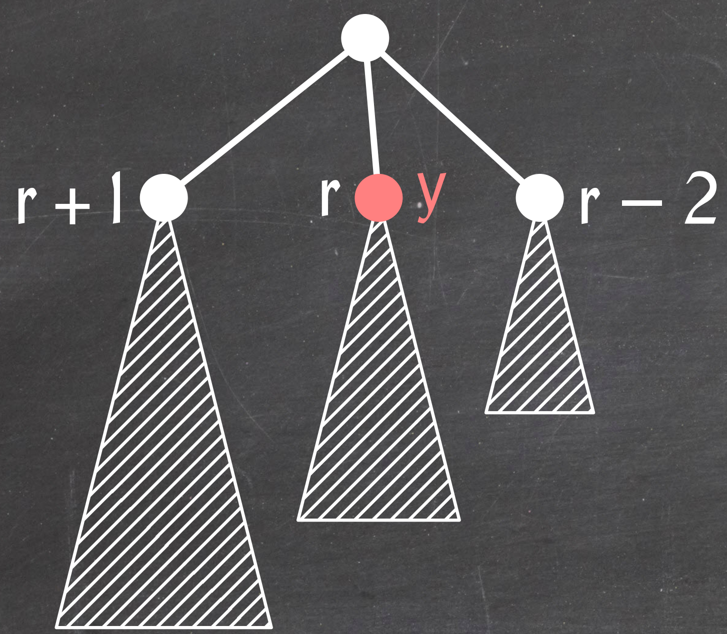

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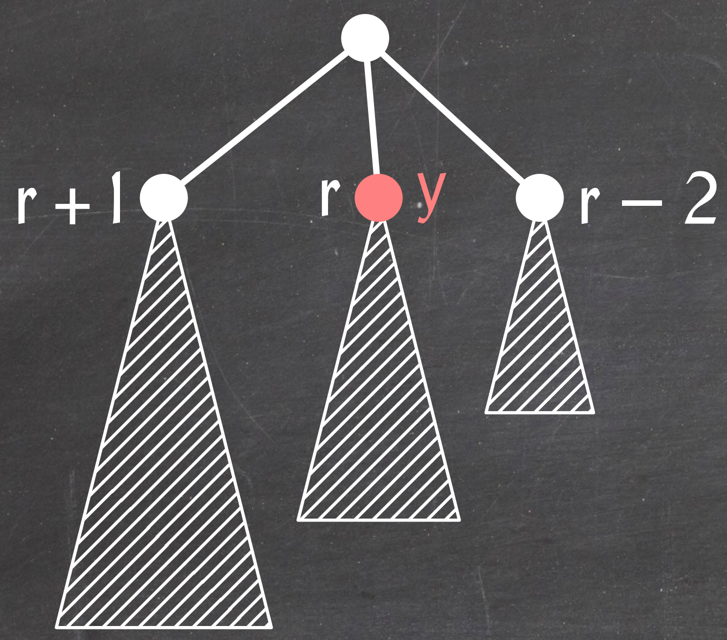

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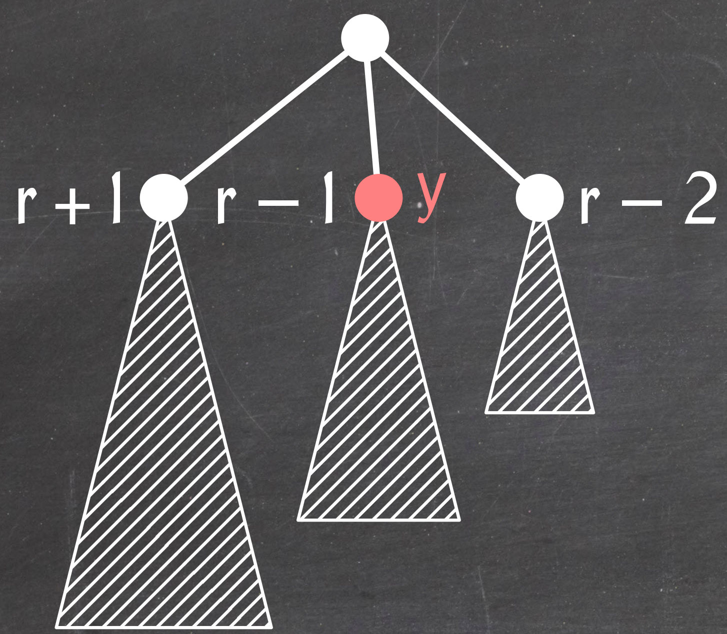

If y is thin, then

- decrease its rank by one and

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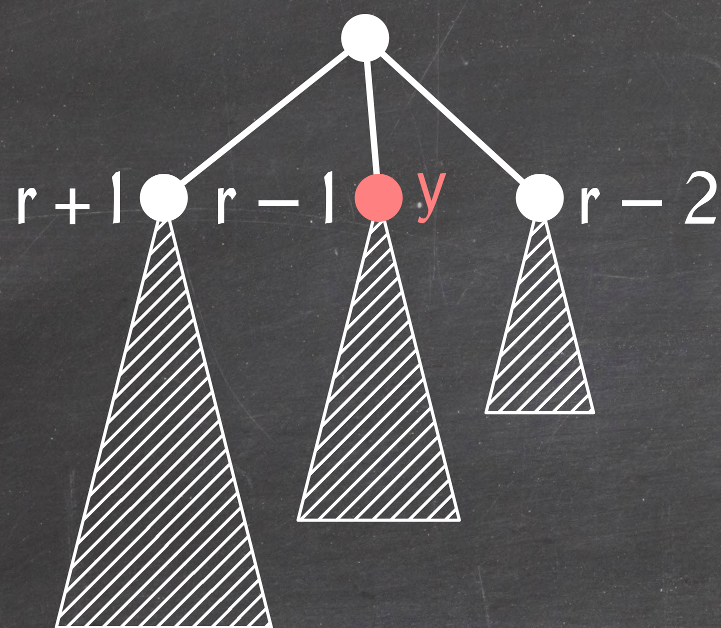

If y is thin, then

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- fix violation at y.leftSibOrParent.

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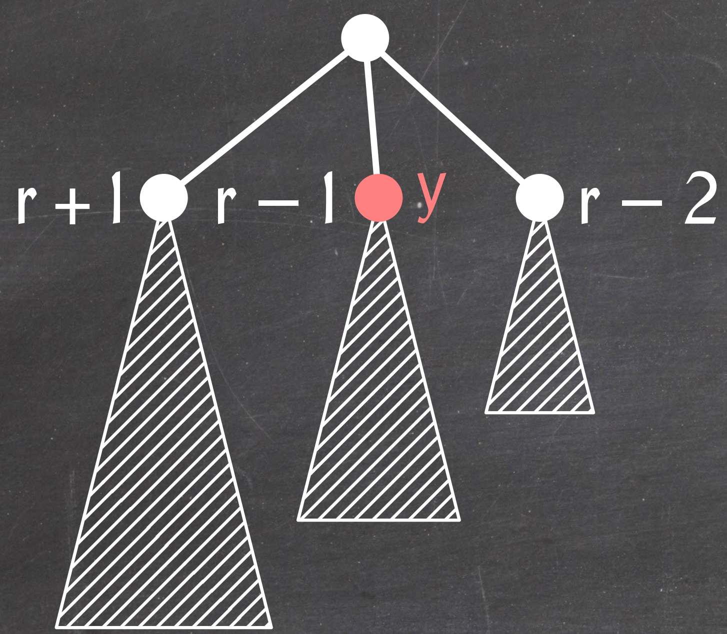

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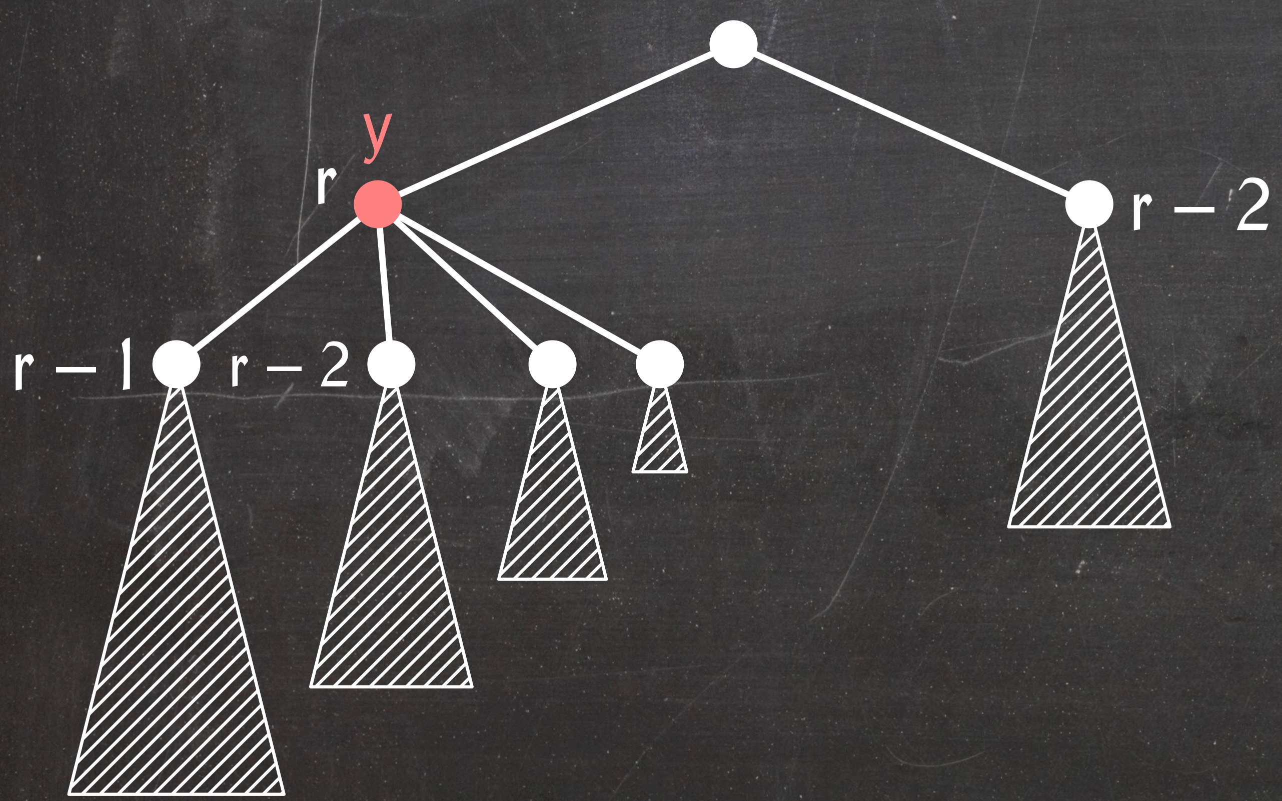

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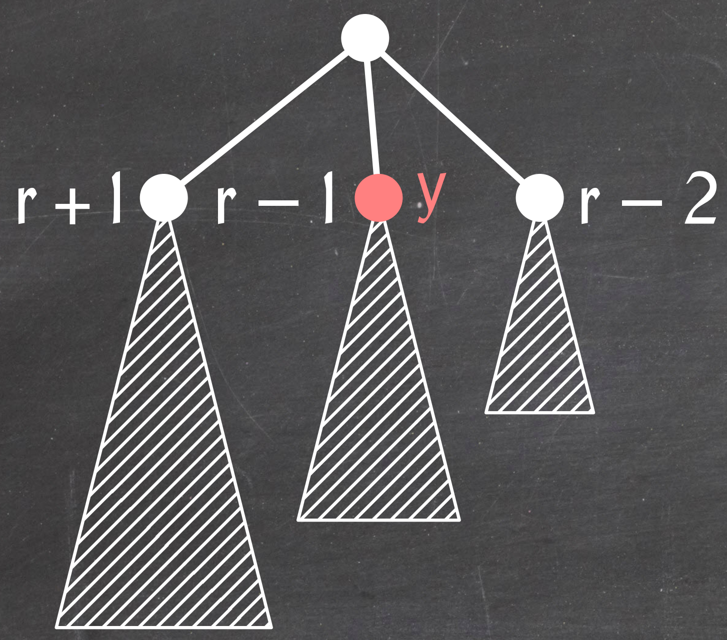

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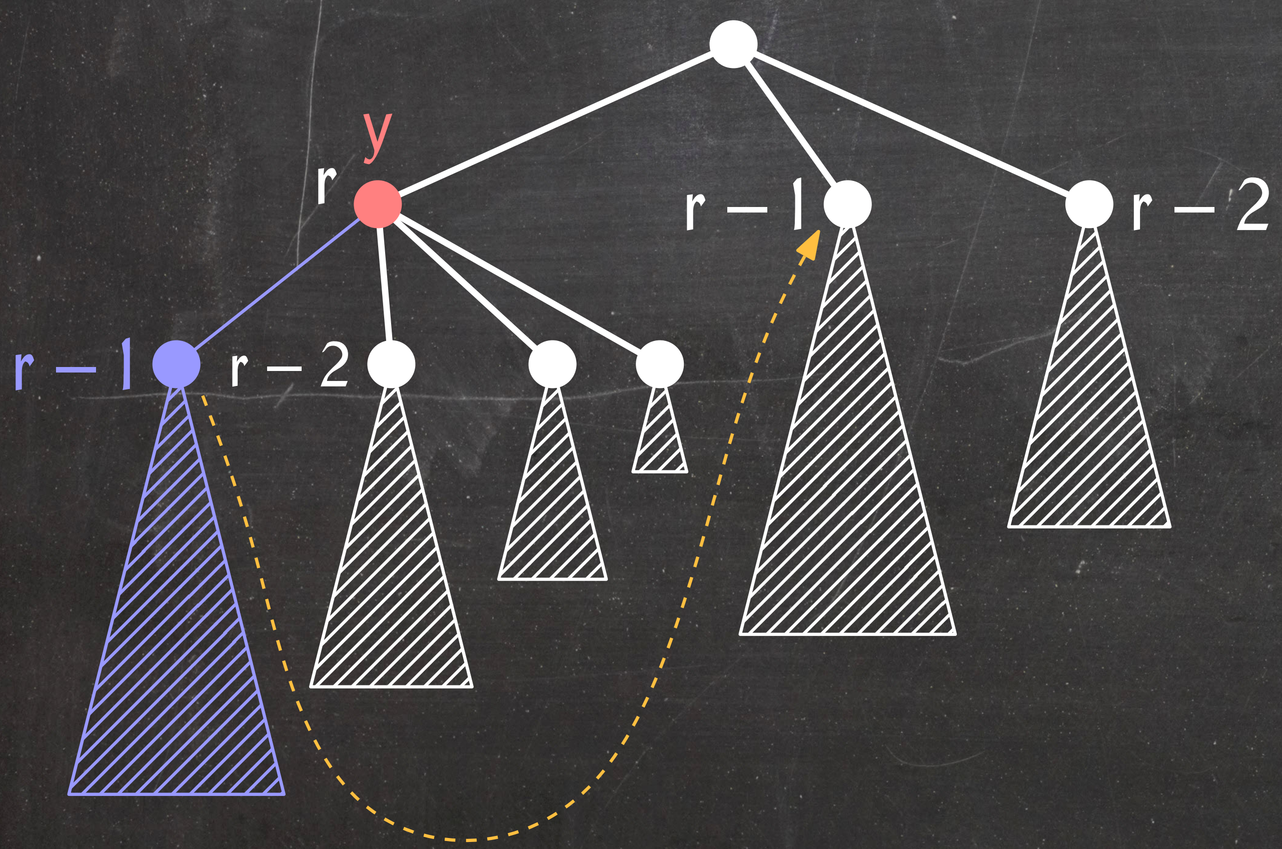
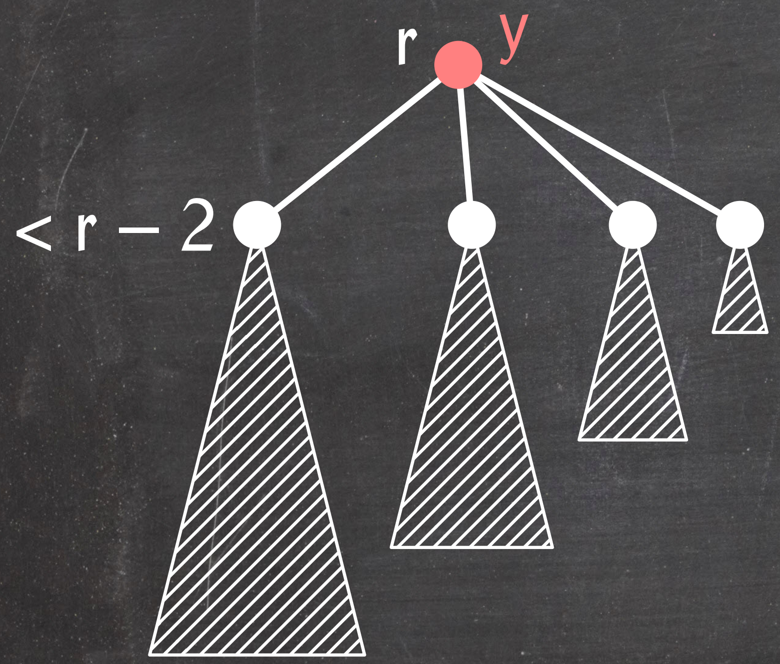


If y is thick, then make y.child y's right sibling.

Parent Violation



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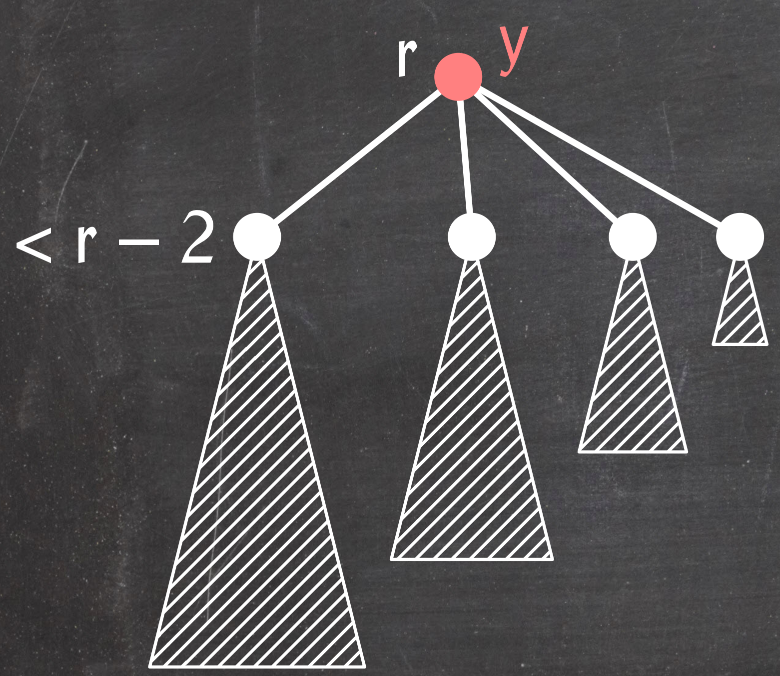

If y is a root, then set y.rank = y.child.rank + 1.

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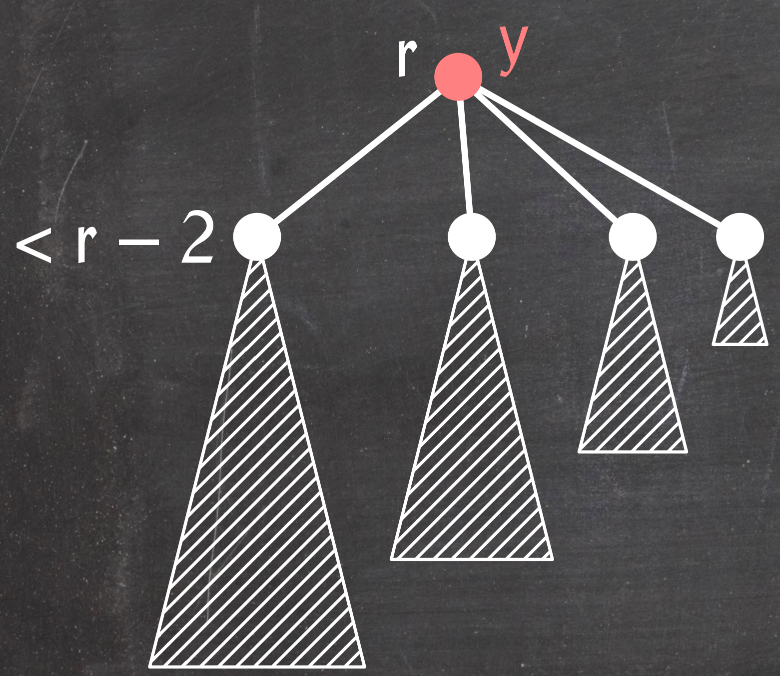

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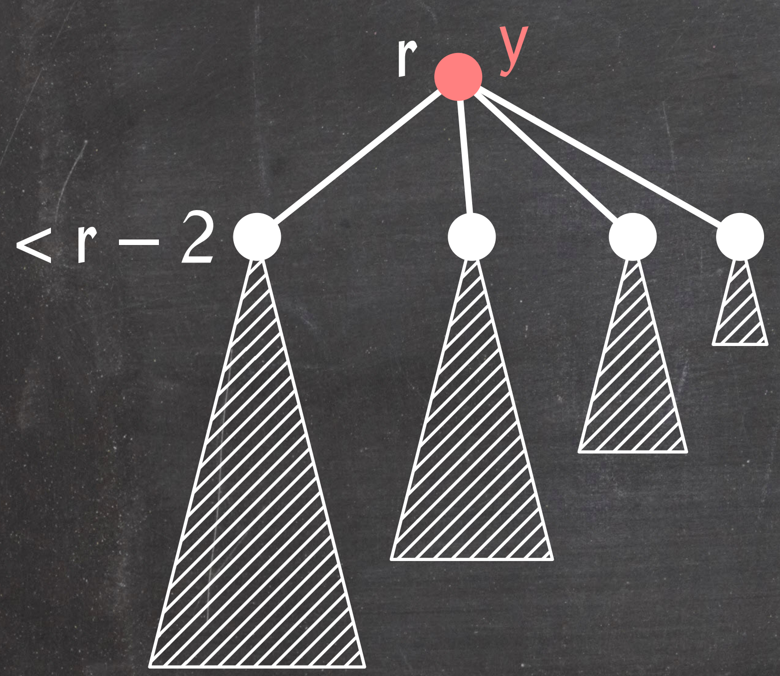

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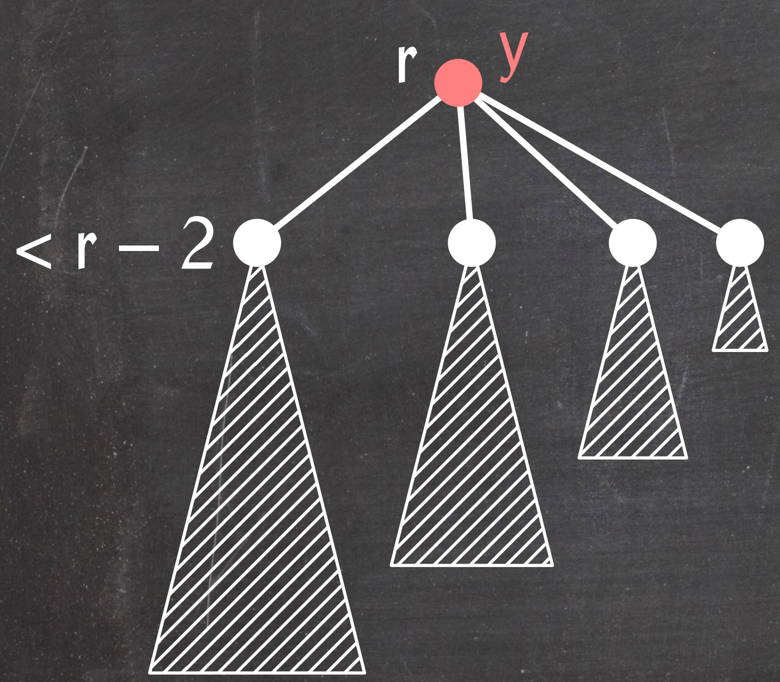

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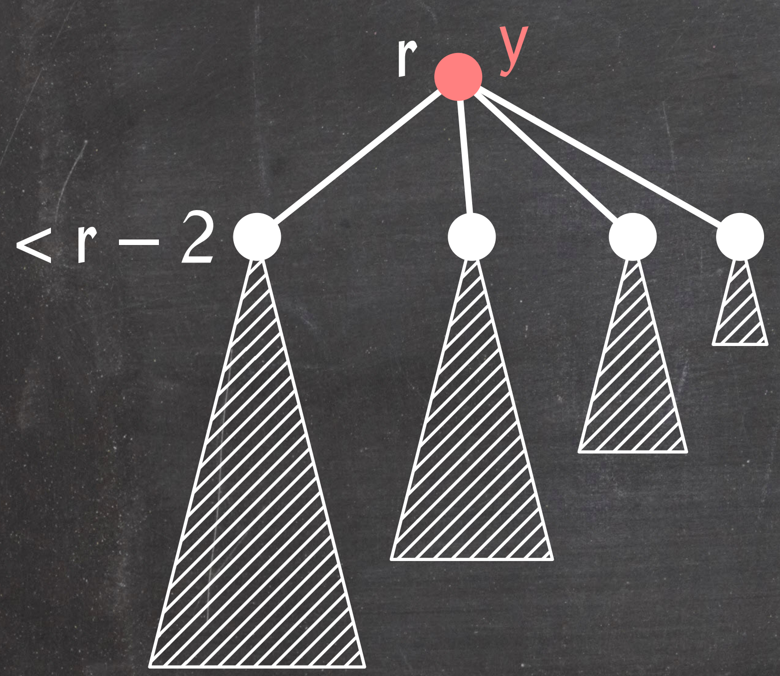

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These costs are completely fictitious but must satisfy an important condition to be useful:

$$\sum_{i=1}^{m} c_i \le \sum_{i=1}^{m} \hat{c}_i$$

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$$\sum_{i=1}^{m} \hat{c}_i = \sum_{i=1}^{m} (c_i + \Phi_i - \Phi_{i-1}) = \sum_{i=1}^{m} c_i + \Phi_m - \Phi_0 \geq \sum_{i=1}^{m} c_i$$

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Intuition:

- The potential captures parts of the data structure that can make operations expensive.
- If operations that take long eliminate these "expensive" parts of the data structure, then there can't be many expensive operations without lots of operations that create these expensive parts.
- These operations can "pay" for the cost of the expensive operations.

Amortized Analysis: Stack with MultiPop Operation

Operations:

S.push(x)	Push element x on the stack
S.pop()	Pop the topmost element from the stack
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What can make operations expensive?

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Our goal is to prove that the amortized cost per operation is constant.

What can make operations expensive?

MultiPop becomes expensive if k is large and there are lots of elements on the stack.

Amortized Analysis: Stack with MultiPop Operation

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$$\Phi = |S|$$

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Amortized Analysis: Binary Counter

Consider a binary counter initially set to 0.

The only operation we want to support is Increment.

0 1 1 0 0 1 1 1 1

↓

0 1 1 0 1 0 0 0 0

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$\downarrow$

0

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What makes increment operations expensive?

Lots of 1s that need to be flipped into 0s.

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$$\begin{array}{ccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$

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Φ = #1s in the current counter value

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A Potential Function for Thin Heap

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$$\Phi = 2 \cdot \text{number of thin nodes} + \text{number of roots}$$

Amortized Cost of Insert, FindMin, and Delete

Insert:

- $c \in O(1)$
- $\Delta\Phi = +1$:
  - Δ(number of roots) = +1
  - Δ(number of thin nodes) = 0

$\Rightarrow \hat{c} \in O(1)$

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Delete:

- We show that $\hat{c}(\text{DecreaseKey}) \in O(1)$.
- We show that $\hat{c}(\text{DeleteMin}) \in O(\lg n)$.

$\Rightarrow \hat{c} \in O(\lg n)$

Amortized Cost of DeleteMin

Actual cost: $O(\lg n + \text{number of roots} + \text{number of children of } Q.min)$

- $O(\lg n)$ for initializing R
- $O(1)$ per addition to R
- $O(1)$ per link operation
- $O(\lg n)$ to collect final list of roots from R
- Number of additions to R = number of roots and children of Q.min
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Amortized cost:

$\hat{c} = c + \Delta\Phi = O(\lg n + \text{number of roots}) + 2\lg n - \text{number of roots} \in O(\lg n).$

Amortized Cost of DecreaseKey

Make affected element x a root (if it isn't already a root):

- $c \in O(1)$
- Δ(number of roots) ≤ 1
- Δ(number of thin nodes) ≤ 1:
  - x's parent becomes thin if it was thick and x is the leftmost child.

$\Rightarrow \Delta\Phi \leq 3$

$\Rightarrow \hat{c} \in O(1)$

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- Fixing the last violation has constant amortized cost,
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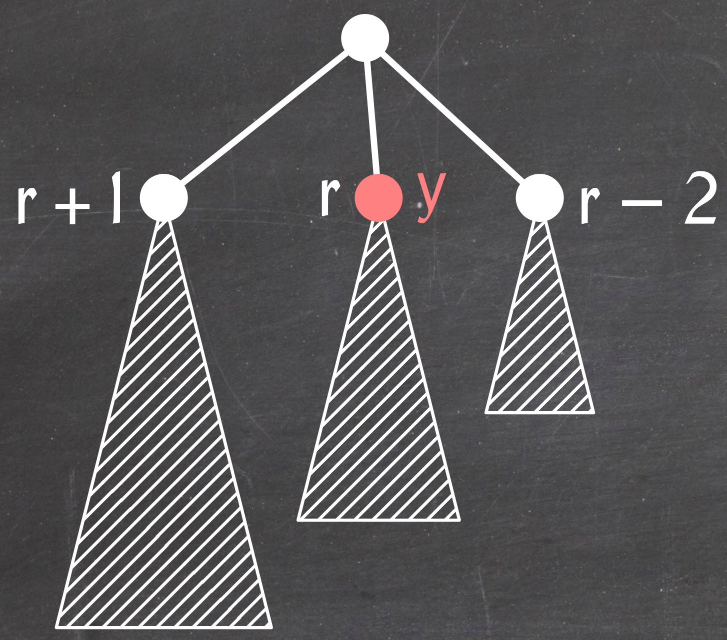
$\Rightarrow \hat{c}(\text{DecreaseKey}) \in O(1)$.

Amortized Cost of Fixing Sibling Violations



If y is thin,

- $c \in O(1)$
- Δ(number of thin nodes) = -1
- Δ(number of roots) = 0

$\Rightarrow \Delta\Phi = -2$

$\Rightarrow \hat{c} = 0$

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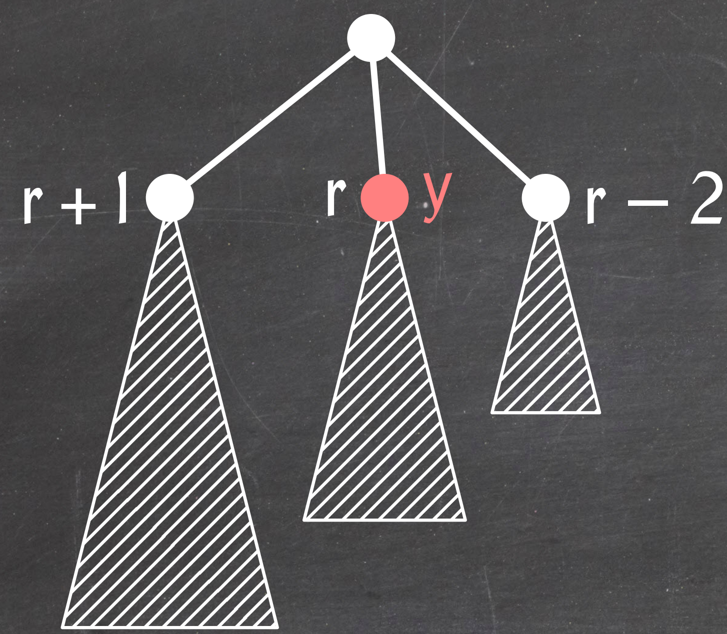

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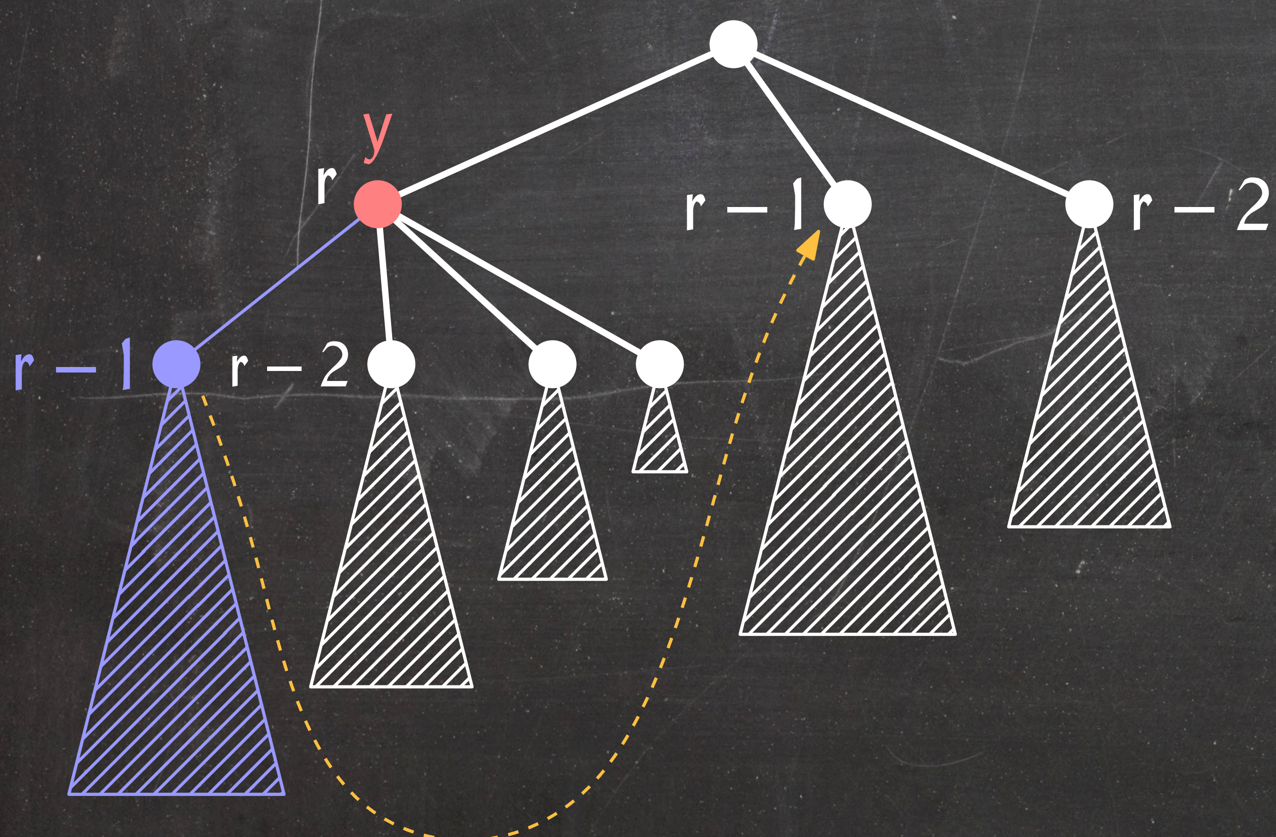


If y is thick,

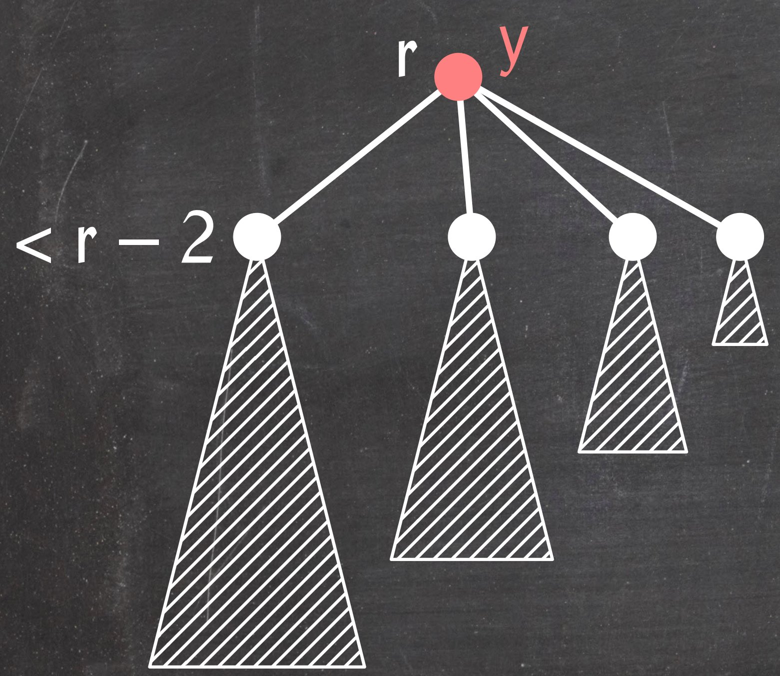
- $c \in O(1)$
- Δ(number of thin nodes) = $+1$
- Δ(number of roots) = 0

$\Rightarrow \Delta\Phi = +2$

$\Rightarrow \hat{c} \in O(1)$

After this, we're done!

Amortized Cost of Fixing Parent Violations



If y is a root, then

- $c \in O(1)$
- $\Delta(\text{number of roots}) = 0$
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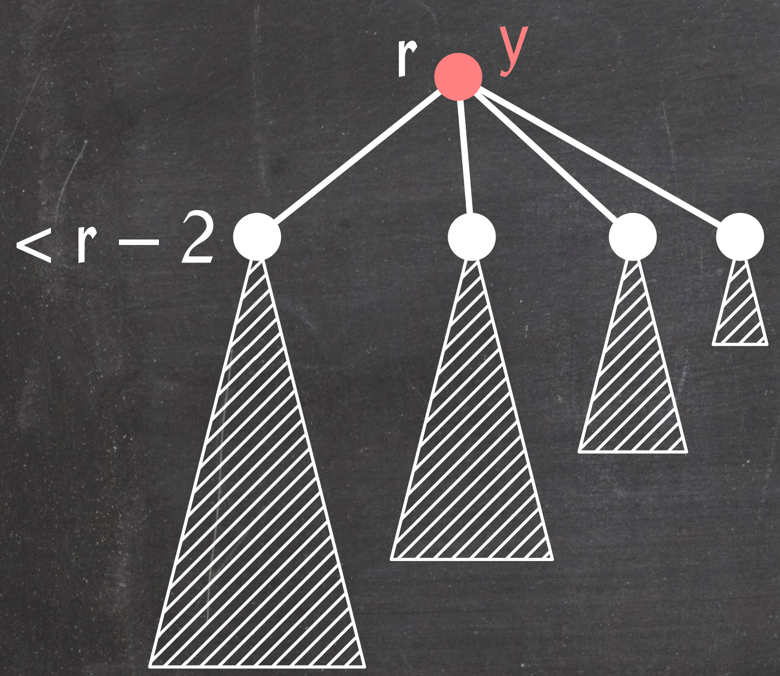

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If y is not a root and is not the leftmost child of its parent, then

- $c \in O(1)$
- Δ(number of roots) = $+1$
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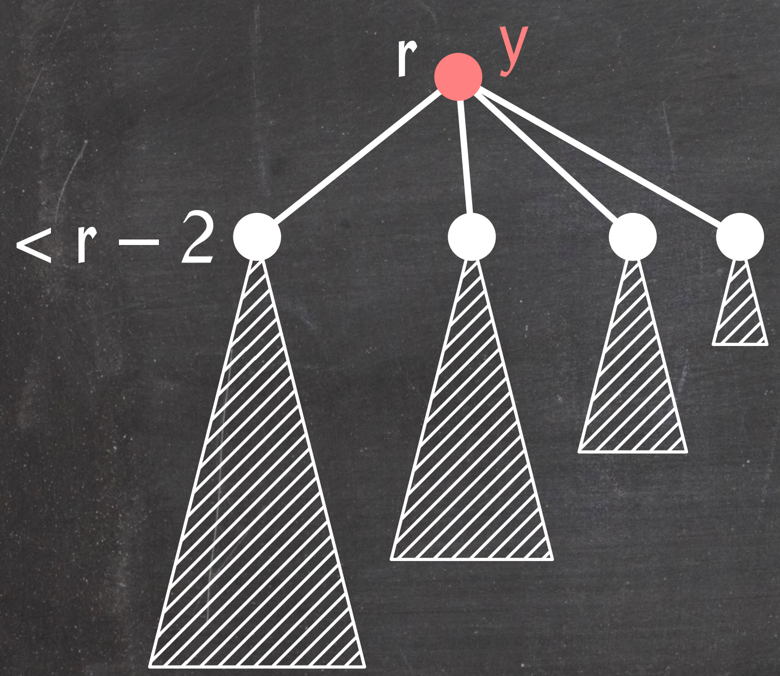
Amortized Cost of Fixing Parent Violations



If y is not a root and is the leftmost child of its parent, and its parent is thin, then

- $c \in O(1)$
- $\Delta(\text{number of roots}) = +1$
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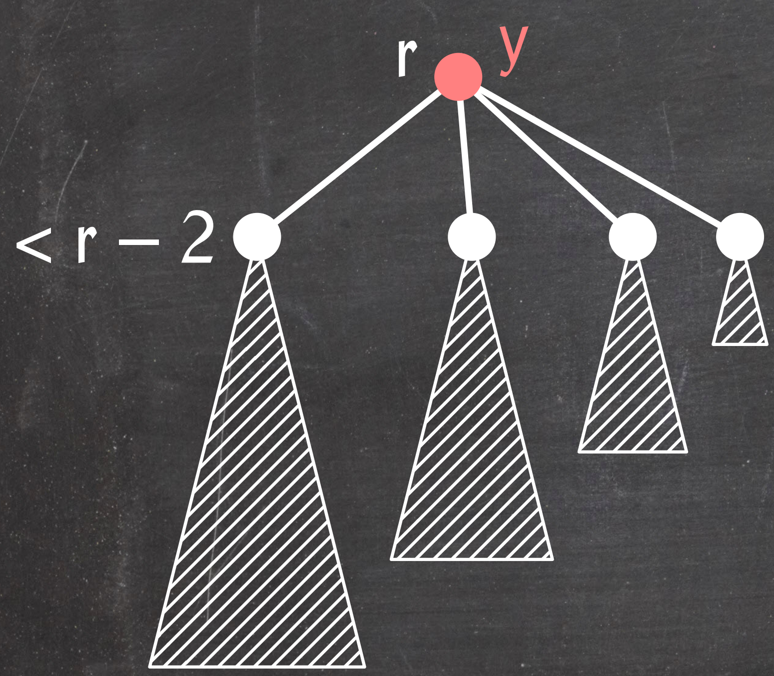
Amortized Cost of Fixing Parent Violations



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- Δ(number of roots) = +1
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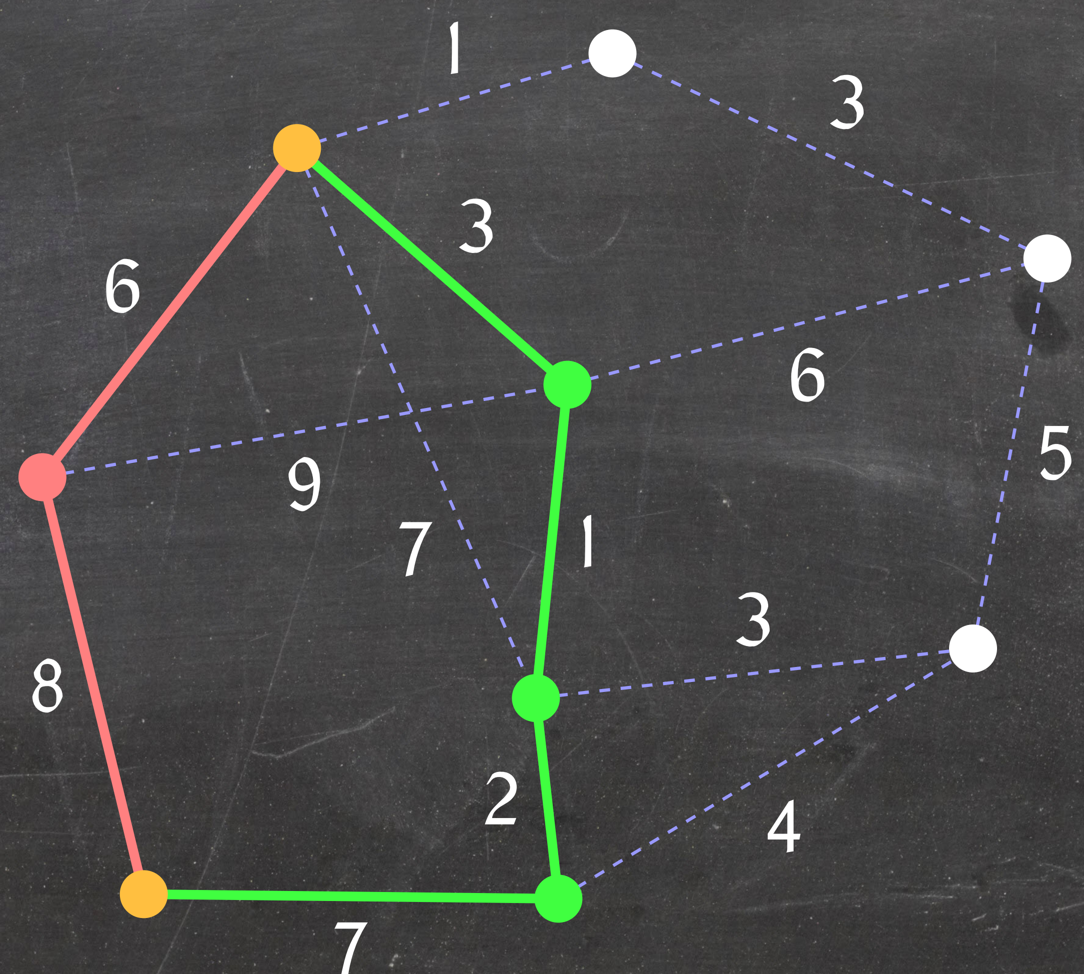
$\Rightarrow \Delta\Phi = +1$

$\Rightarrow \hat{c} \in O(1)$

After this, we're done!

Shortest Path

Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a shortest path from u to v is a path from u to v with minimum total edge weight among all paths from u to v.



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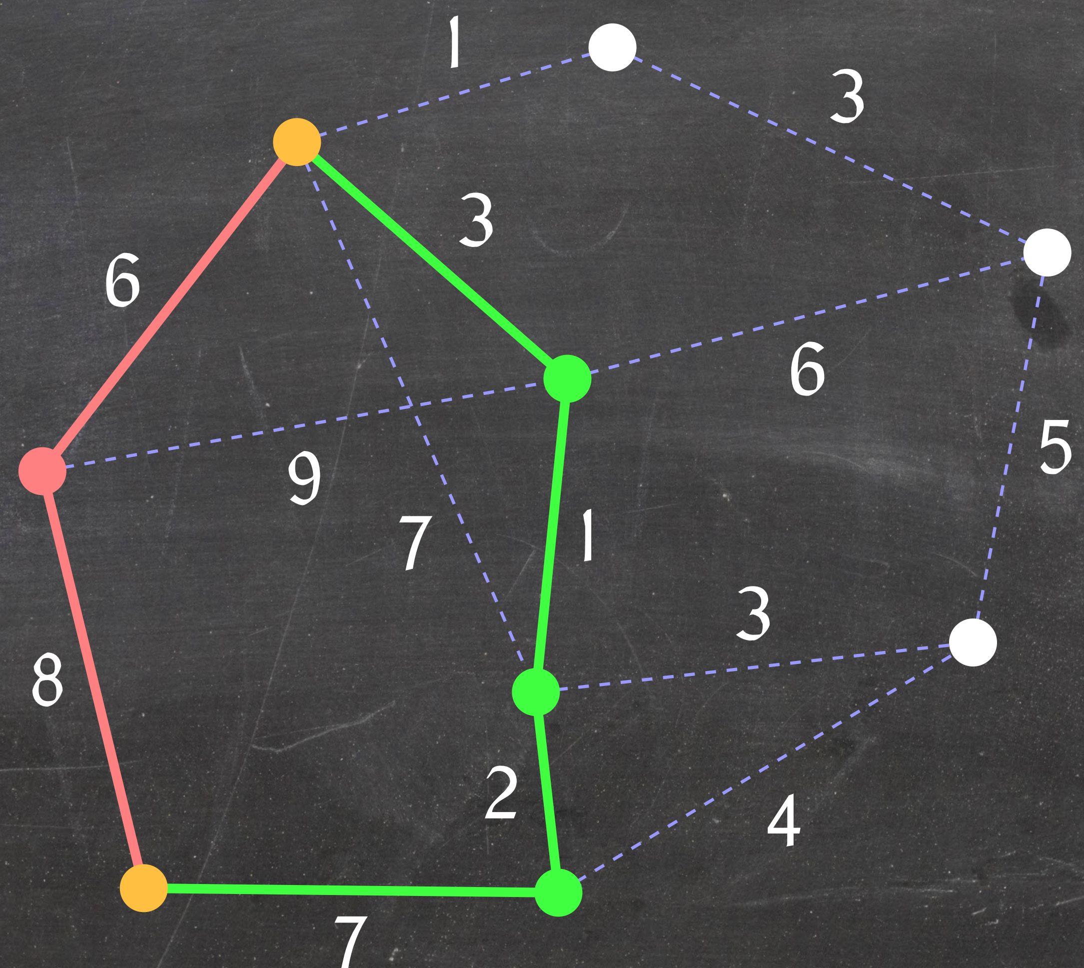
Let the distance dist(s, w) from s to v be the length of a shortest path from s to v.



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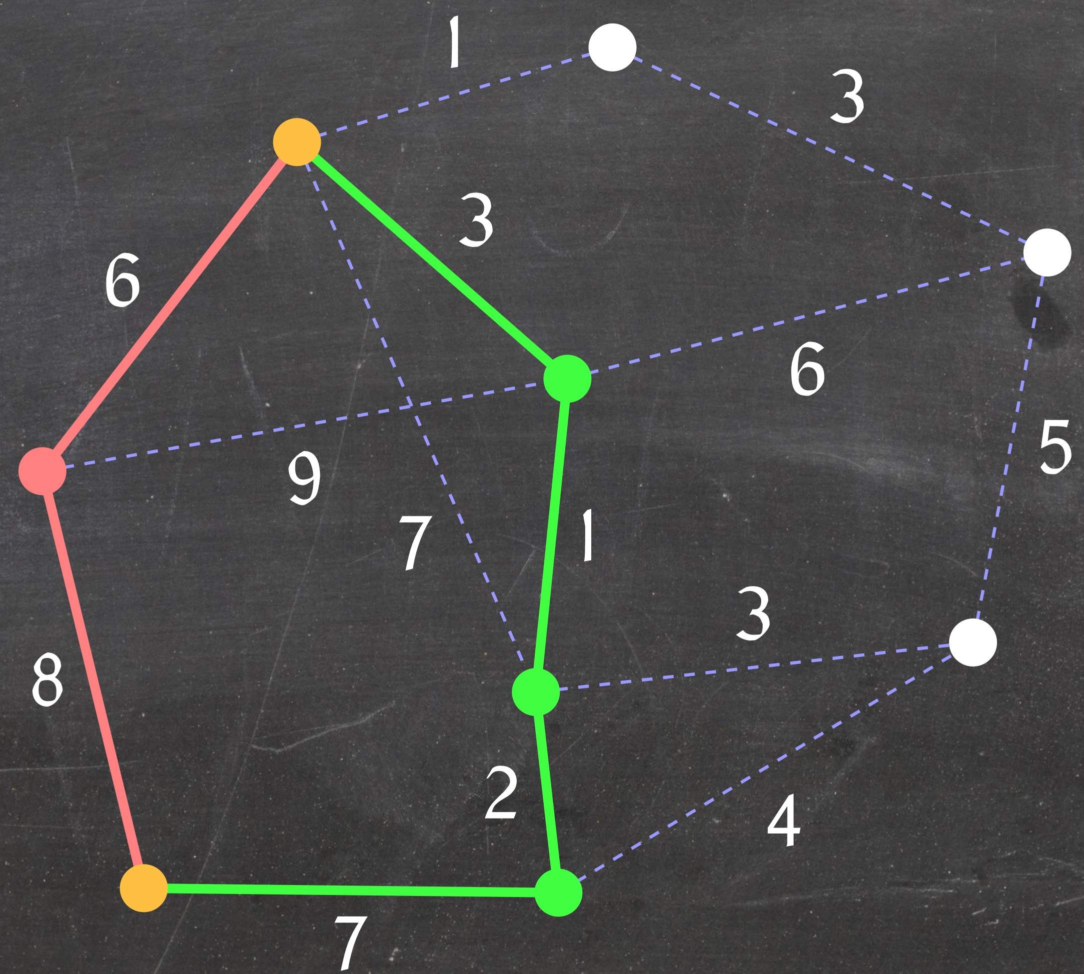

This is well-defined only if there is no negative cycle (cycle with negative total edge weight) that has a vertex on a path from u to v.

Optimal Substructure of Shortest Paths

For a path P and two vertices u and w in P, let P[u, w] be the subpath of P from u to w.

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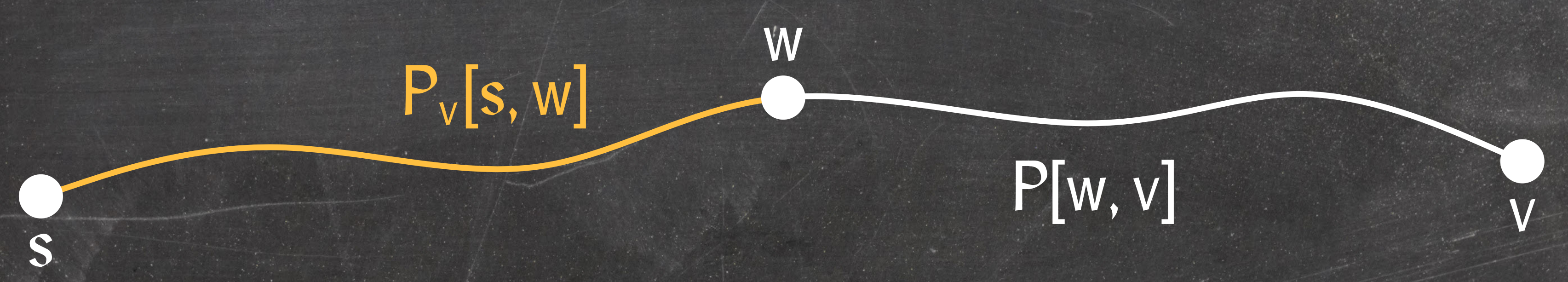
Lemma: If P_v is a shortest path from s to v and w is a vertex in P_v, then $P_v[s, w]$ is a shortest path from s to w.



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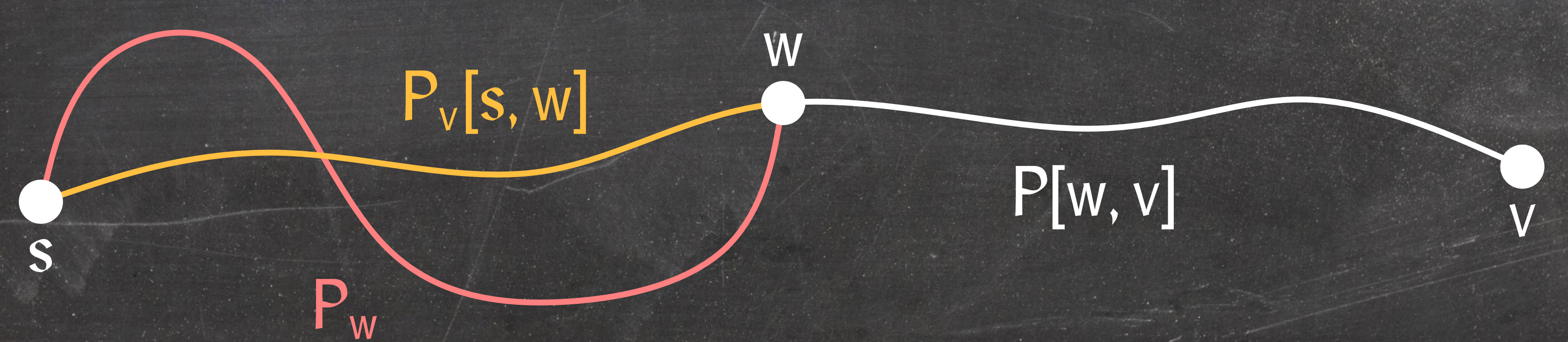
Lemma: If P_v is a shortest path from s to v and w is a vertex in P_v, then $P_v[s, w]$ is a shortest path from s to w.

Assume there exists a path P_w from s to w with $w(P_w) < w(P_v[s, w])$.

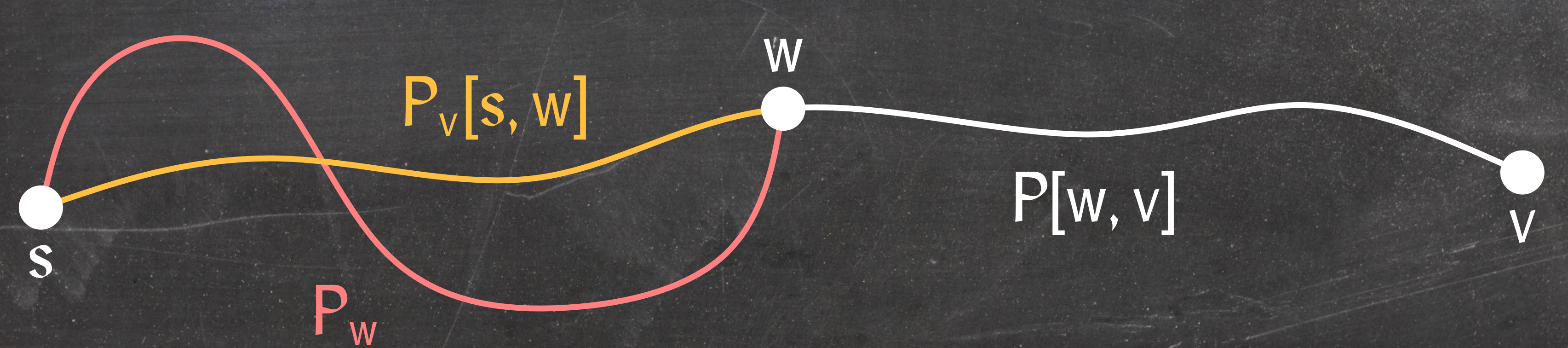


Optimal Substructure of Shortest Paths

For a path P and two vertices u and w in P, let $P[u, w]$ be the subpath of P from u to w.

Lemma: If P_v is a shortest path from s to v and w is a vertex in P_v, then $P_v[s, w]$ is a shortest path from s to w.

Assume there exists a path P_w from s to w with $w(P_w) < w(P_v[s, w])$.

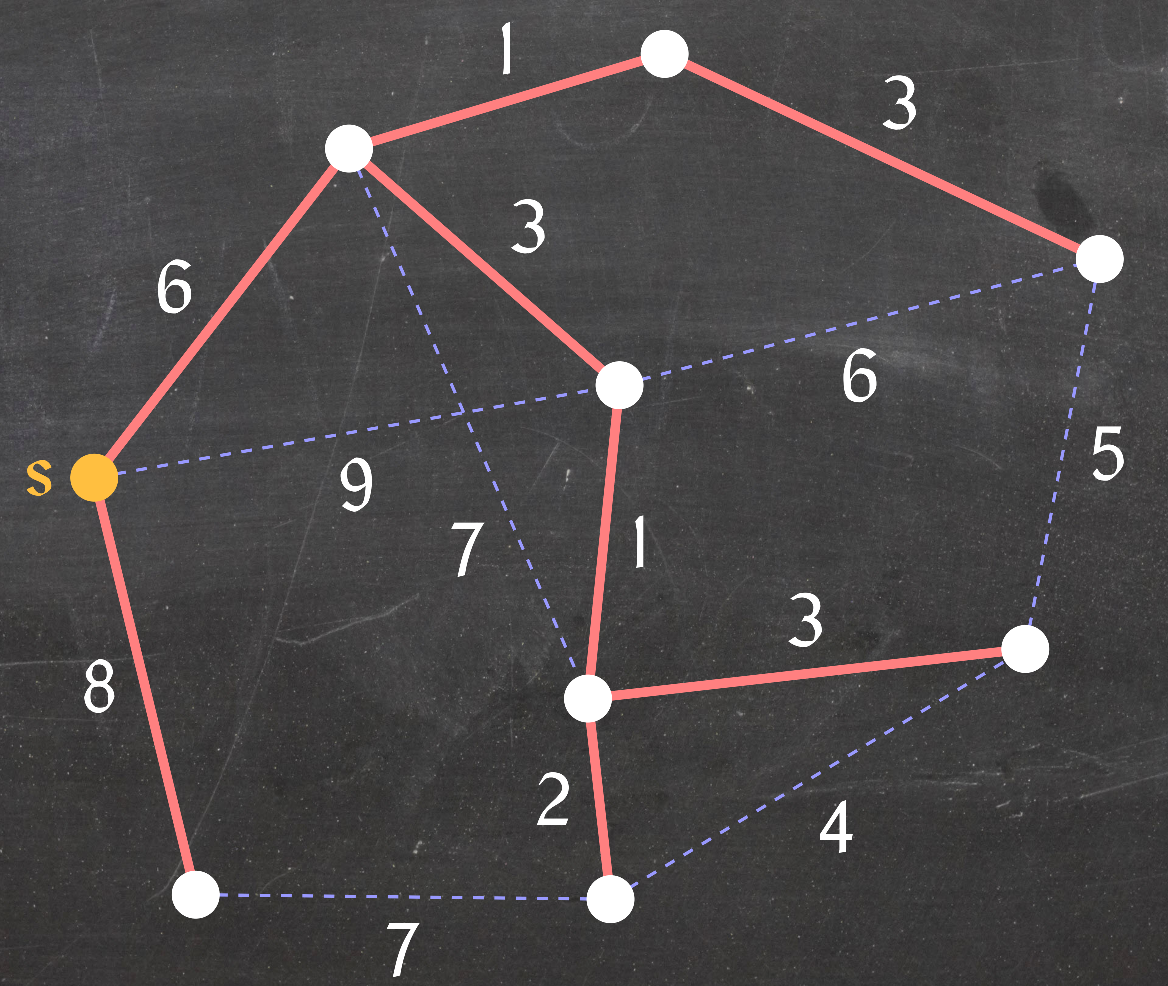


Then $w(P_w \circ P_v[w, v]) < w(P_v[s, w] \circ P_v[w, v]) = w(P_v)$, a contradiction because P_v is a shortest path from s to v.

Shortest Path Tree

For a vertex $s \in G$, let $R(s)$ be the set of vertices reachable from s: for every vertex $v \in R(s)$, there exists a path from s to v.

Lemma: For every node $s \in G$, there exists a collection of paths $\mathbb{S} = \{P_v \mid v \in R(s)\}$ such that P_v is a shortest path from s to v and $\bigcup_{v \in R(s)} P_v$ is a tree.



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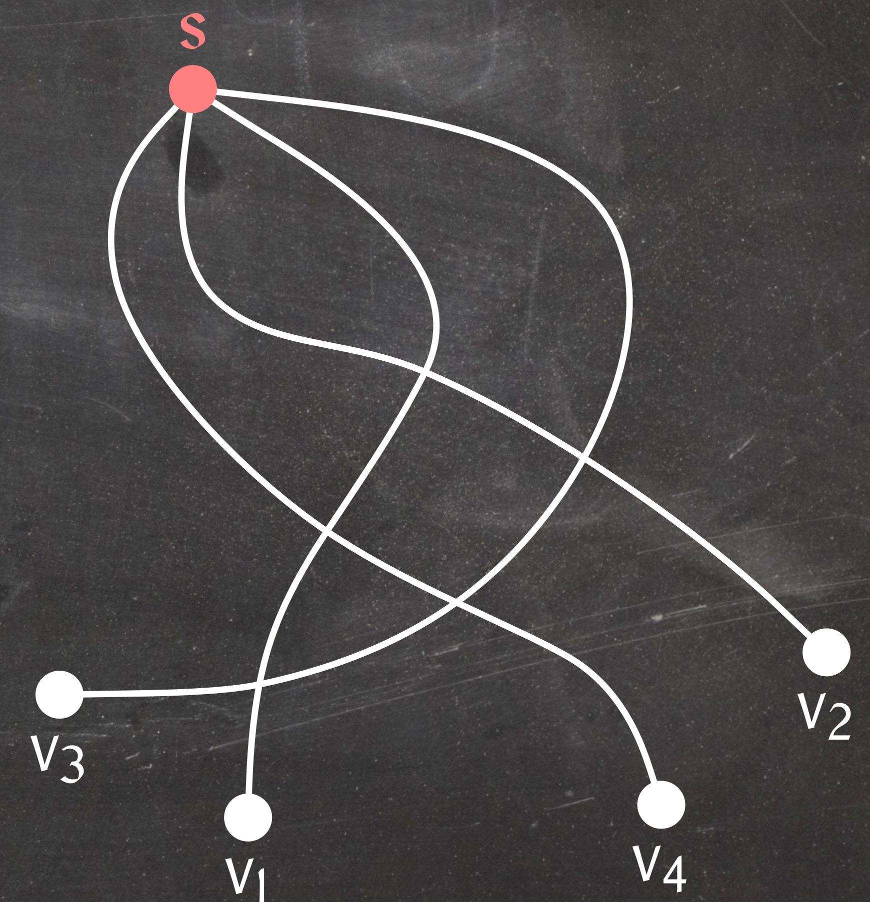
Let $R(s) = \{v_1, v_2, \ldots, v_t\}$ and let $\{P'_{v_1}, P'_{v_2}, \ldots, P'_{v_t}\}$ be a collection of shortest paths from s to these vertices.

We define a sequence of trees $\langle T_1, T_2, \ldots, T_t \rangle$ and shortest paths $\langle P_{v_1}, P_{v_2}, \ldots, P_{v_t} \rangle$ as follows:



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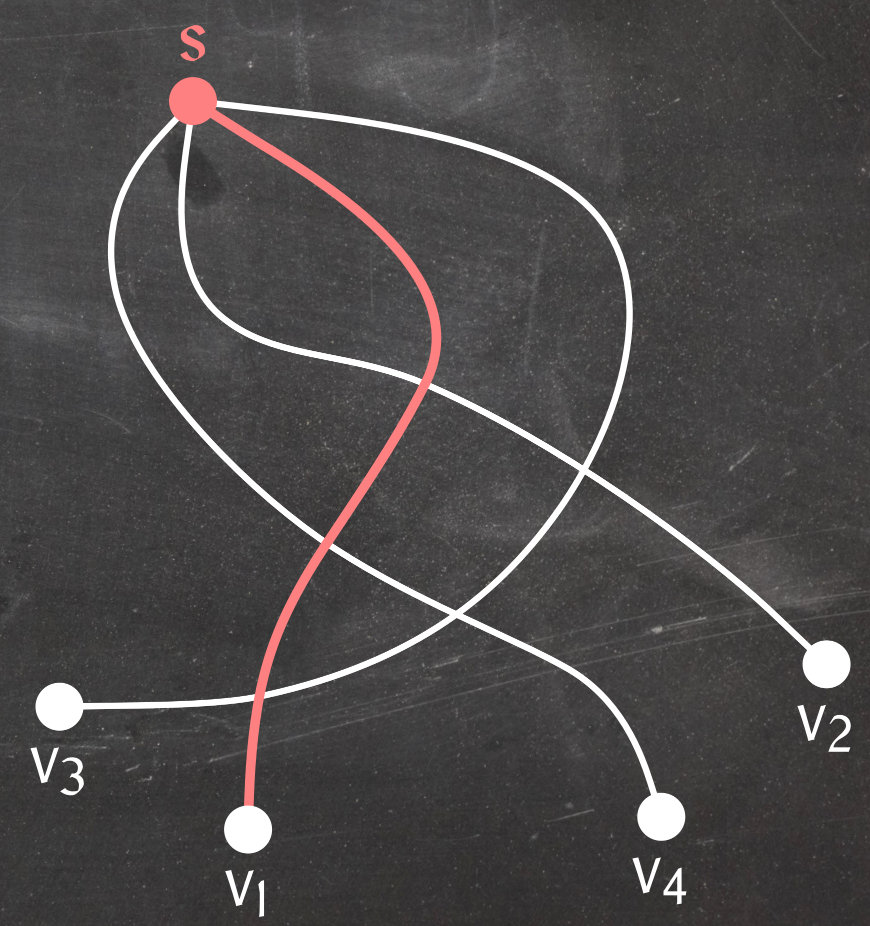

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  - $P_{v_i} = T[s, w] \circ P'_{v_i}[w, v_i]$
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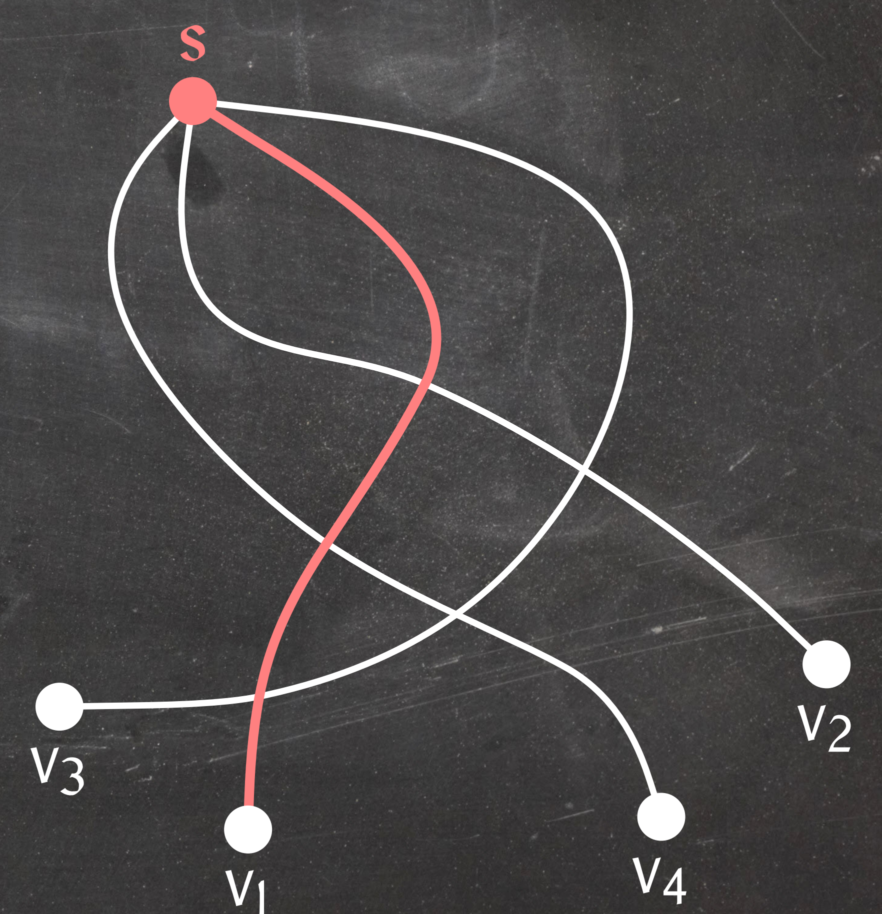

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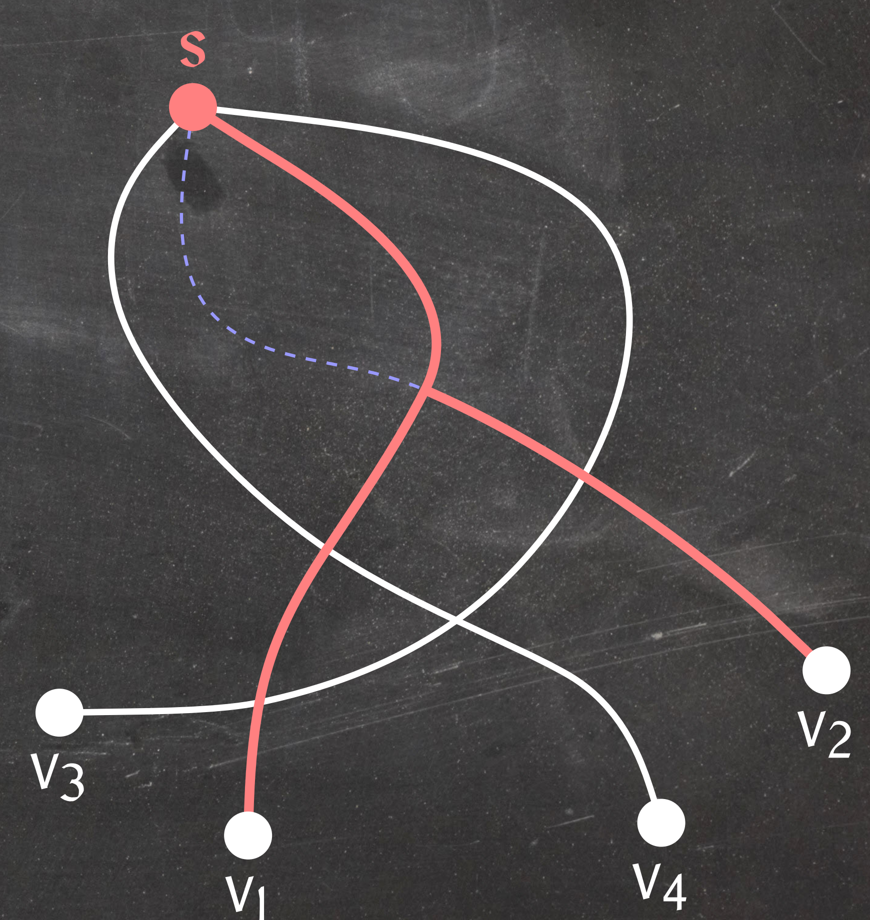

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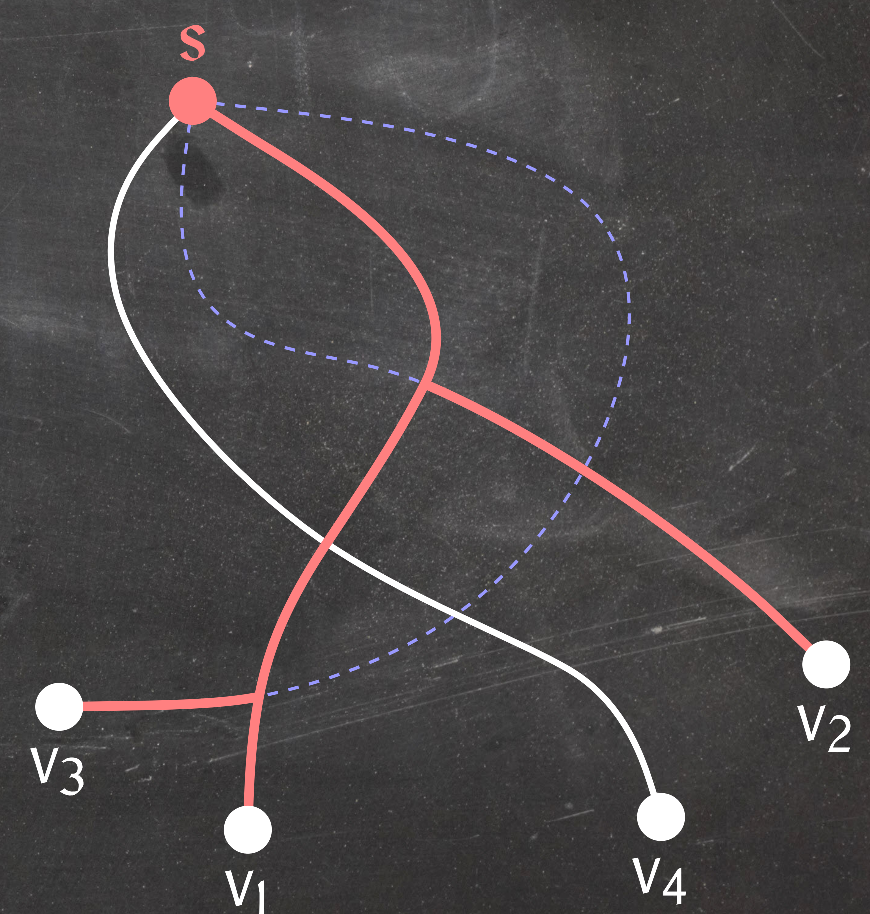

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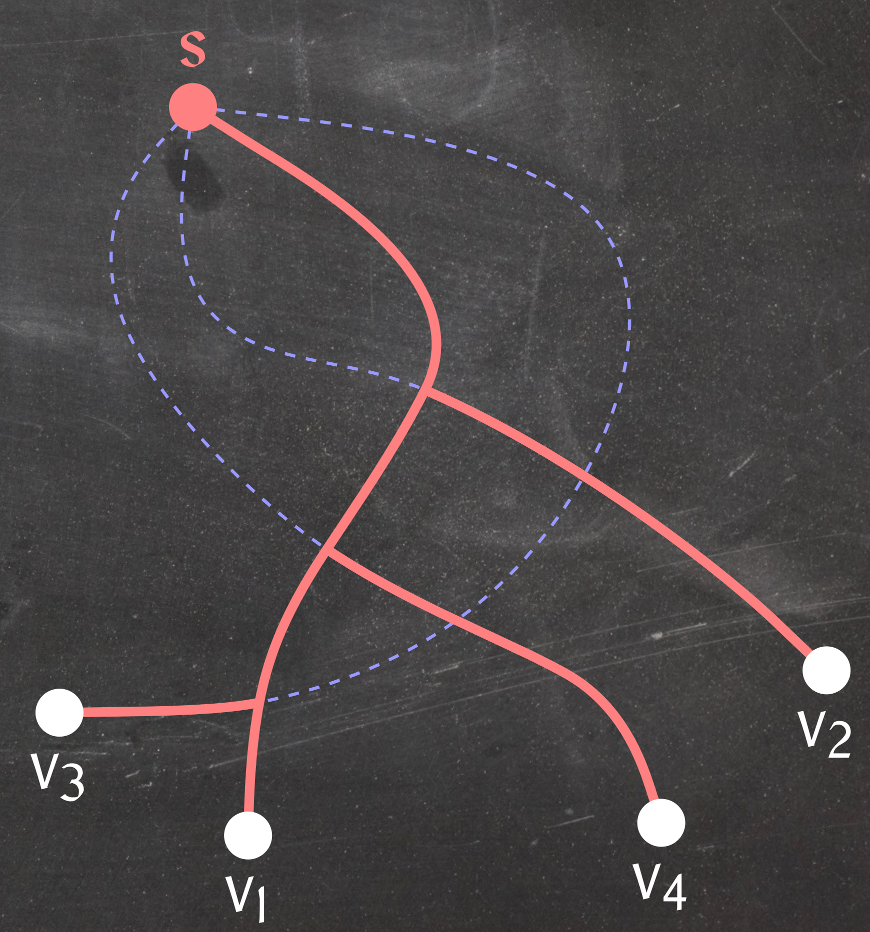

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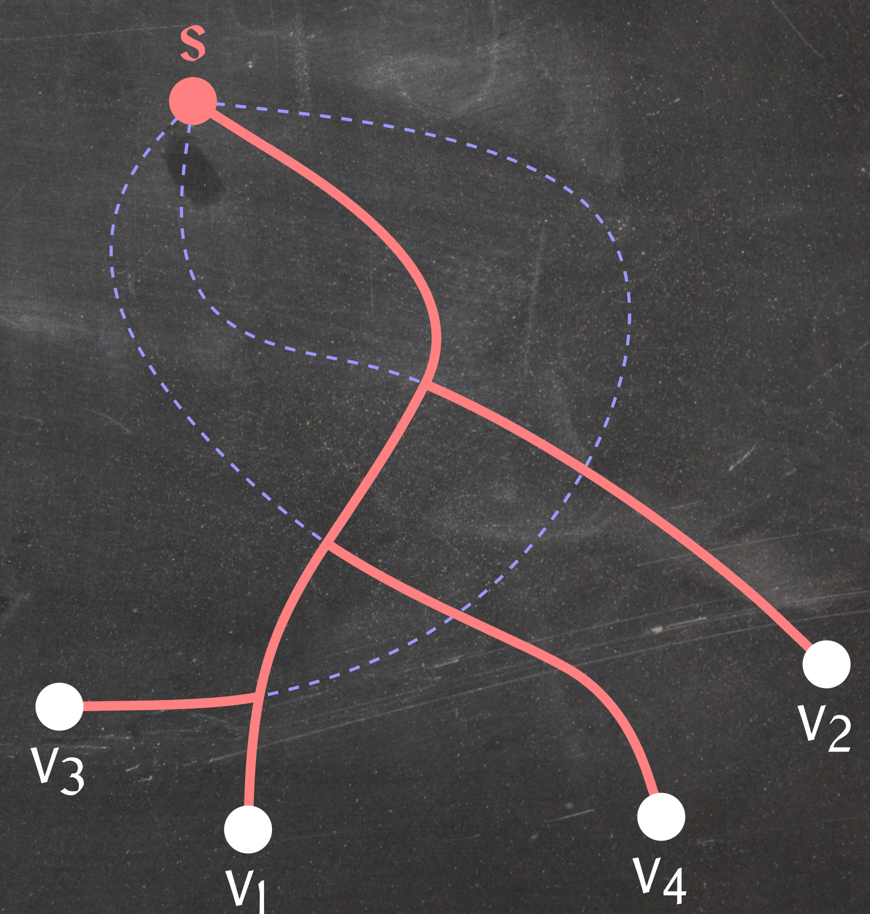
$T_t = \bigcup_{v \in R(s)} P_v$



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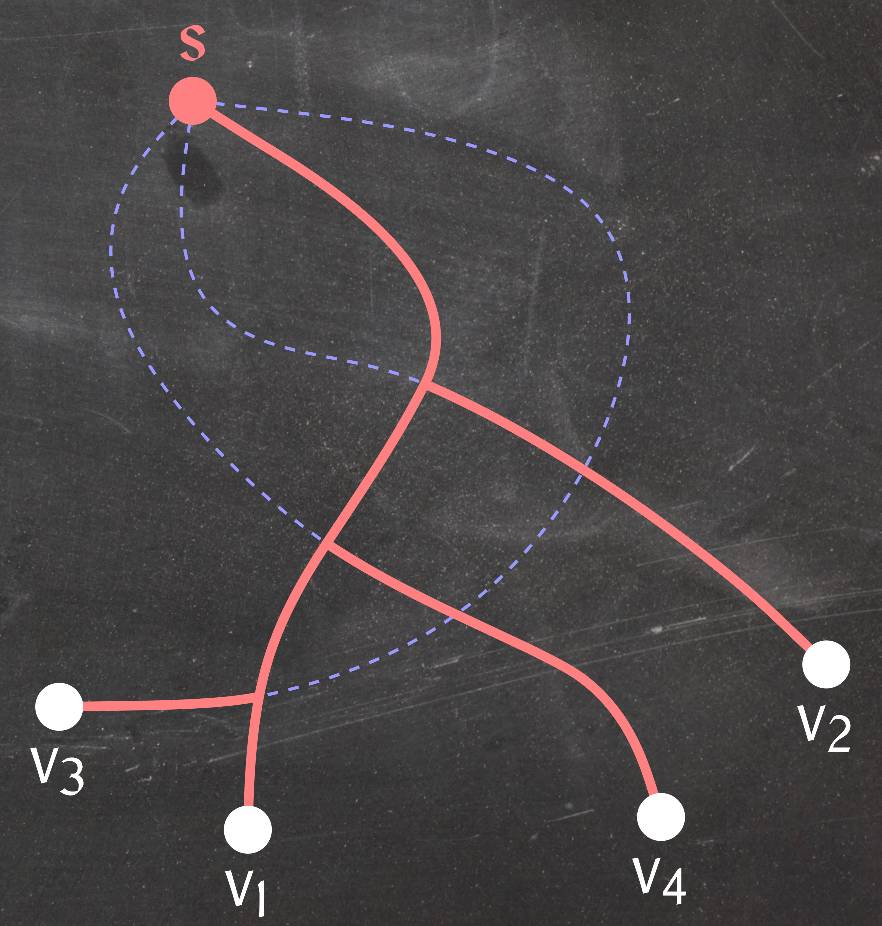
T_t is a tree:

- T_1 is a tree.
- T_i is obtained by adding a path to T_{i-1} that shares only one vertex with T_{i-1}.
- To create a cycle, the added path would have to share two vertices with T_{i-1}.



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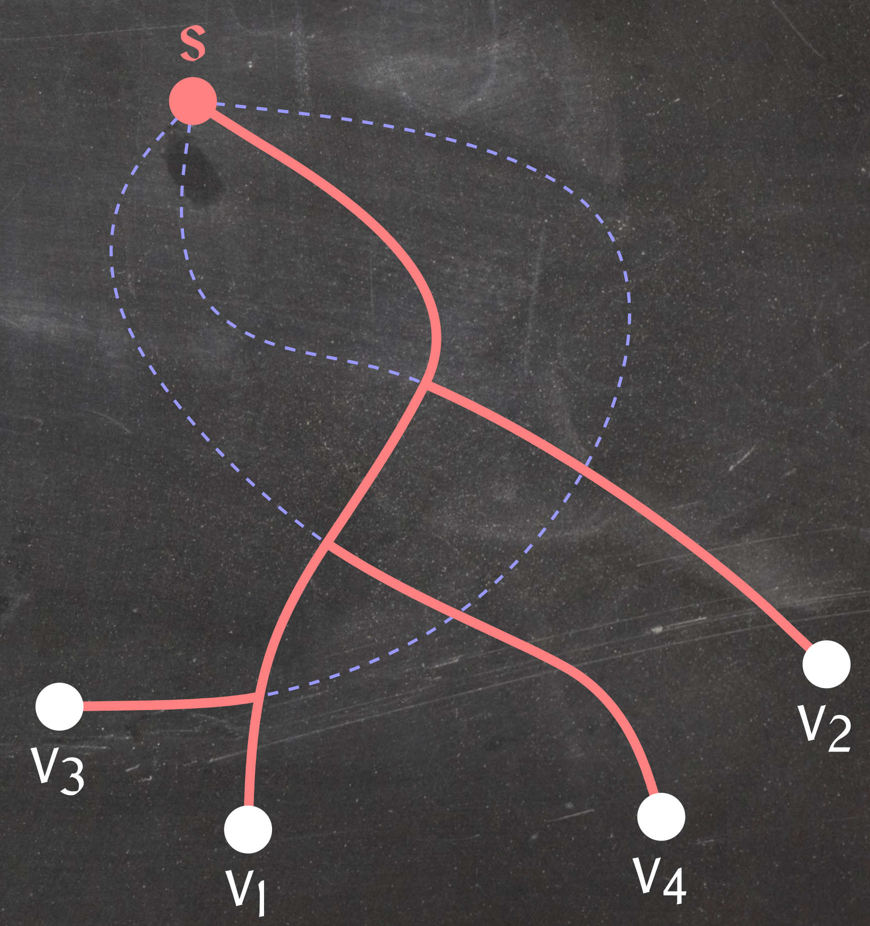
P_v is a shortest path from s to v, for all $v \in R(s)$.



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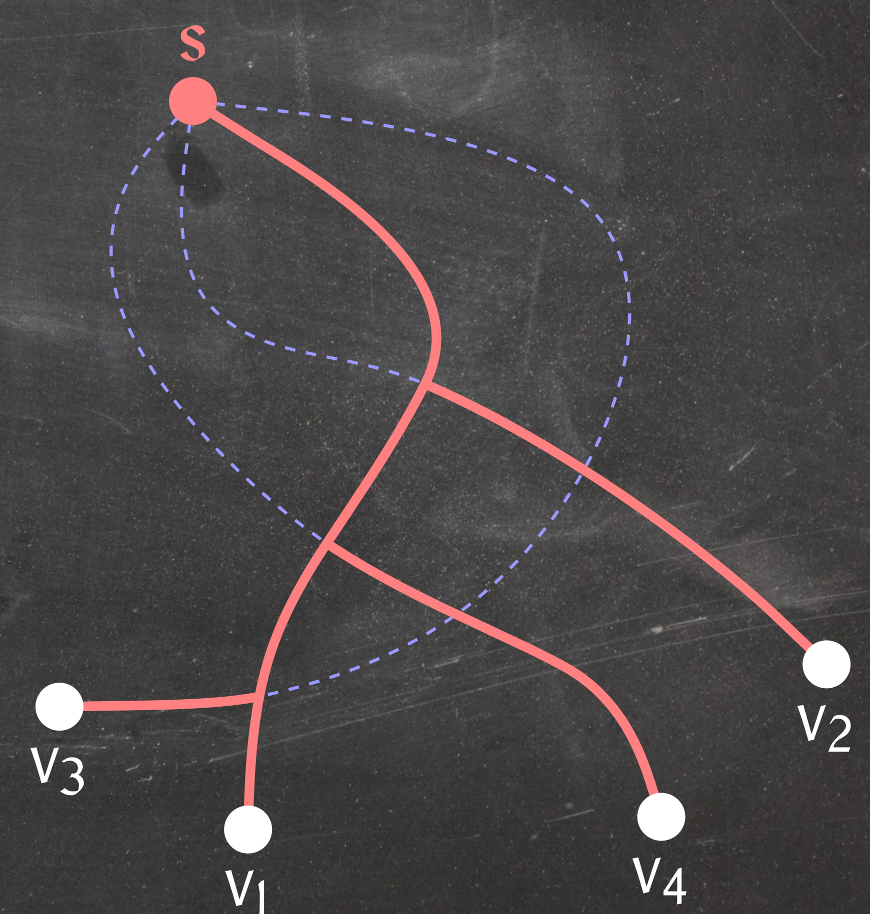
Prove by induction on i that $T_i[s, v]$ is a shortest path from s to v, for all $v \in T_i$.



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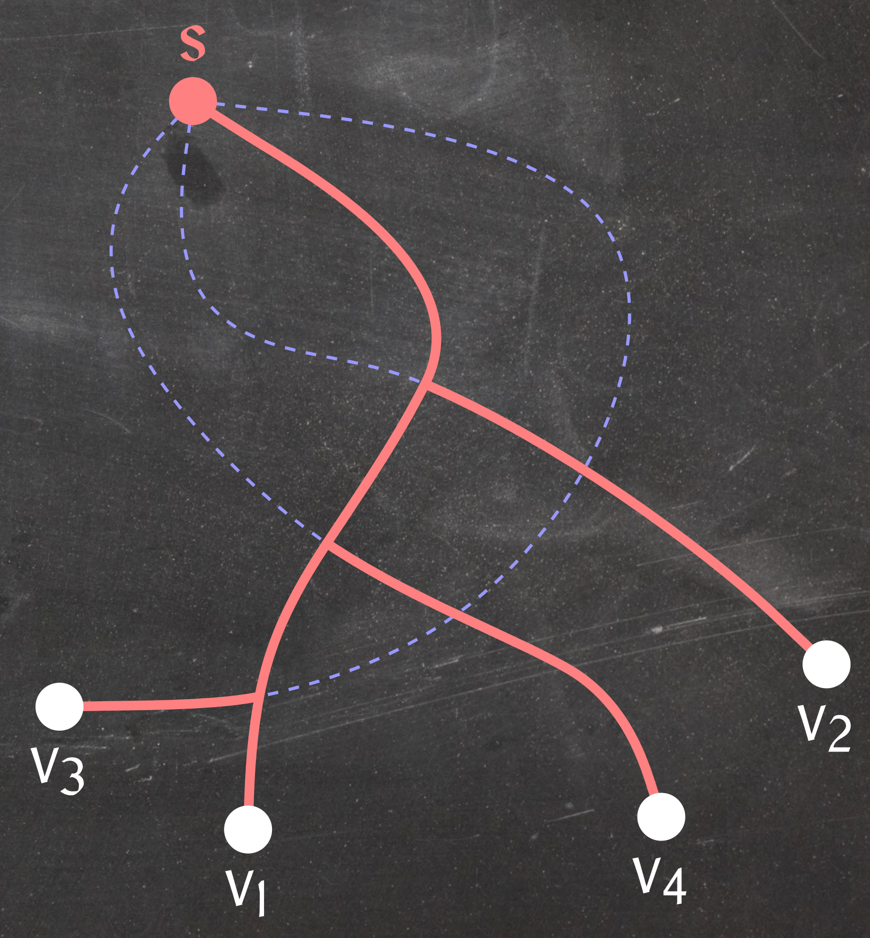
For $i = 1$, $T_1 = P_{v_1} = P'_{v_1}$ is a shortest path from s to v_1. By optimal substructure, $T_1[s, v] = P'_{v_1}[s, v]$ is a shortest path from s to v for all $v \in T_1$.



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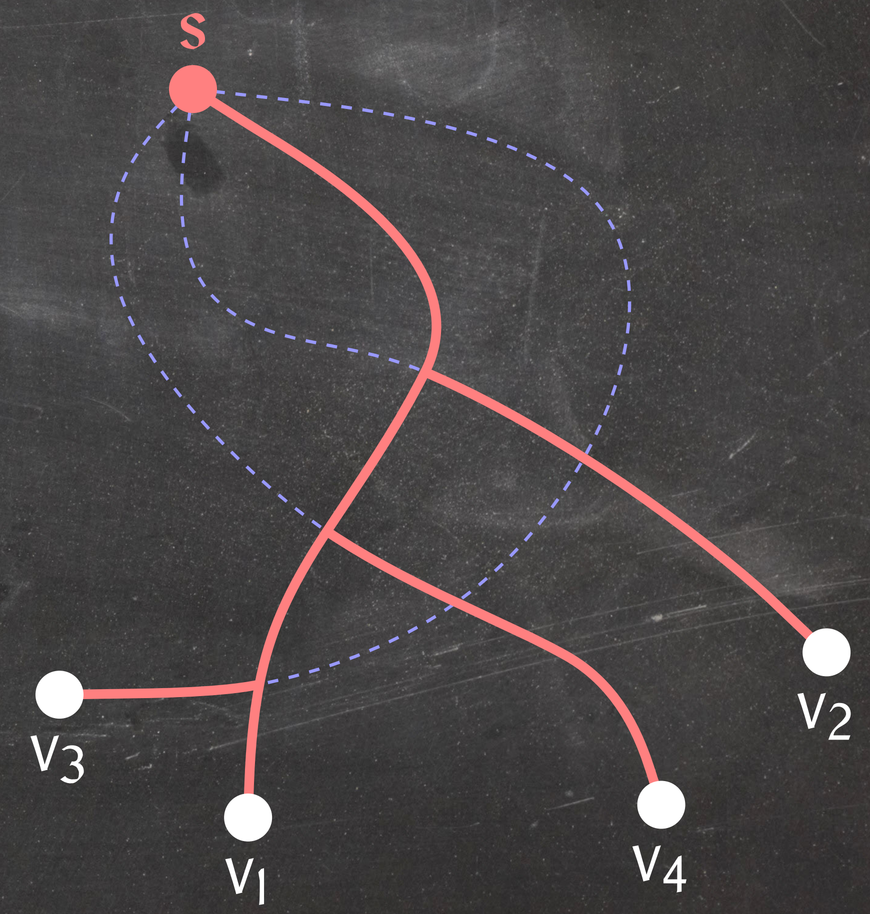
For $i > 1$, $T_{i-1}[s, v]$ is a shortest path from s to v for all $v \in T_{i-1}$, by the inductive hypothesis.



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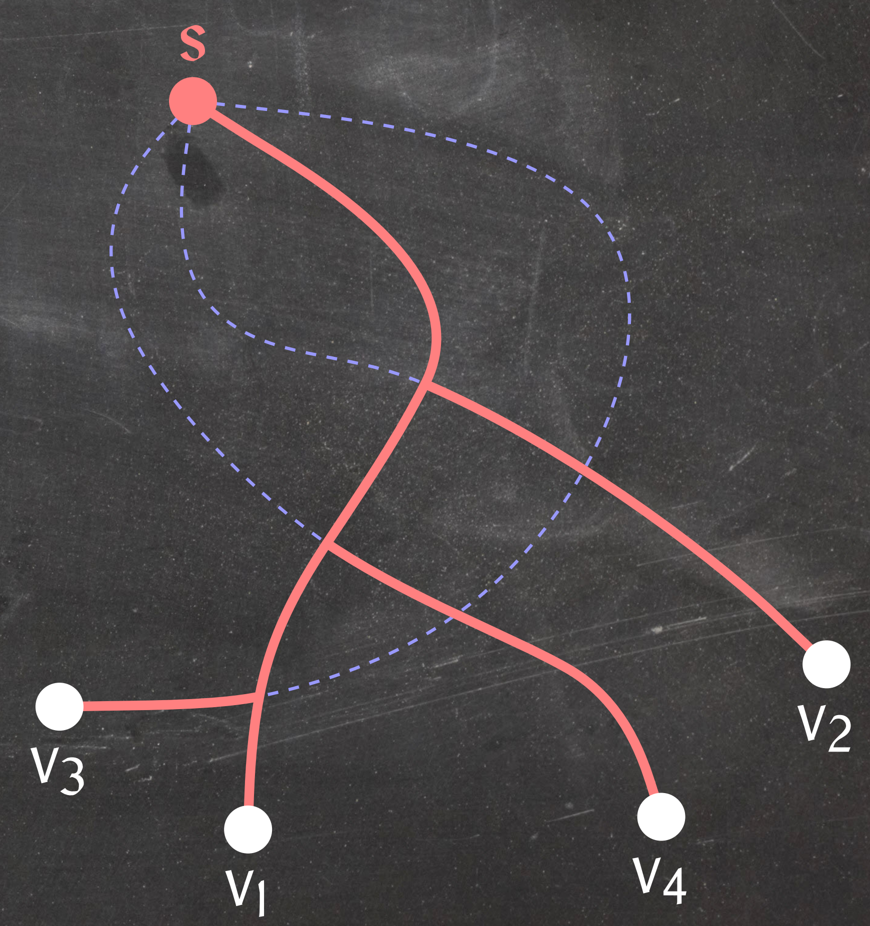
Thus, $w(T_{i-1}[s, w]) \le w(P'_{v_i}[s, w])$ and therefore $w(P_{v_i}) = w(T_{i-1}[s, w]) + w(P'_{v_i}[w, v_i]) \le w(P'_{v_i})$.



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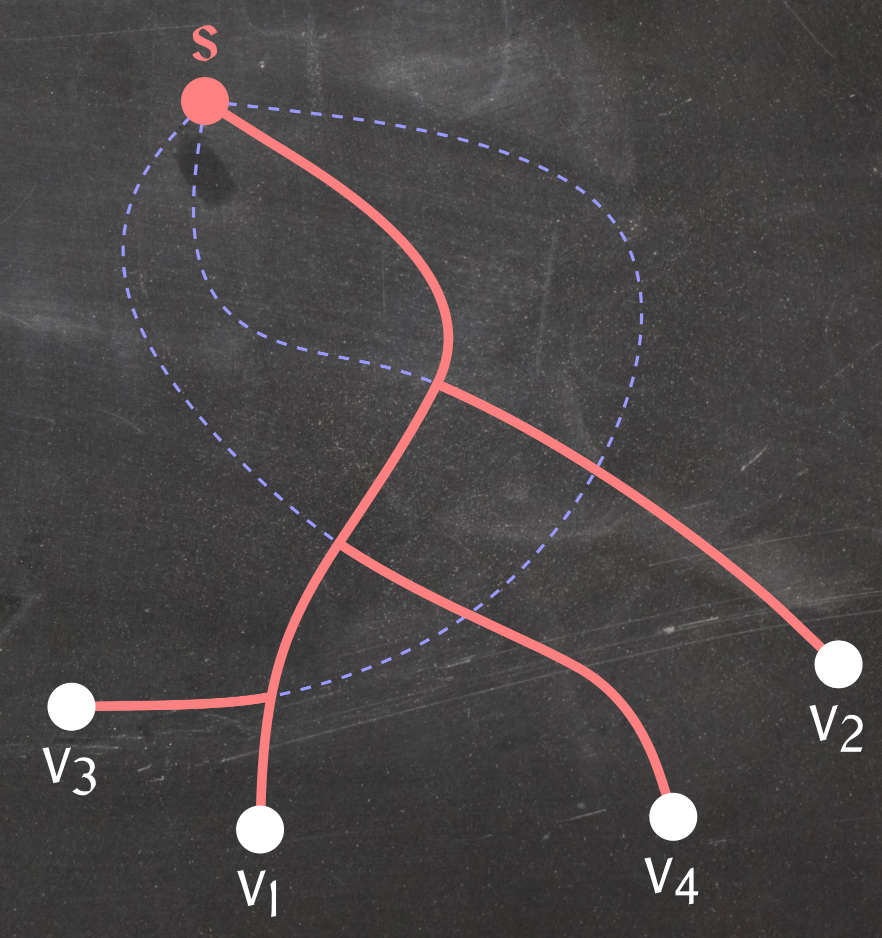
Since P'_{v_i} is a shortest path from s to v_i, so is P_{v_i}.



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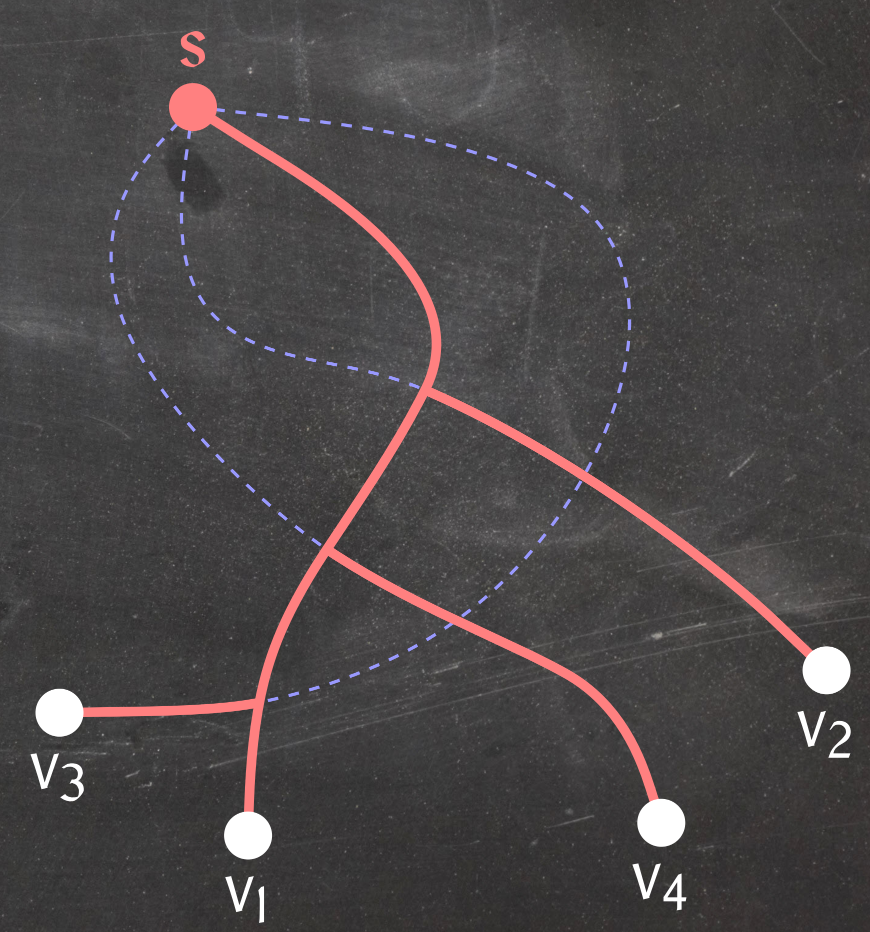
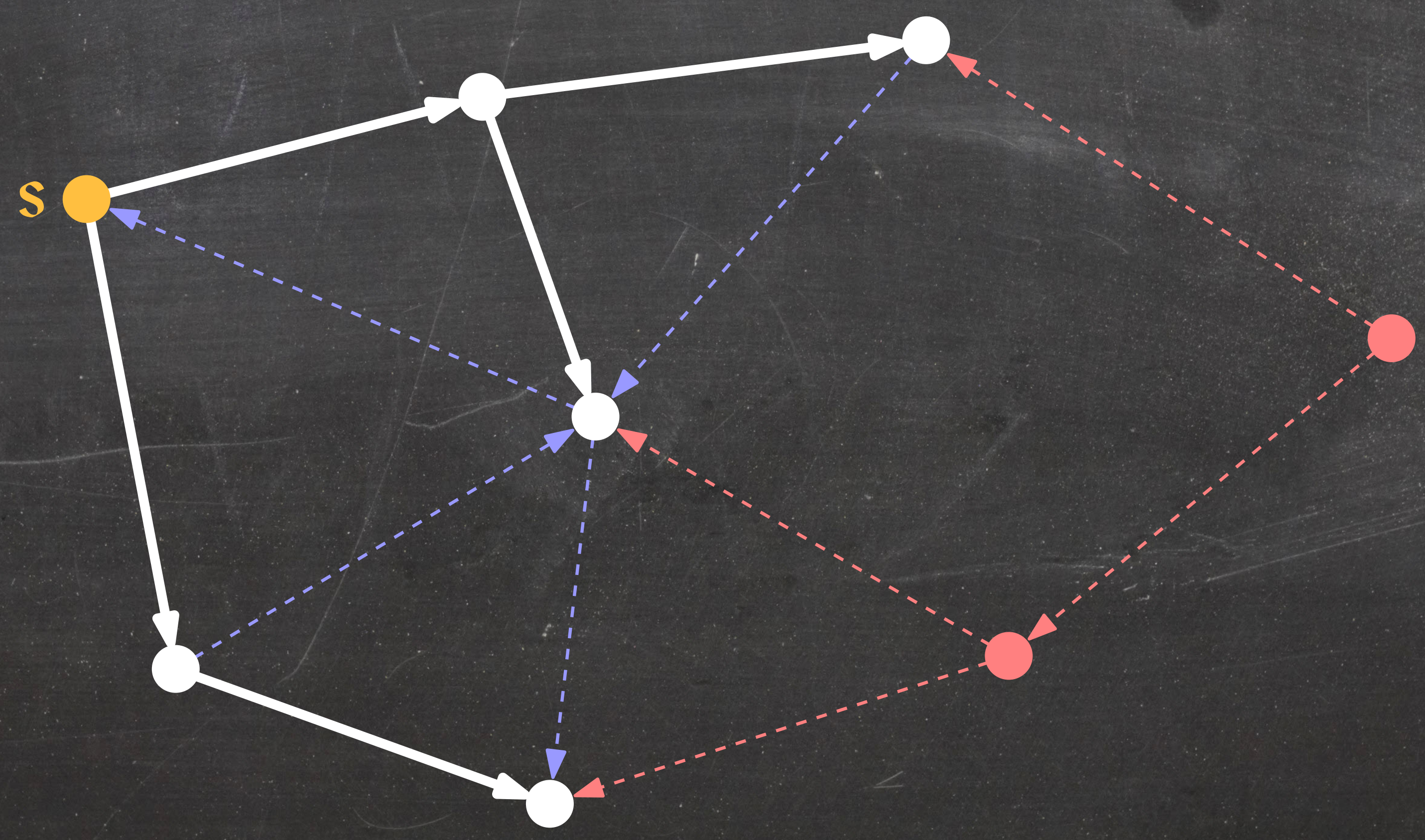


A Characterization of Shortest Path Trees

An out-tree of s is a spanning tree T of $G[R(s)] = (R(s), E[R(s)])$, where $E[R(s)] = \{(v, w) \in E \mid v, w \in R(s)\}$, such that there exists a path from s to v in T, for all $v \in R(s)$.



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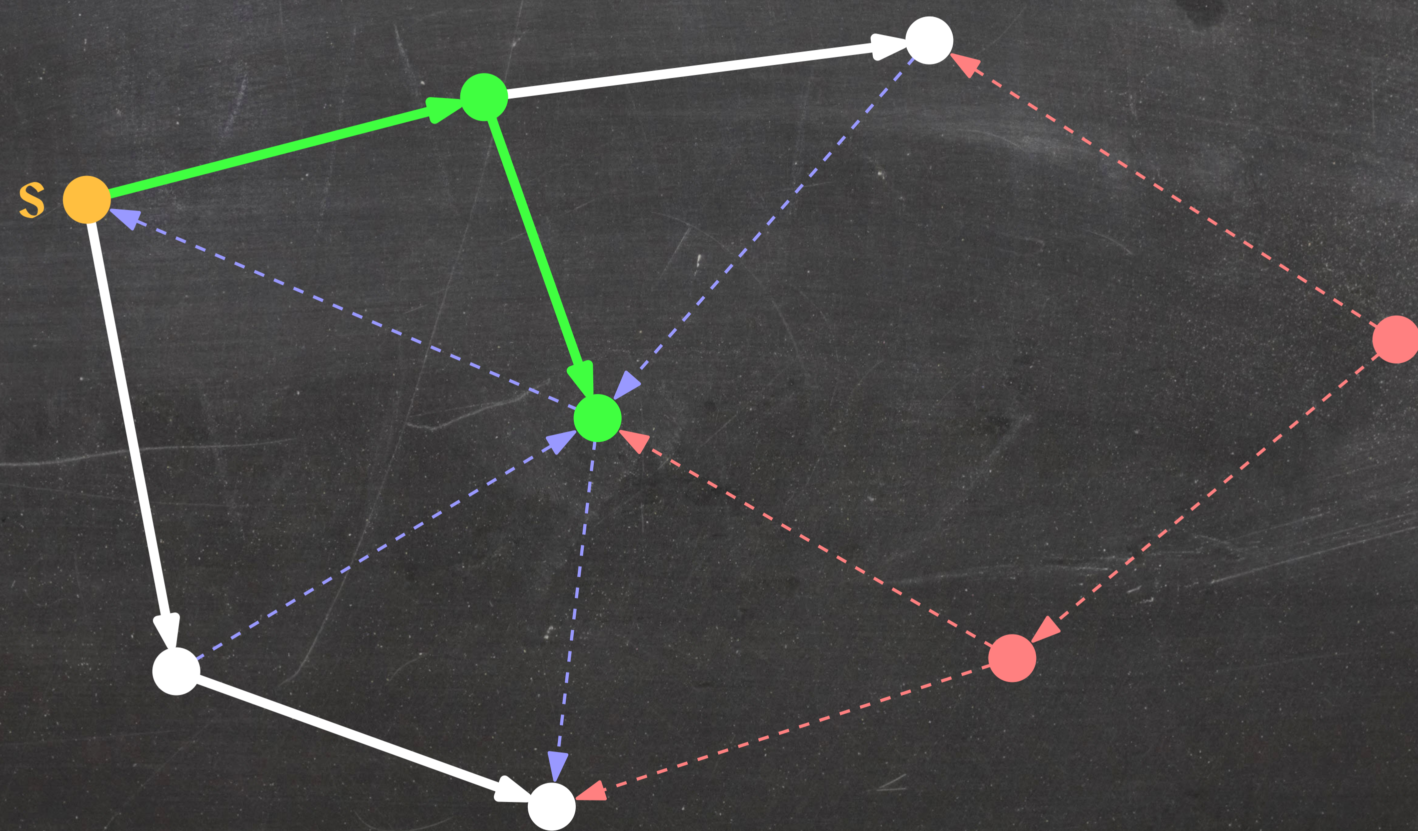
For an out-tree T of s and every $v \in T$, let $d_T(v) = w(T[s, v])$.



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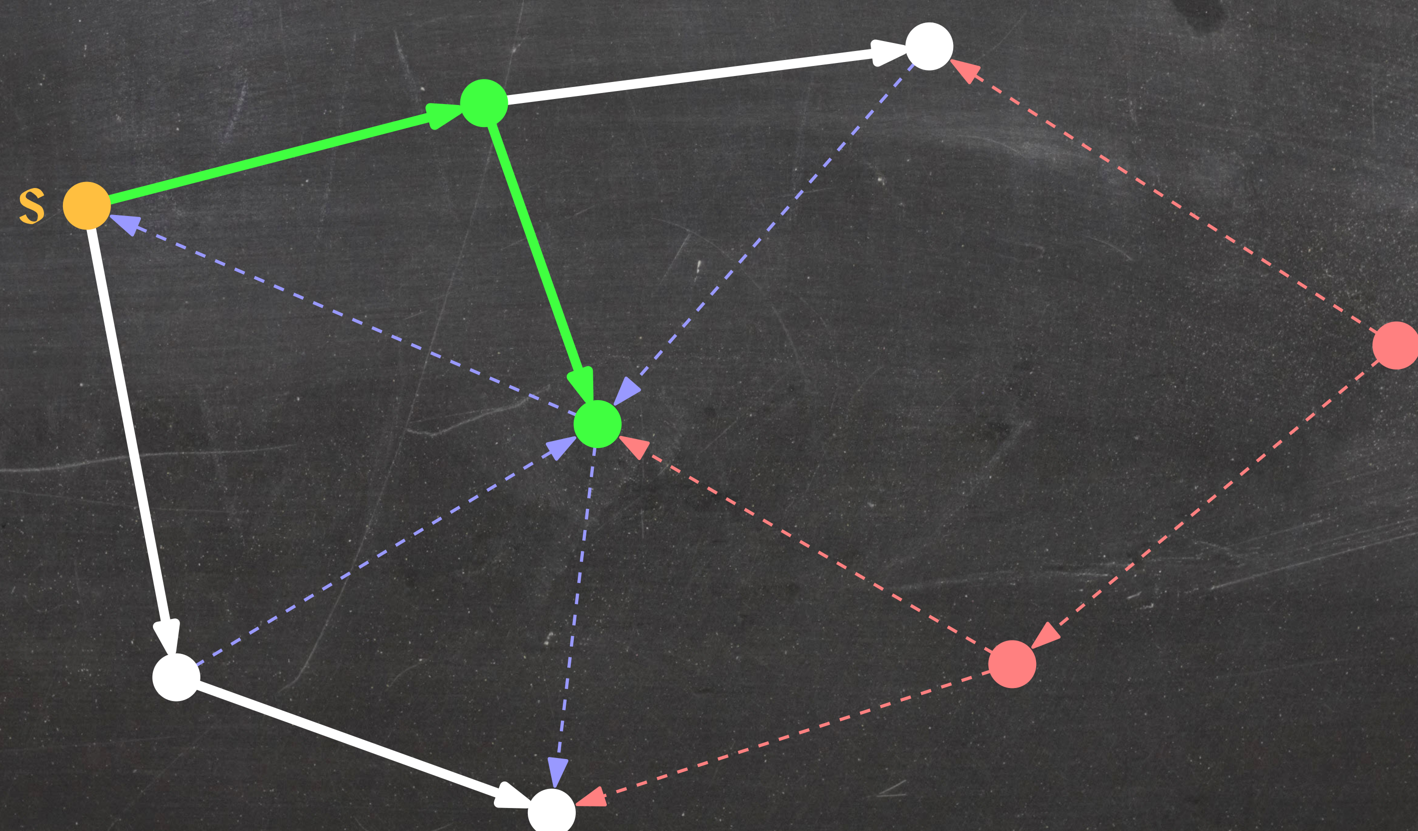
Let $D(T) = \sum_{v \in R(s)} d_T(v)$.



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For an out-tree T of s and every $v \in T$, let $d_T(v) = w(T[s, v])$.

Let $D(T) = \sum_{v \in R(s)} d_T(v)$.

Lemma: An out-tree T of s is a shortest path tree if and only if D(T) is minimal among all out-trees of s.

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Lemma: An out-tree T of s is a shortest path tree if and only if $D(T)$ is minimal among all out-trees of s.

Let T and T' be two out-trees of s such that

- T is a shortest path tree and
- $D(T')$ is minimal among all out-trees of s. In particular, $D(T') \leq D(T)$.

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If $D(T') < D(T)$, there exists some vertex $v \in R(s)$ such that $d_{T'}(v) < d_T(v)$.

A Characterization of Shortest Path Trees

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For an out-tree T of s and every $v \in T$, let $d_T(v) = w(T[s, v])$.

Let $D(T) = \displaystyle\sum_{v \in R(s)} d_T(v)$.

Lemma: An out-tree T of s is a shortest path tree if and only if $D(T)$ is minimal among all out-trees of s.

Let T and T' be two out-trees of s such that

- T is a shortest path tree and
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If $D(T') < D(T)$, there exists some vertex $v \in R(s)$ such that $d_{T'}(v) < d_T(v)$.

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$\Rightarrow$ $D(T) = D(T')$.

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$\Rightarrow$ T' is a shortest path tree.

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Build a shortest-path tree by starting with s and adding vertices in R(s) one by one.

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Dijkstra(G, s)

$T = (\{s\}, \emptyset)$
while some vertex in T has an out-neighbour not in T
  do choose an edge (u, v) such that
    - $u \in T$,
    - $v \notin T$, and
    - $d_T(u) + w(u, v)$ is minimized.
  add v and (u, v) to T
return T

Dijkstra's Algorithm

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```
T = (V, ∅)
mark every vertex of G as unexplored
set d(v) = +∞ and e(v) = nil for every vertex v ∈ G
mark s as explored and set d(v) = 0
Q = an empty priority queue
for every edge (s, v) incident to s
    do Q.insert(v, w(s, v))
       d(v) = w(s, v)
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while not Q.isEmpty()
    do u = Q.deleteMin()
       mark u as explored
       add e(u) to T
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⇒ Dijkstra's algorithm takes $O(n \lg n + m)$ time.

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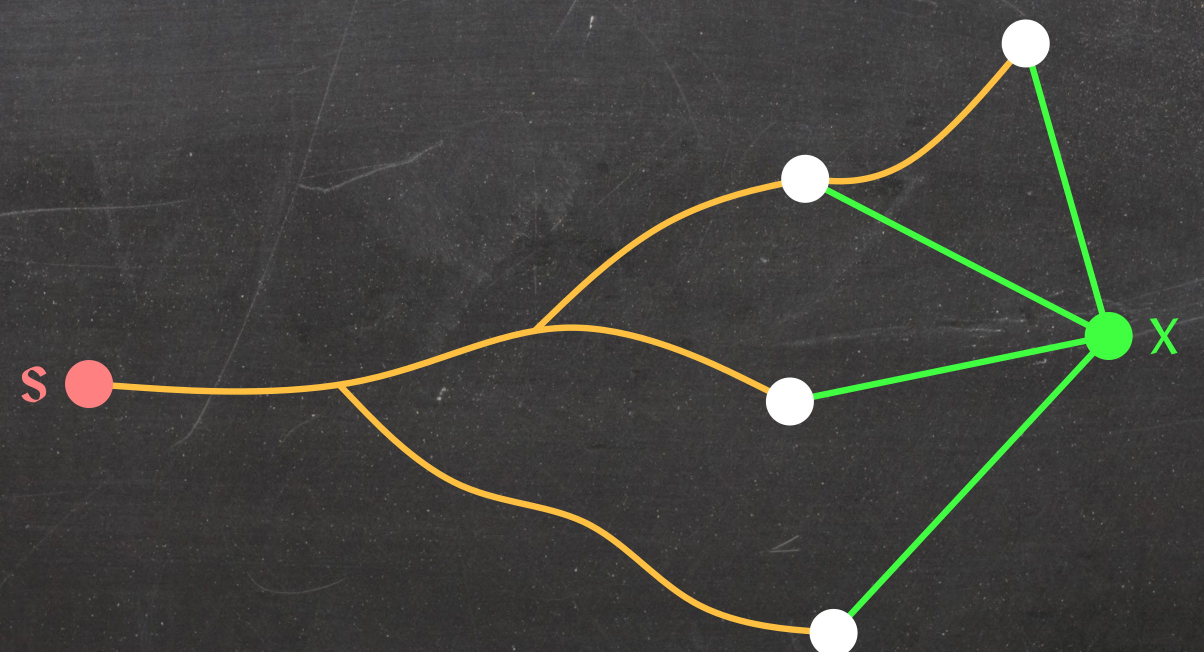
For every vertex $x \notin T$, we have

$$d(x) = \min_{\substack{(u,x) \in E \\ u \in T}} d(u) + w(u, x) = \min_{\substack{(u,x) \in E \\ u \in T}} \mathrm{dist}(s, u) + w(u, x).$$



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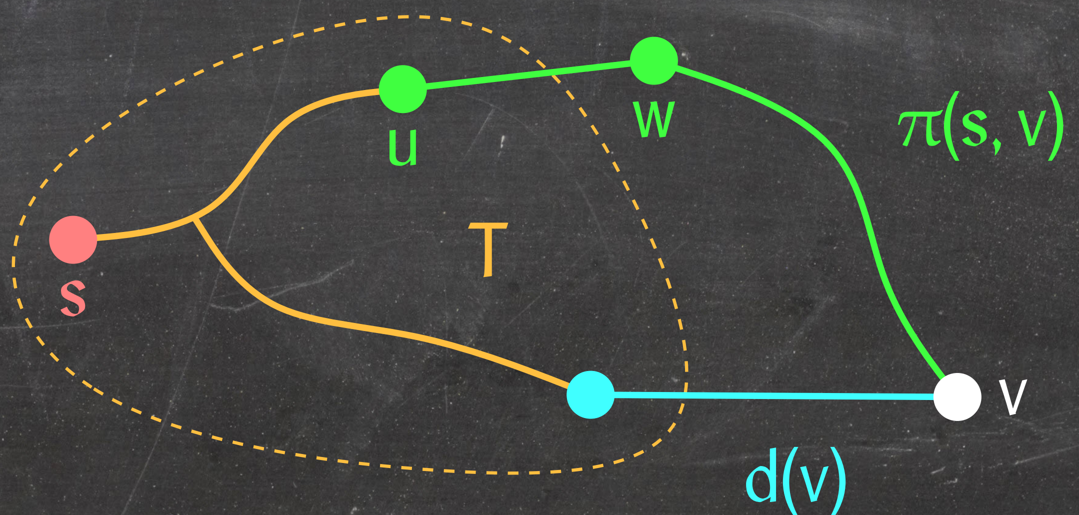
The shortest path $\pi(s, v)$ from s to v must include a vertex $w \notin T$ whose predecessor u in $\pi(s, v)$ belongs to T.



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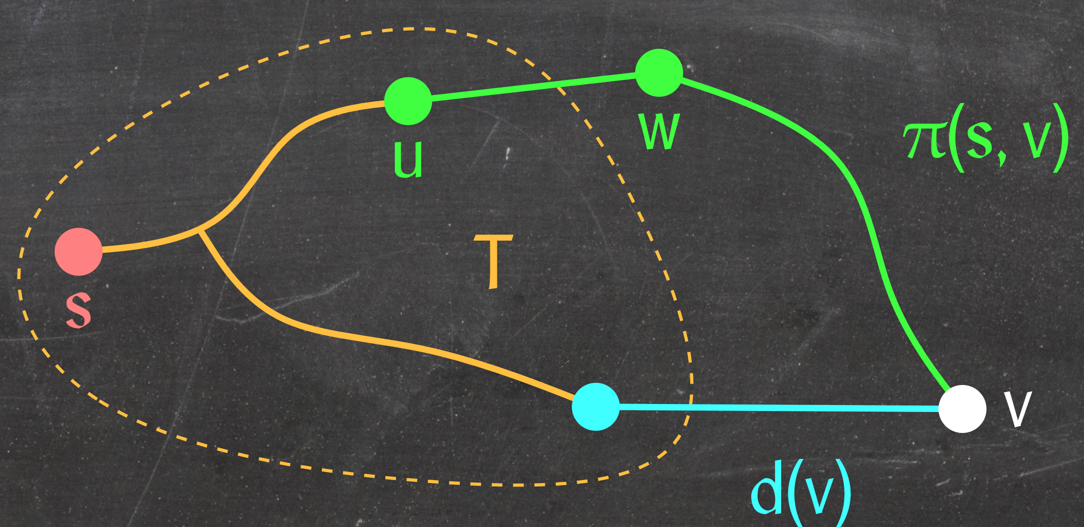

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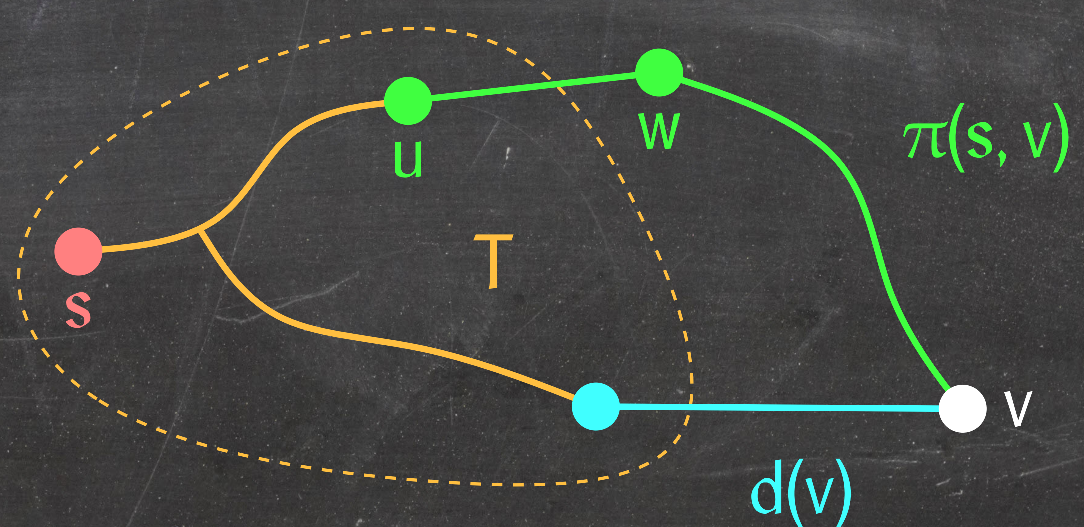

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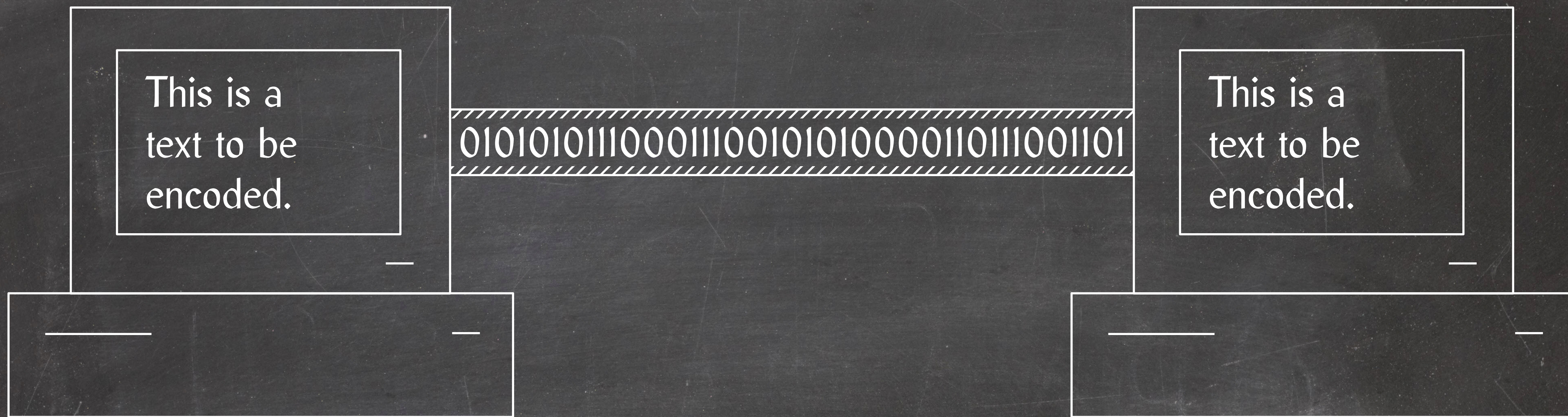
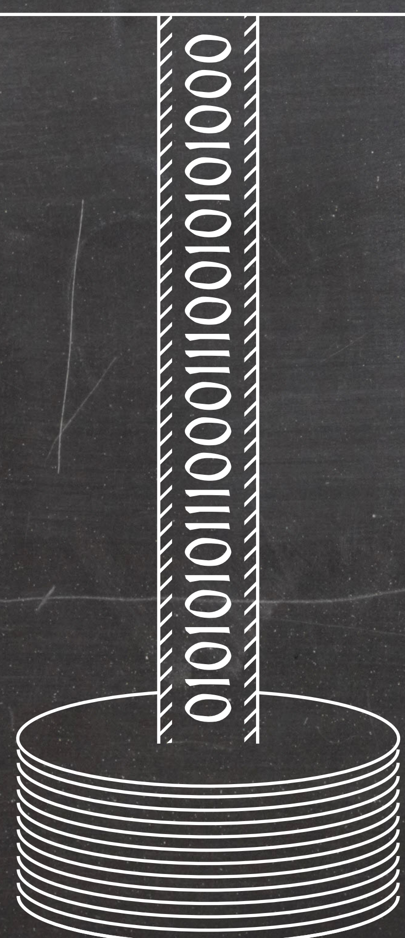
$\Rightarrow$ v is not the next vertex we add to T, a contradiction.

Minimum Length Codes





Goal:

- Encode a given text using as few bits as possible:
  - Limit amount of disk space required to store the text.
  - Send the text over a potentially slow network.
  - ...

Codes That Can Be Decoded

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Non-prefix-free codes cannot always be decoded uniquely!

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$$\Rightarrow \quad C(\langle x_1, x_2, \ldots, x_{i-1} \rangle) = C(\langle y_1, y_2, \ldots, y_{i-1} \rangle) \text{ and}$$
$$C(\langle x_i, x_{i+1}, \ldots, x_m \rangle) = C(\langle y_i, y_{i+1}, \ldots, y_n \rangle).$$

$C(T)$	$C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$	$C(x_i)$	$C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$
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Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

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Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

Since both $C(x_i)$ and $C(y_i)$ are prefixes of $C(\langle x_i, x_{i+1}, \ldots, x_m \rangle)$, $C(x_i)$ must be a prefix of $C(y_i)$, a contradiction.

$C(T)$	$C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$	$C(x_i)$	$C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$
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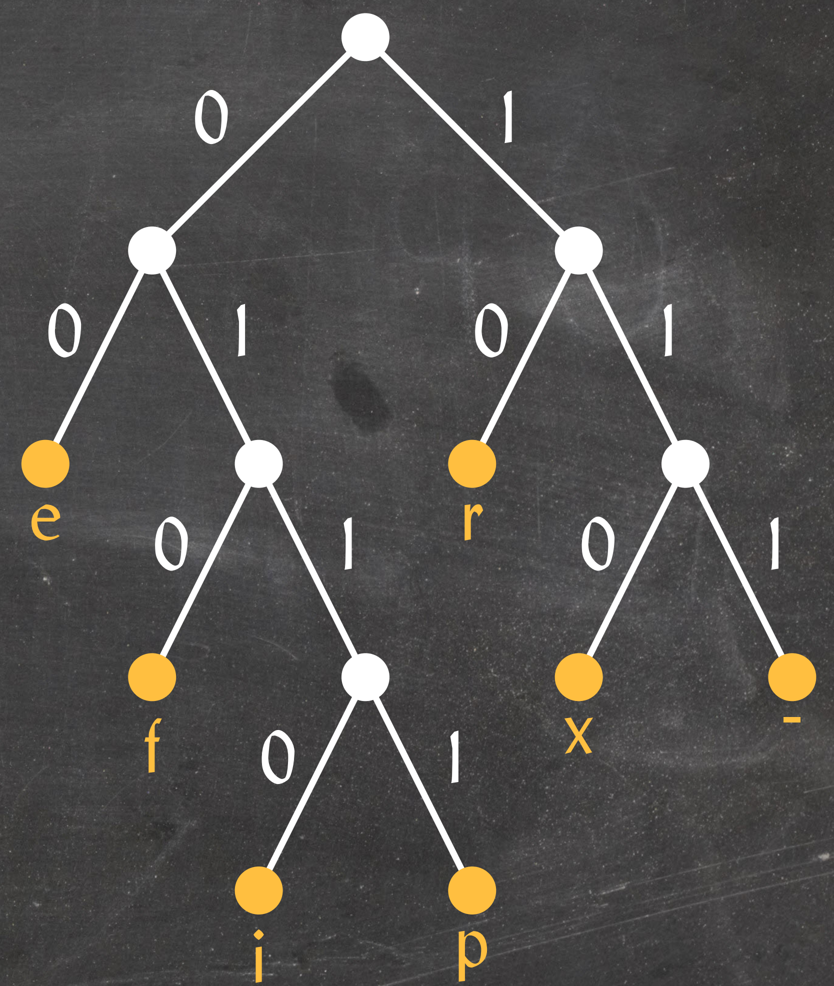
Prefix Codes and Binary Trees

Observation: Every prefix-free code $C(\cdot)$ can be represented as a binary tree $\mathcal{T}_C$ whose leaves correspond to the letters in the alphabet.

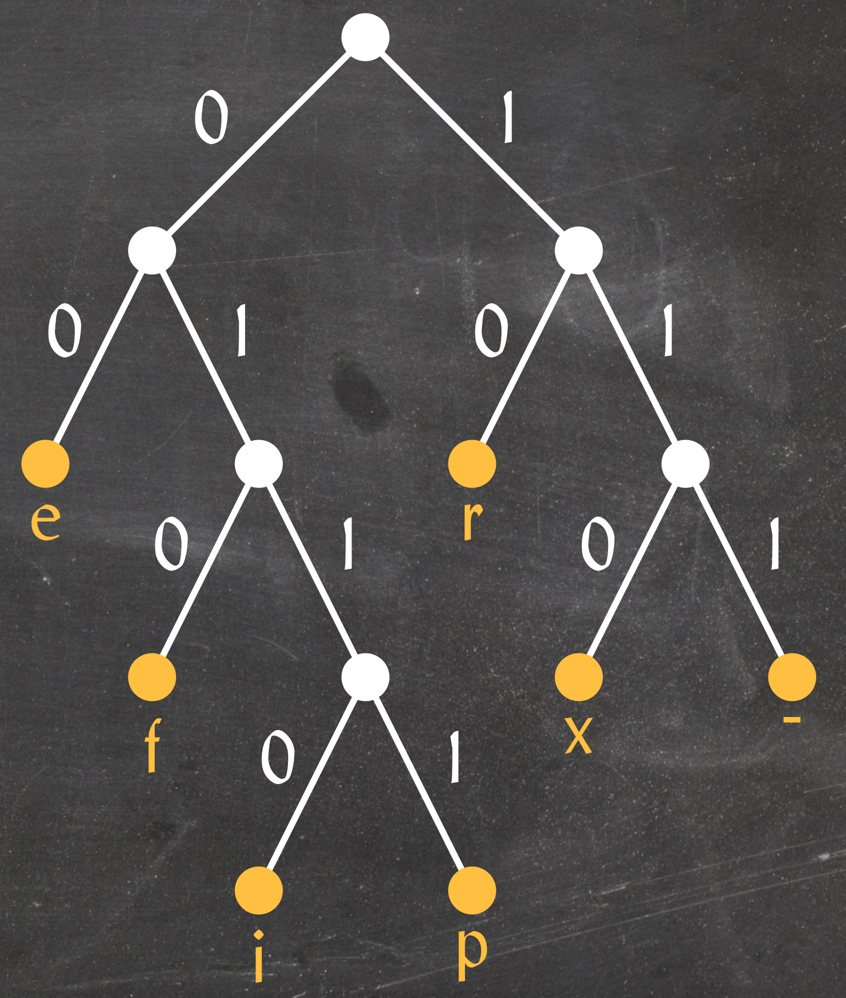
	e	f	i	p	r	x	-
C	00	010	0110	0111	10	110	111



Prefix Codes and Binary Trees

Observation: Every prefix-free code $C(\cdot)$ can be represented as a binary tree $\mathcal{T}_C$ whose leaves correspond to the letters in the alphabet.

	e	f	i	p	r	x	-
C	00	010	0110	0111	10	110	111



The depth of character x in $\mathcal{T}_C$ is the number of bits $|C(x)|$ used to encode x using $C(\cdot)$.

Optimal Prefix Codes and Binary Trees

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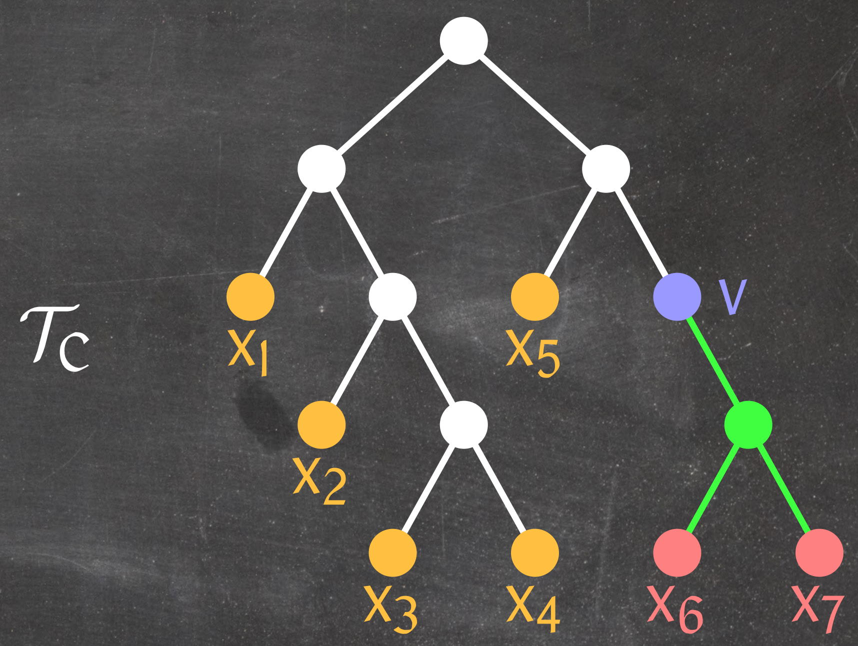

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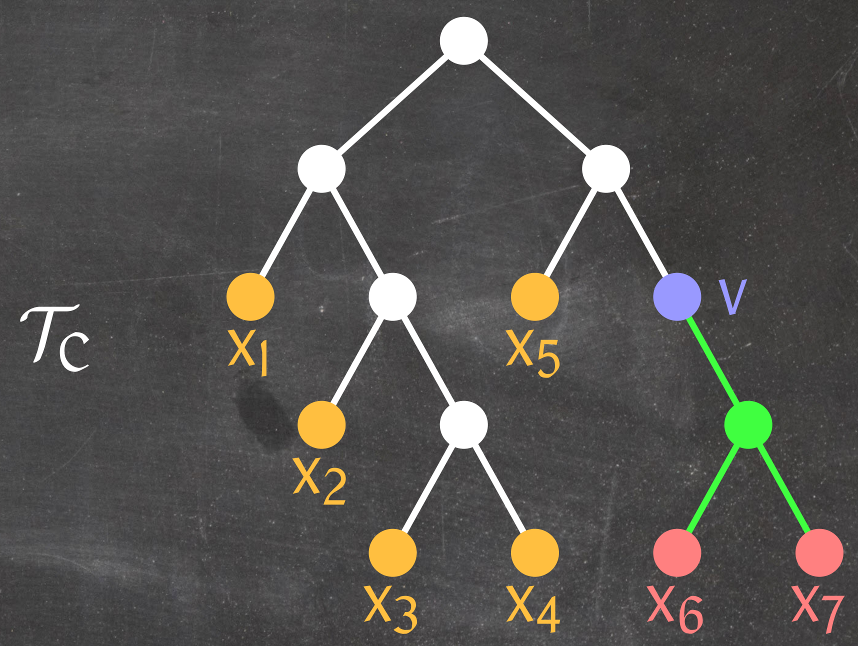

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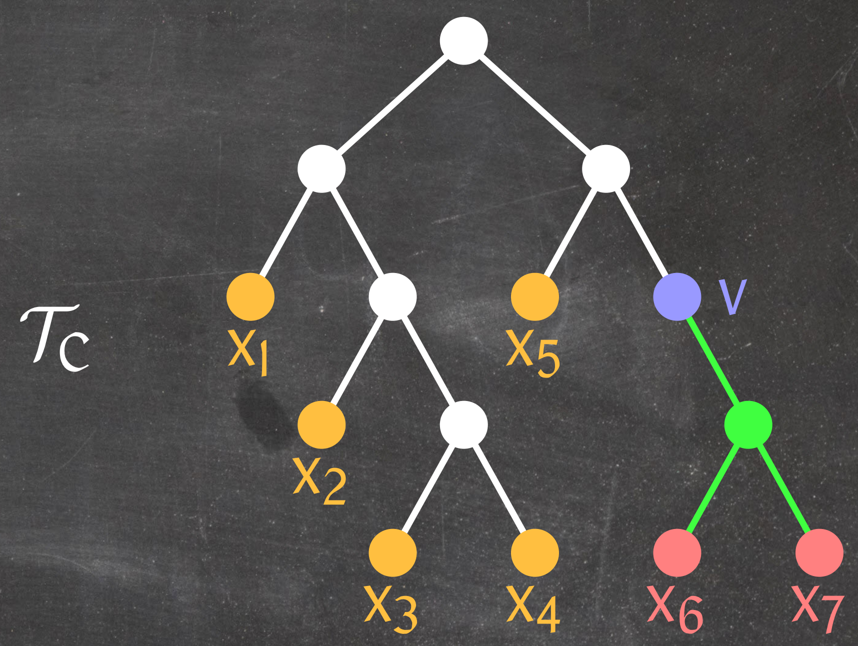

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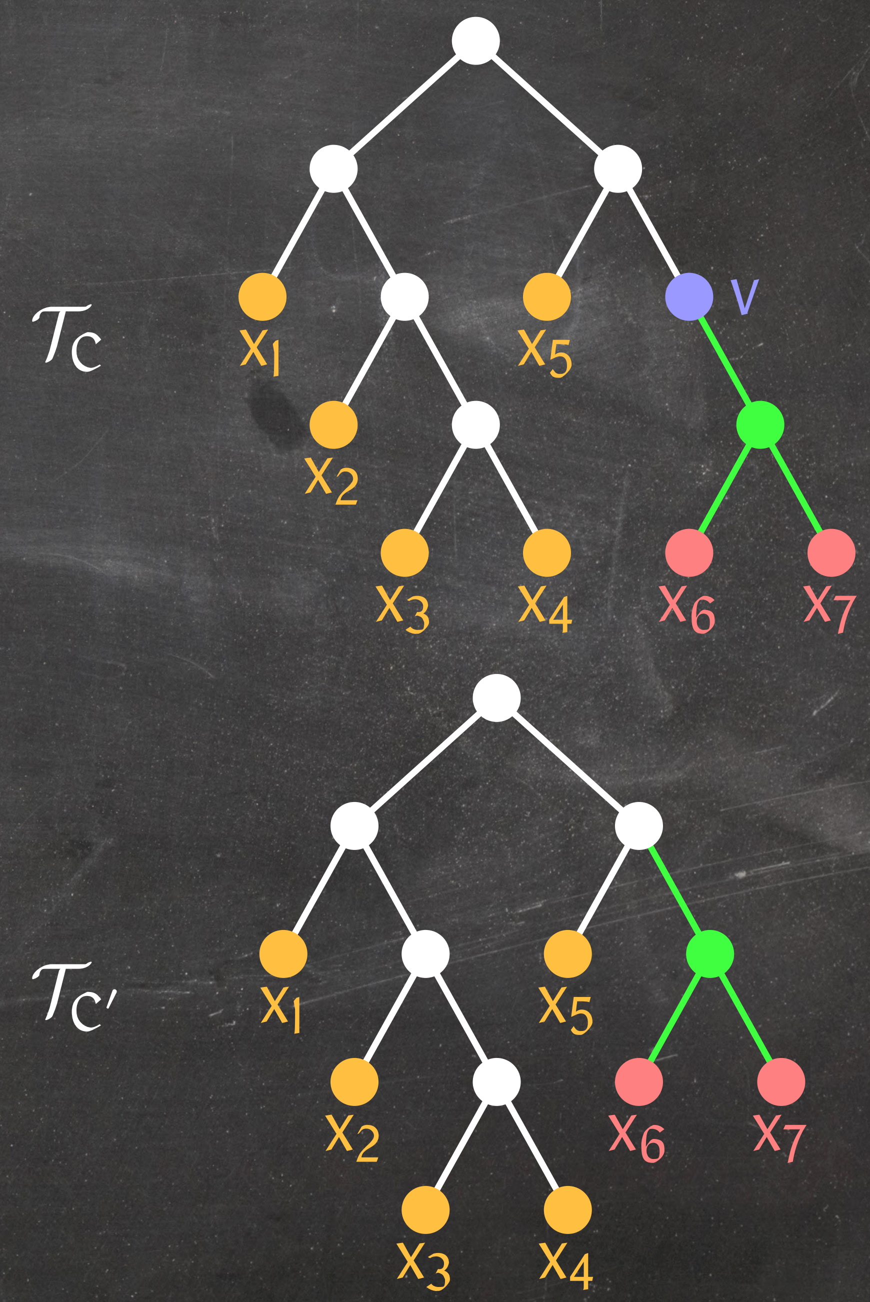

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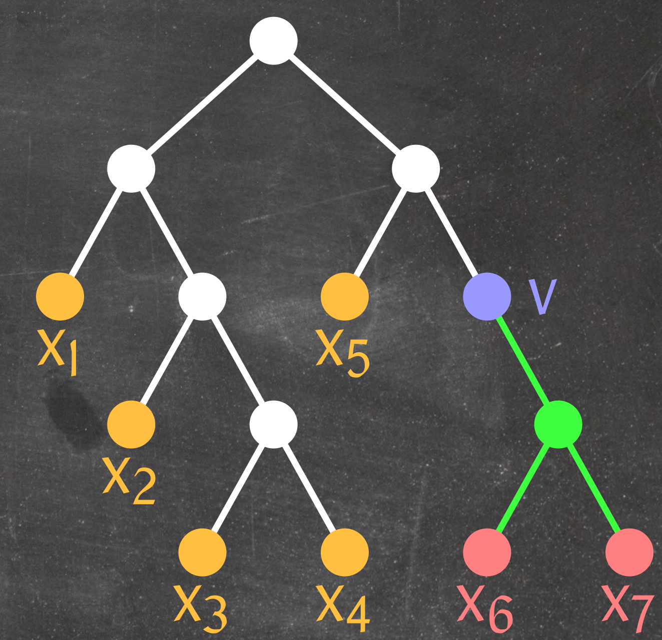
$\mathcal{T}_C$



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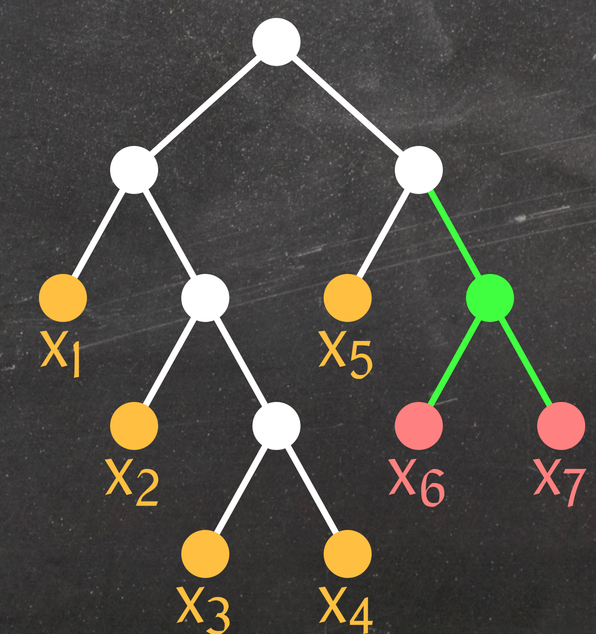

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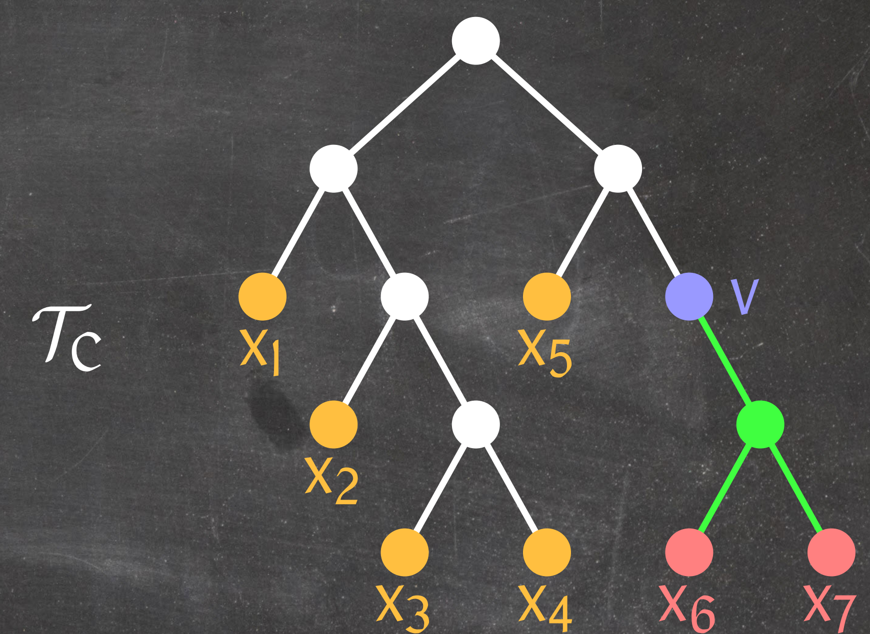
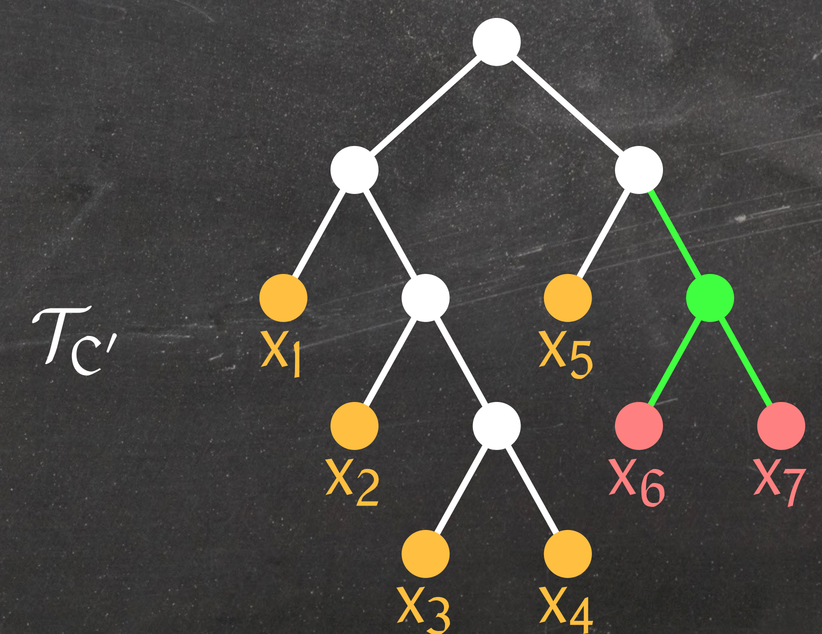
$\Rightarrow$ $|C'(T)| \le |C(T)|$, contradicting the choice of C.





A Greedy Choice for Optimal Prefix Codes

We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.

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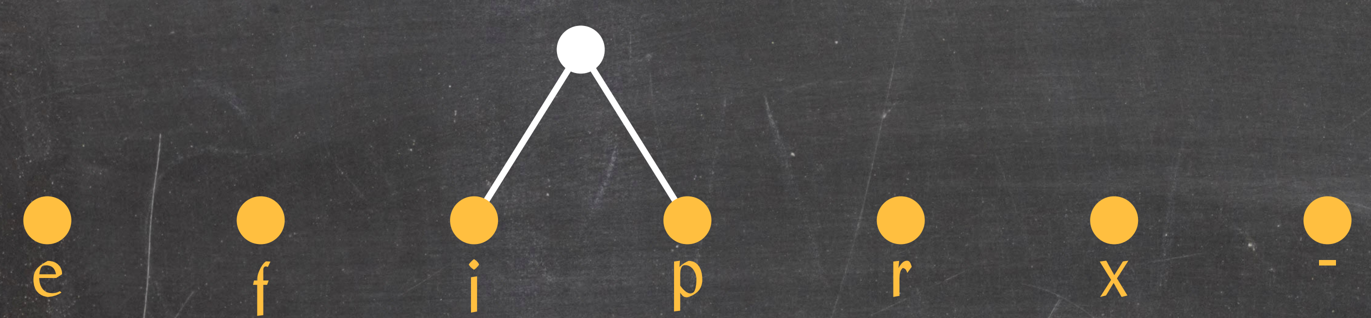

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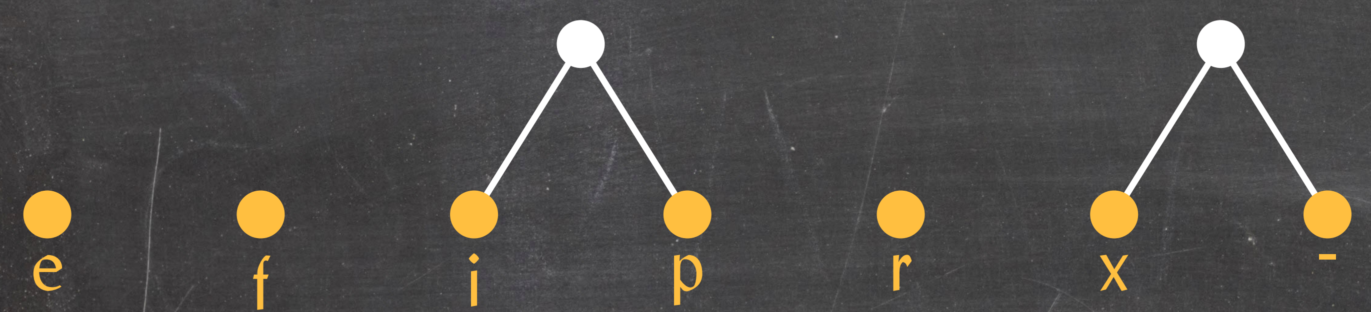

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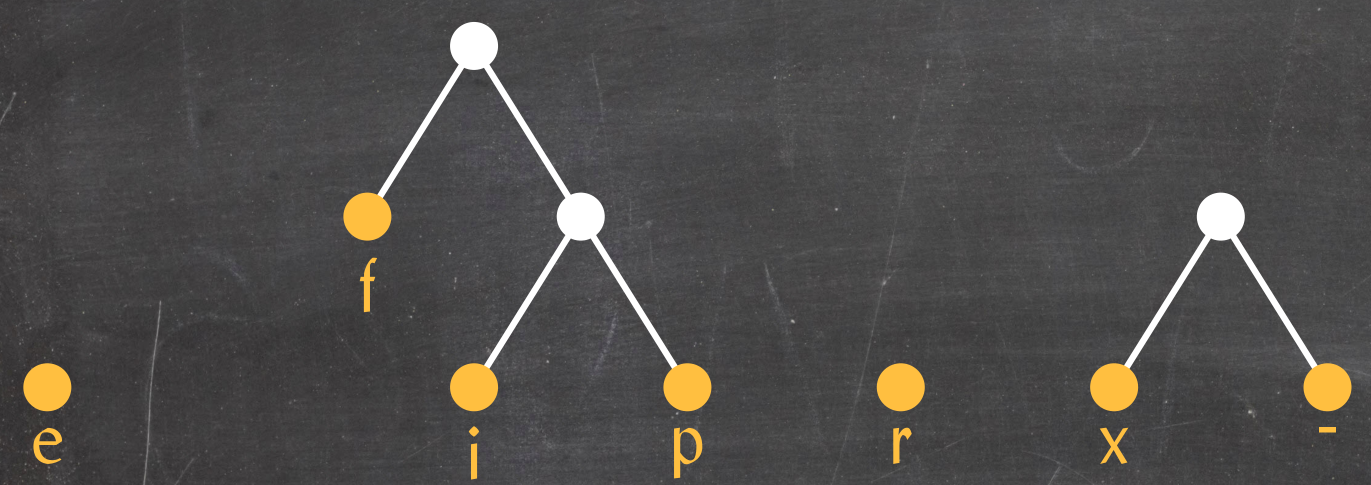

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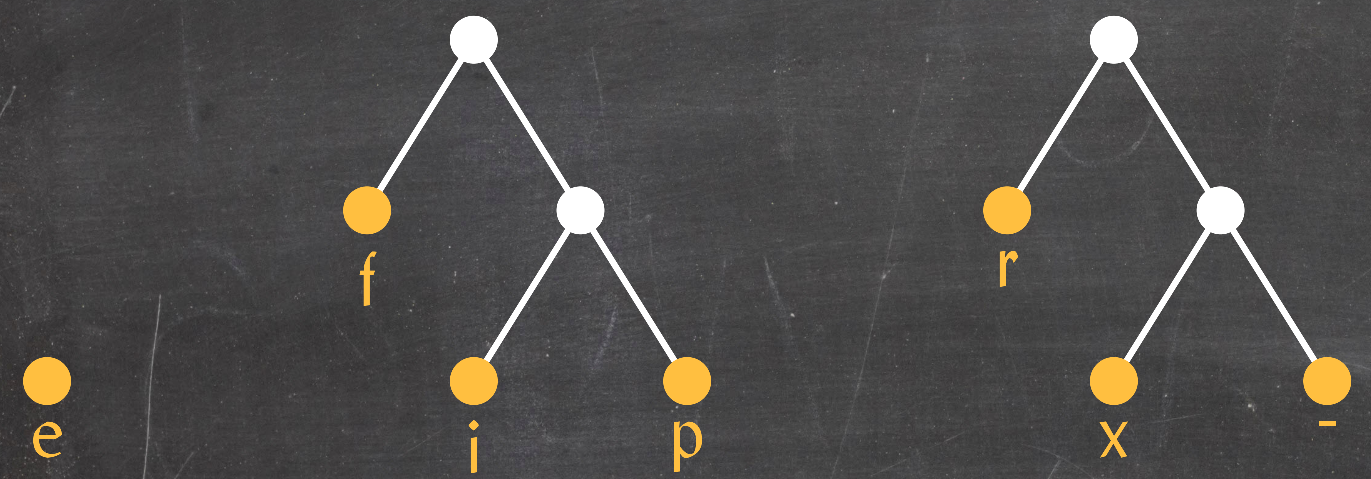

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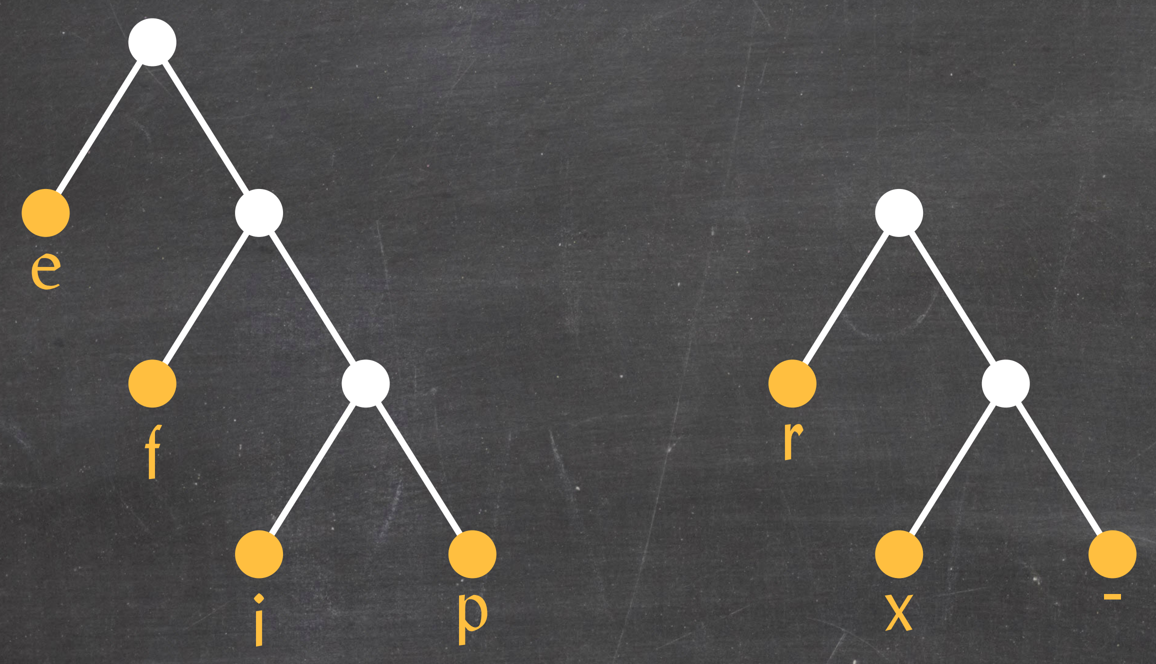

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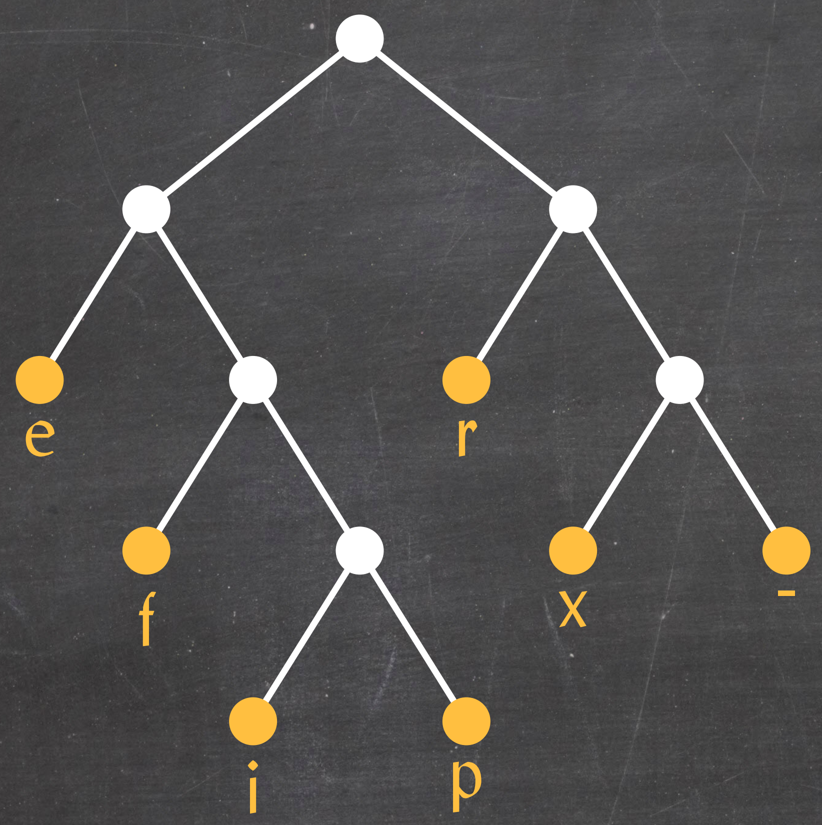

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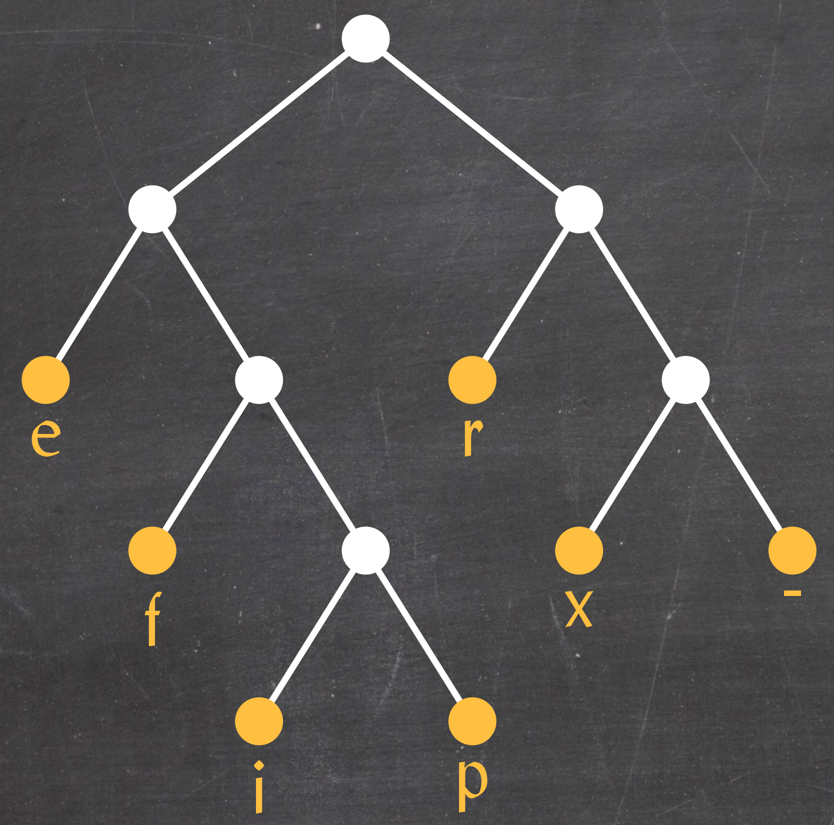

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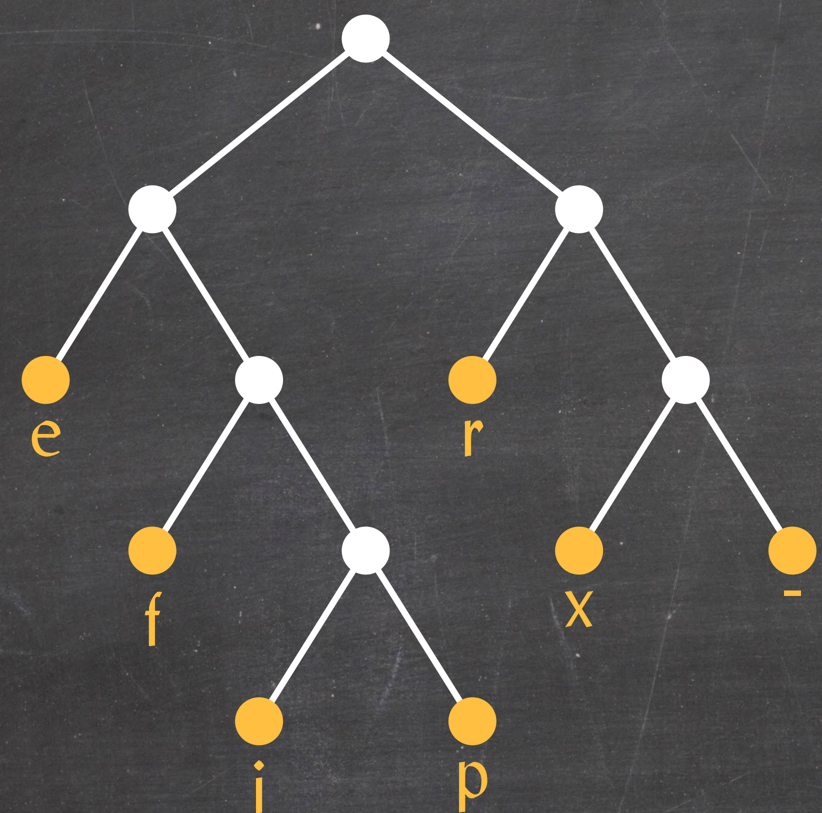

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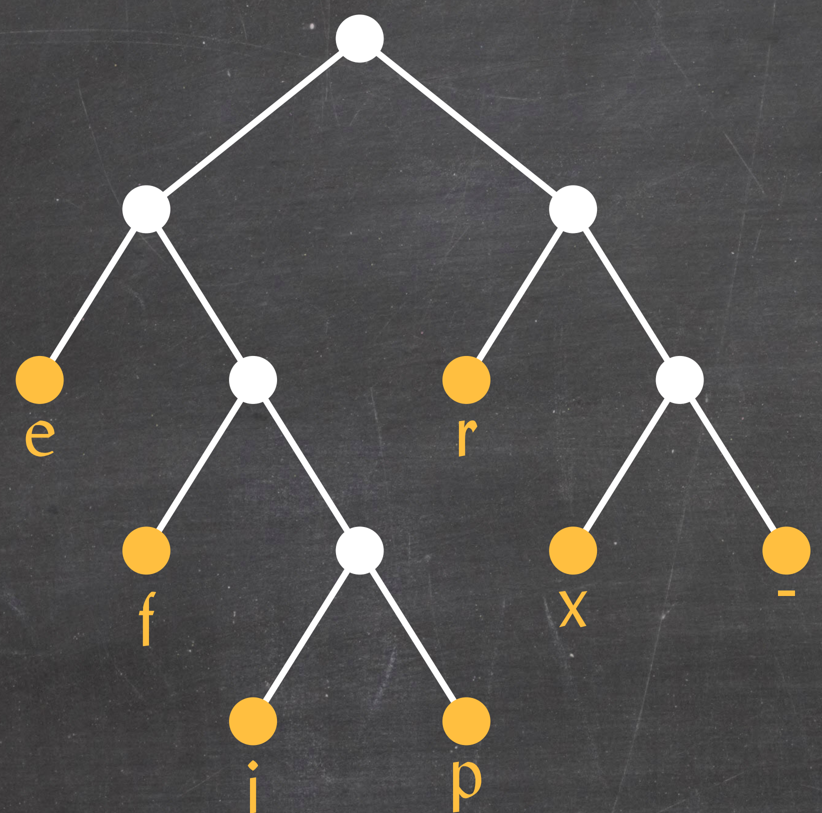

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"prefix-free"

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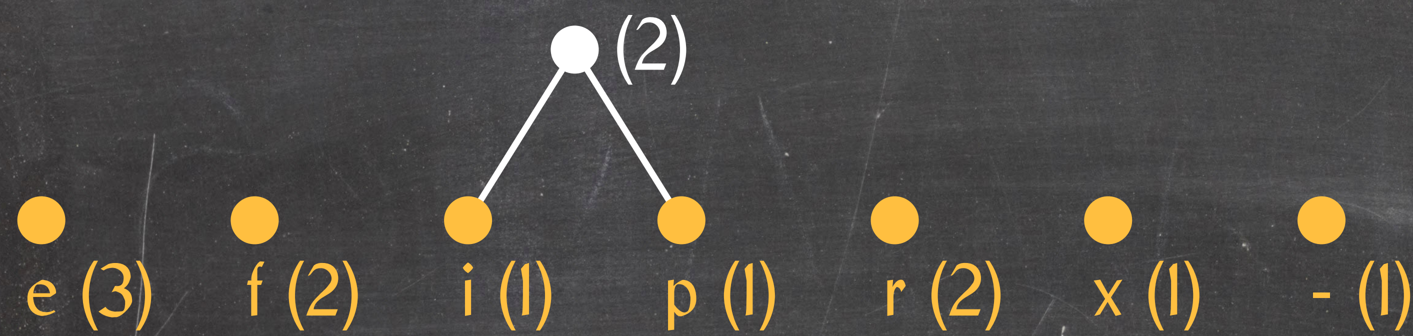

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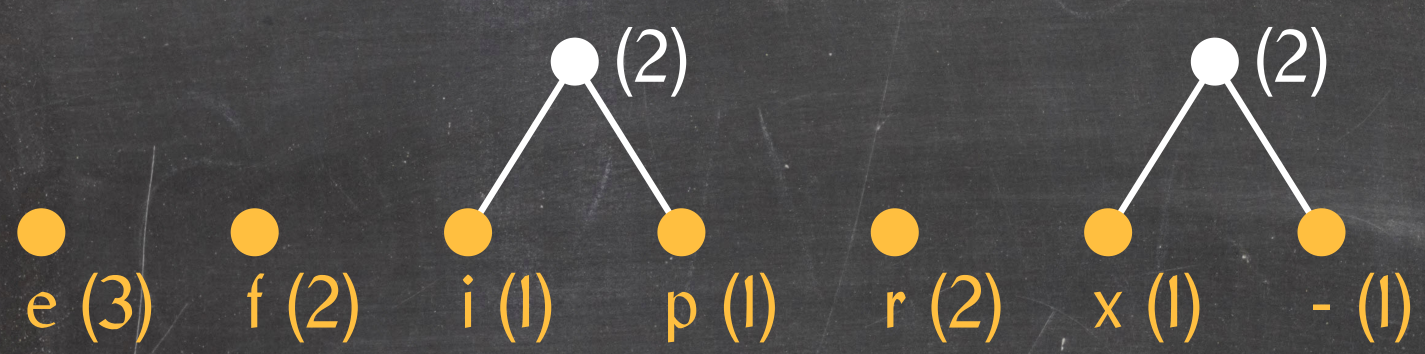

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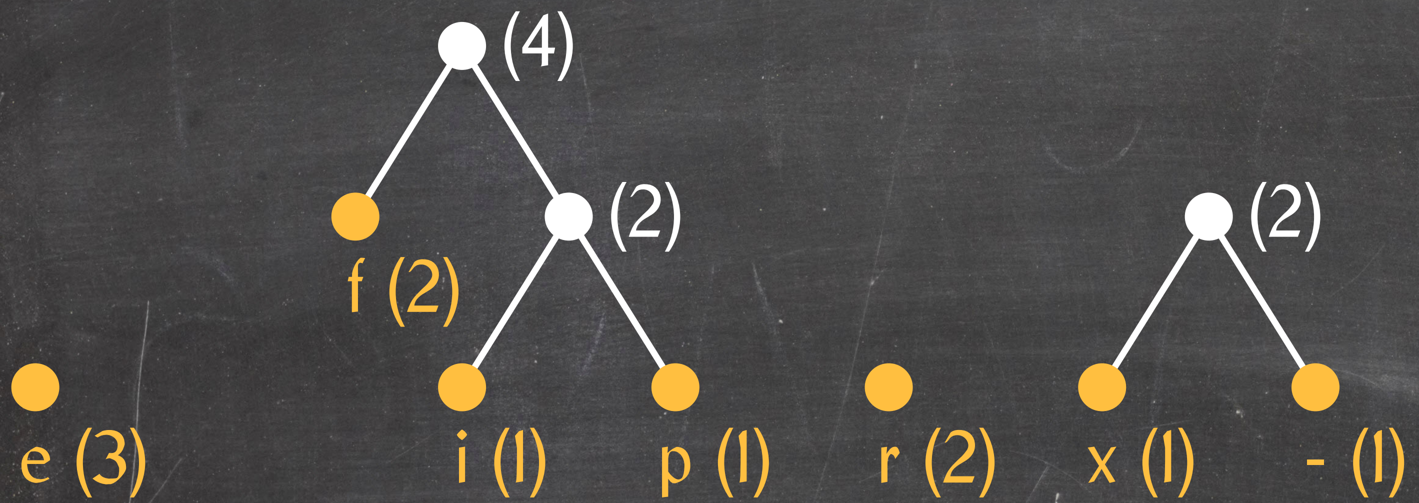

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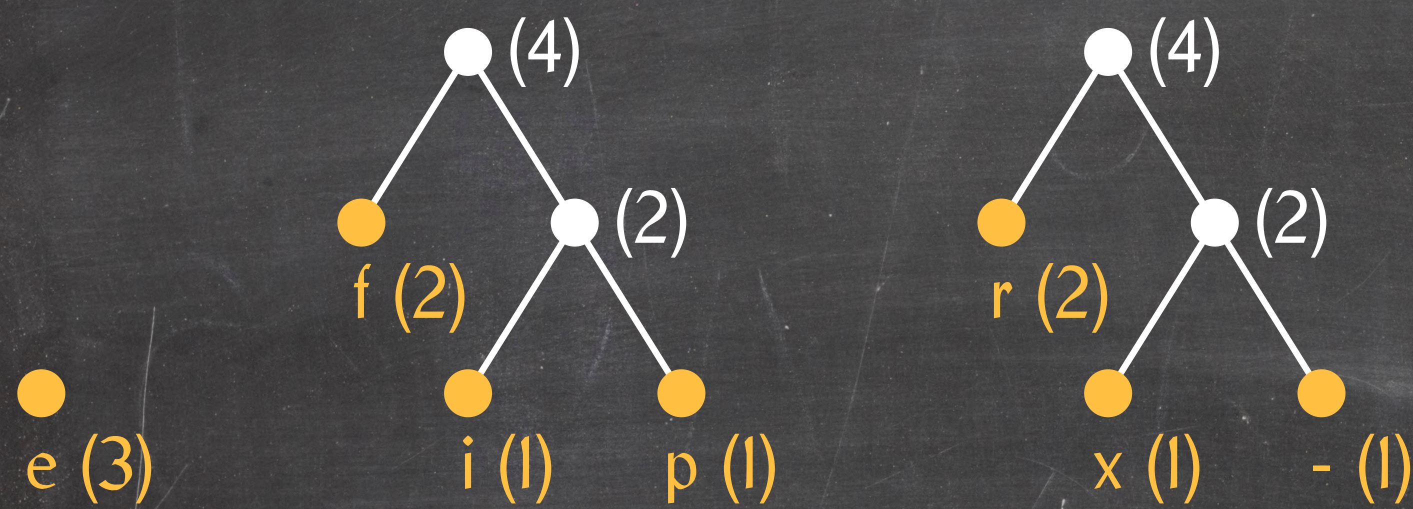

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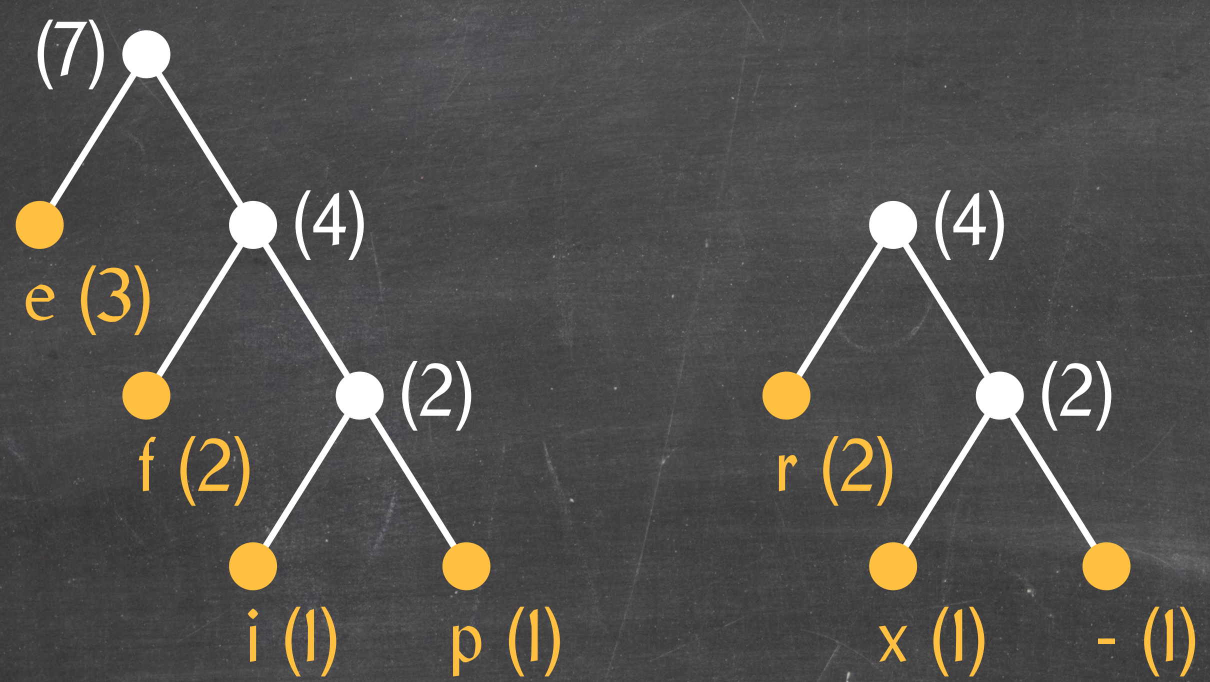

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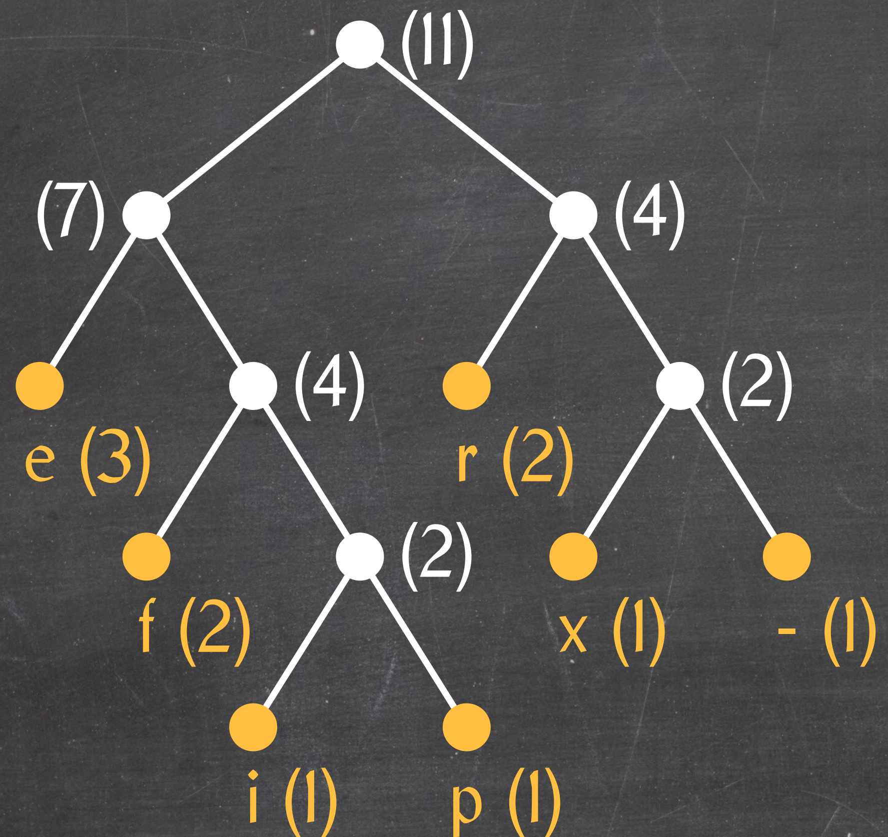

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Huffman's Algorithm

```
Huffman(T)

determine the set A of characters that occur in T and their frequencies
Q = an empty priority queue
for every character x ∈ A
    do create a node v associated with x and define f(v) = f(x)
       Q.insert(v, f(v))
while |Q| > 1
    do v = Q.deleteMin()
       w = Q.deleteMin()
       u = a new node with frequency f(u) = f(v) + f(w)
       make v and w children of u
       Q.insert(u, f(u))
return Q.deleteMin()
```

Lemma: Huffman's algorithm runs in $O(m \lg n)$ time, where $m = |T|$ and n is the size of the alphabet.

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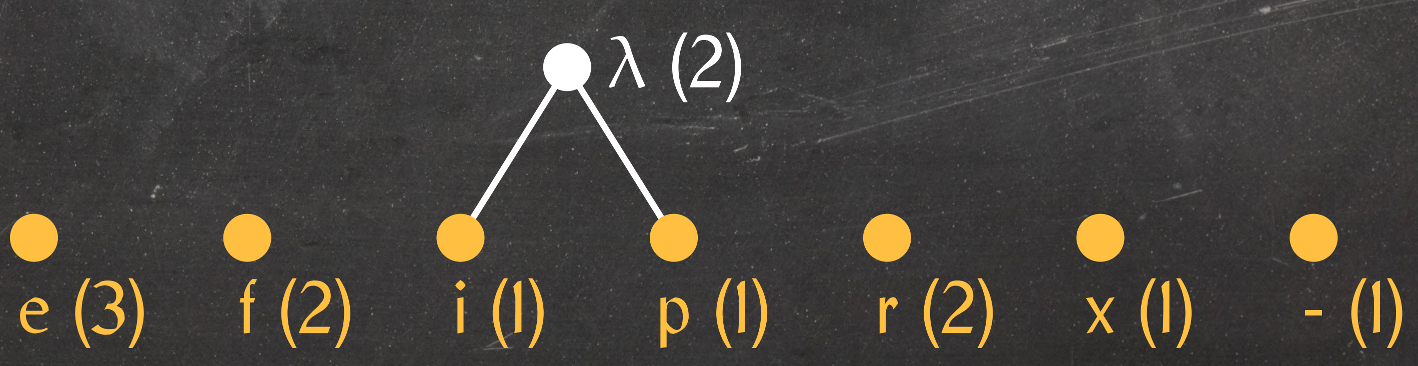

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$\Downarrow$

“zrefzx-free”

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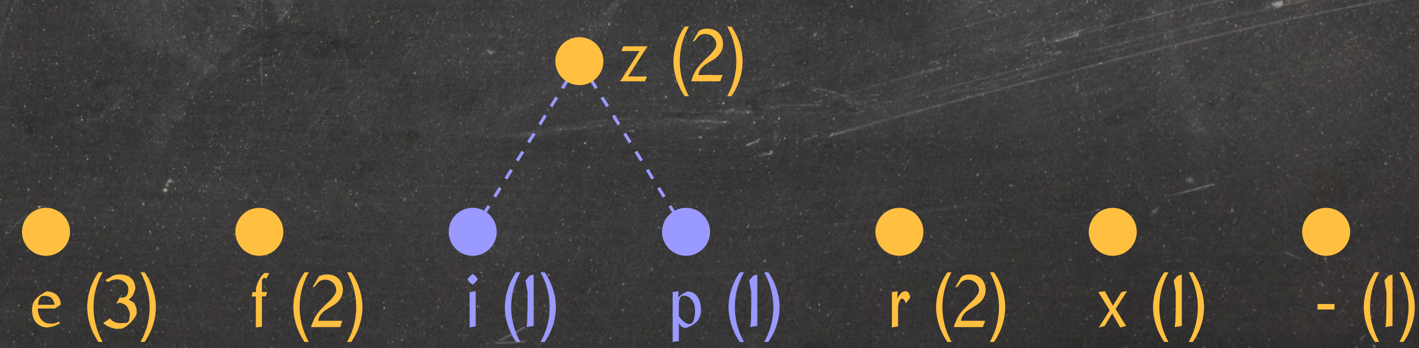
Replacing a and b with z in T produces a new text T' over an alphabet of size $n - 1$ where z has frequency f(z).



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"prefix-free"

⇓

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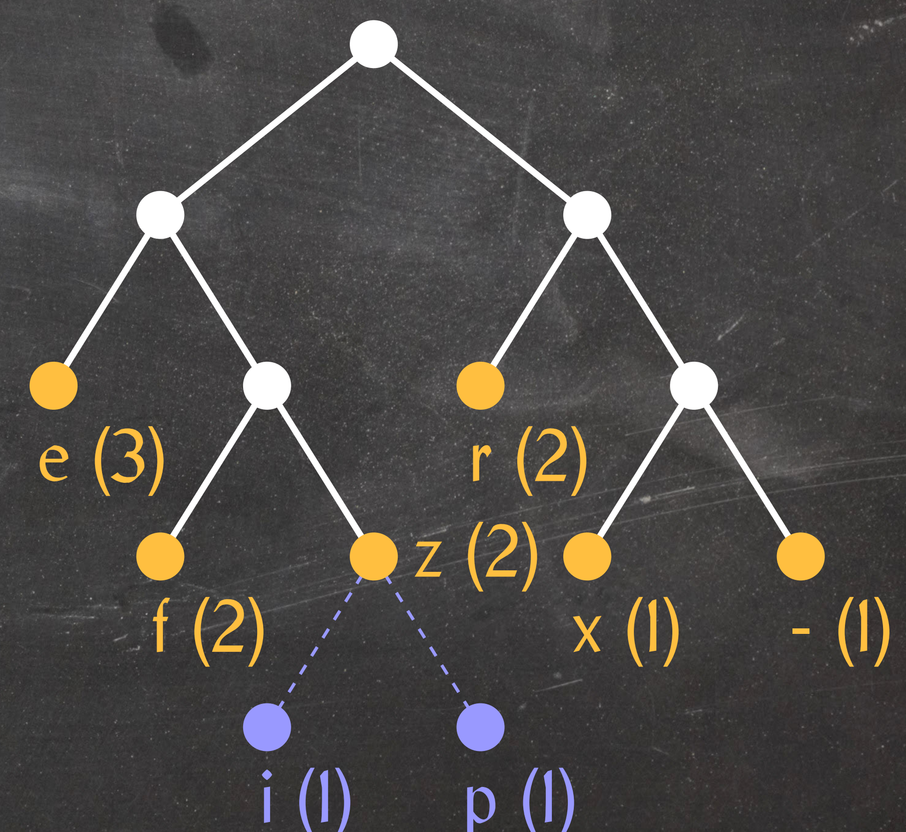

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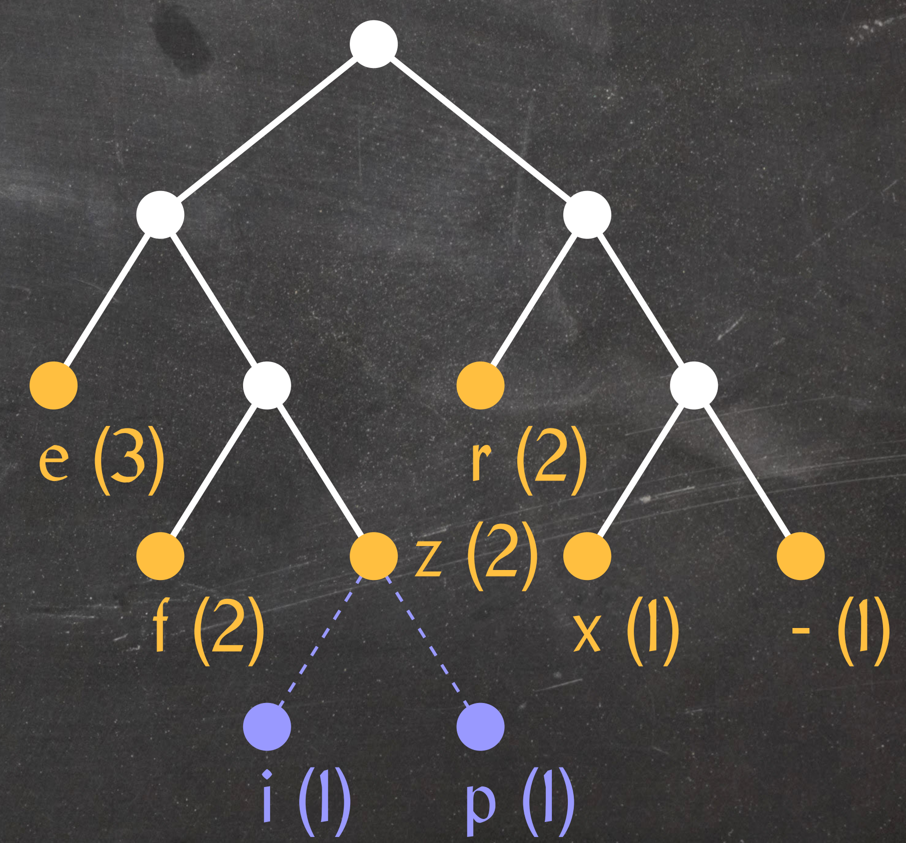
By induction, it produces an optimal code $C'(\cdot)$ for T'.



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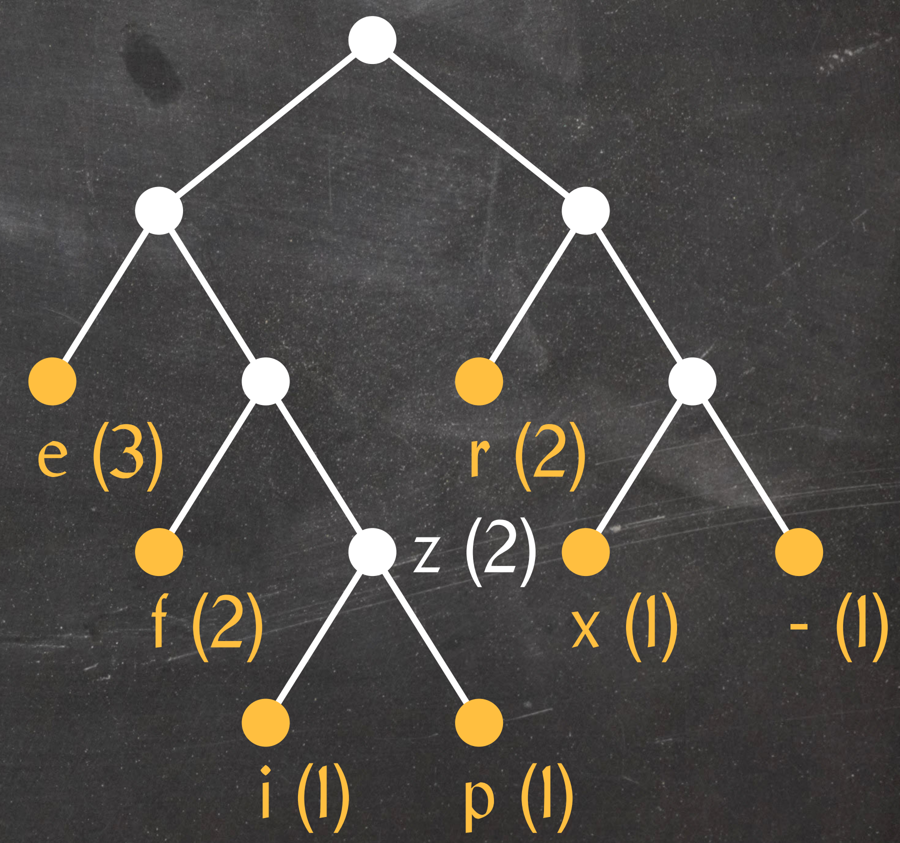
"prefix-free"



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Let $C''(\cdot)$ be the code for T' defined as

$$C''(x) = \begin{cases} C^*(x) & x \neq z \\ \sigma & x = z \text{ and } C^*(a) = \sigma 0 \end{cases}$$

"prefix-free"

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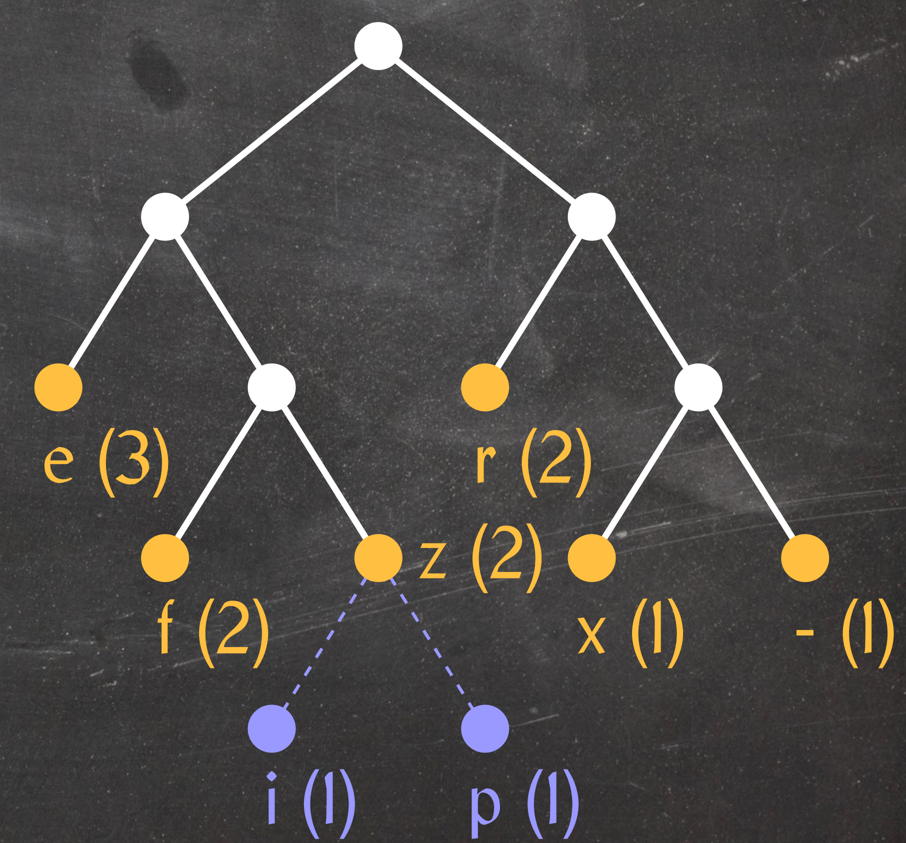

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Claim: There exists an optimal prefix-free code $C(\cdot)$ for T such that the two least frequent characters a and b in T are siblings in $\mathcal{T}_C$.

$\Rightarrow$ Huffman's algorithm produces an optimal prefix-free code for T.

Assume there exists a better code $C^*(\cdot)$ such that a and b are siblings in $\mathcal{T}_{C^*}$, that is, $|C^*(T)| < |C(T)|$.

"prefix-free"

$\Downarrow$

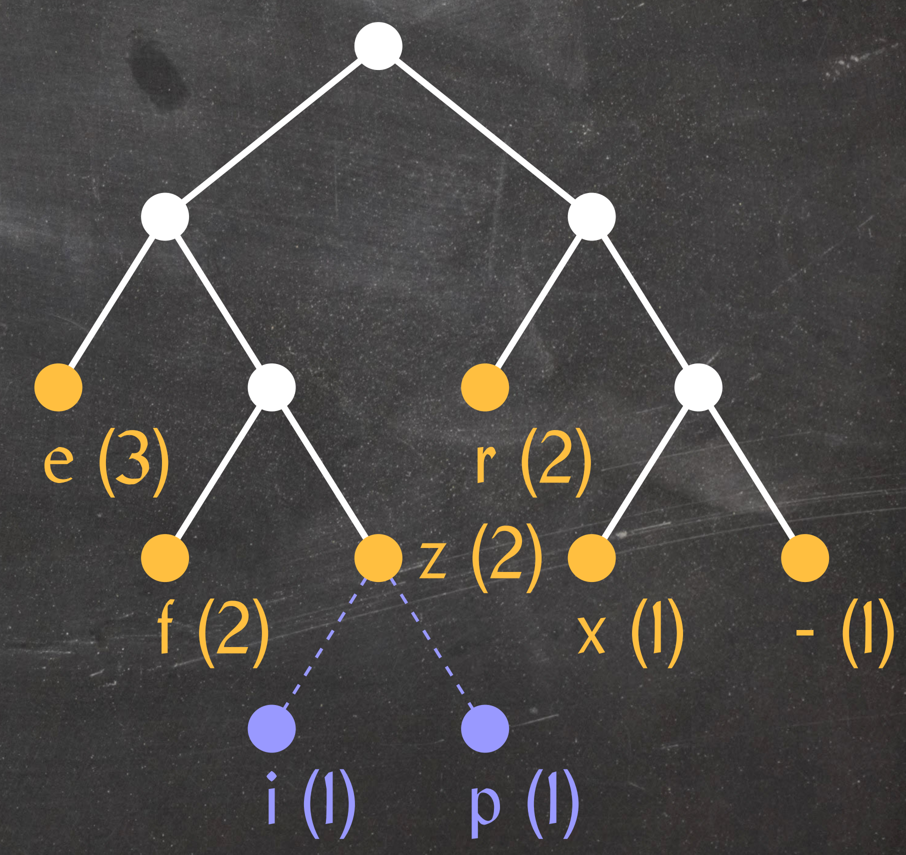
"zrefzx-free"

Let $C''(\cdot)$ be the code for T' defined as

$$C''(x) = \begin{cases} C^*(x) & x \neq z \\ \sigma & x = z \text{ and } C^*(a) = \sigma 0 \end{cases}$$

We also have

$$C'(x) = \begin{cases} C(x) & x \neq z \\ \sigma & x = z \text{ and } C(a) = \sigma 0 \end{cases}$$



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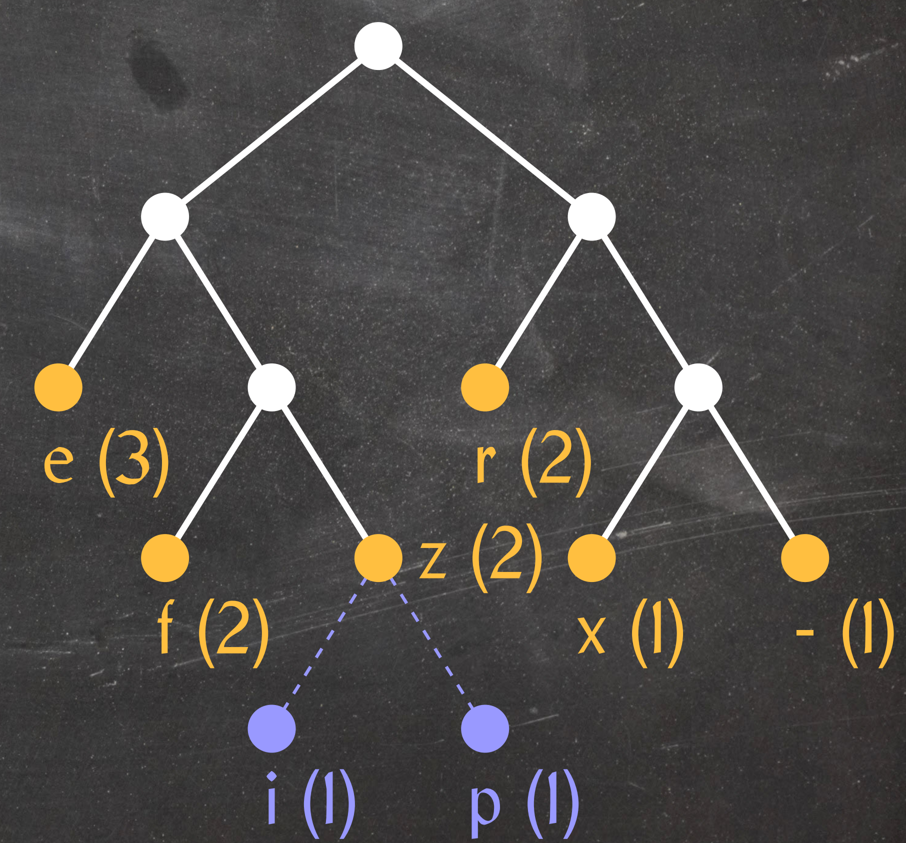

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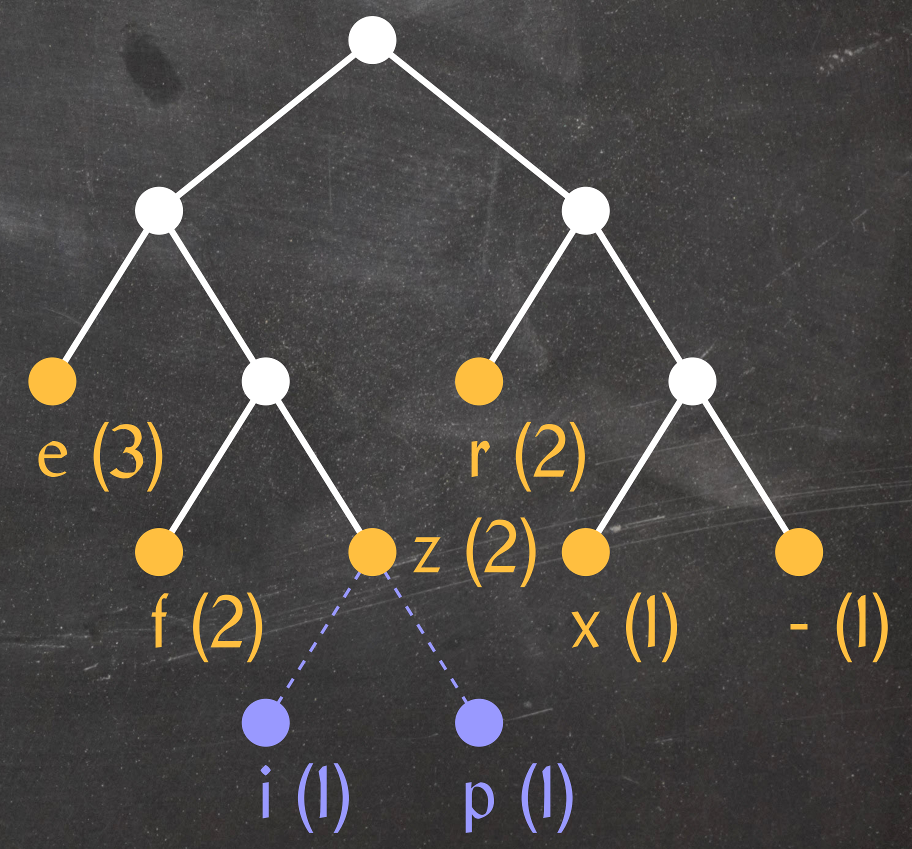

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$\Rightarrow$ $|C''(T')| < |C'(T')|$, a contradiction because $C'(\cdot)$ is optimal for T'.

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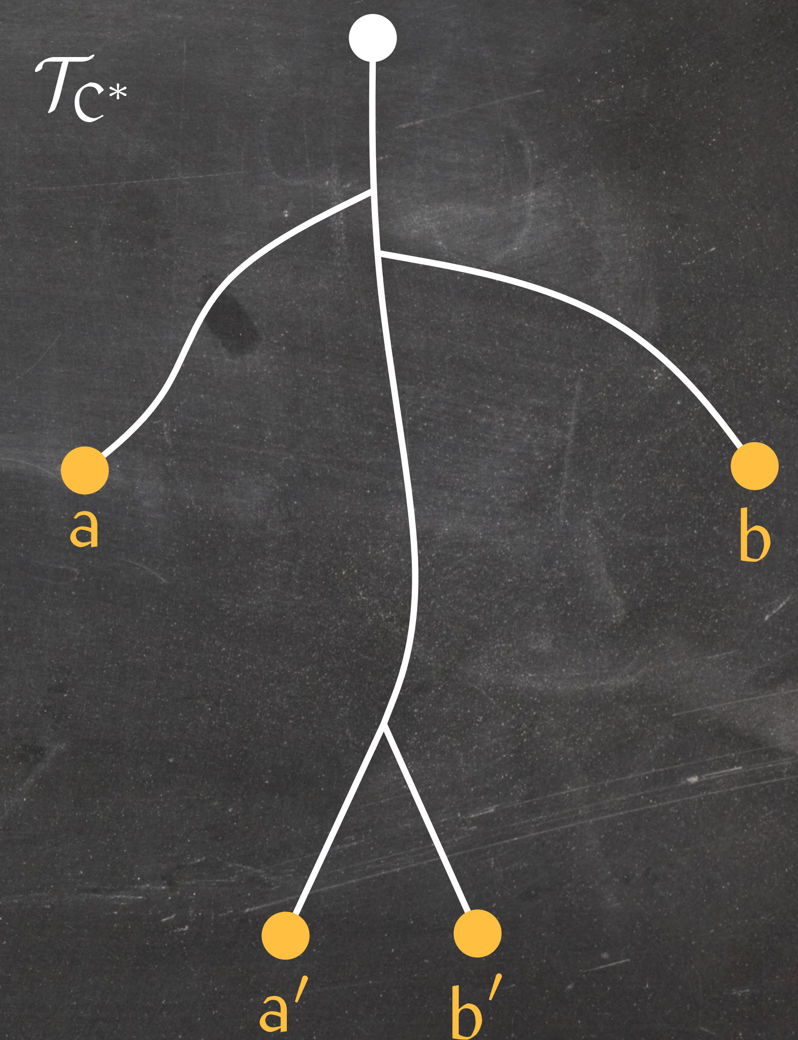
Let $C^*(\cdot)$ be an optimal code for T.

The sibling b' of the deepest leaf a' in $\mathcal{T}_{C^*}$ is also a leaf.



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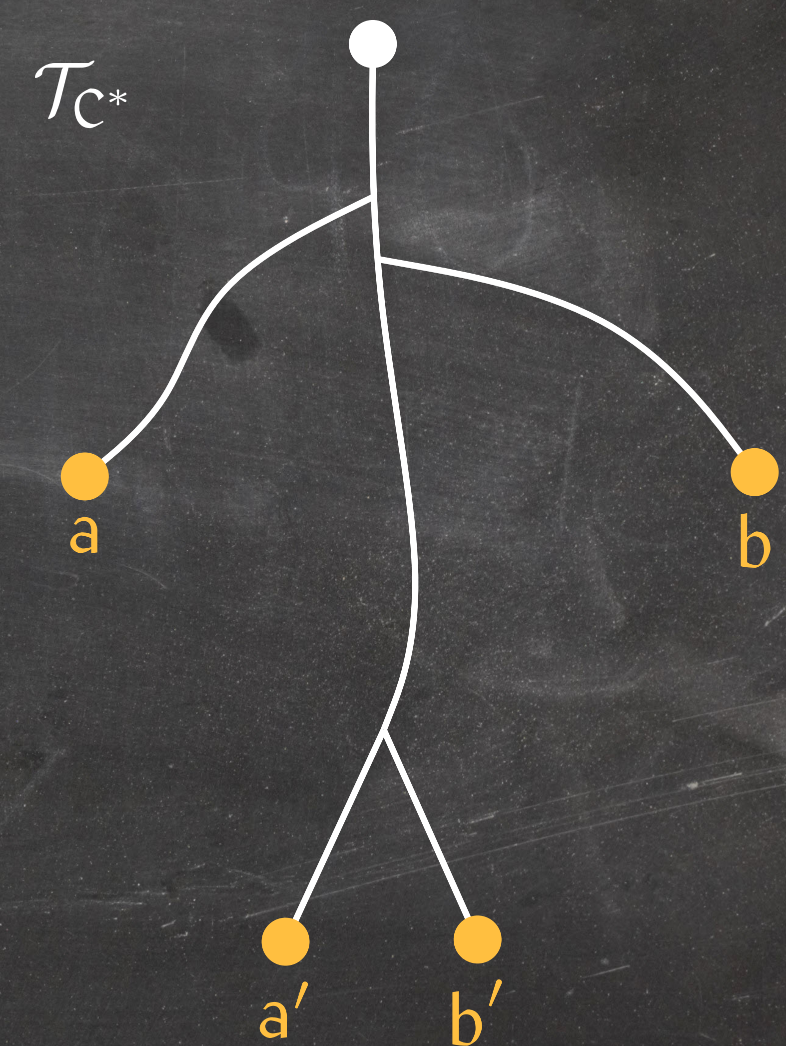
We have $|C^*(a)| \le |C^*(a')|$ and $|C^*(b)| \le |C^*(b')|$.



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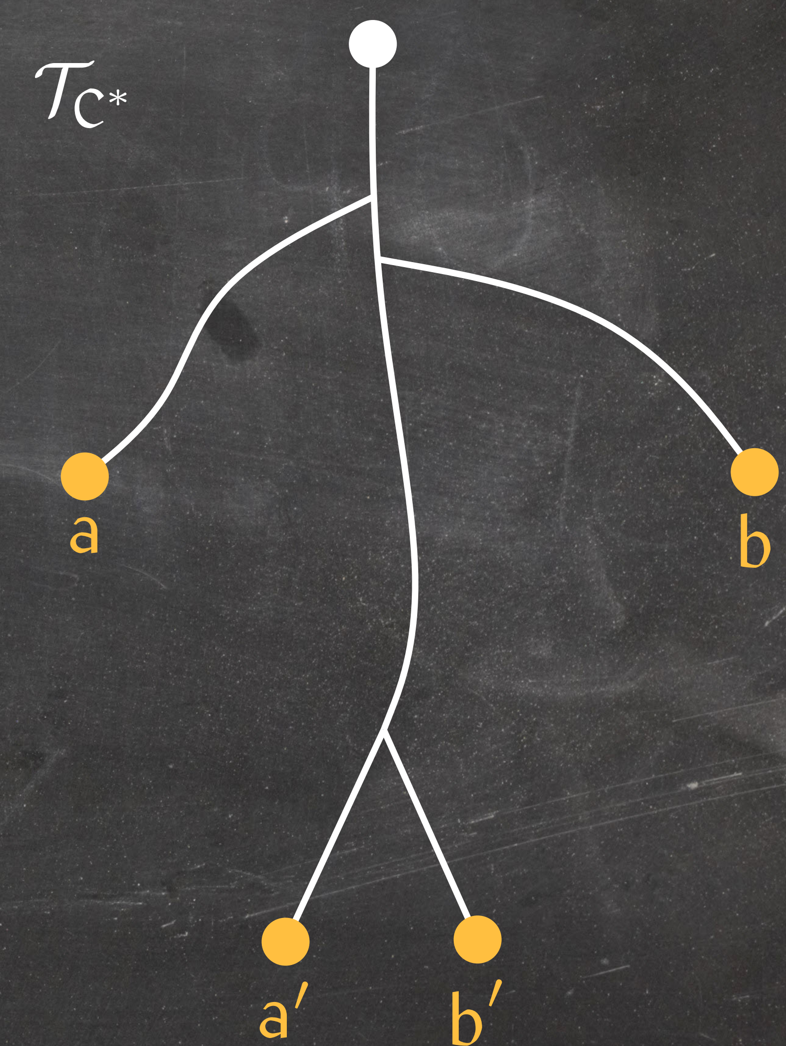

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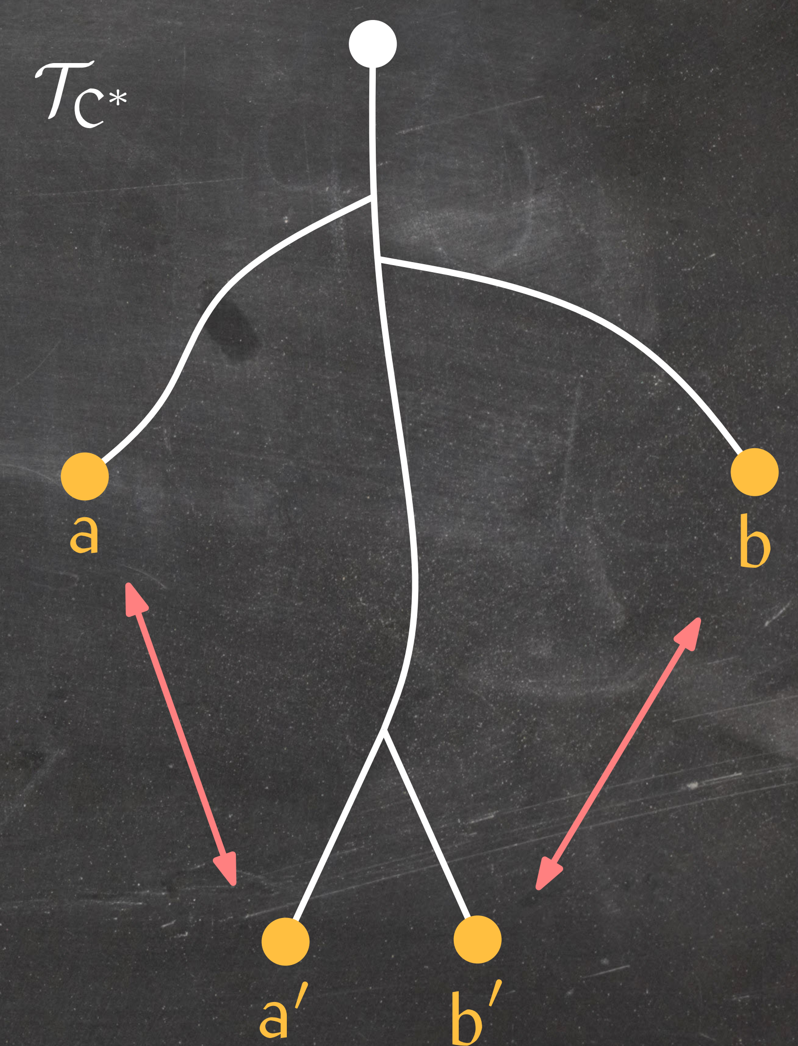

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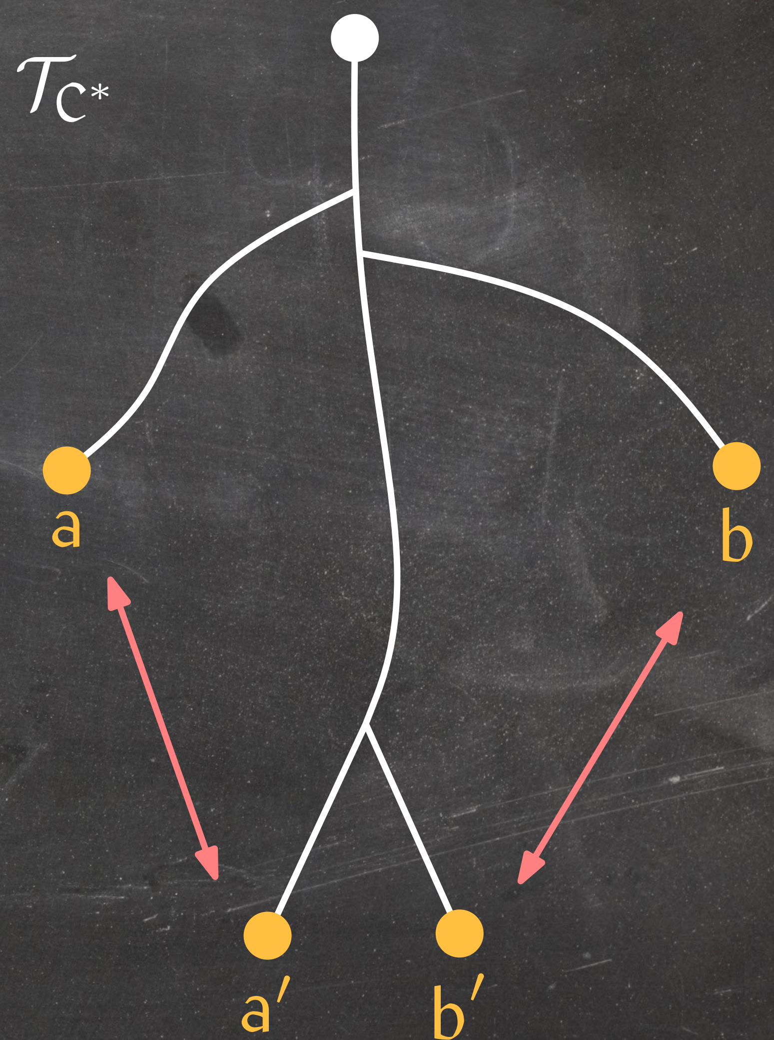
We prove that $|C(T)| \le |C^*(T)|$, that is, $C(\cdot)$ is an optimal prefix-free code for T.



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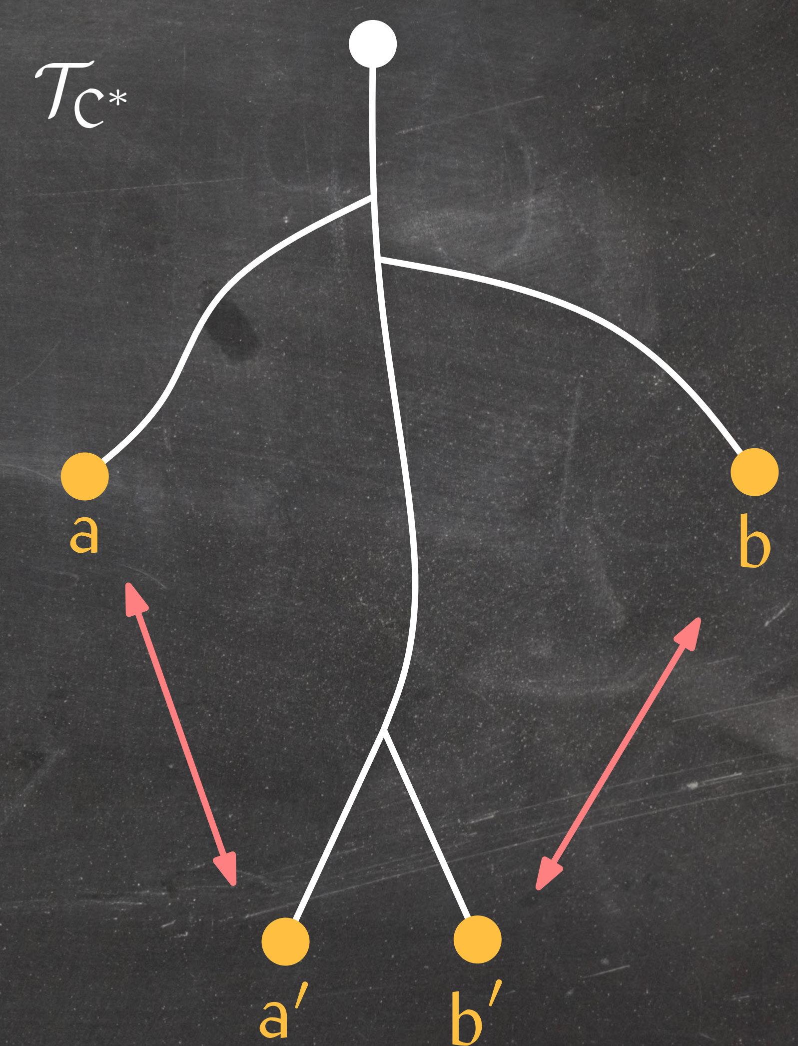
Since a and b are siblings in $\mathcal{T}_C$, this proves the claim.



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$$\begin{aligned}|C(T)| - |C^*(T)| = {} & f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\ & f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')|\end{aligned}$$

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&\quad\ f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= (f(a) - f(a'))\,(|C^*(a')| - |C^*(a)|) + (f(b) - f(b'))\,(|C^*(b')| - |C^*(b)|)
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&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= \underbrace{(f(a) - f(a'))}_{\le 0}\underbrace{(|C^*(a')| - |C^*(a)|)}_{\ge 0} + \underbrace{(f(b) - f(b'))}_{\le 0}\underbrace{(|C^*(b')| - |C^*(b)|)}_{\ge 0}
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&\le 0
\end{aligned}$$

Summary

Greedy algorithms make natural local choices in their search for a globally optimal solution.

Many good heuristics are greedy:

- Simple
- Work well in practice

Proof that a greedy algorithm finds an optimal solution:

- Induction
- Exchange argument

Useful data structures:

- Union-find data structure
- Thin Heap

Analysis of a sequence of data structure operations:

- Amortized analysis
- Potential functions