Greedy Algorithms

Textbook Reading

Chapters 16, 17, 21, 23 & 24

Overview

Design principle:

Make progress towards a globally optimal solution by making locally optimal choices, hence the name.

Problems:

- Interval scheduling
- Minimum spanning tree
- Shortest paths
- Minimum-length codes

Proof techniques:

- Induction
- The greedy algorithm "stays ahead"
- Exchange argument

Data structures:

- Priority queue
- Union-find data structure

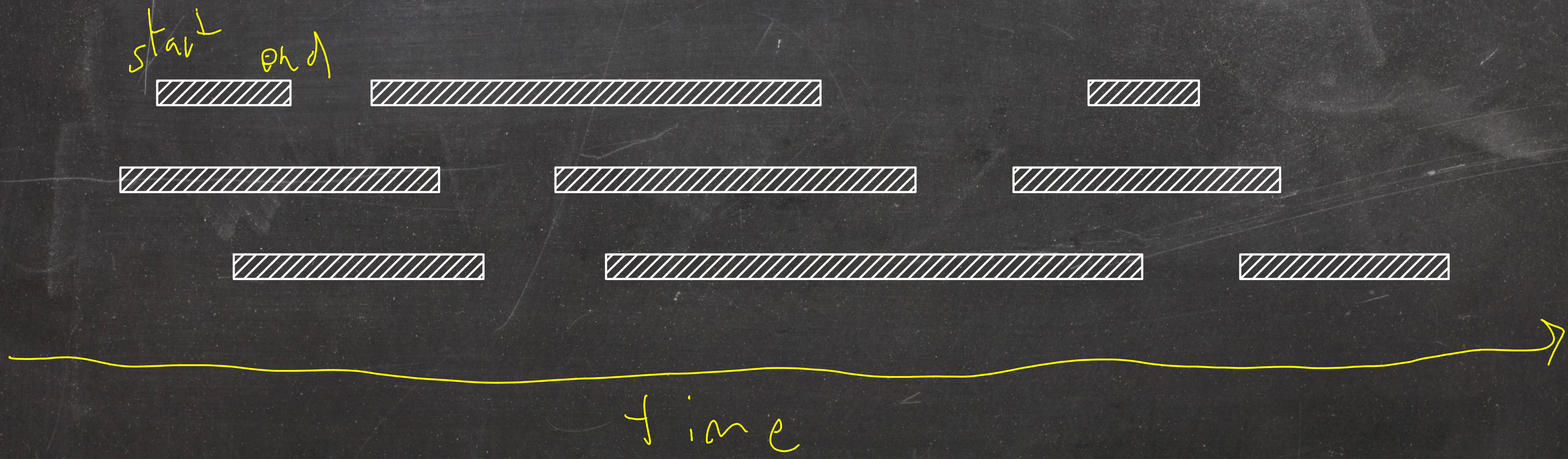
Interval Scheduling

Given:

A set of activities competing for time intervals on a certain resource
(E.g., classes to be scheduled competing for a classroom)

Goal:

Schedule as many non-conflicting activities as possible



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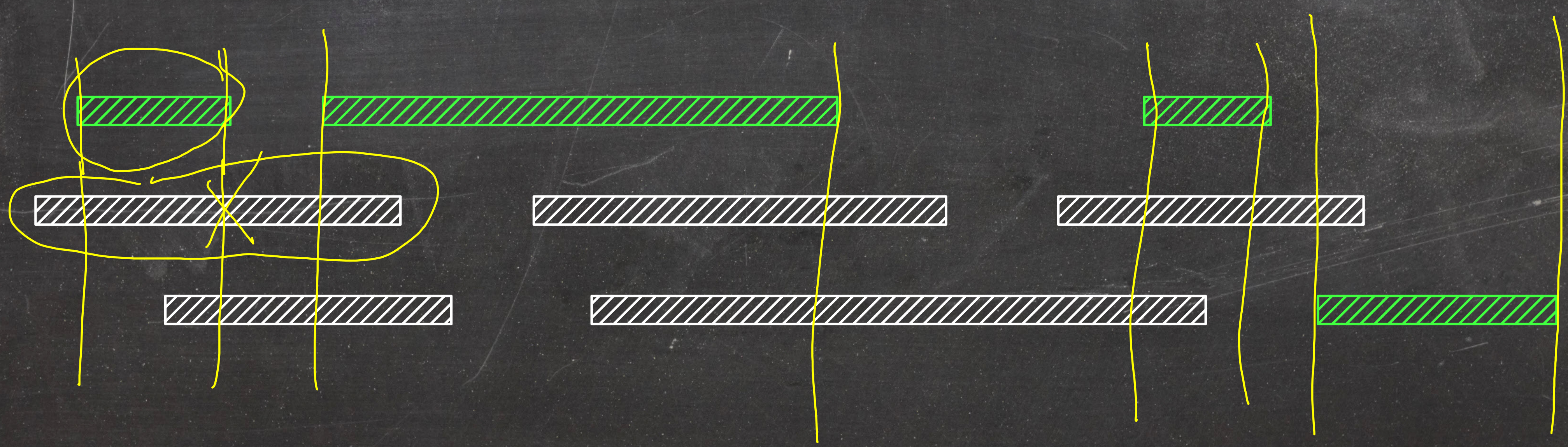

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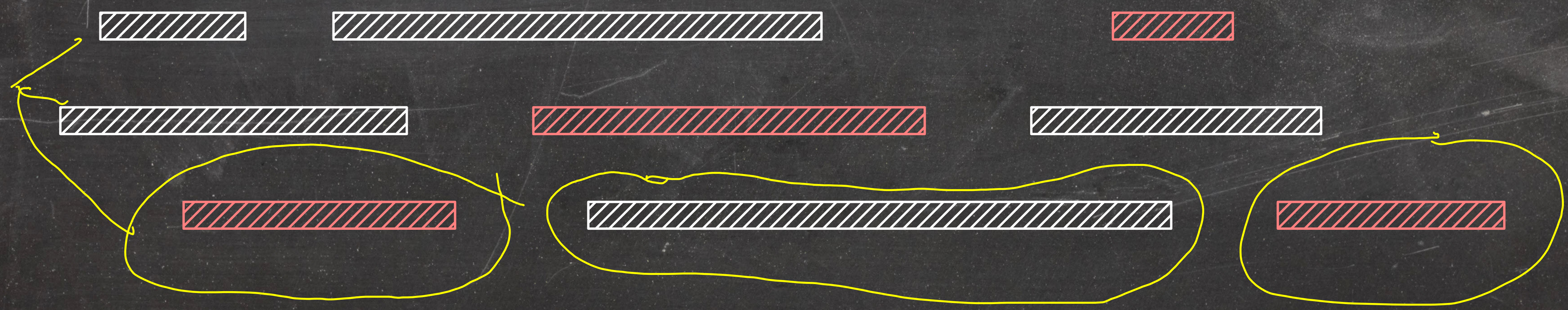

A Greedy Framework for Interval Scheduling

S = set of intervals

S′ = output schedule

FindSchedule(S)

```
S′ = ∅
while S is not empty
    do pick an interval I in S
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Main questions:

- Can we choose an arbitrary interval I in each iteration?
- How do we choose interval I in each iteration?

Greedy Strategies for Interval Scheduling

options: shortest I

median interval

least conflicts

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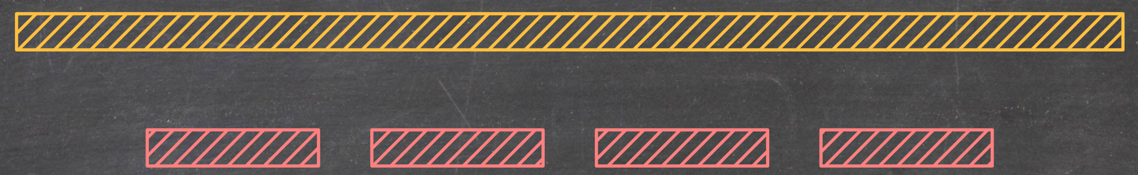

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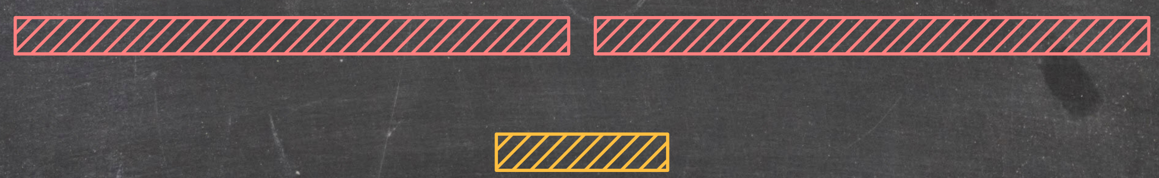

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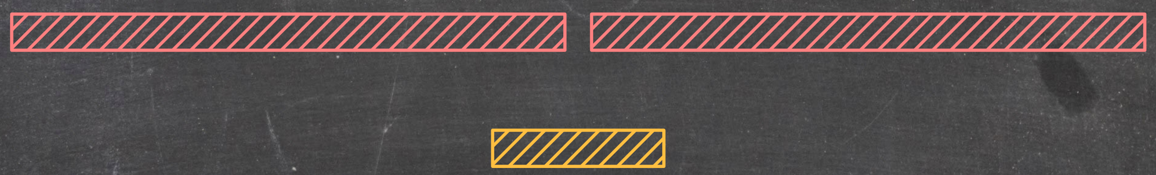

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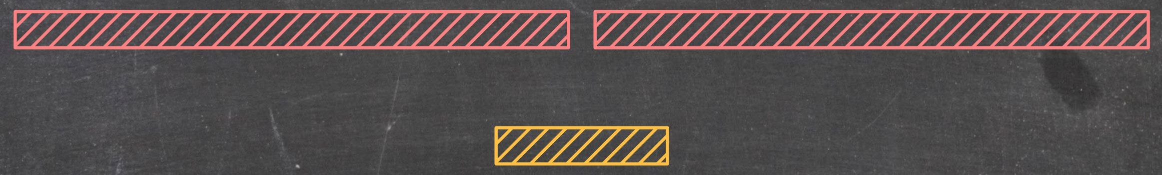

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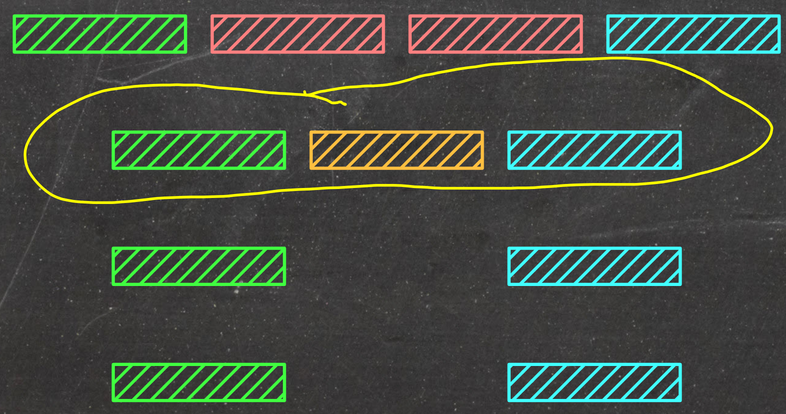

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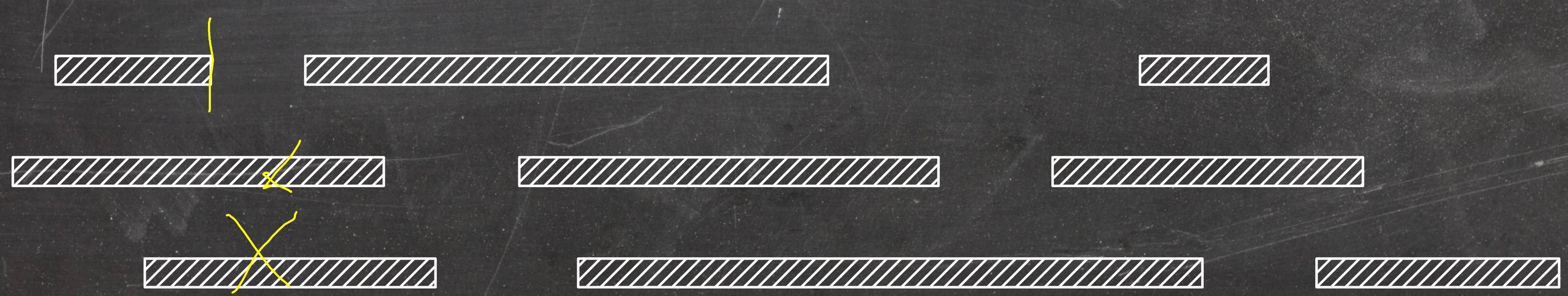

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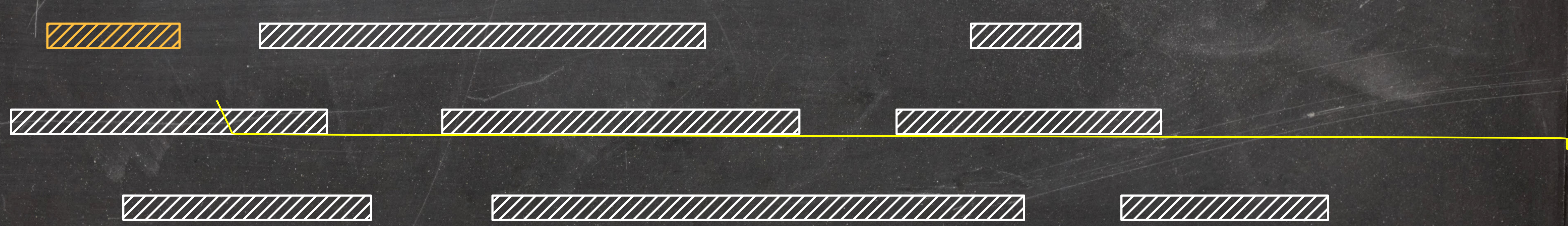

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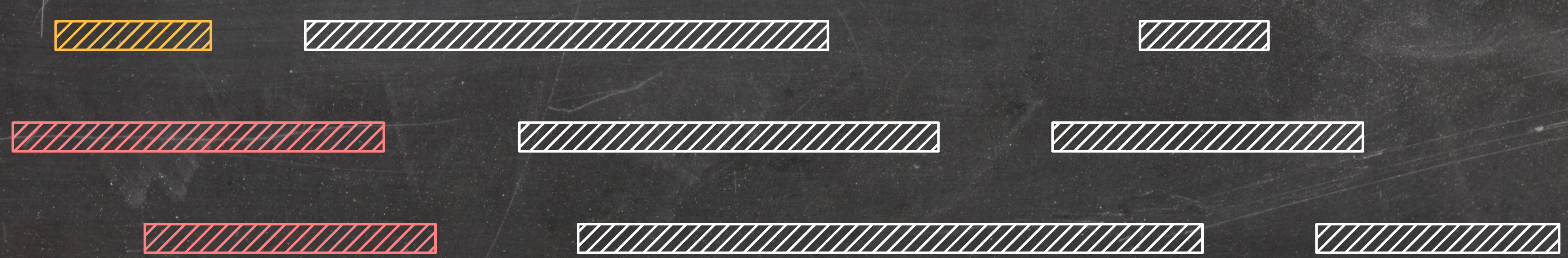

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Let $I_1 \prec I_2 \prec \cdots \prec I_k$ be the schedule we compute.

Let $O_1 \prec O_2 \prec \cdots \prec O_m$ be an optimal schedule.

Prove by induction on j that I_j ends no later than O_j.

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$\Rightarrow$ Since O_{j+1} starts after O_j ends, it also starts after I_j ends.

$\Rightarrow$ If $k < m$, FindSchedule inspects O_{k+1} after I_k and thus would have added it to its output, a contradiction.

The Greedy Algorithm Stays Ahead

Lemma: FindSchedule finds a maximum-cardinality set of conflict-free intervals.

Proof by induction:

Base case(s): Verify that the claim holds for a set of initial instances.

Inductive step: Prove that, if the claim holds for the first k instances, it holds for the (k + 1)st instance.

The Greedy Algorithm Stays Ahead

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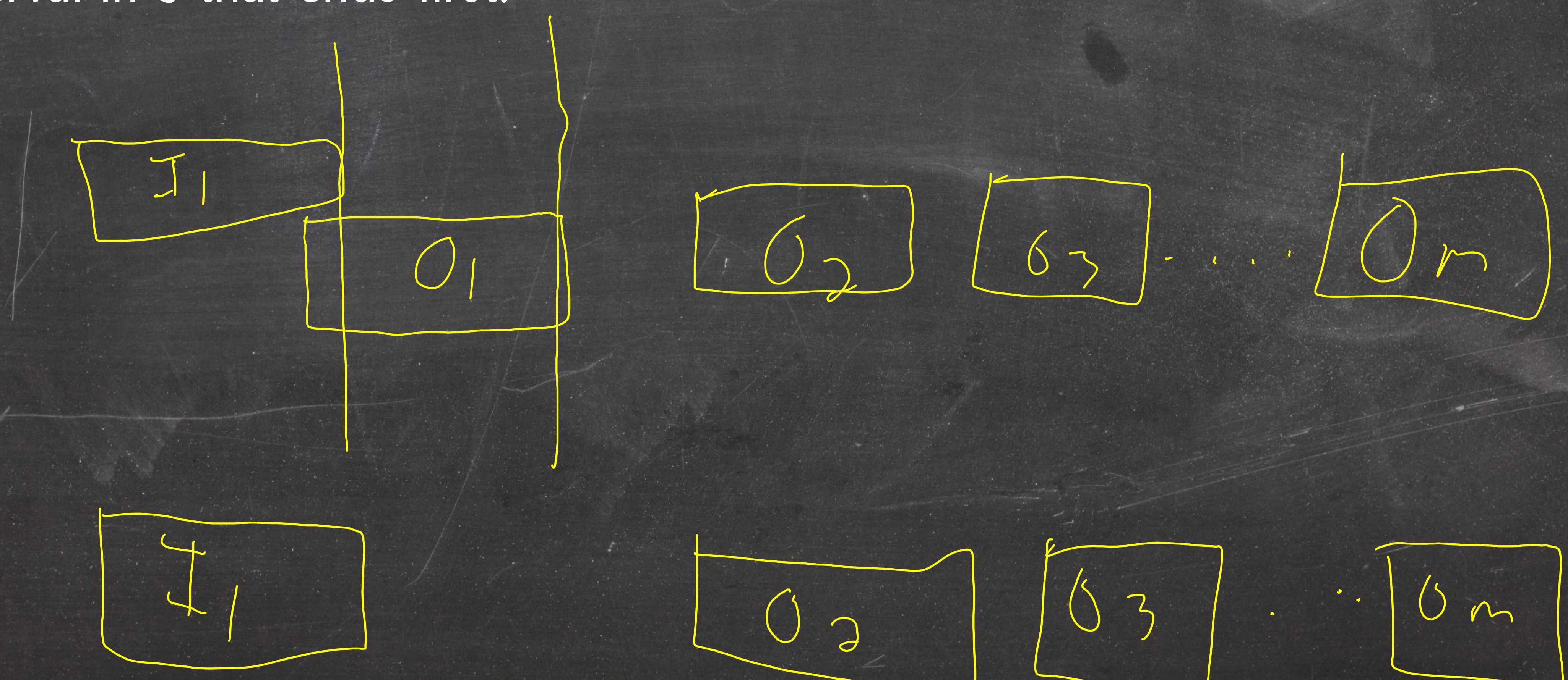
Base case: I_1 ends no later than O_1 because both I_1 and O_1 are chosen from S and I_1 is the interval in S that ends first.



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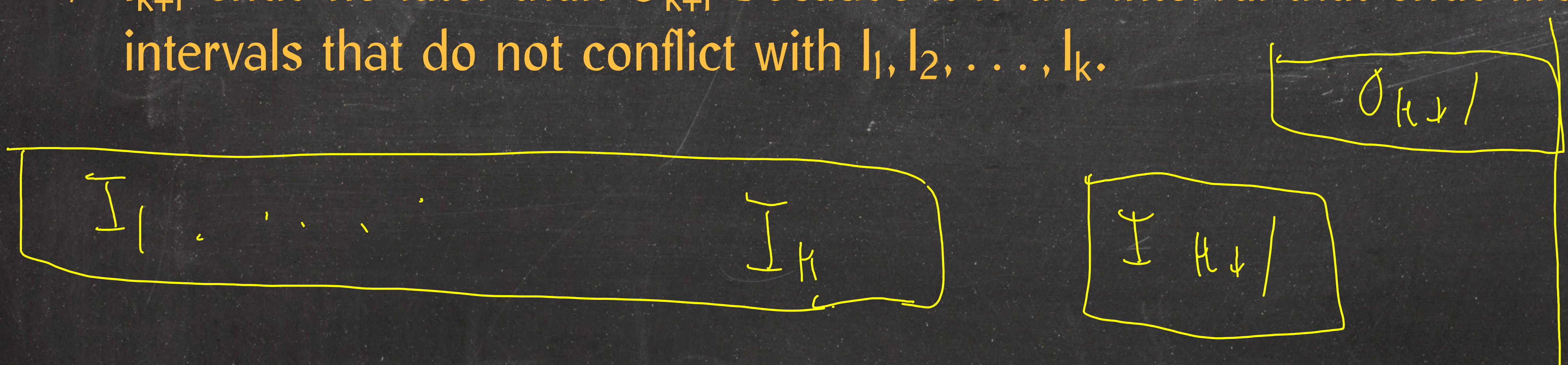
$\Rightarrow$ I_{k+1} ends no later than O_{k+1} because it is the interval that ends first among all intervals that do not conflict with $I_1, I_2, \ldots, I_k$.



Implementing The Algorithm

FindSchedule(S)

```
S' = []
sort the intervals in S by increasing finish times
S'.append(S[1])
f = S[1].f          → finish time of set S'
for i = 2 to |S|
    do if S[i].s > f
            then S'.append(S[i])
                 f = S[i].f
return S'
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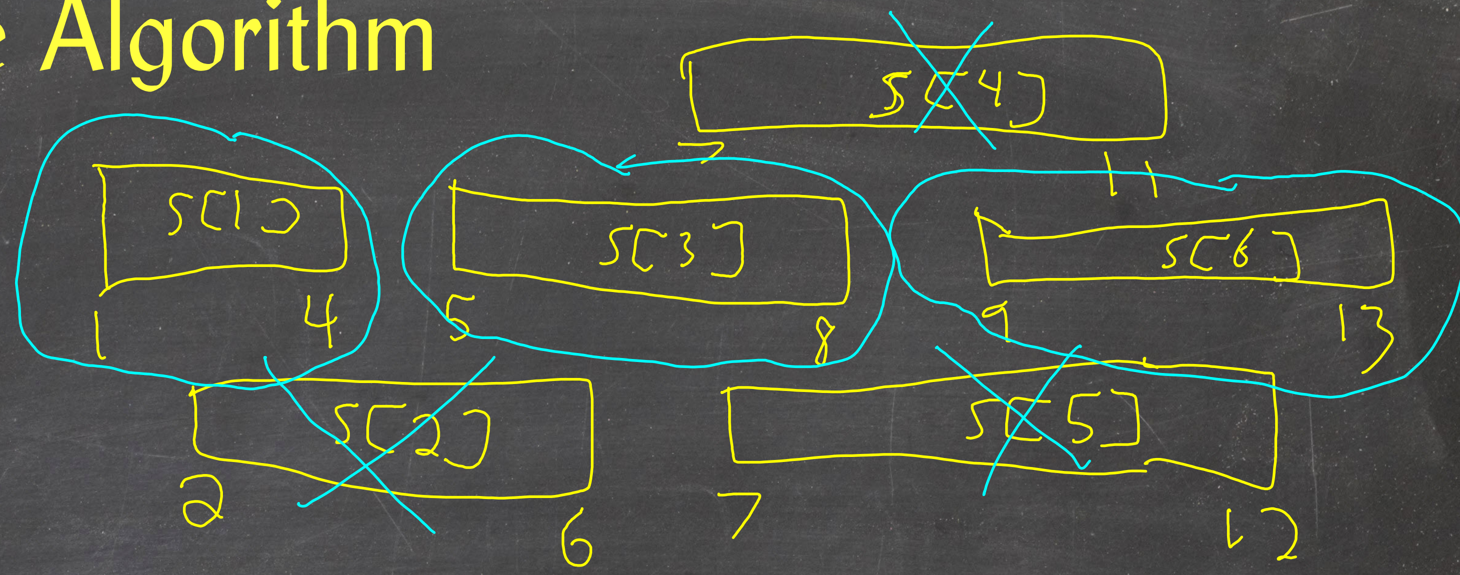

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Lemma: A maximum-cardinality set of non-conflicting intervals can be found in $O(n \lg n)$ time.

Minimum Spanning Tree

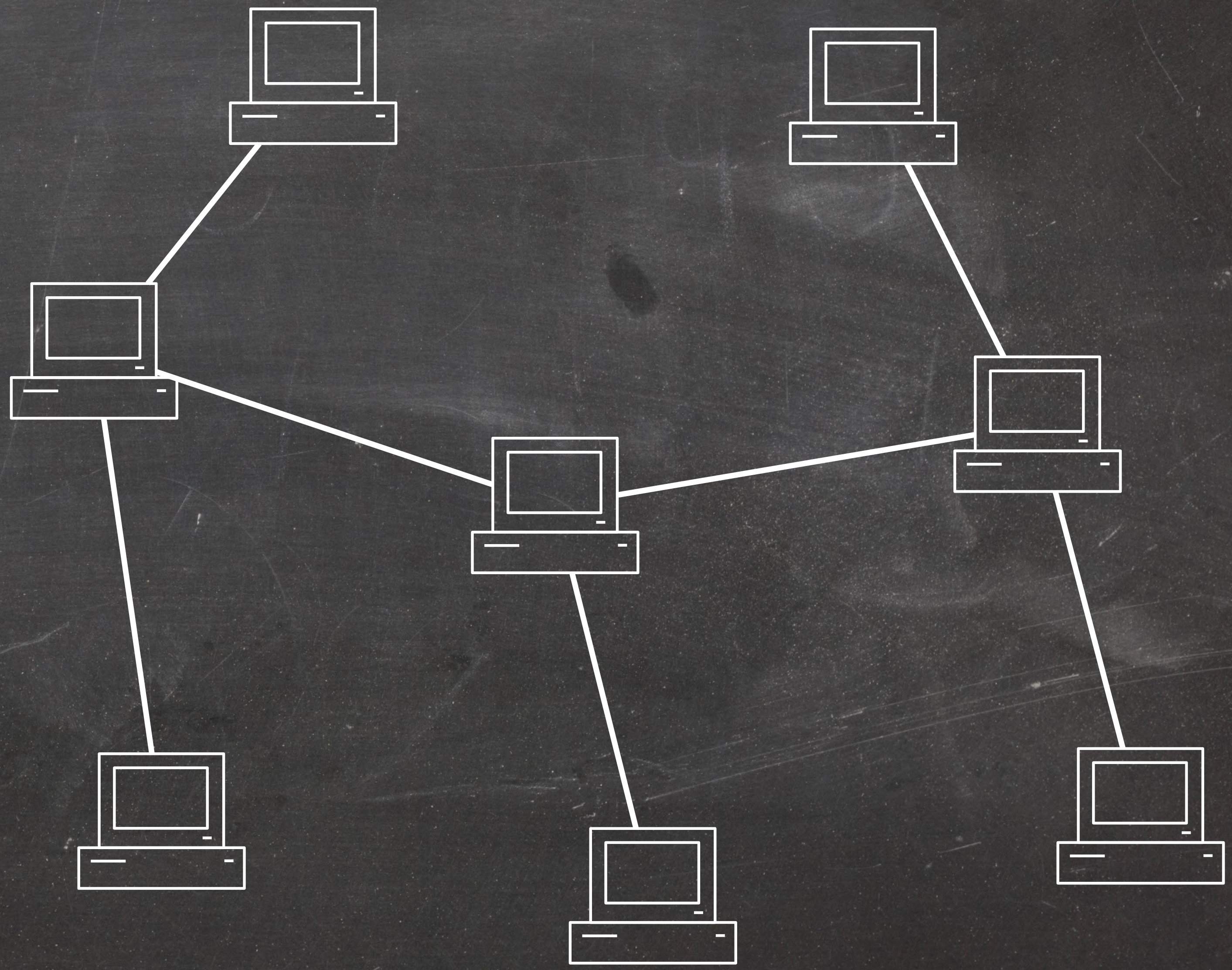
Given: n computers

Goal: Connect them so that every computer can communicate with every other computer.

We don't care whether the connection between any pair of computers is short.

We don't care about fault tolerance.

Every foot of cable costs us $1.

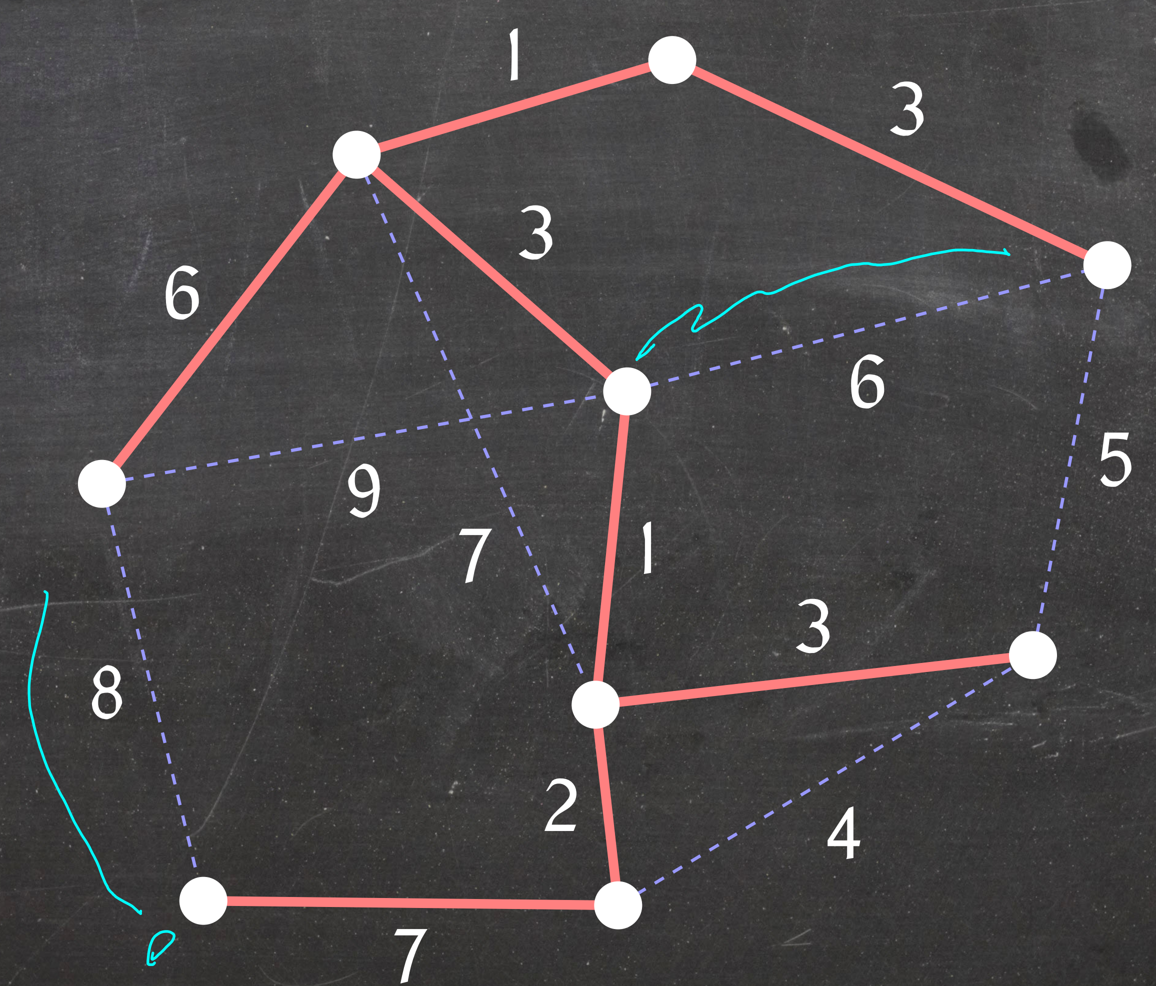


⇒ We want the cheapest possible network.

Minimum Spanning Tree

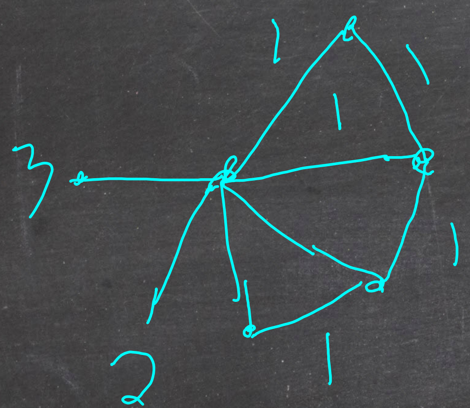
Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a minimum spanning tree (MST) T of G is a spanning tree with minimum total weight

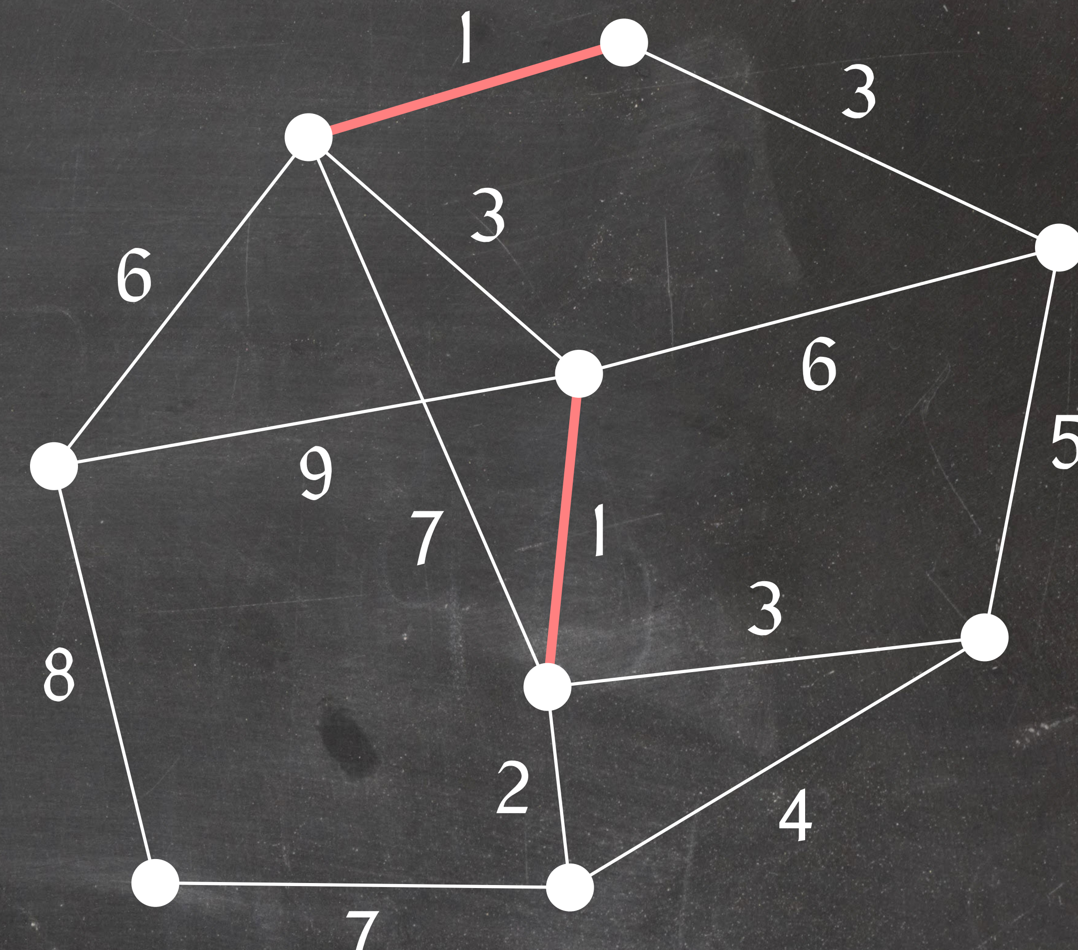
$$w(T) = \sum_{e \in T} w(e).$$



Kruskal's Algorithm

Greedy choice: Pick the shortest edge





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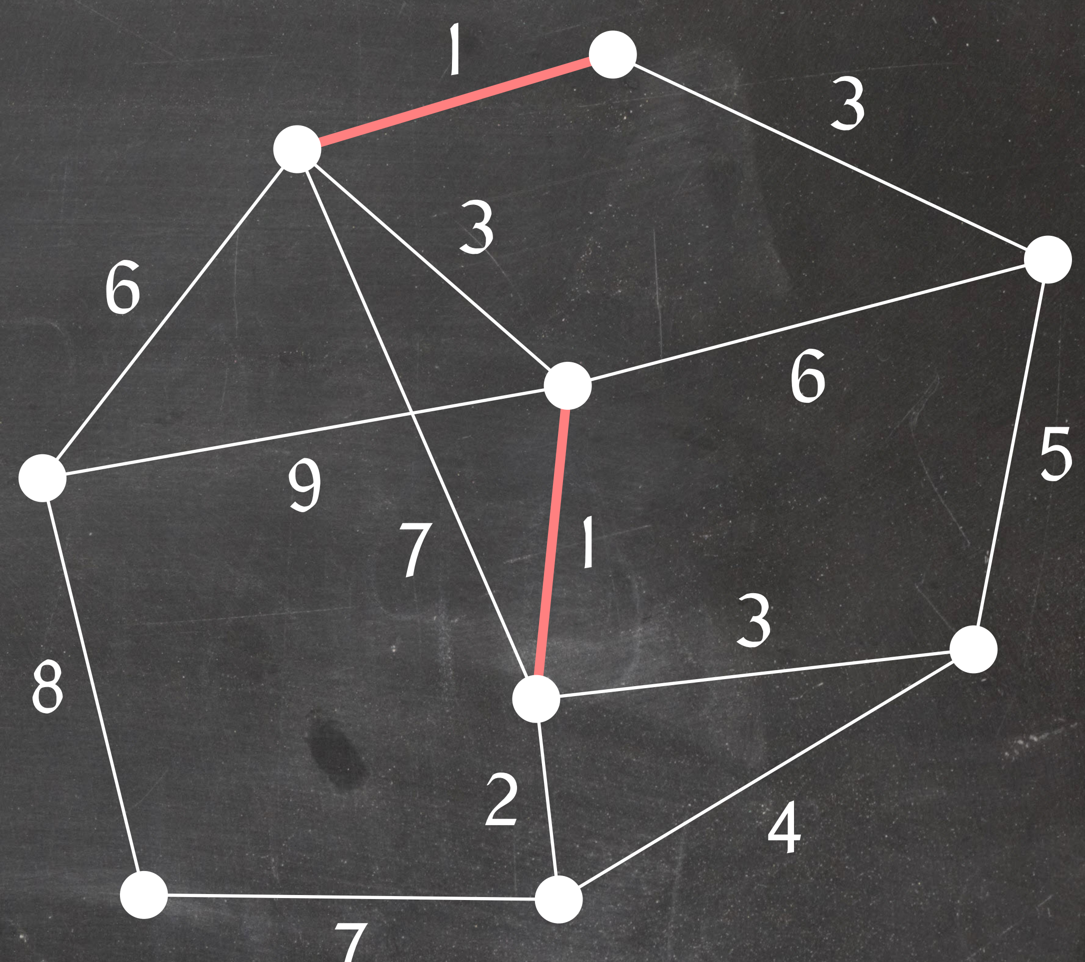

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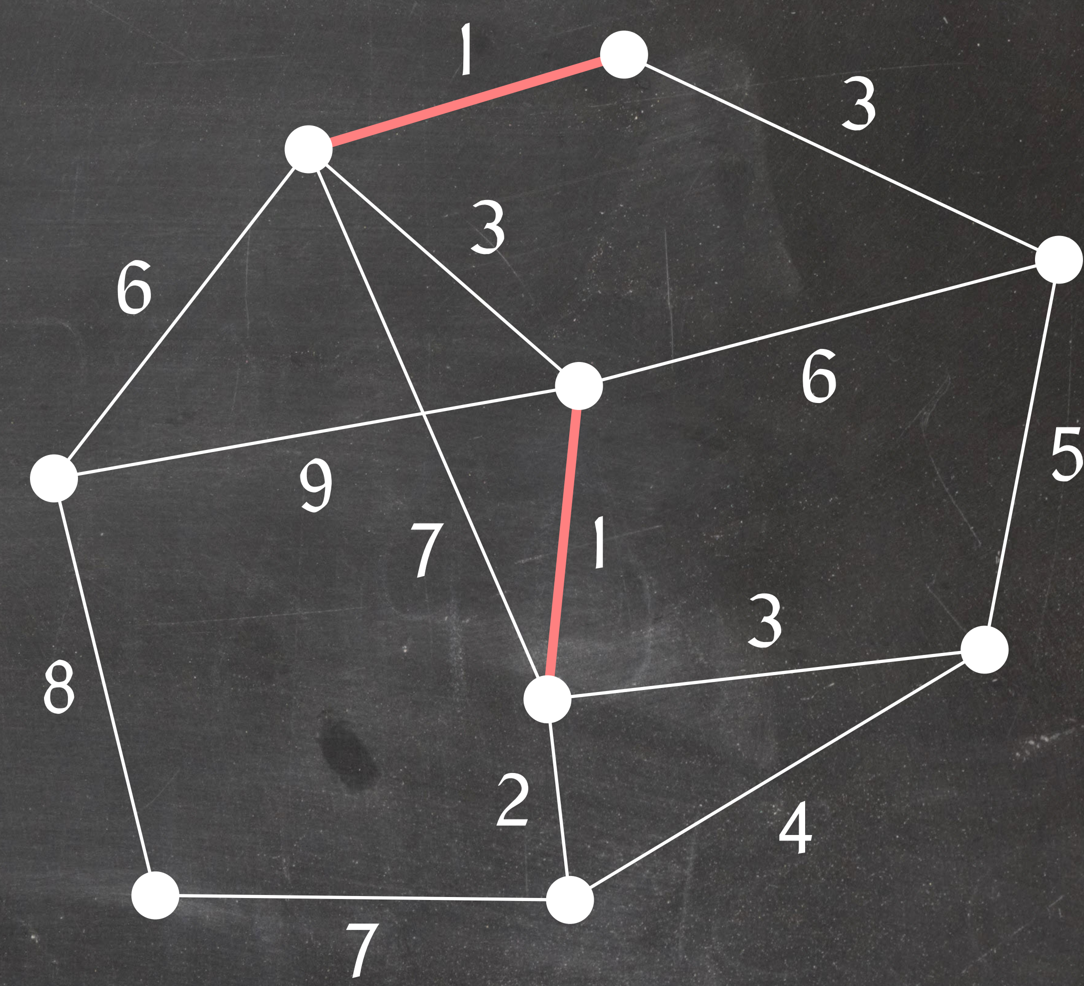

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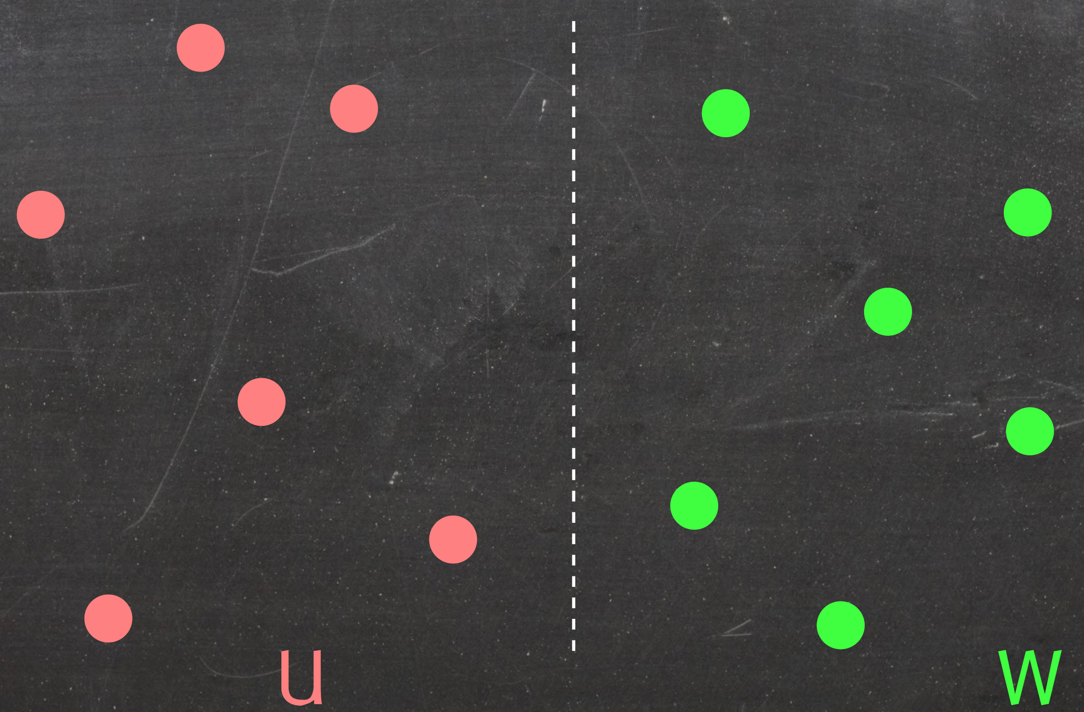
A Cut Theorem

A cut is a partition (U, W) of V into two non-empty subsets: $\emptyset \subset U \subset V$ and $W = V \setminus U$.



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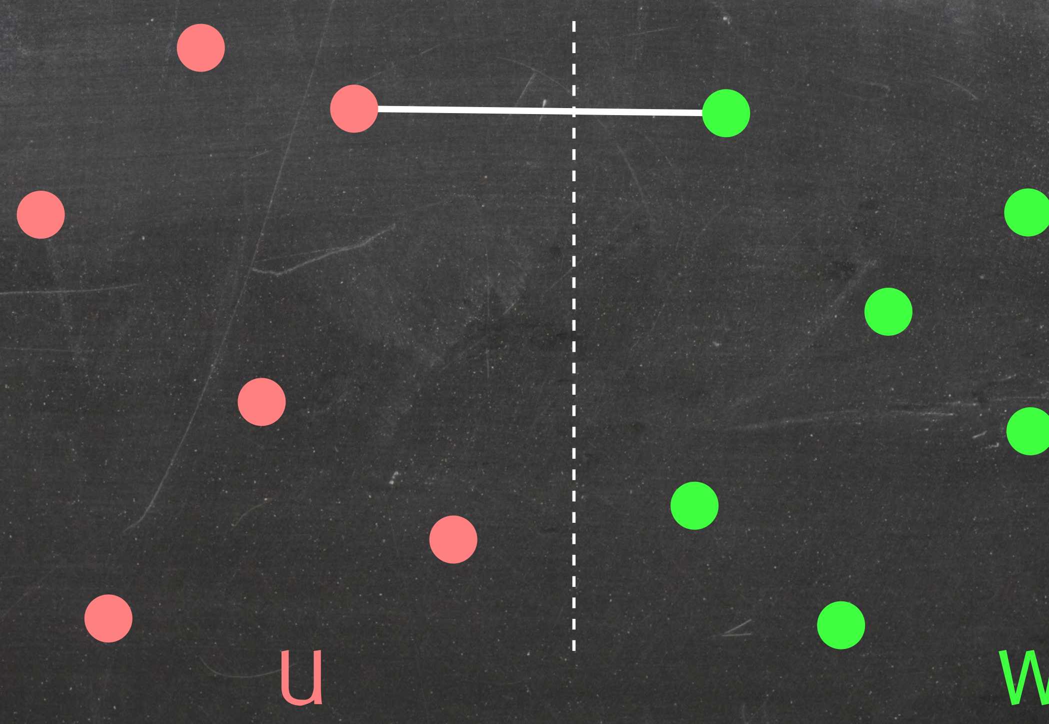

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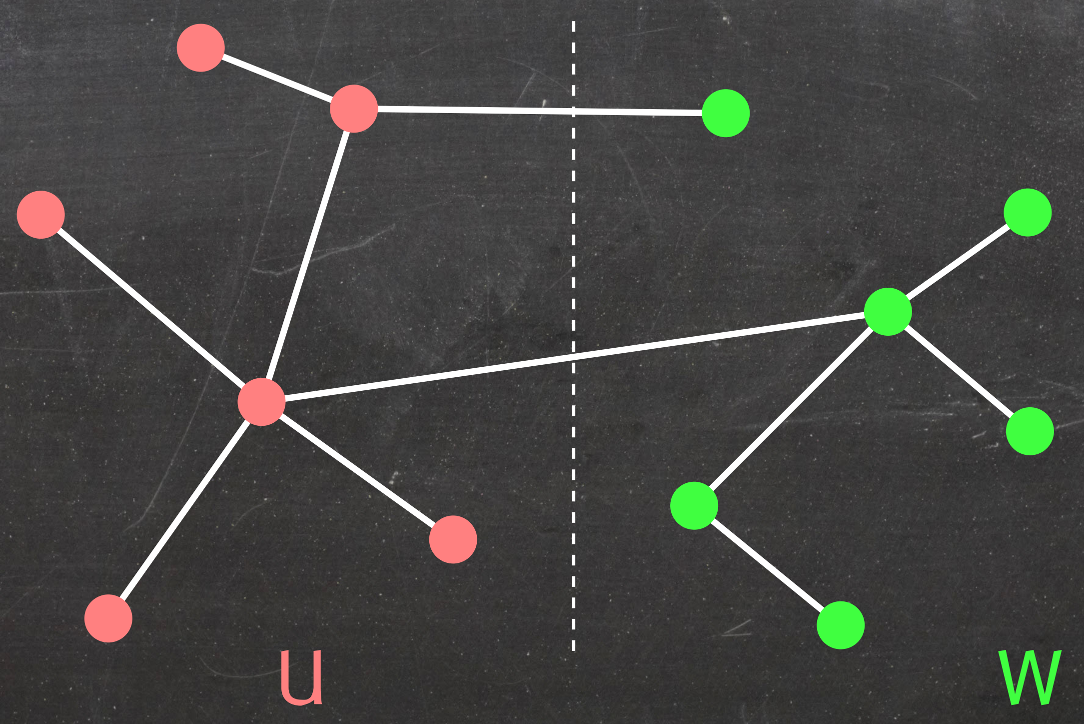
Theorem: Let T be a minimum spanning tree, let (U, W) be an arbitrary cut, and let e be the cheapest edge crossing the cut. Then there exists a minimum spanning tree that contains e and all edges of T that do not cross the cut.



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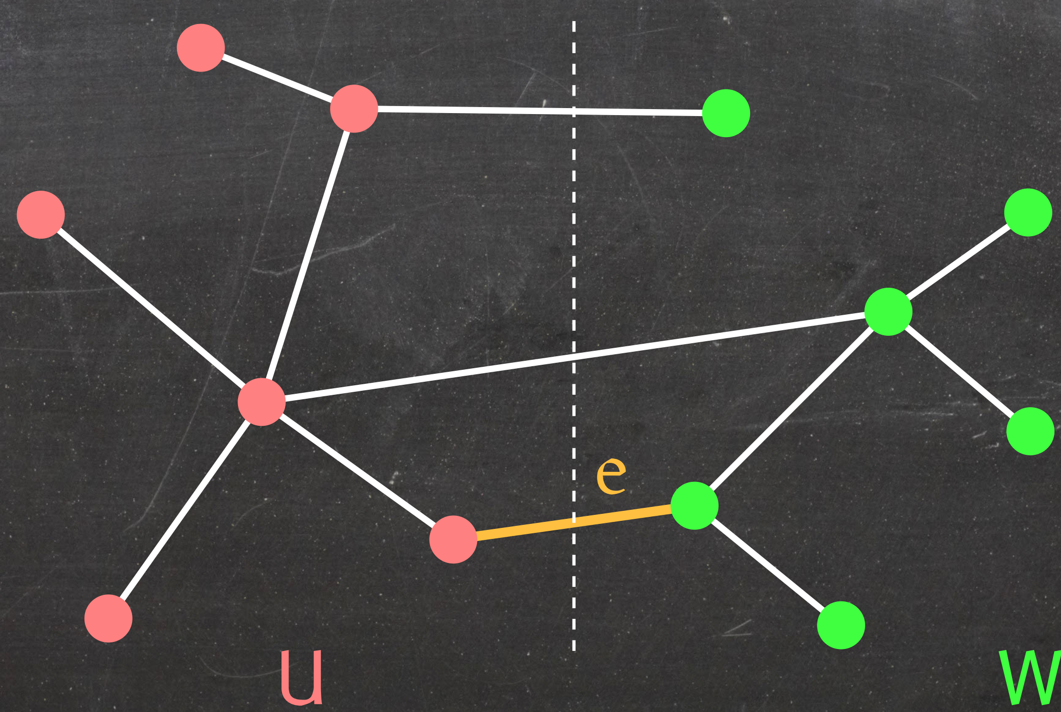

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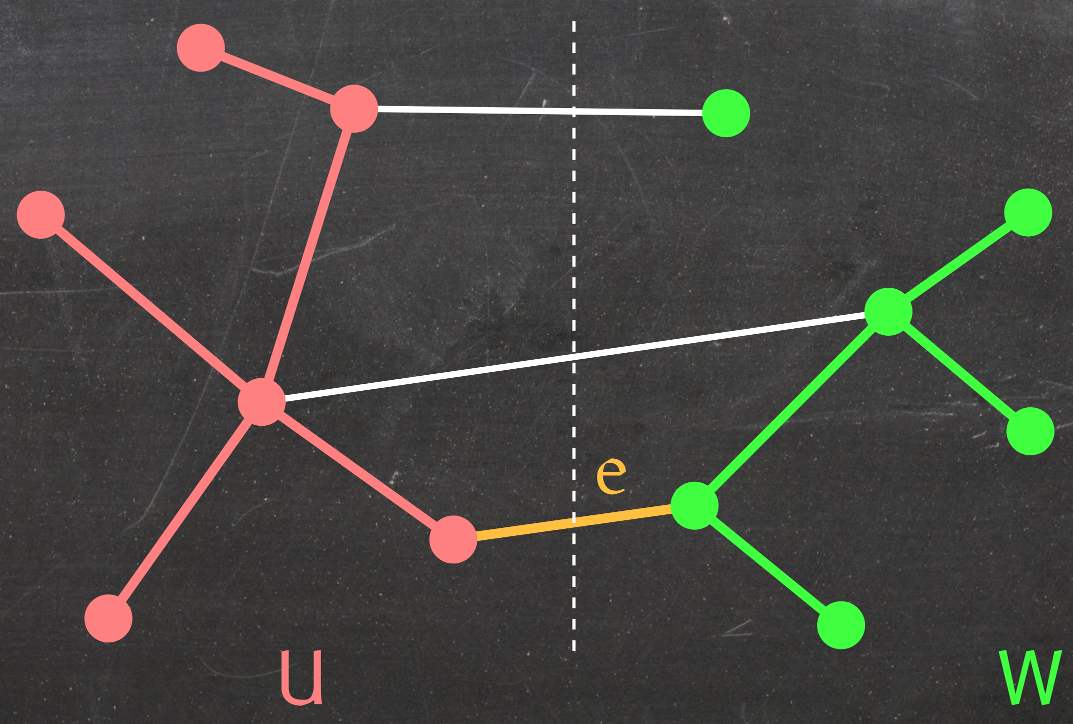

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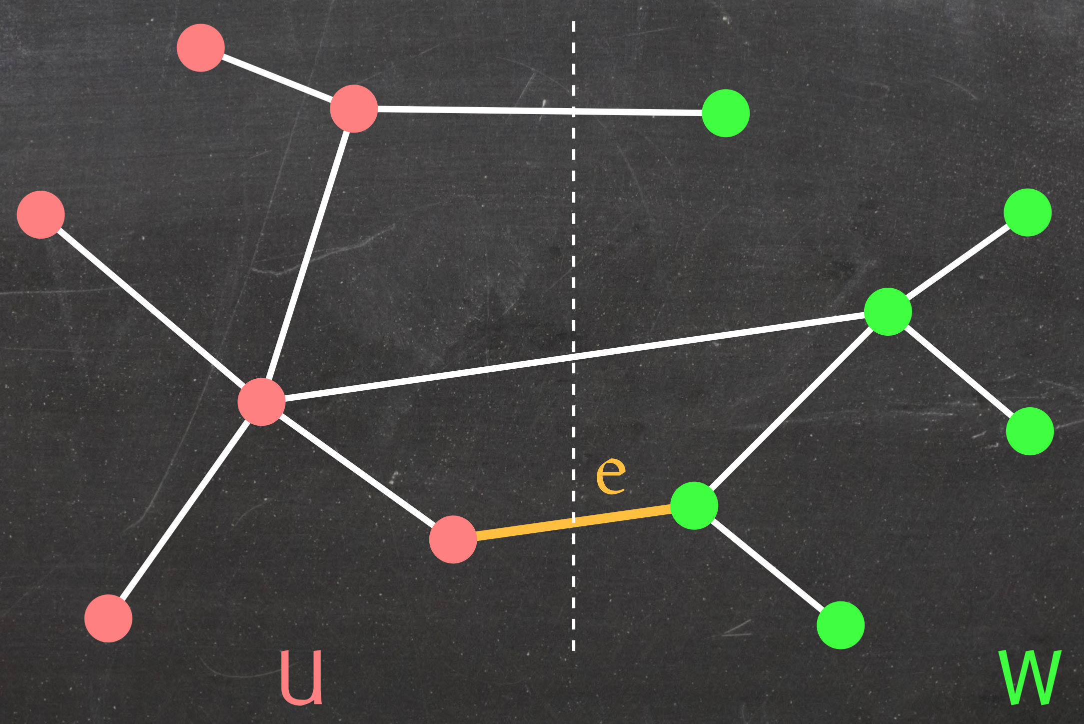

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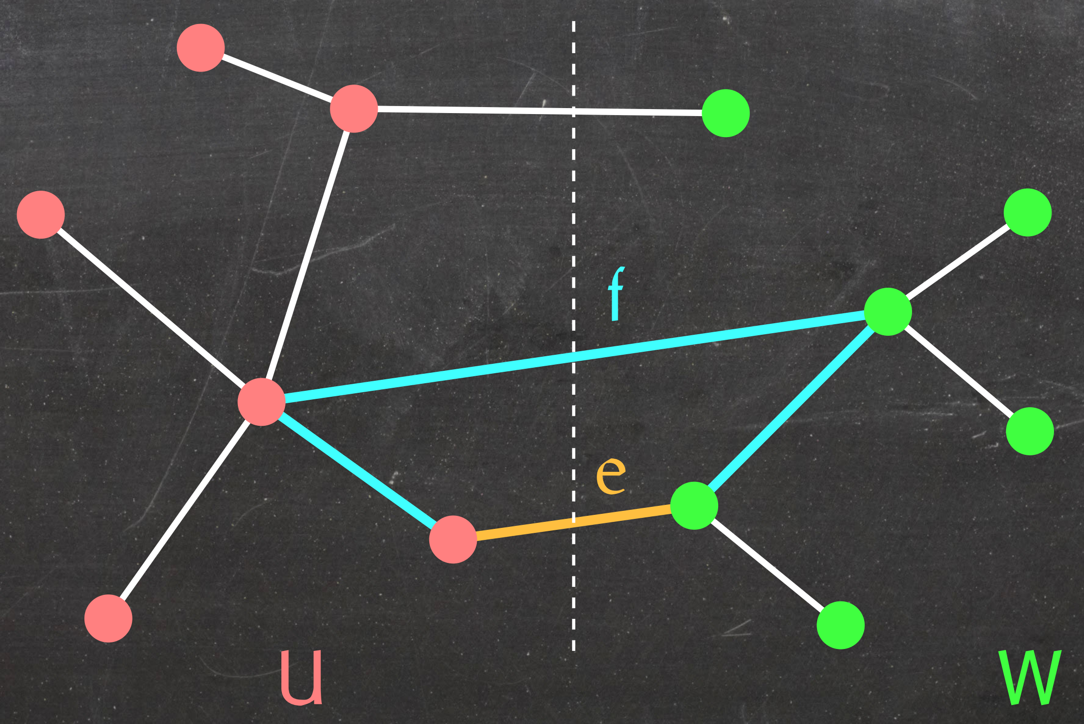

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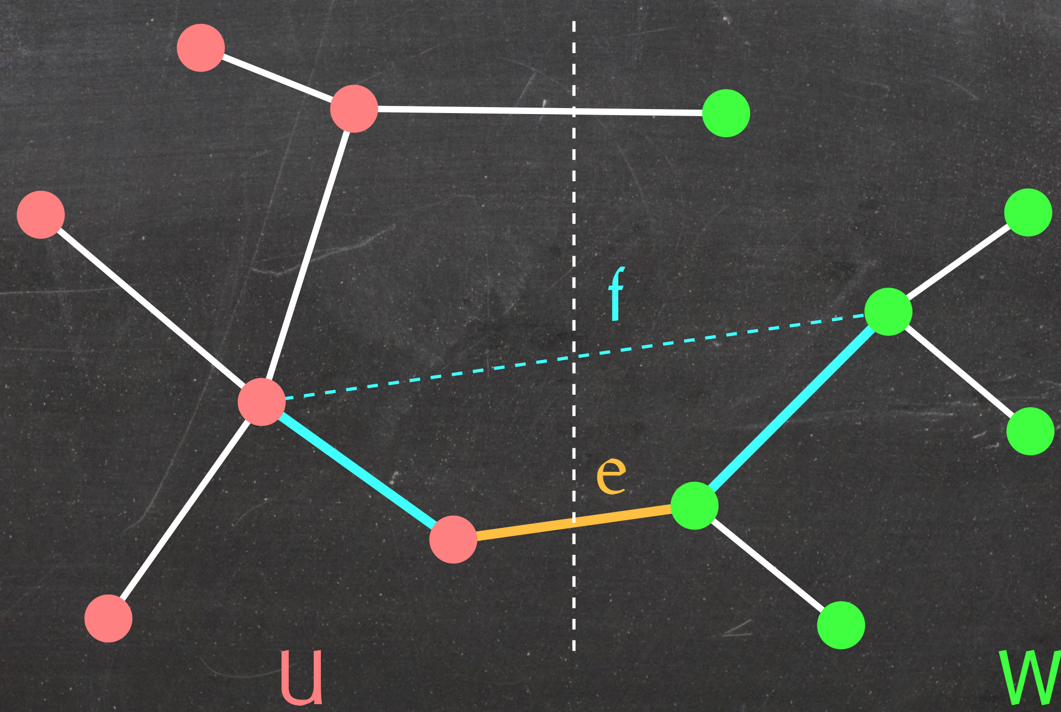

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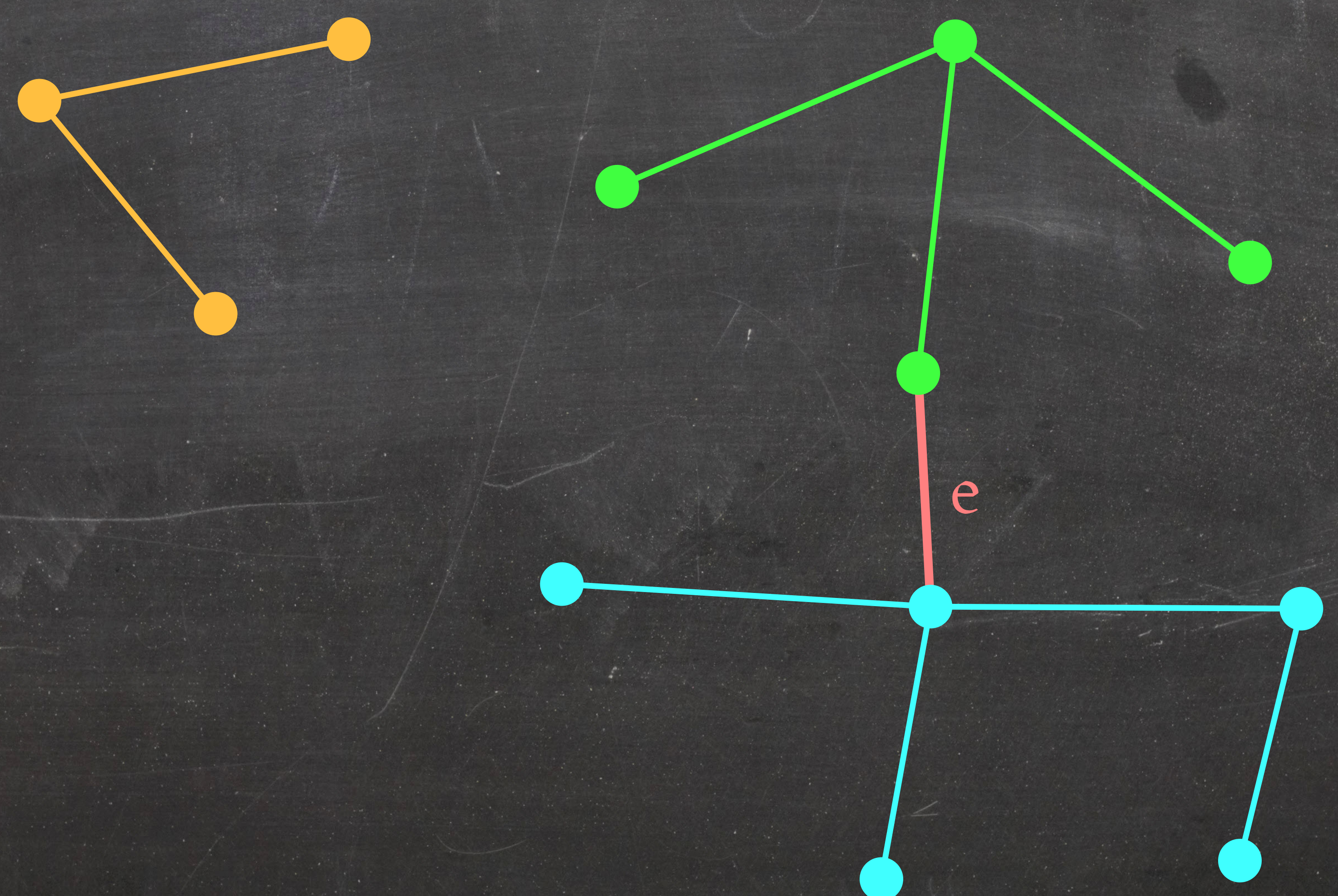

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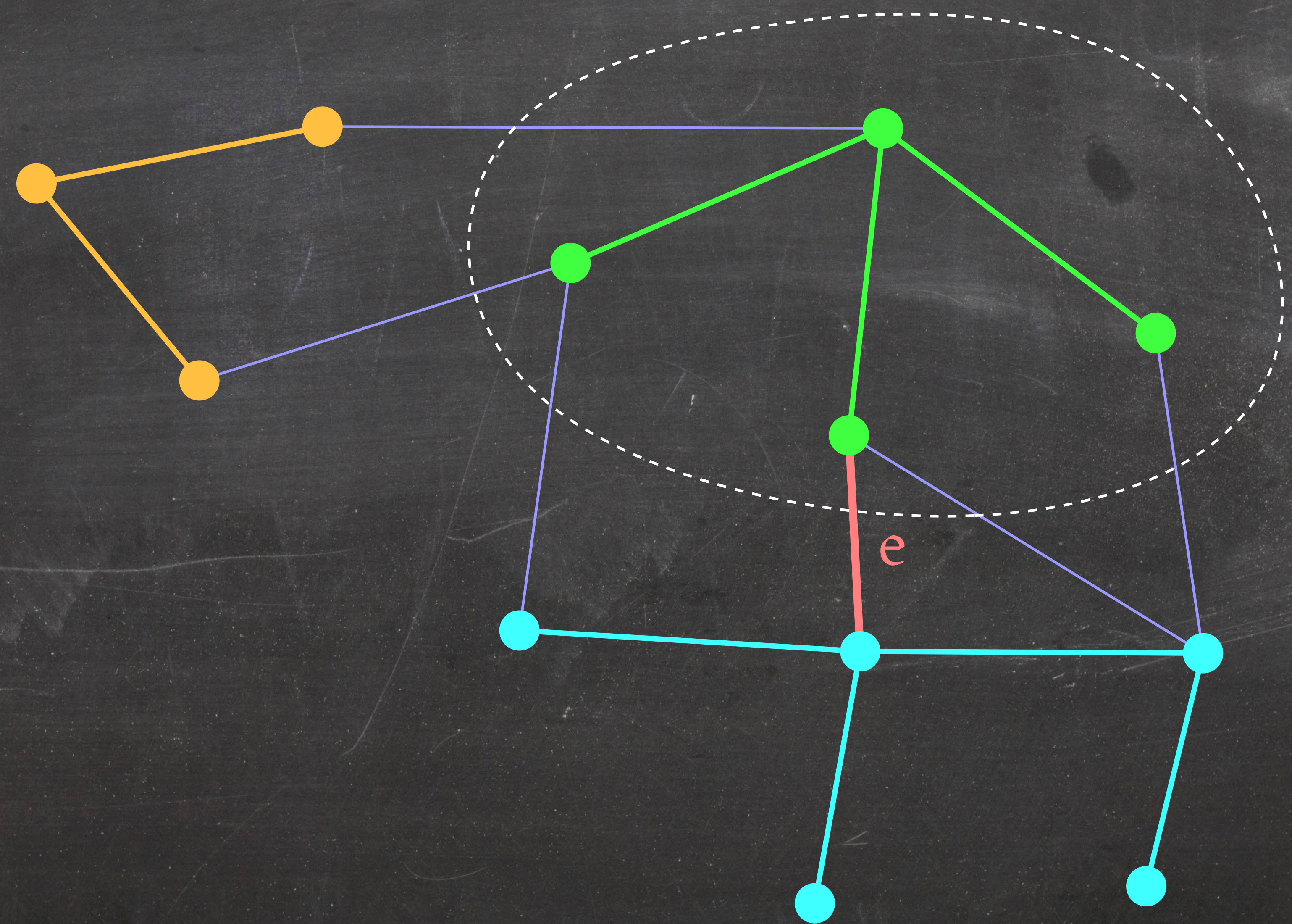

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$$\Downarrow$$

Kruskal(G)

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T = (V, ∅)
sort the edges in G by increasing weight
for every edge (v, w) of G, in sorted order
    do if v and w belong to different connected components of T
            then add (v, w) to T
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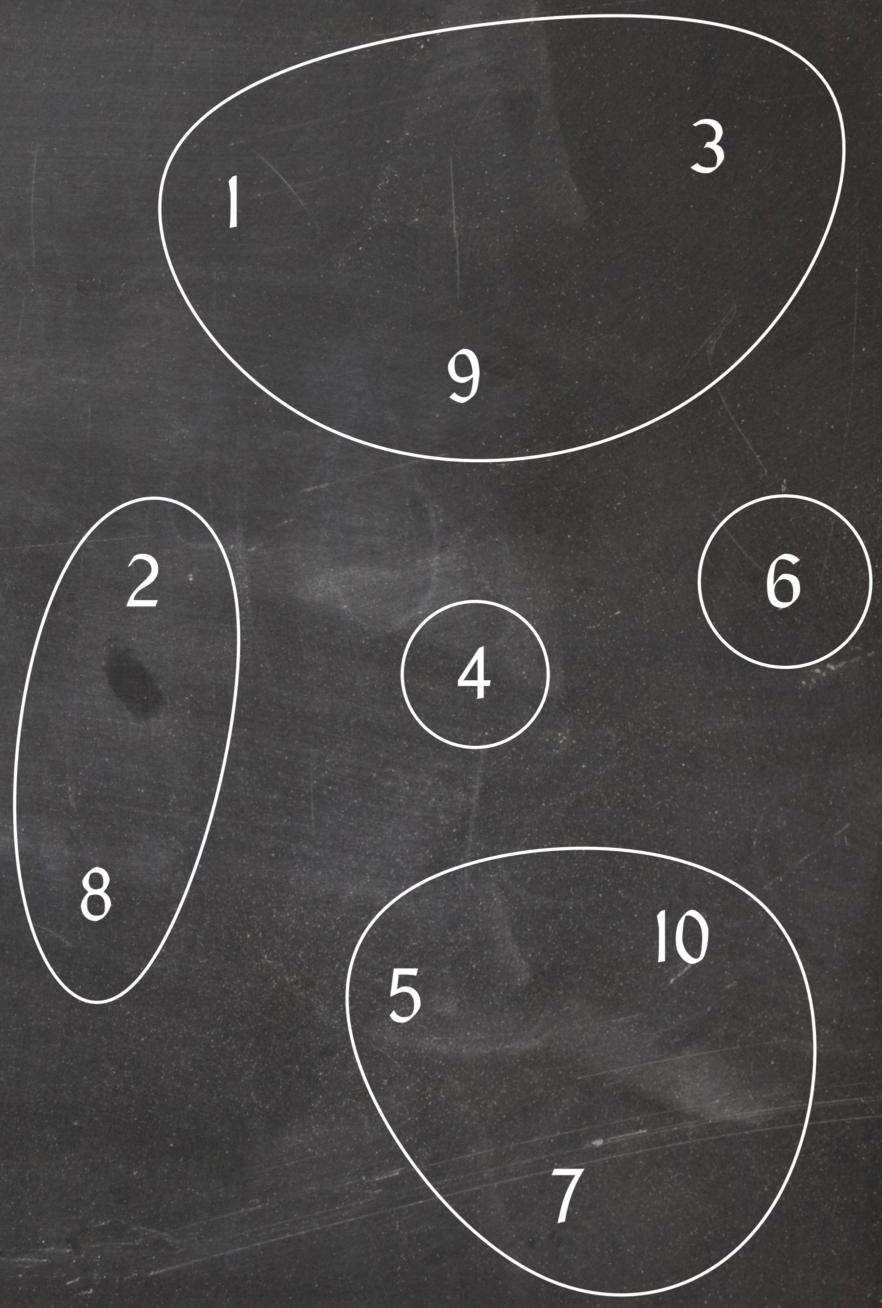
A Union-Find Data Structure

Given a set S of elements, maintain a partition of S into subsets $S_1, S_2, \ldots, S_k$.



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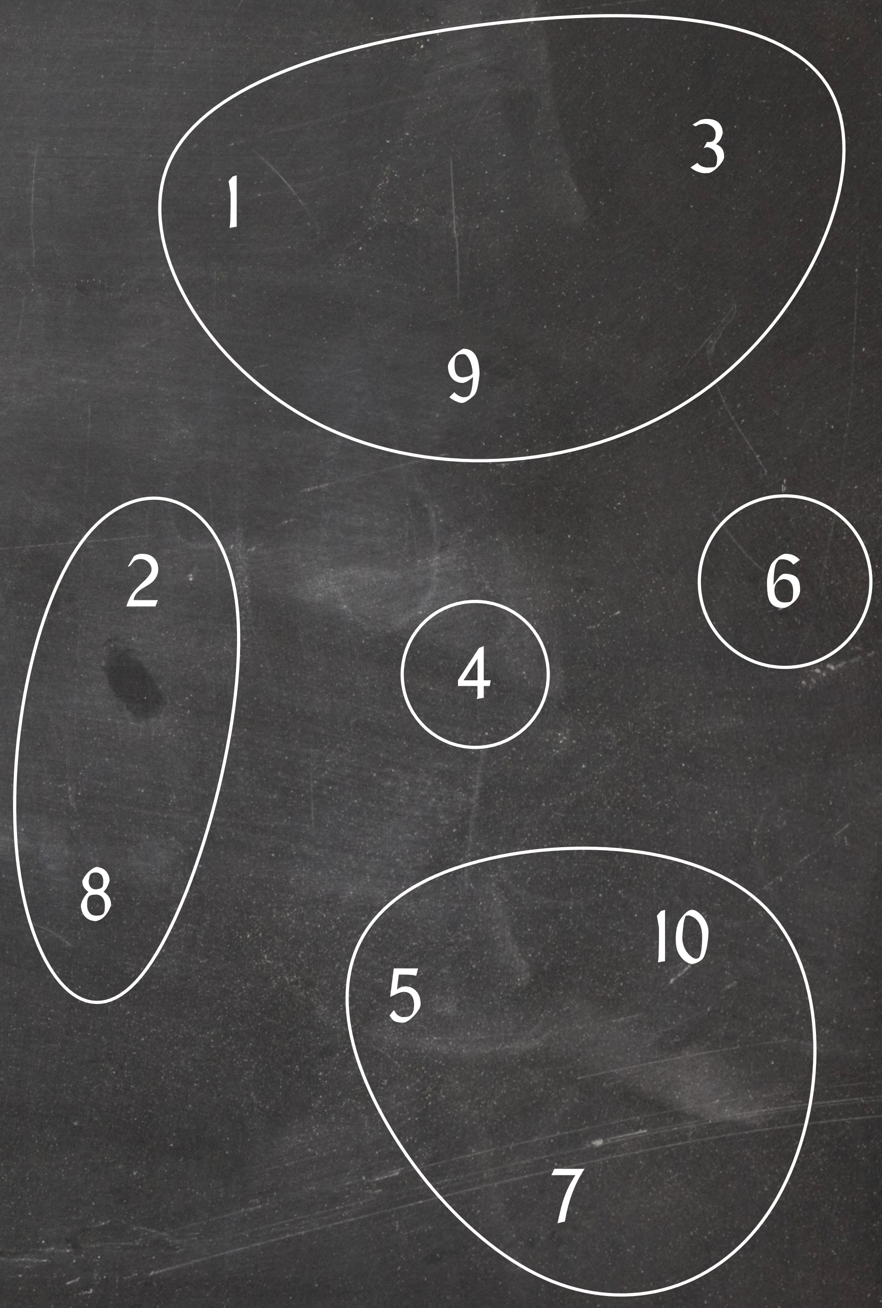
Union(x, y): Replace sets S_i and S_j in the partition with $S_i \cup S_j$, where $x \in S_i$ and $y \in S_j$.



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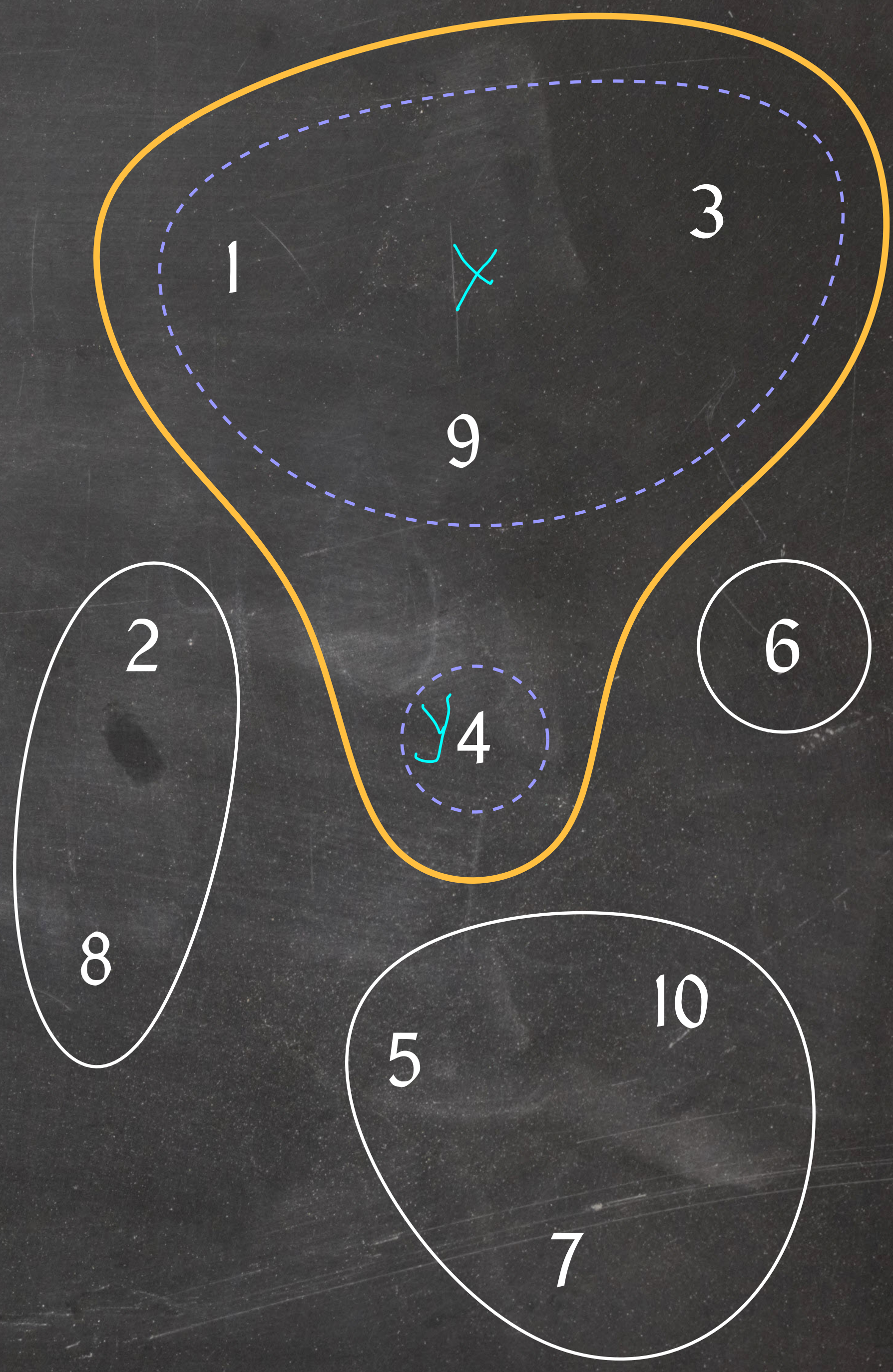

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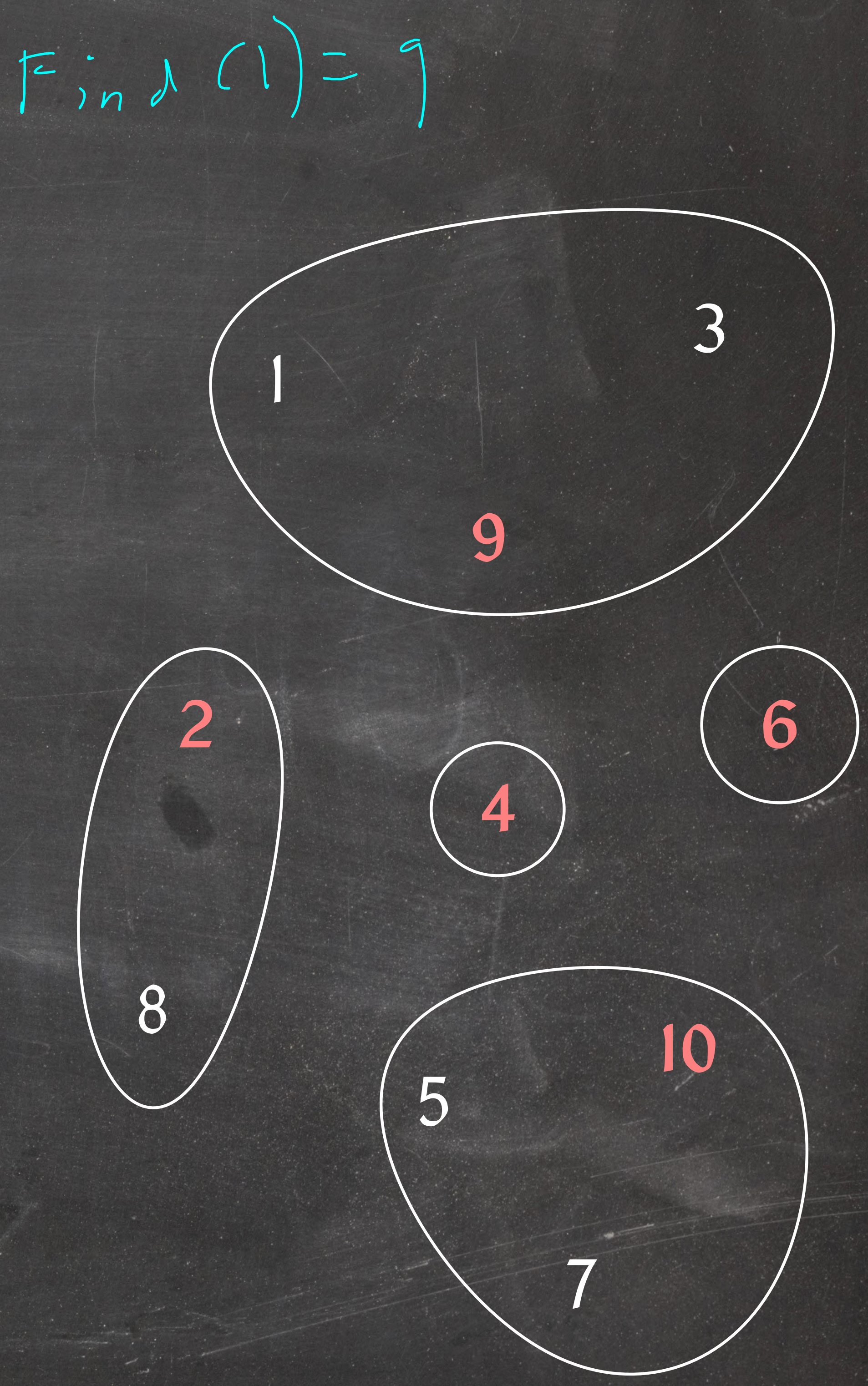
Union(x, y): Replace sets S_i and S_j in the partition with $S_i \cup S_j$, where $x \in S_i$ and $y \in S_j$.

Find(x): Return a representative $r(S_i) \in S_i$ of the set S_i that contains x.



A Union-Find Data Structure

Given a set S of elements, maintain a partition of S into subsets $S_1, S_2, \ldots, S_k$.

Support the following operations:

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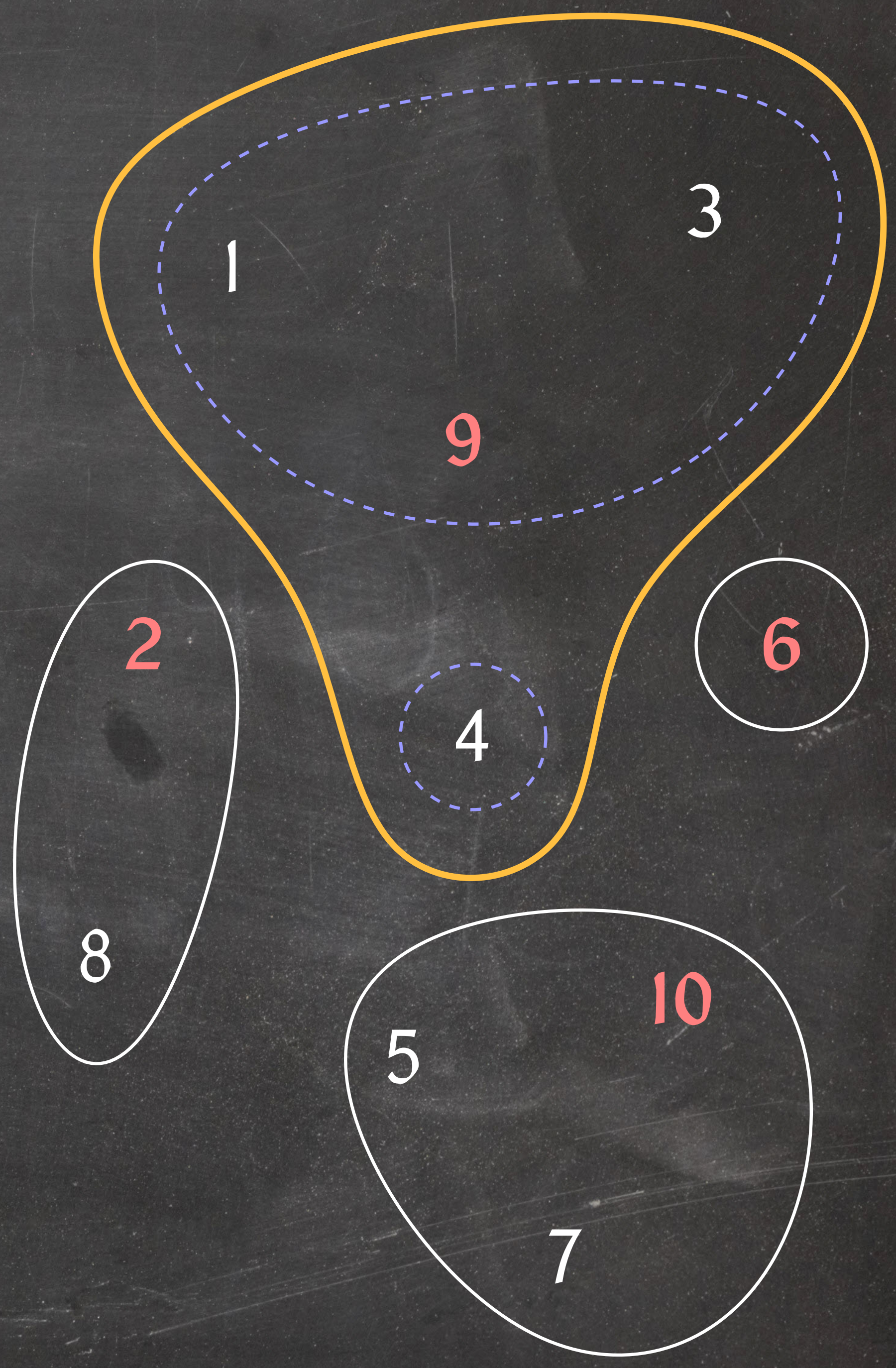

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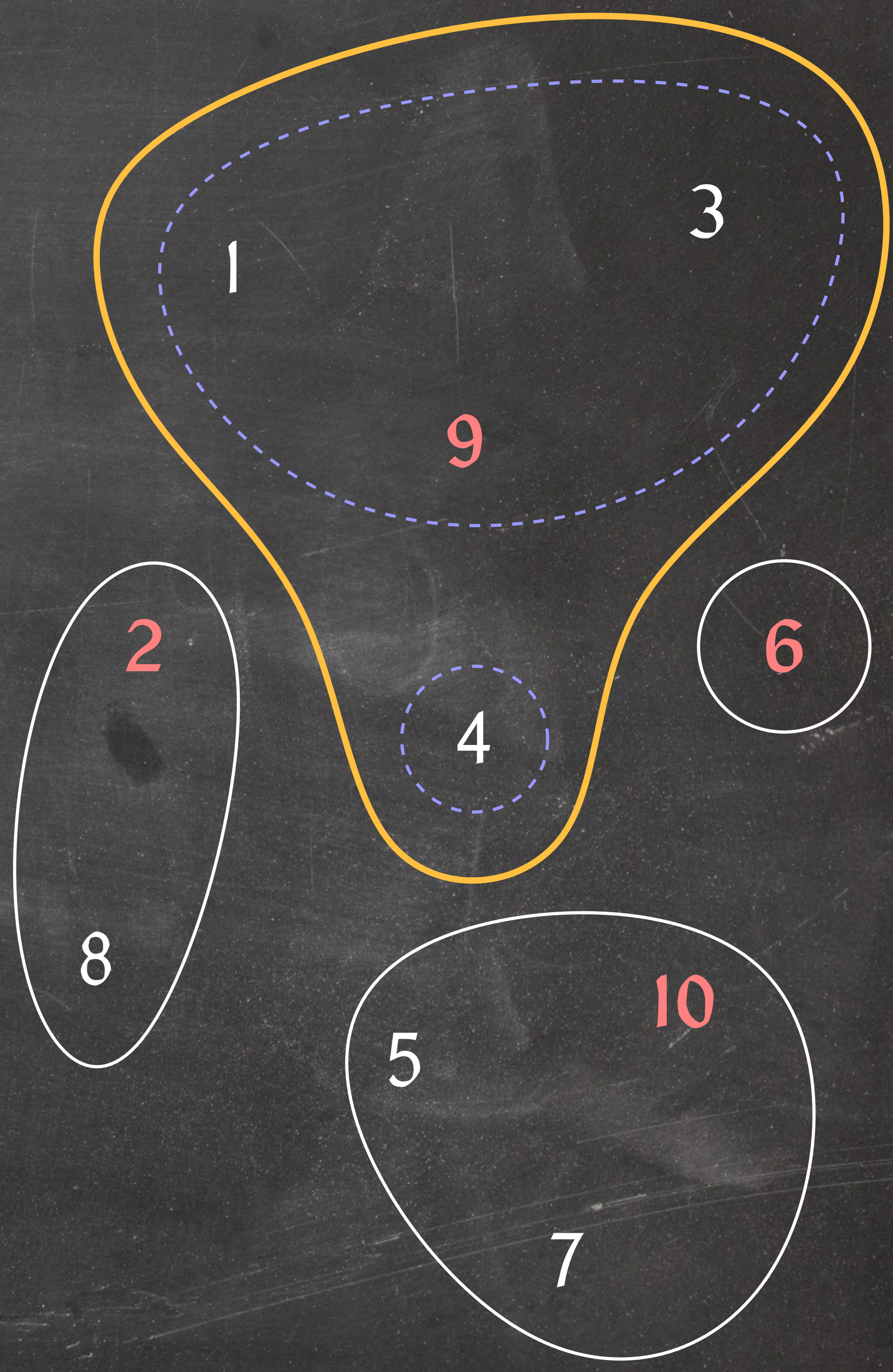
In particular, Find(x) = Find(y) if and only if x and y belong to the same set.



Kruskal's Algorithm Using Union-Find

Idea: Maintain a partition of V into the vertex sets of the connected components of T.

Kruskal(G)

```
T = (V, ∅)
initialize a union-find structure D for V with every vertex v ∈ V in its own set
sort the edges in G by increasing weight
for every edge (v, w) of G, in sorted order
    do if D.find(v) ≠ D.find(w)
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               D.union(v, w)
return T
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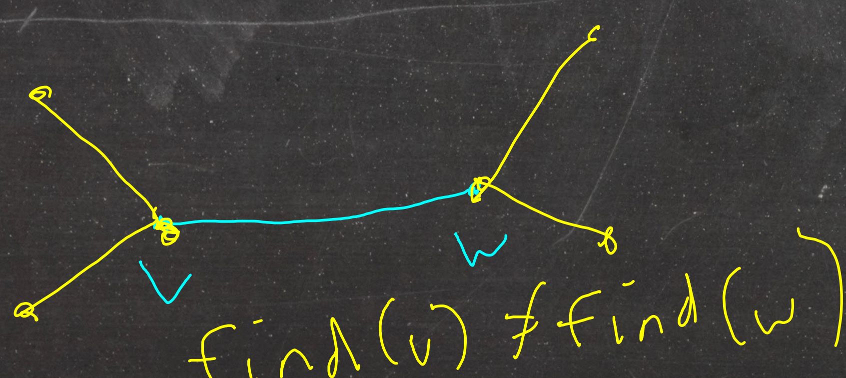

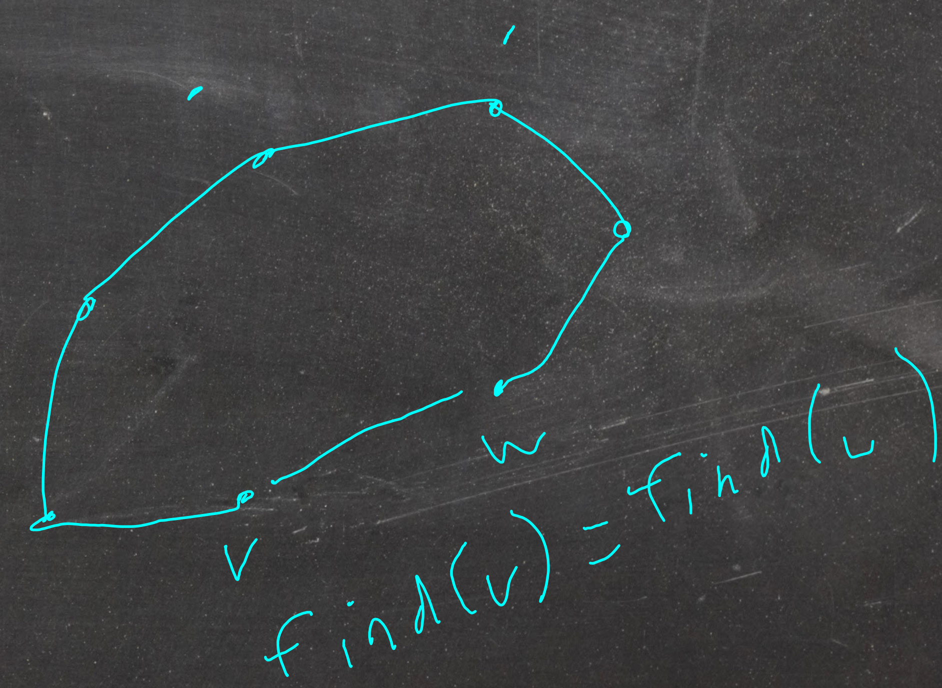

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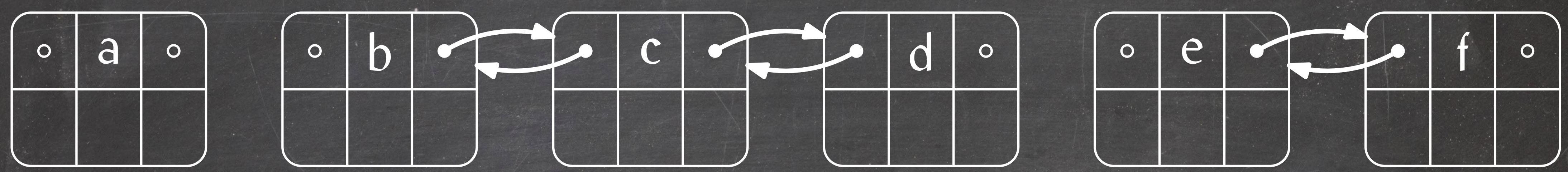
Lemma: Kruskal's algorithm takes $O(m \lg m)$ time plus the cost of $2m$ Find and $n - 1$ Union operations.

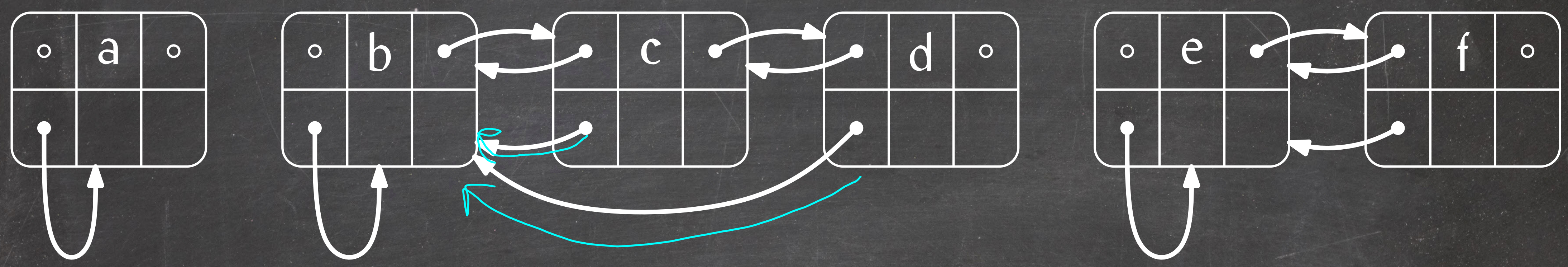
A Simple Union-Find Structure



List node:

- A set element
- Pointers to predecessor and successor
- Pointer to head of the list
- Pointer to tail of the list (only valid for head node)
- Size of the list (only valid for head node)

A Simple Union-Find Structure



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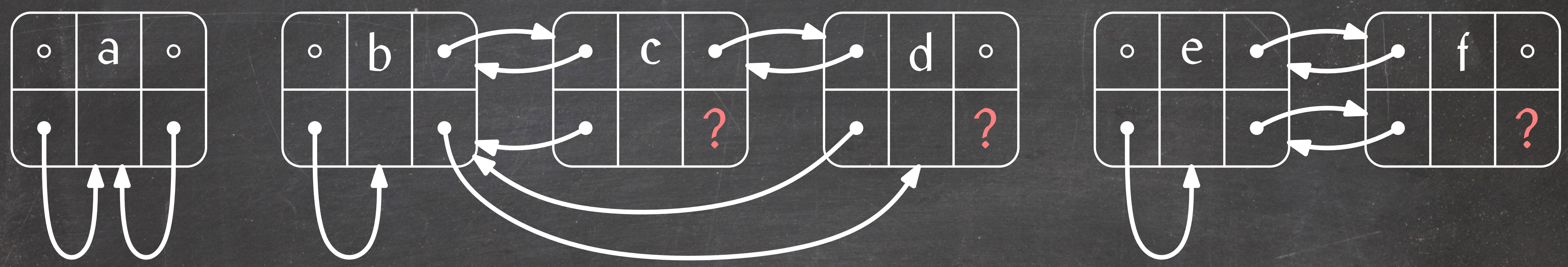

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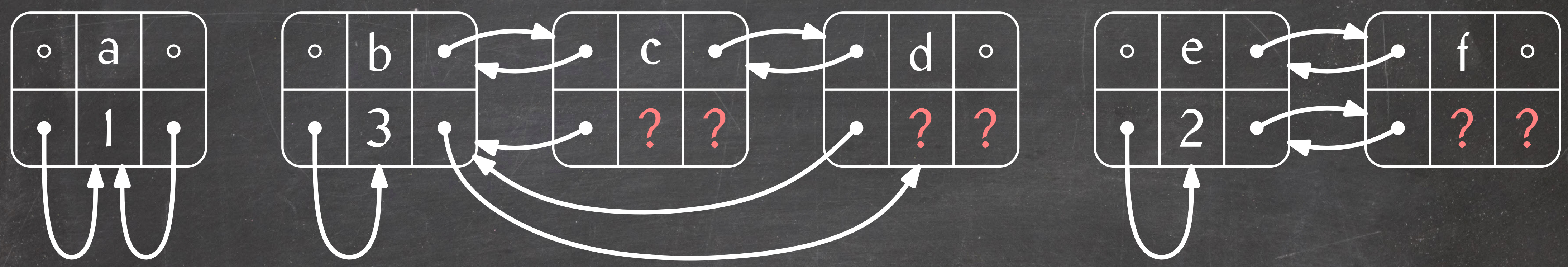

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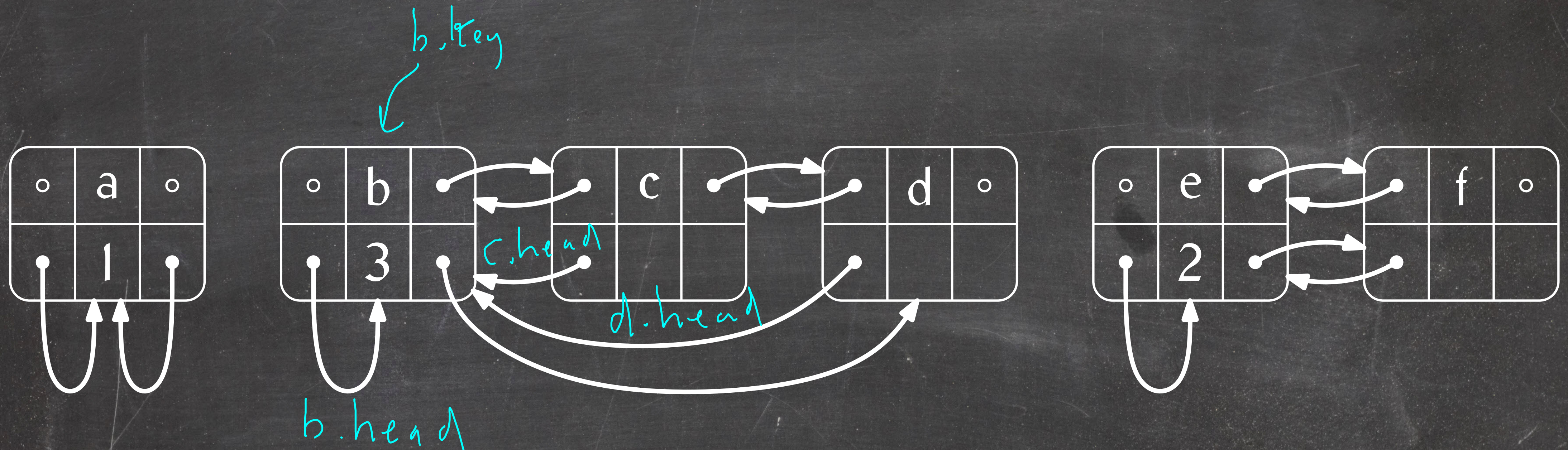
Find

D.find(x)

```
return x.head.key
```



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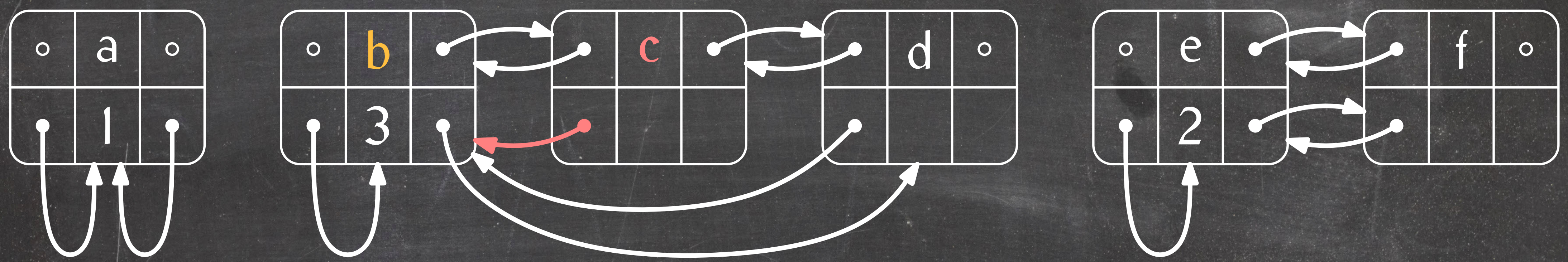

D.find(c) = b

Find

D.find(x)

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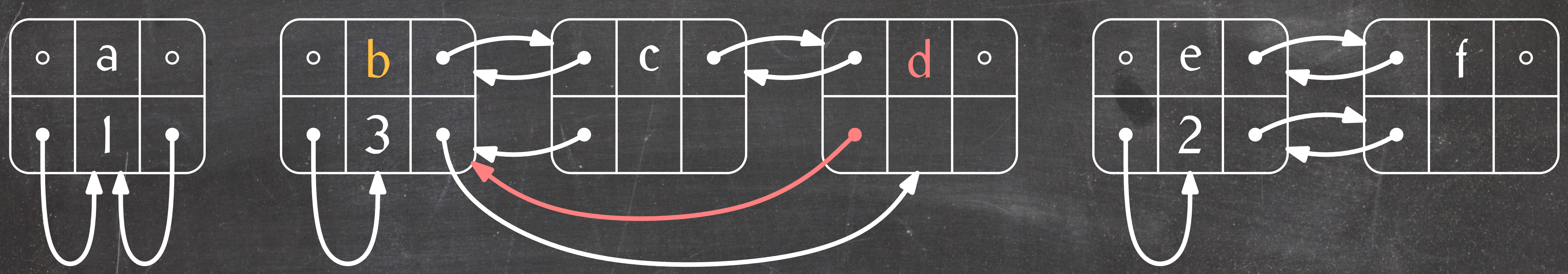

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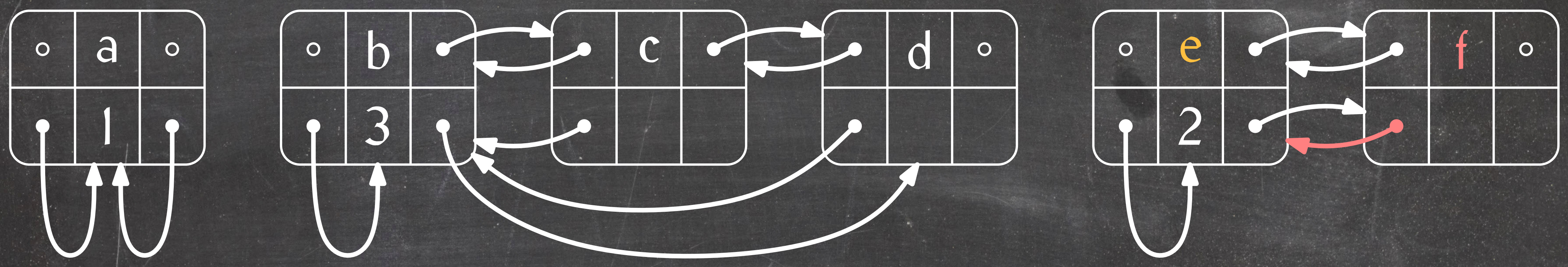

D.find(c) = b

D.find(d) = b

D.find(f) = e

Union

D.union(x, y)

```
1   if x.head.listSize < y.head.listSize
2       then swap x and y
3   y.head.pred = x.head.tail
4   x.head.tail.succ = y.head
5   x.head.listSize = x.head.listSize + y.head.listSize
6   x.head.tail = y.head.tail
7   z = y.head
8   while z ≠ null
9       do z.head = x.head
10          z = z.succ
```

→ after 2) x.head.listSize ≥ y.head.listSize

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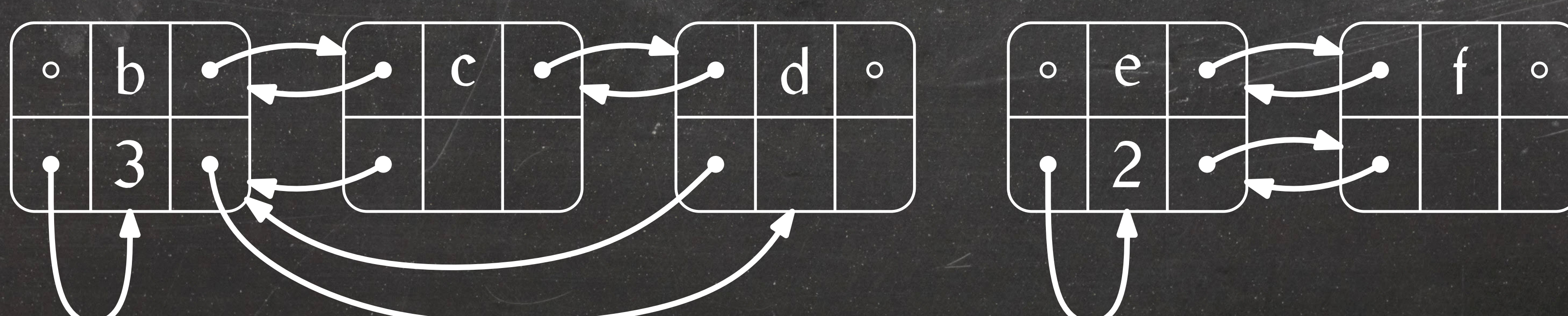

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```

3 < 2 no
don't swap

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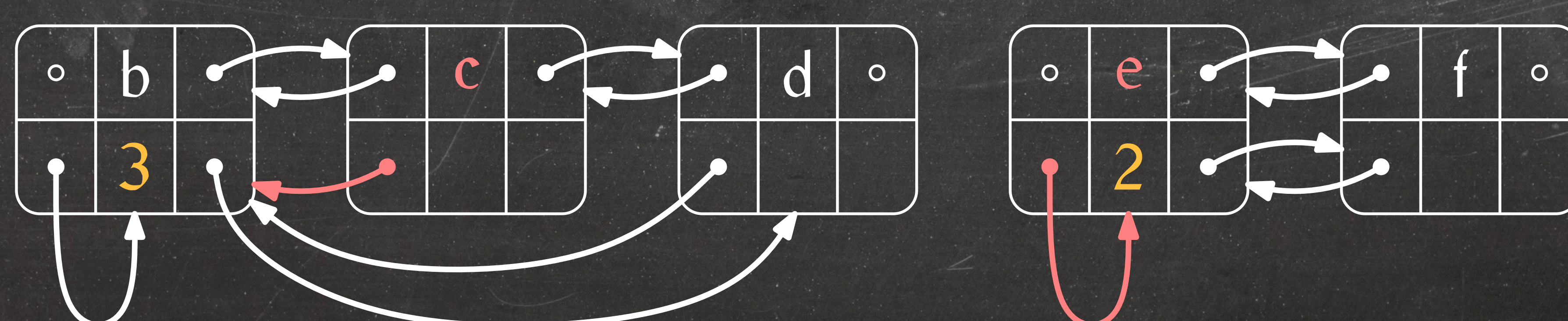

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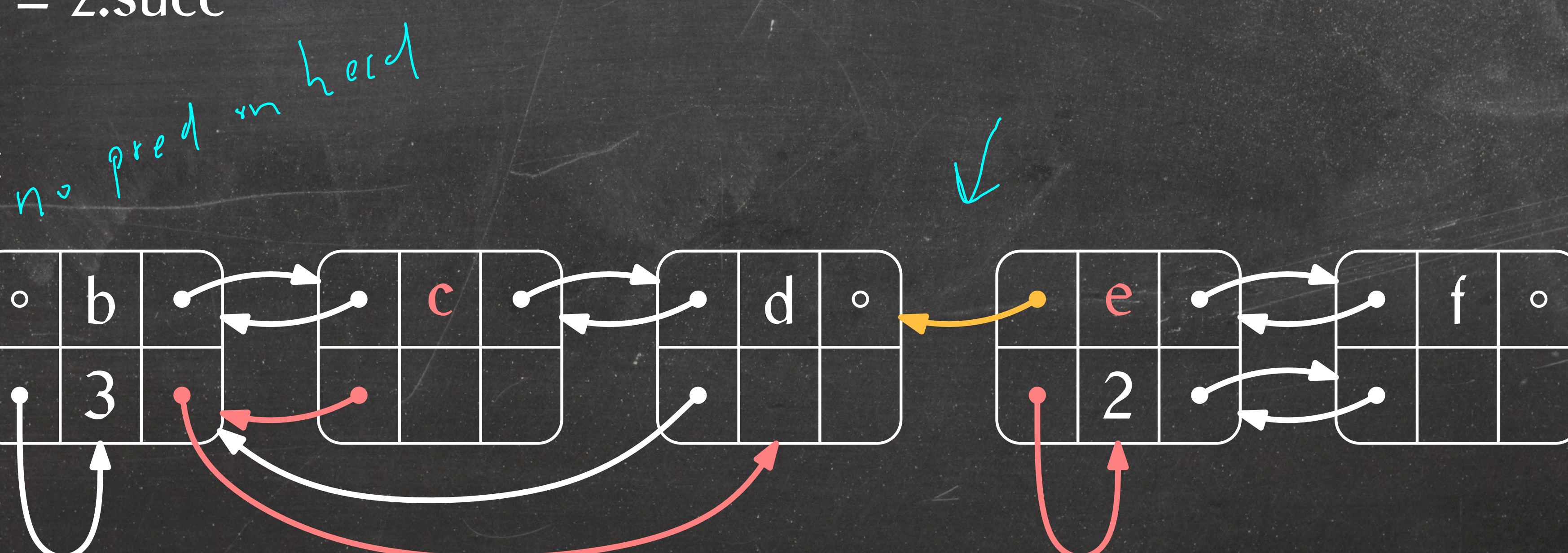

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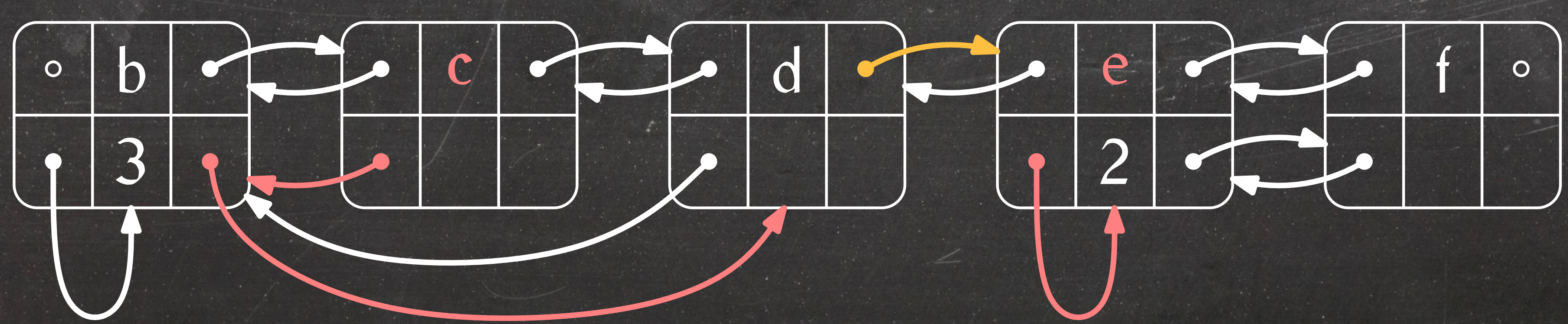

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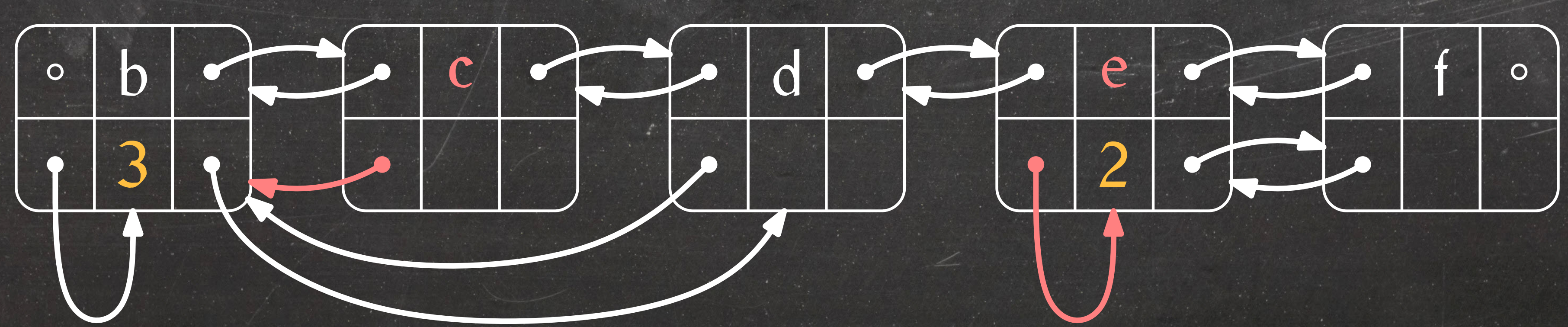

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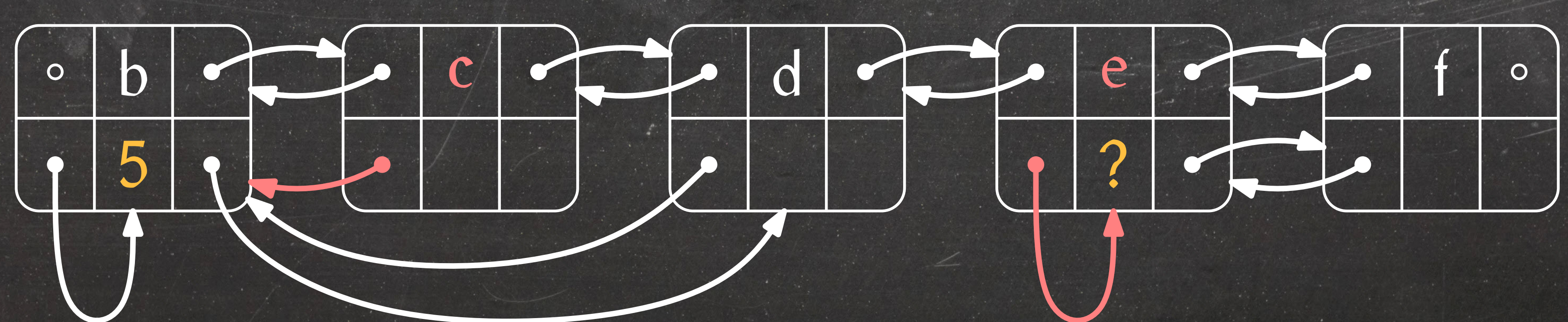

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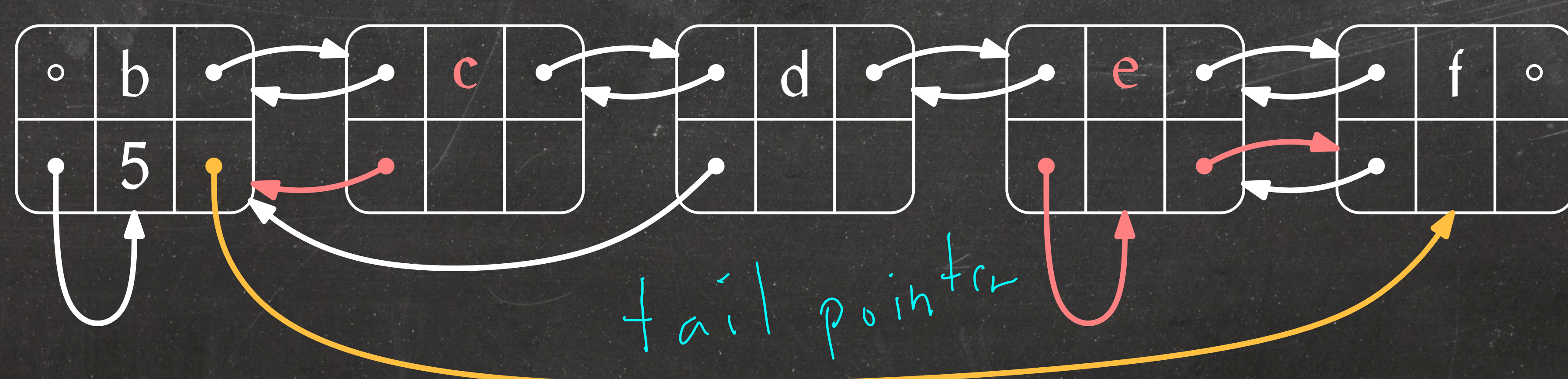

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clear y.head.listSize

also clear y.head.tail

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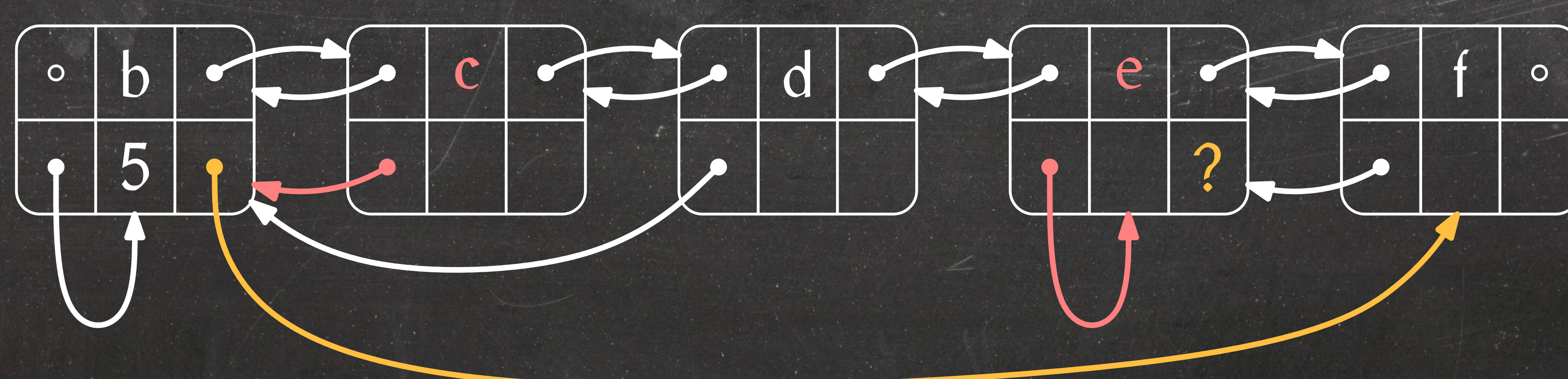

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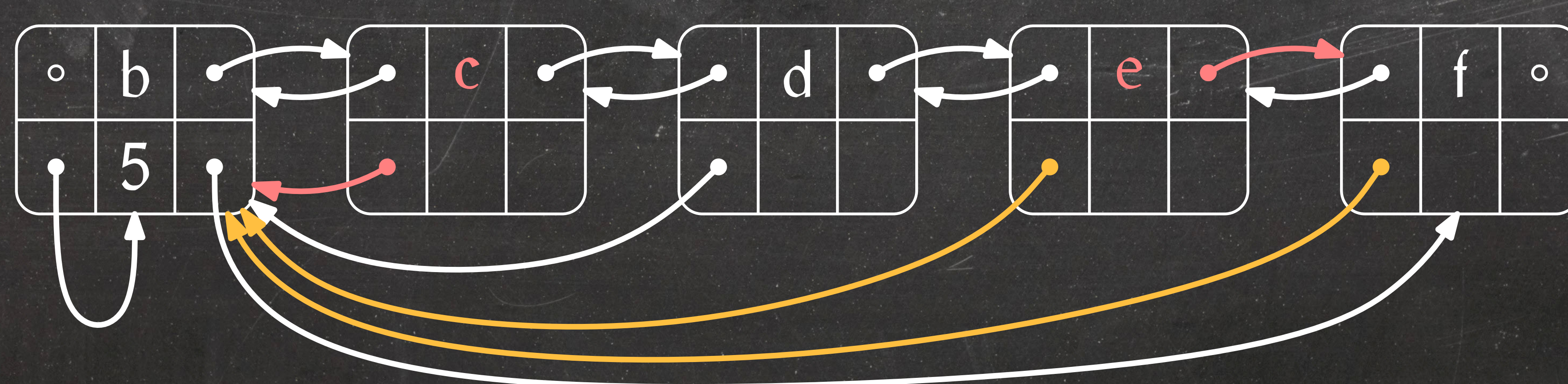

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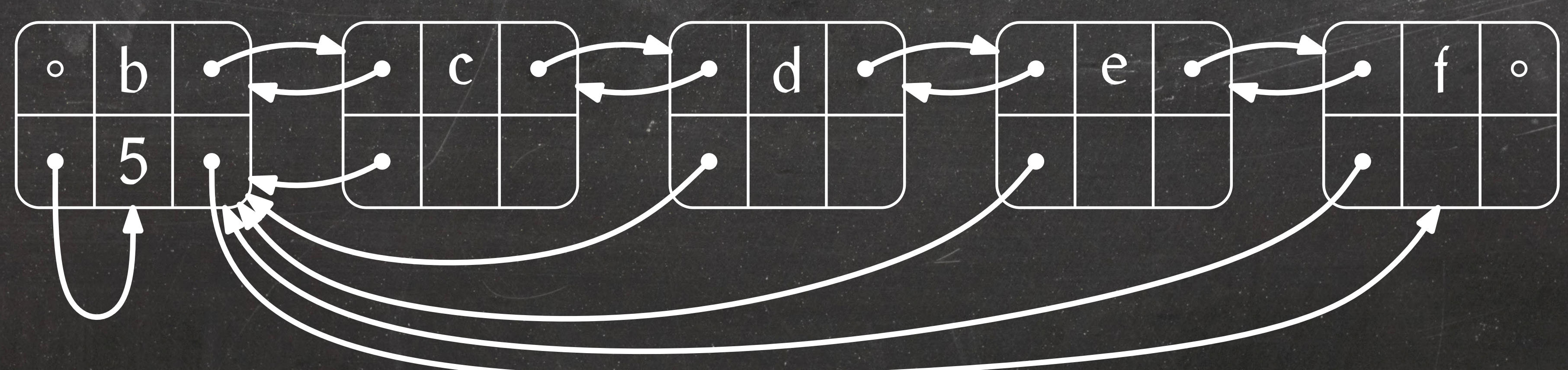

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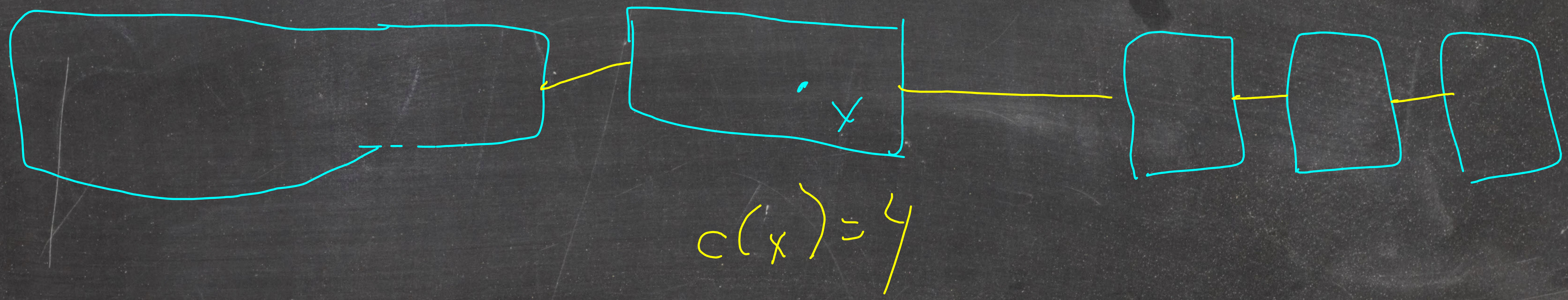

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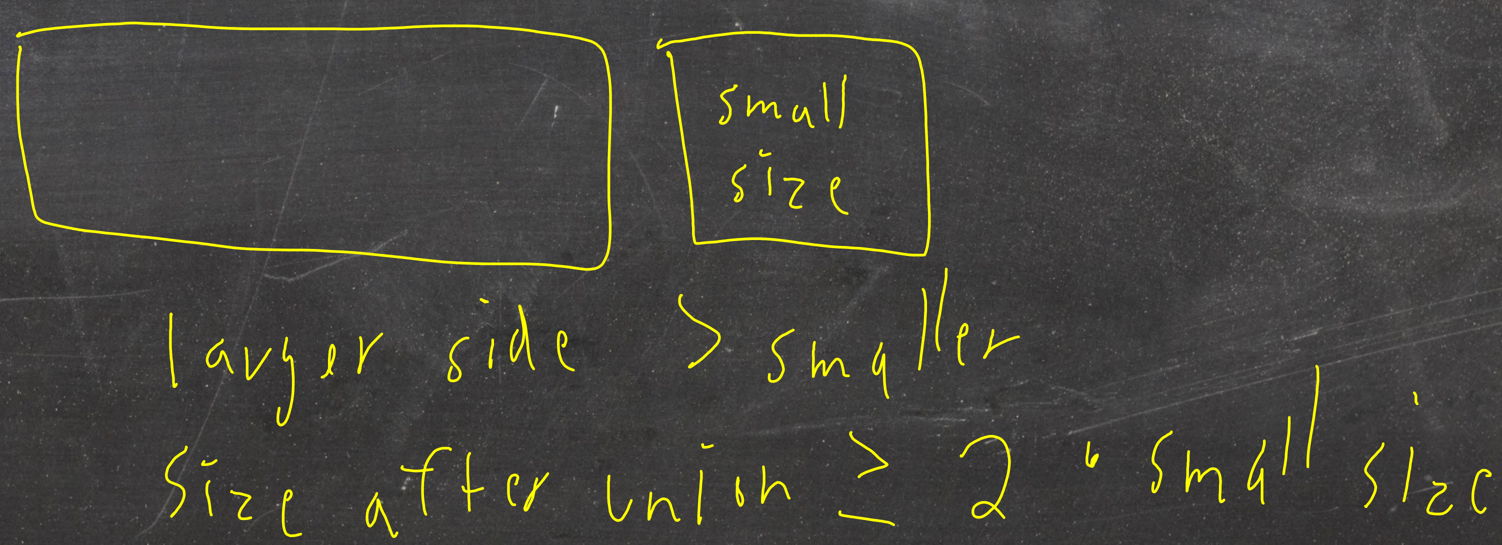

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Inductive step: $i > 0$.

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Corollary: $c(x) \leq \lg n$ for all $x \in S$.

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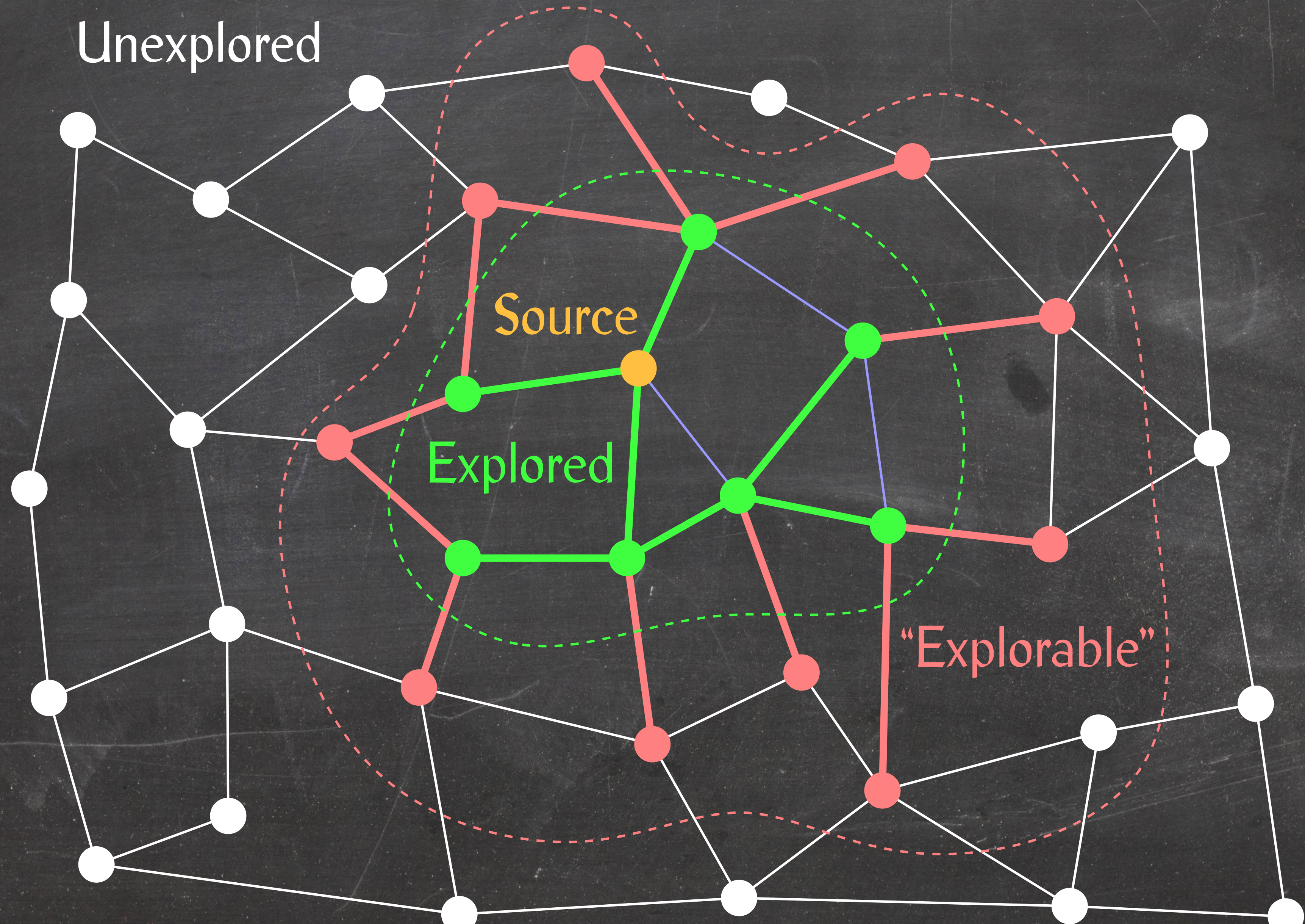
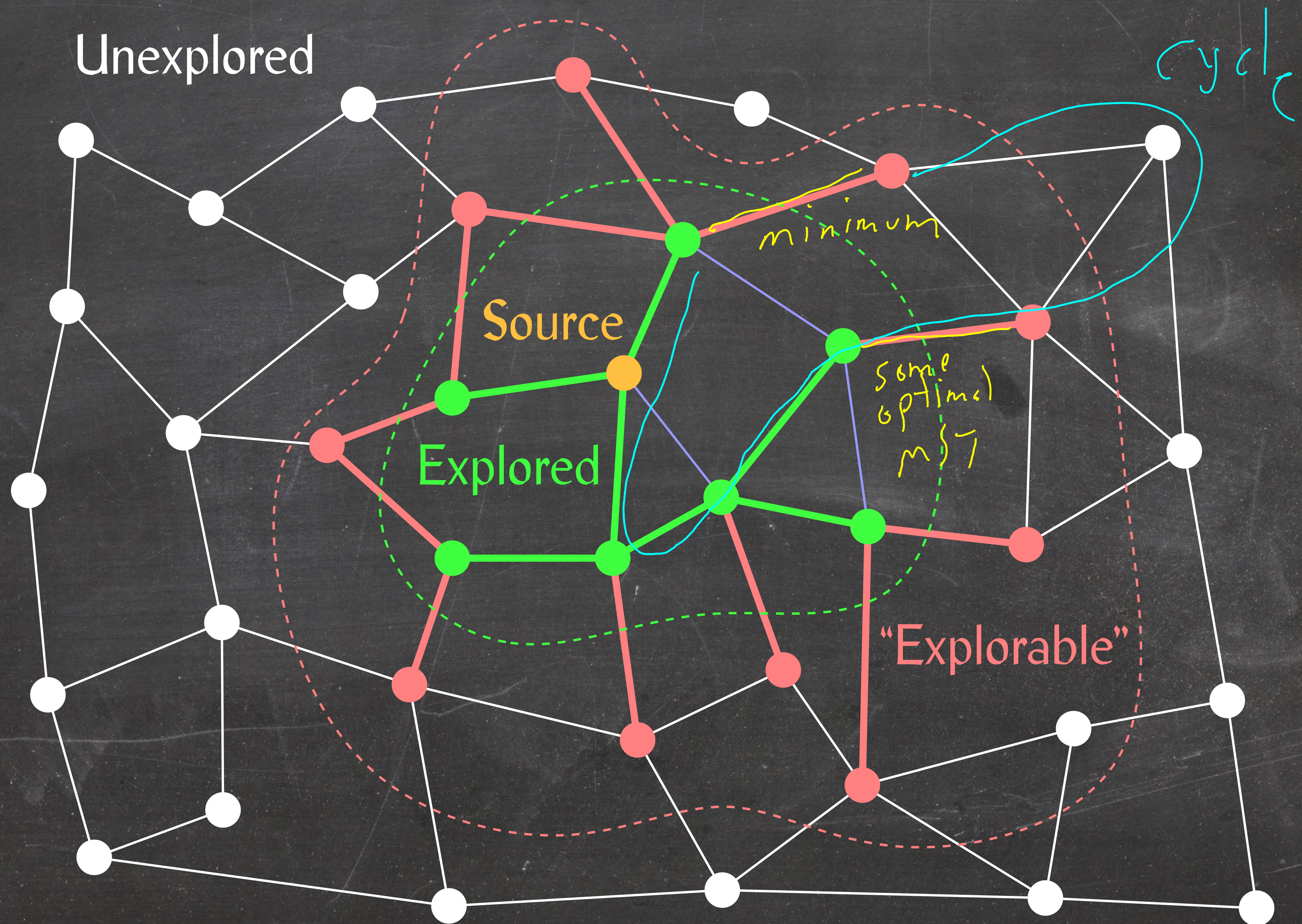
If the graph is connected, then $m \geq n - 1$, so the running time simplifies to $O(m \lg m)$.

The Cut Theorem And Graph Traversal



The Cut Theorem And Graph Traversal

If there exists an MST containing all green edges, then there exists an MST containing all green edges and the cheapest red edge.



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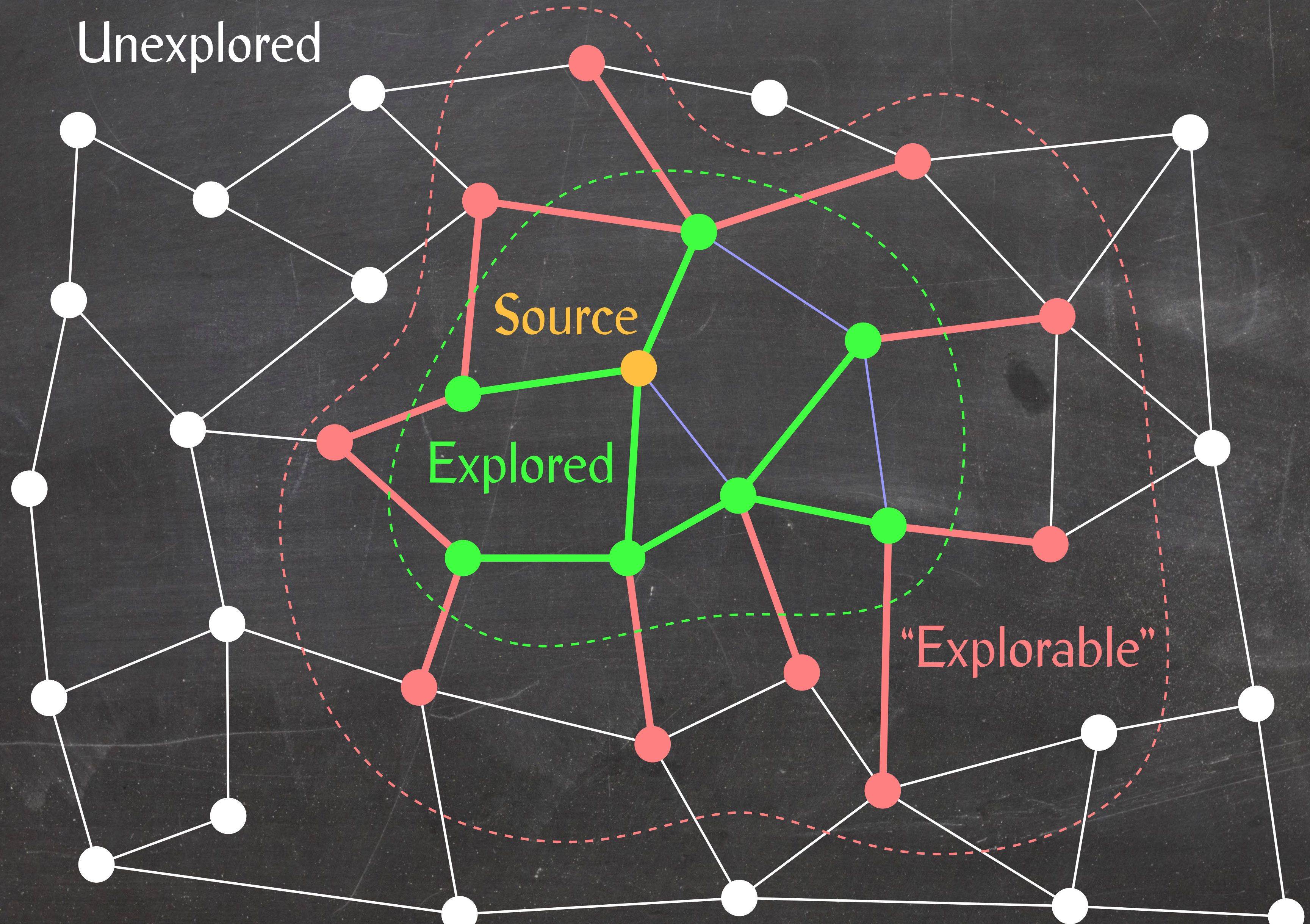

Cut: U = explored vertices, W = V \ U

Prim's Algorithm

Prim(*G*)

```
T = (V, ∅)
mark all vertices of G as unexplored
mark an arbitrary vertex s as explored
while not all vertices are explored
    do pick the cheapest edge e with exactly one unexplored endpoint v
       mark v as explored
       add e to T
return T
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Once T is connected, we have $T^* = T$.

The Abstract Data Type Priority Queue

Operations:

Q.insert(x, p):	Insert element x with priority p
Q.delete(x):	Delete element x
Q.findMin():	Find and return the element with minimum priority
Q.deleteMin():	Delete the element with minimum priority and return it
Q.decreaseKey(x, p):	Change the priority p_x of x to $\min(p, p_x)$

Delete and DecreaseKey assume they're given a pointer to the place in Q where x is stored.

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Example: A binary heap is a priority queue supporting all operations in $O(\lg |Q|)$ time.

Prim's Algorithm Using A Priority Queue

```
Prim(G)
T = (V, ∅)
mark every vertex of G as unexplored
mark an arbitrary vertex s as explored
Q = an empty priority queue
for every edge (s, v) incident to s
    do Q.insert((s, v), w(s, v))
while not Q.isEmpty()
    do (u, v) = Q.deleteMin()
        if v is unexplored
            then mark v as explored
                add edge (u, v) to T
                for every edge (v, w) incident to v
                    do Q.insert((v, w), w(v, w))
return T
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Prim's Algorithm Using A Priority Queue

Invariant: Q contains all edges with exactly one unexplored endpoint.

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Invariant: Q contains all edges with exactly one unexplored endpoint.

⇒ This version of Prim's algorithm computes an MST.

Prim's Algorithm Using A Priority Queue

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$\Rightarrow$ This version of Prim's algorithm computes an MST.

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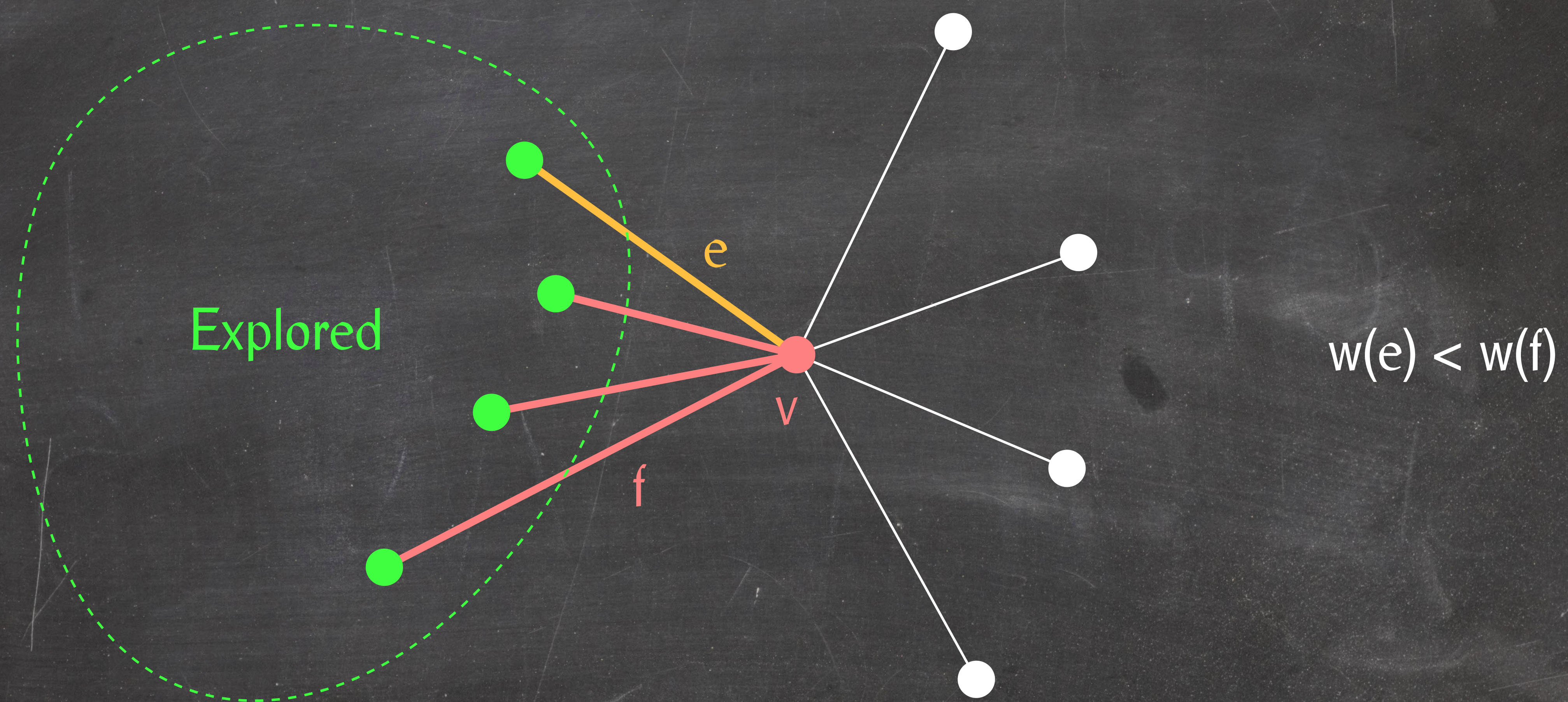
$\Rightarrow$ Every edge is removed from Q once.

$\Rightarrow$ 2m priority queue operations.

Most Edges In Q Are Useless

Observation: Of all the edges connecting an unexplored vertex to explored vertices only the cheapest has a chance of being added to the MST.



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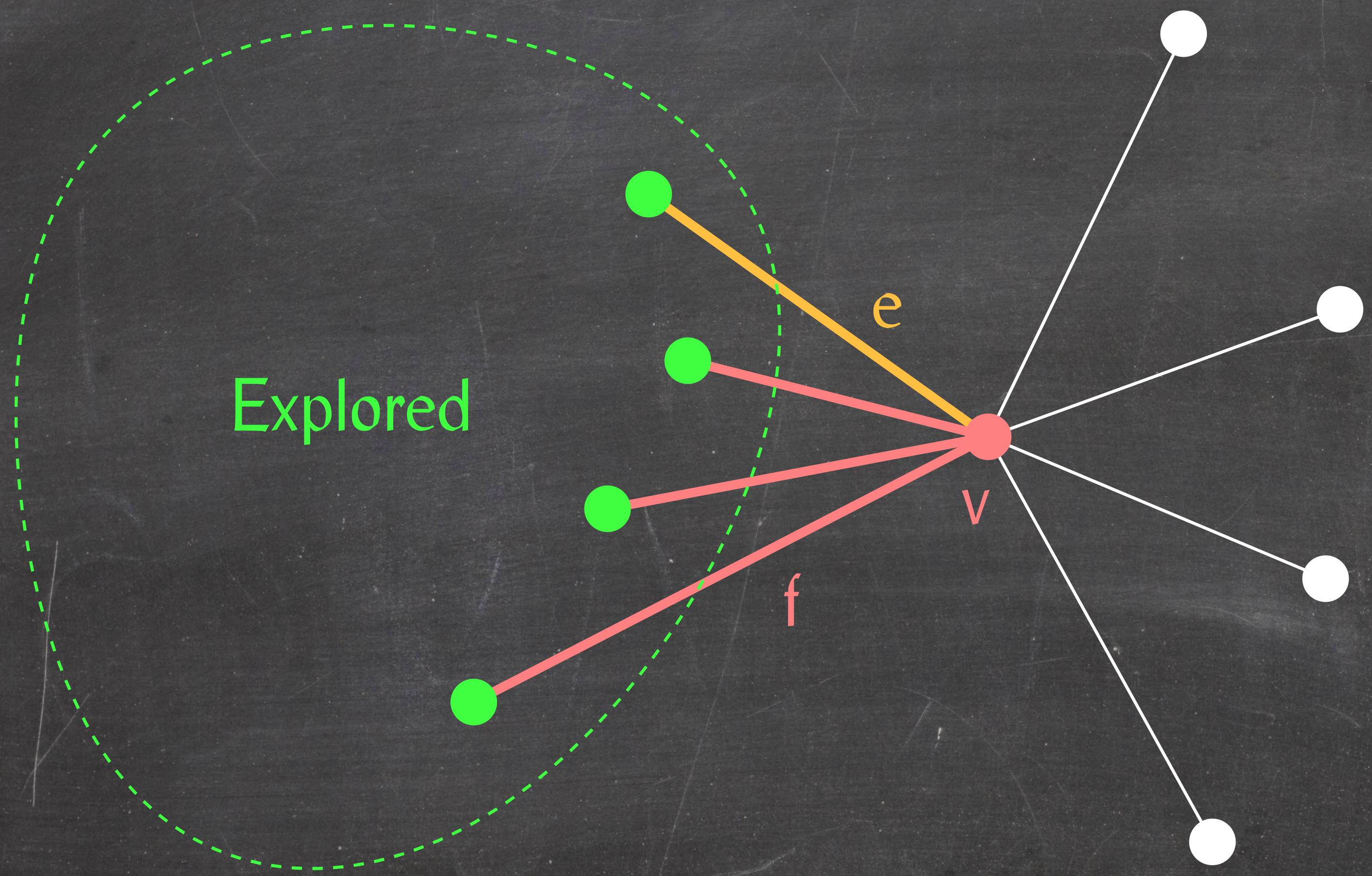

$w(e) < w(f)$

While v is unexplored, all red and orange edges are in Q, so none of the red edges can be the first edge to be removed from Q.

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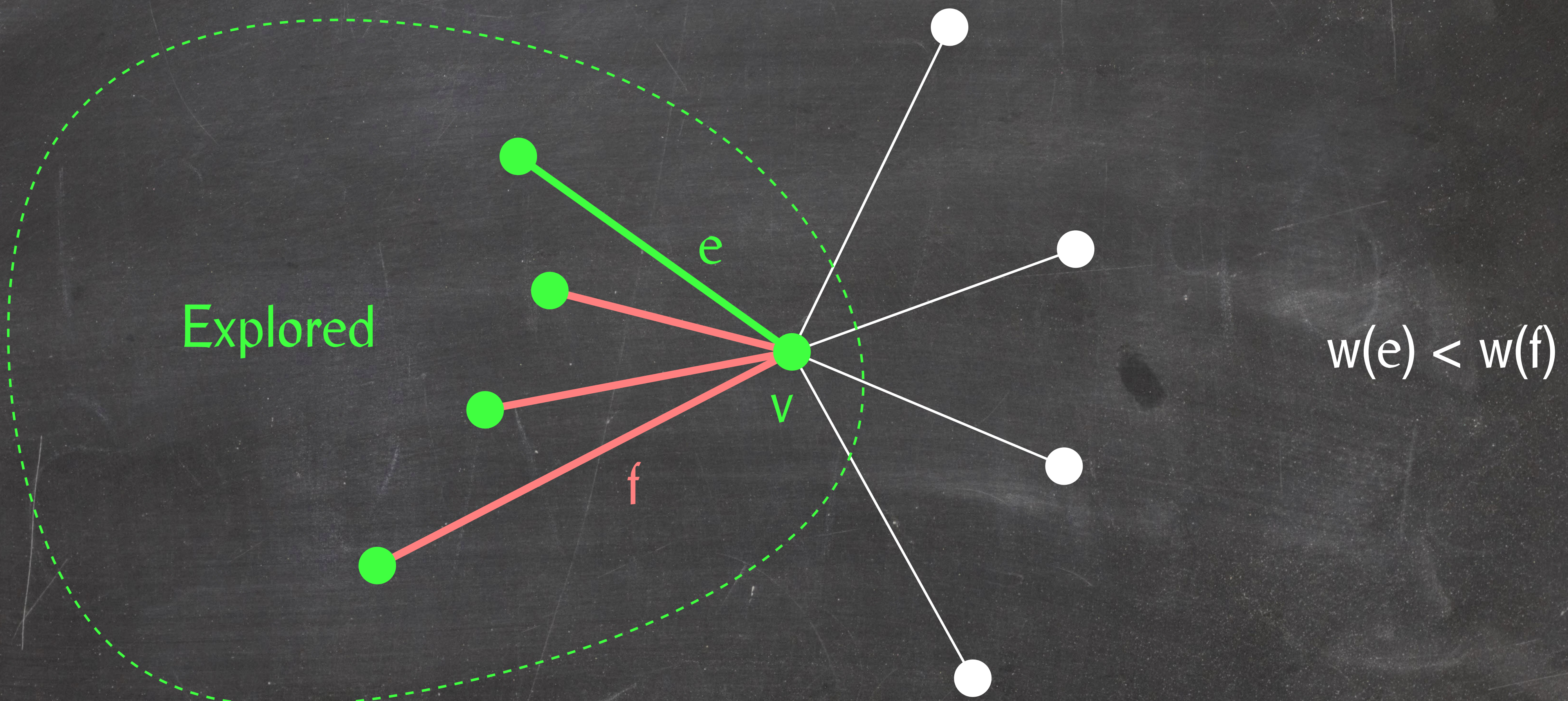

While v is unexplored, all red and orange edges are in Q, so none of the red edges can be the first edge to be removed from Q.

After marking v as explored, both endpoints of red edges are explored, so they cannot be added to T either.

A Faster Version Of Prim's Algorithm

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Prim(G)

T = (V, ∅)
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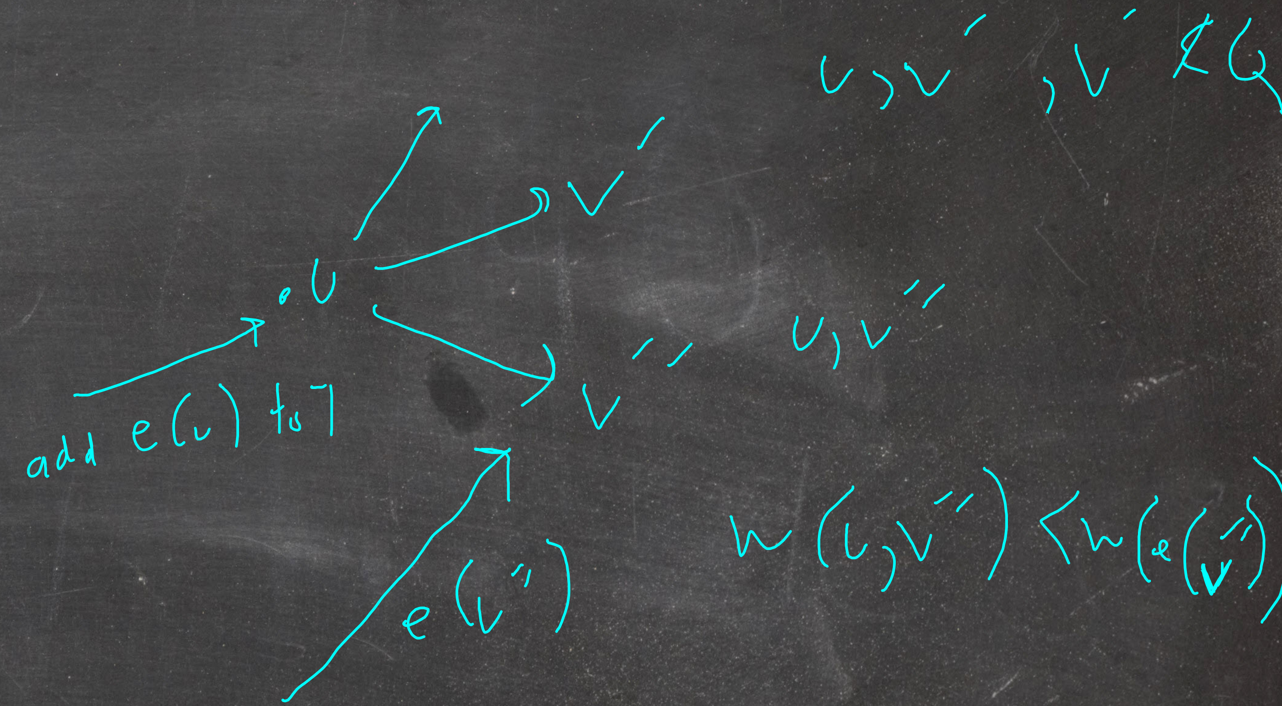

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This version of Prim's algorithm also takes O(m lg m) time:

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- m − n DecreaseKey operations
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⇒ n + m priority queue operations.

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This version of Prim's algorithm also takes $O(m \lg m)$ time:

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Did we gain anything?

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The Thin Heap is a priority queue which supports

- Insert, DecreaseKey, and FindMin in $O(1)$ time and
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Prim's algorithm performs $n + m$ priority queue operations, n of which are DeleteMin operations.

Lemma: Prim's algorithm takes $O(n \lg n + m)$ time.

Thin Tree

A Thin Heap is built from Thin Trees. Thin Trees are defined inductively.

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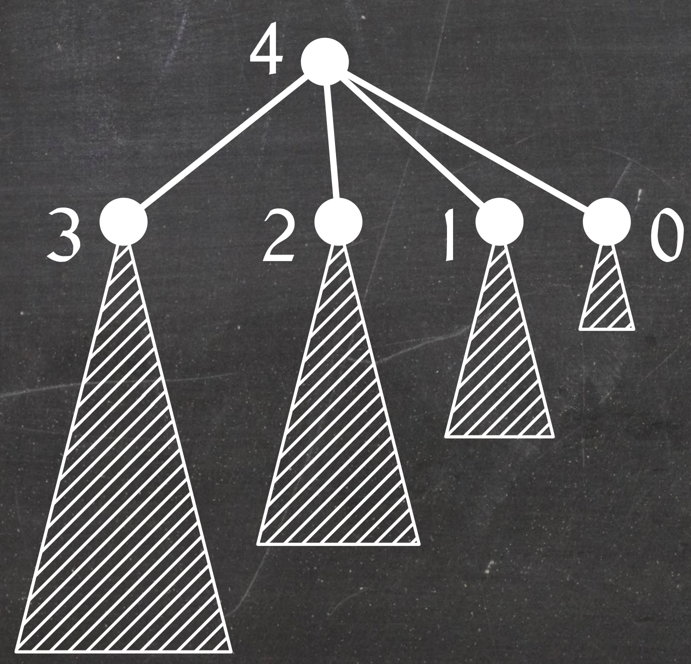
A node of rank $k > 0$ has

Thick node: k children of ranks $k-1, k-2, \ldots, 0$ or
Thin node: $k-1$ children of ranks $k-2, k-3, \ldots, 0$.

Rank 0



Rank 4, thick

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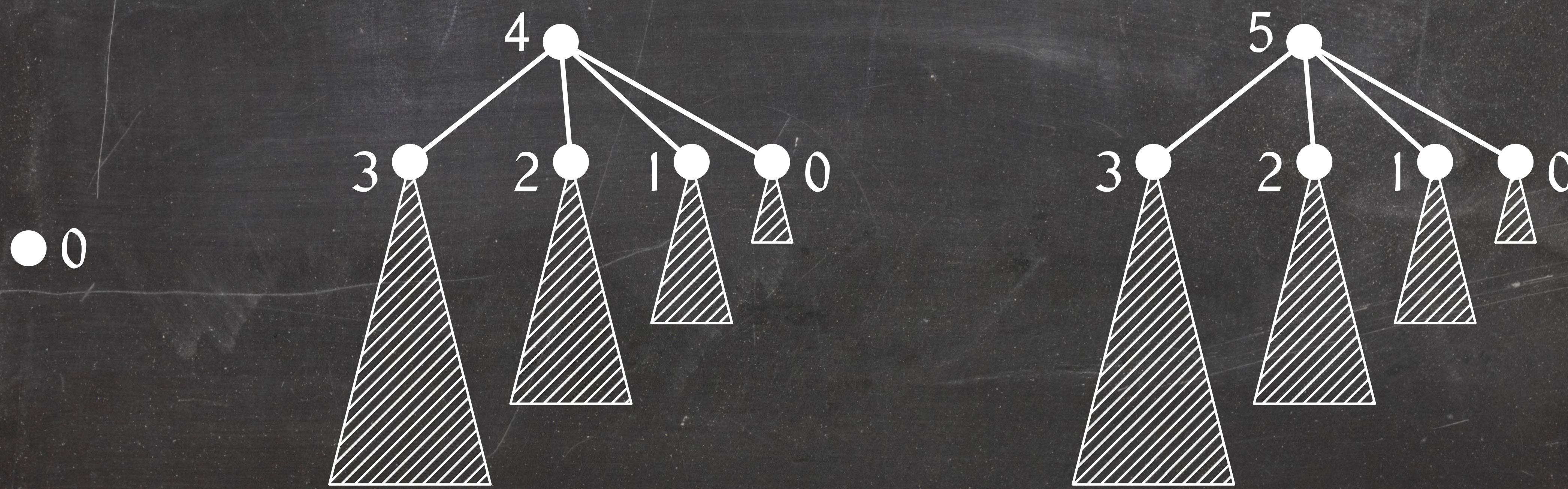
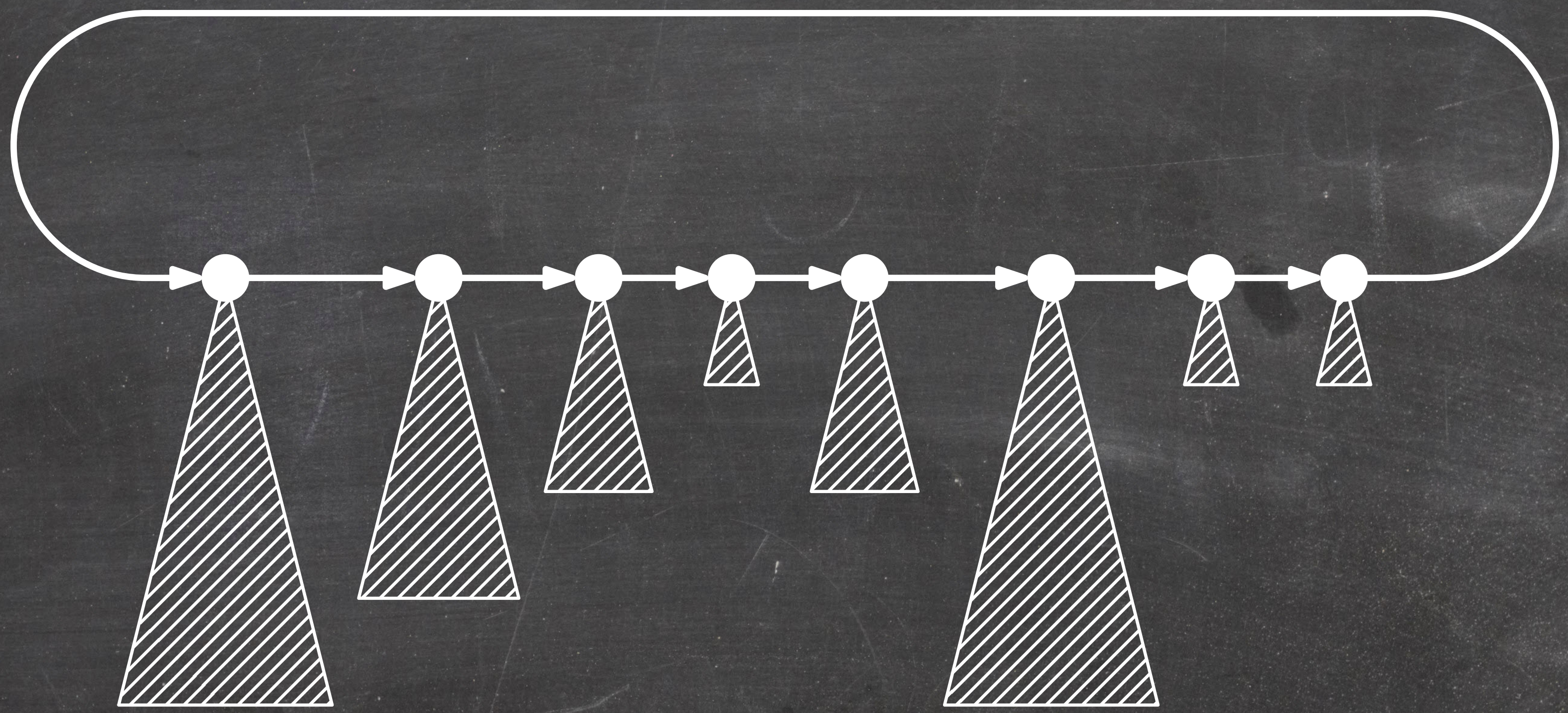


Rank 0 — Rank 4, thick — Rank 5, thin

Thin Heap

A Thin Heap is a circular list of heap-ordered Thin Trees.



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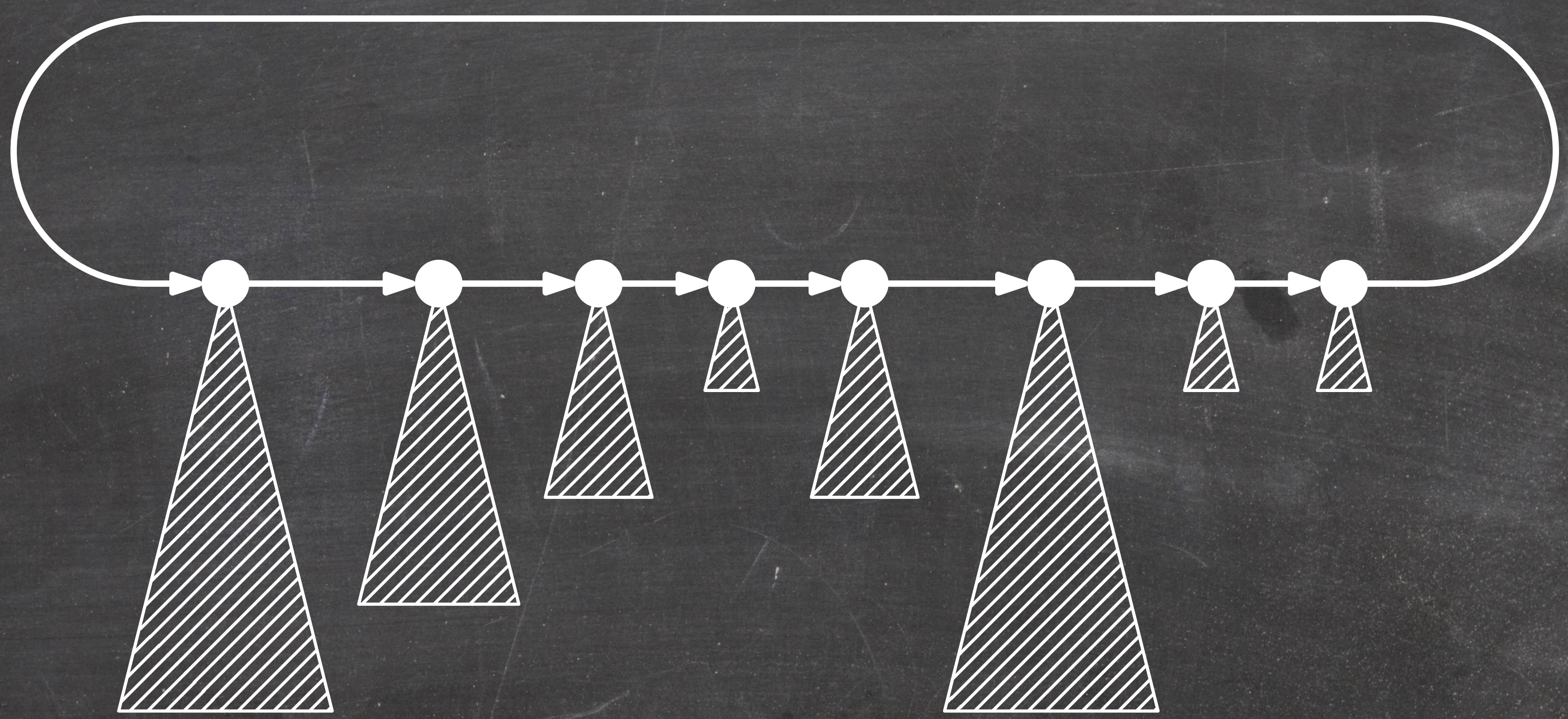
A Thin Heap is a circular list of heap-ordered Thin Trees.

Heap-ordered: Every node stores an element no less than the element stored at its parent.



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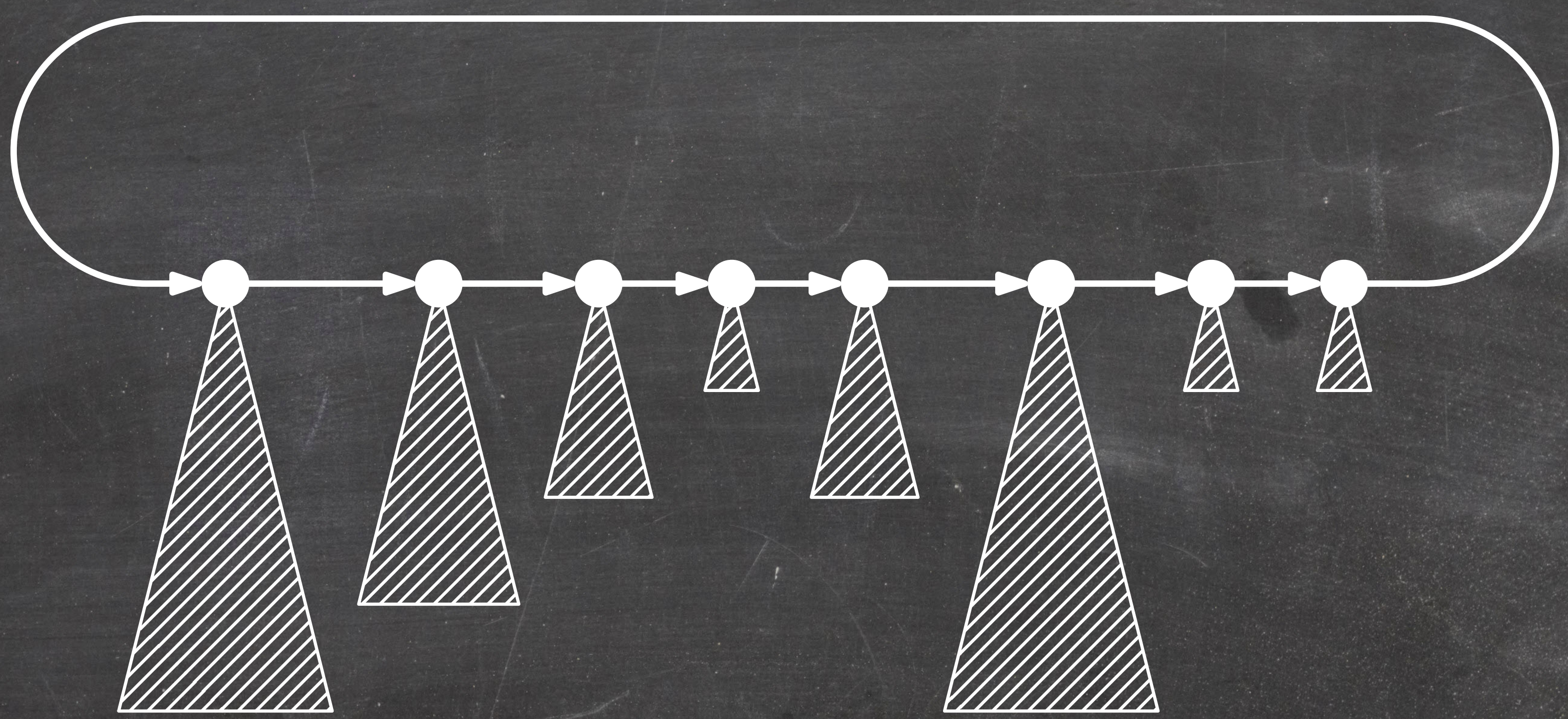

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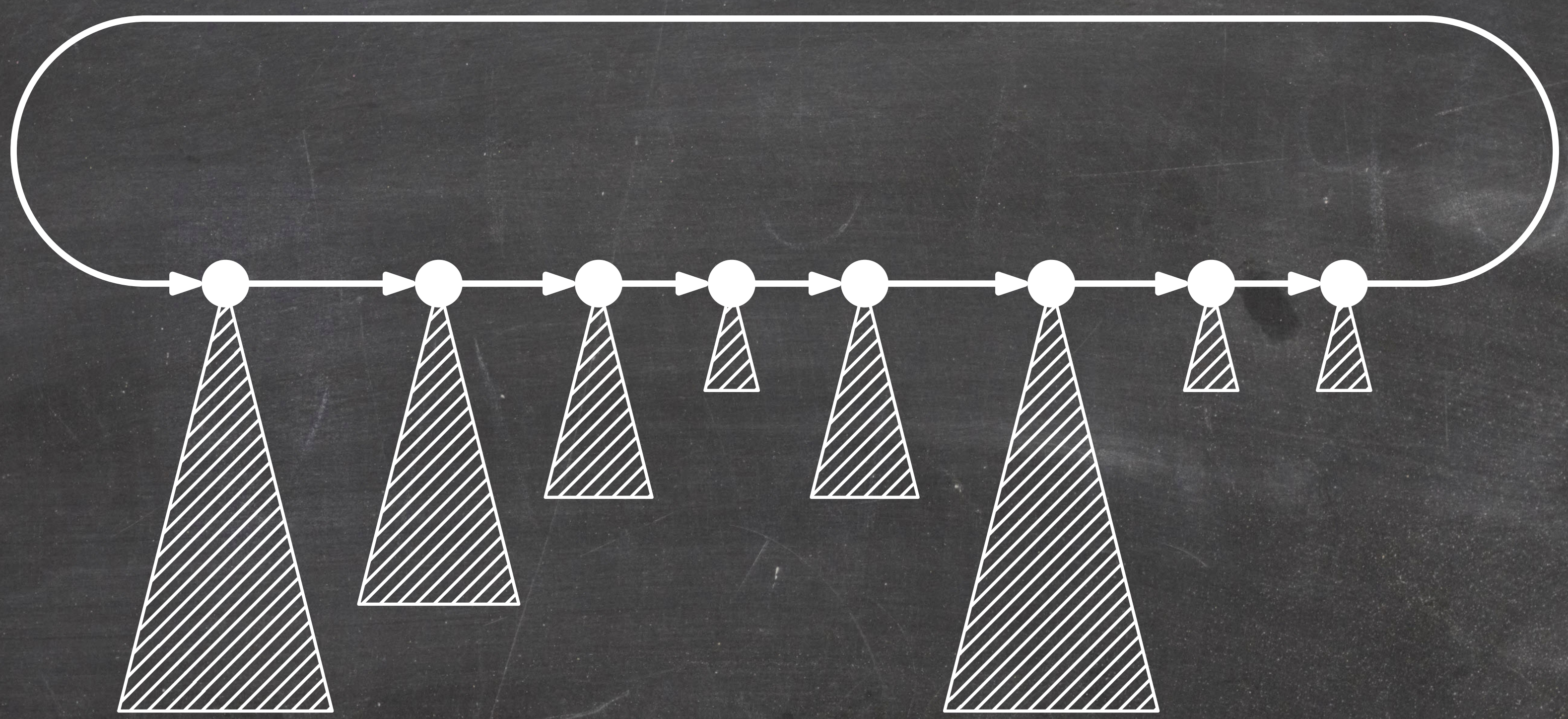

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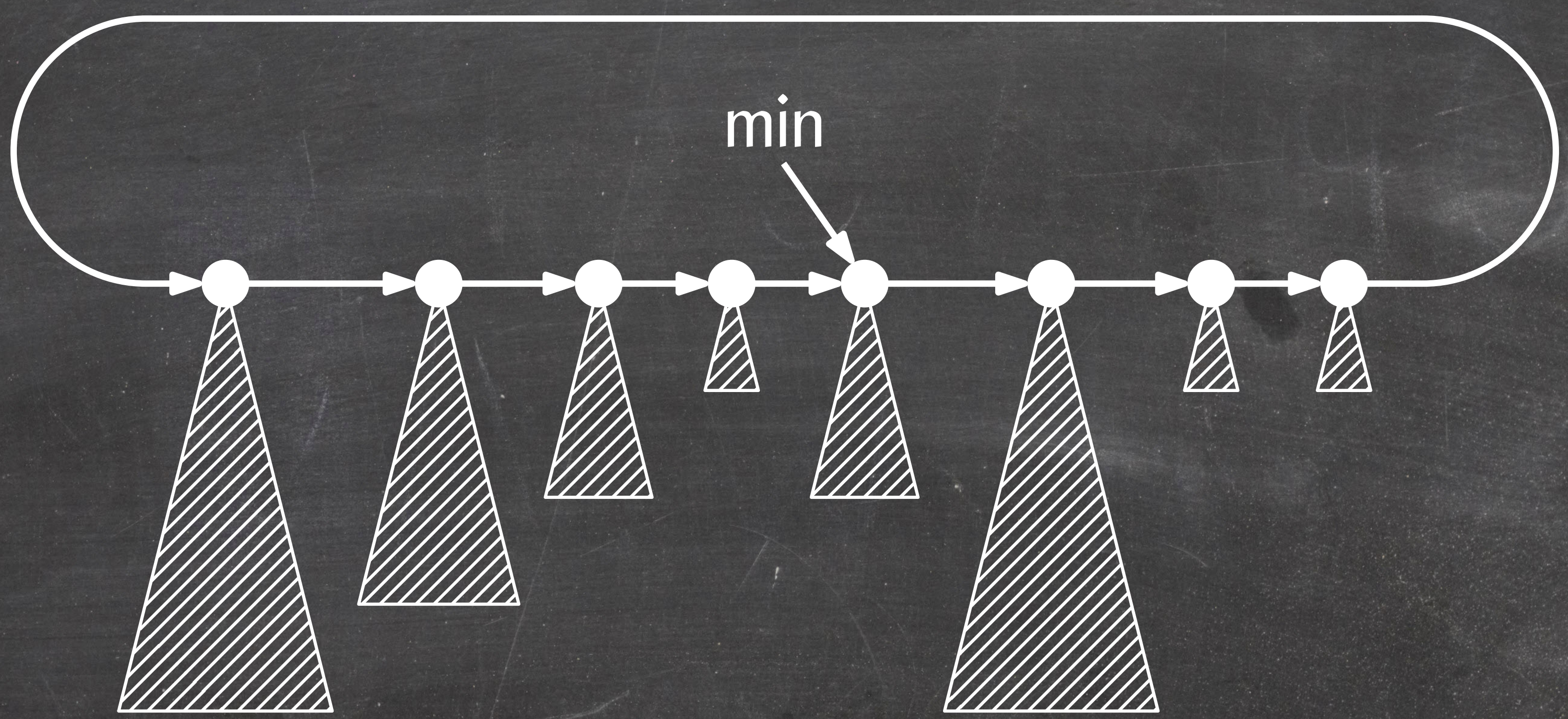


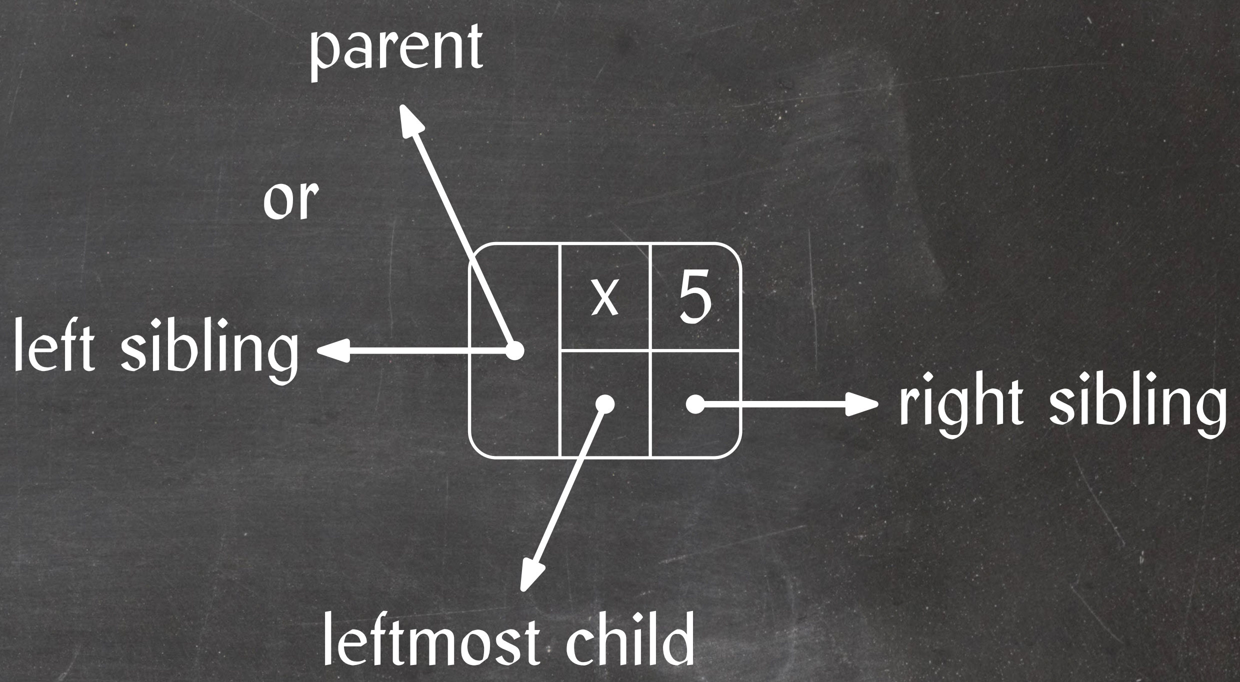
All roots are thick.

The minimum element is stored at one of the roots.

We store a pointer to this root.

Node Representation

- Element stored at the node
- Rank
- Pointer to leftmost child
- Pointer to right sibling
- Pointer to left sibling or parent



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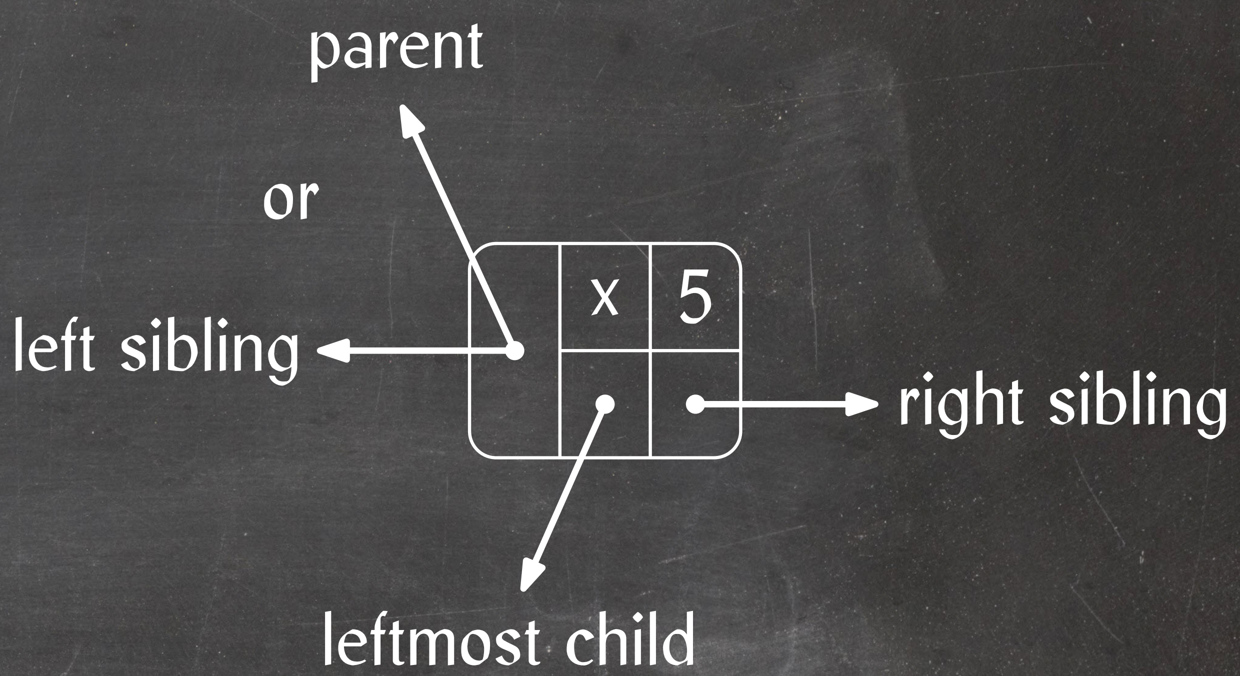
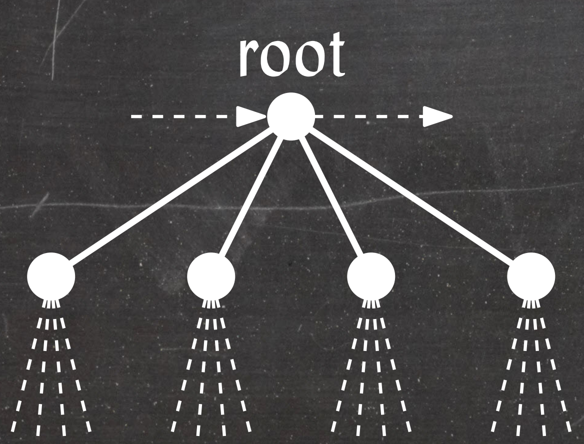
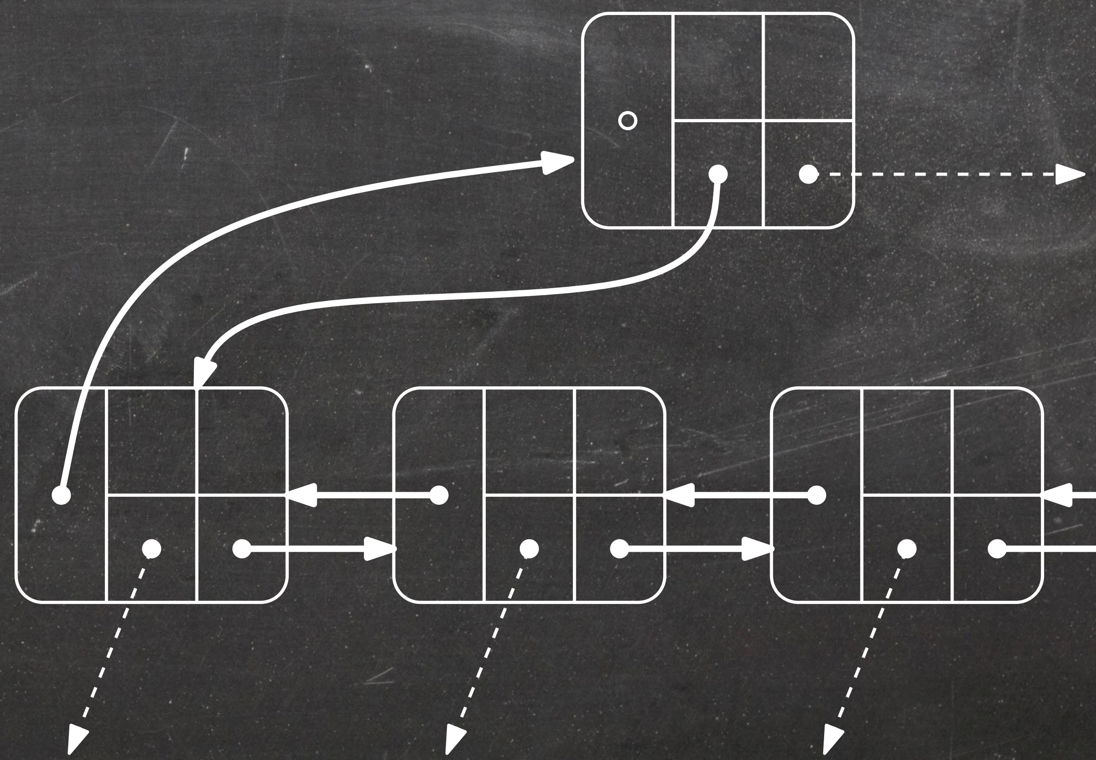
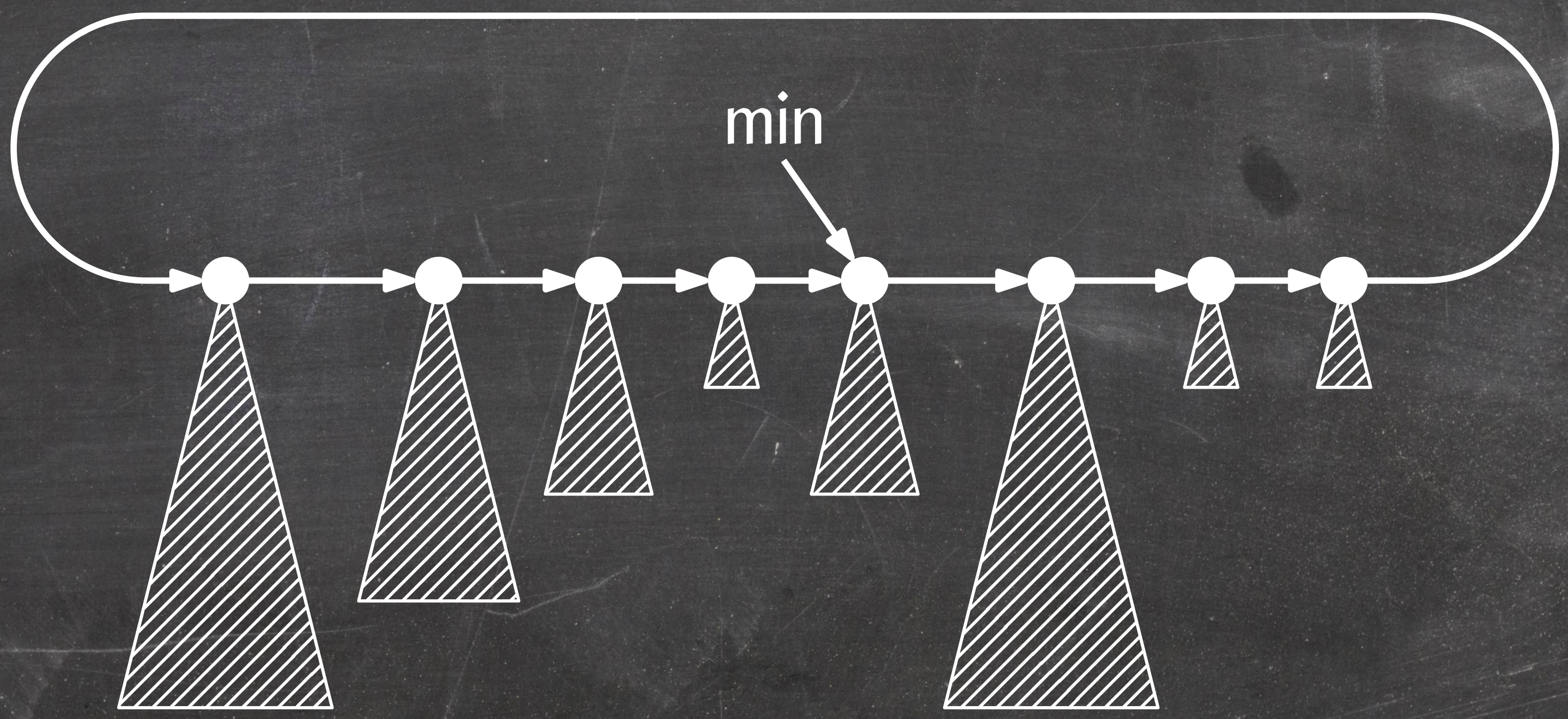






FindMin

... is easy:



Delete

… can be implemented using DecreaseKey and DeleteMin:

Q.delete(x)

```
Q.decreaseKey(x, −∞)
Q.deleteMin()
```

Insert

Insert

If Q is empty:

Insert

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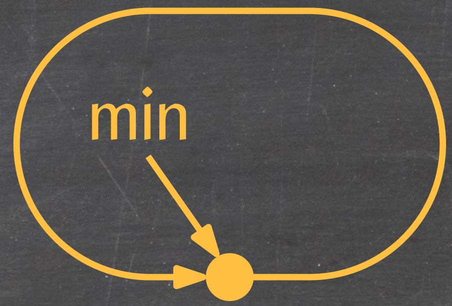

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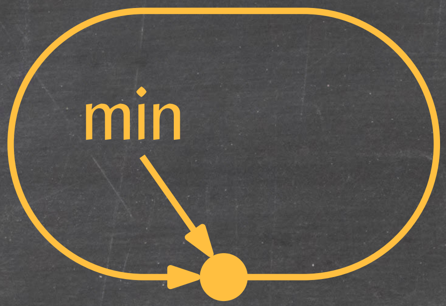

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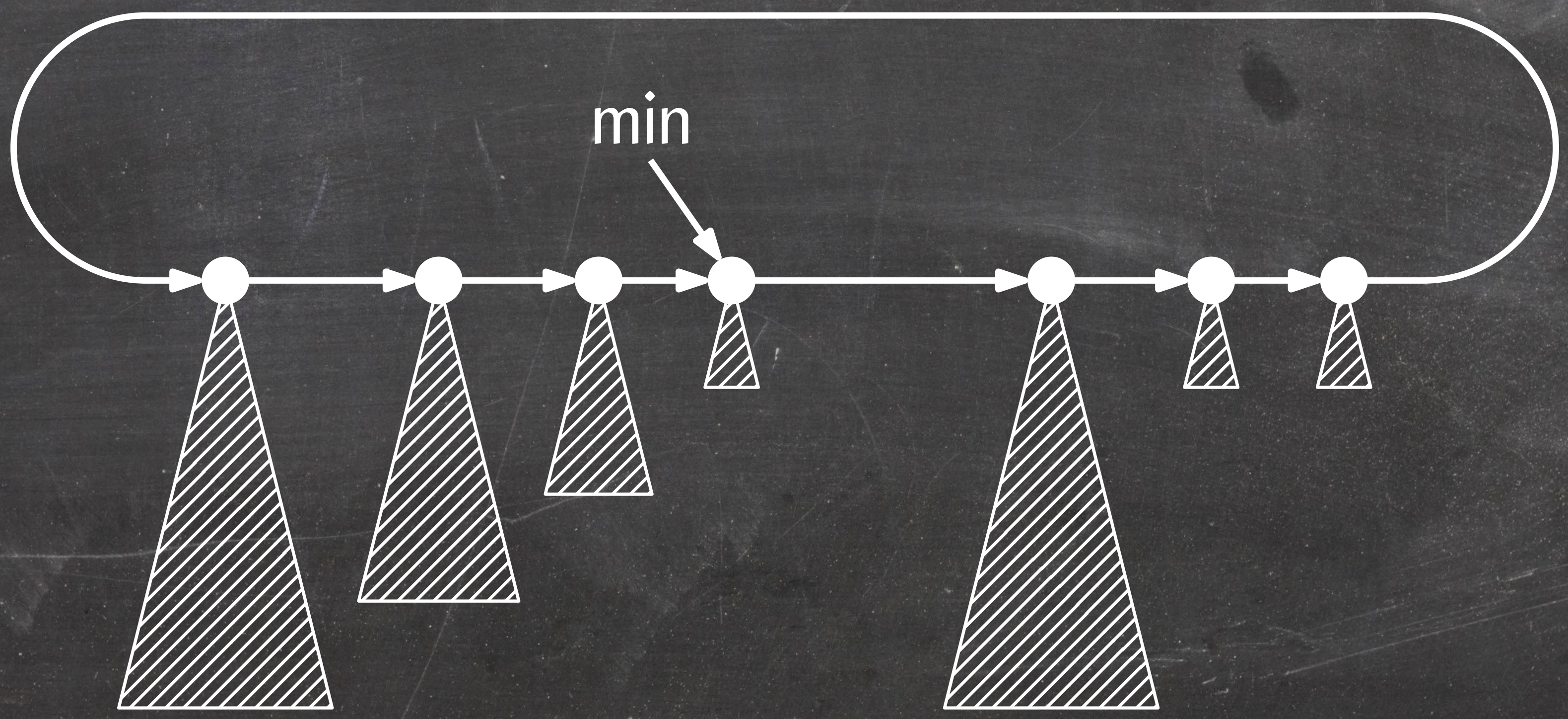

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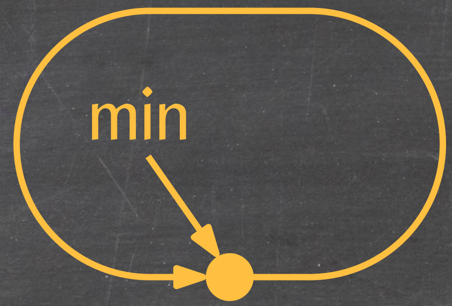

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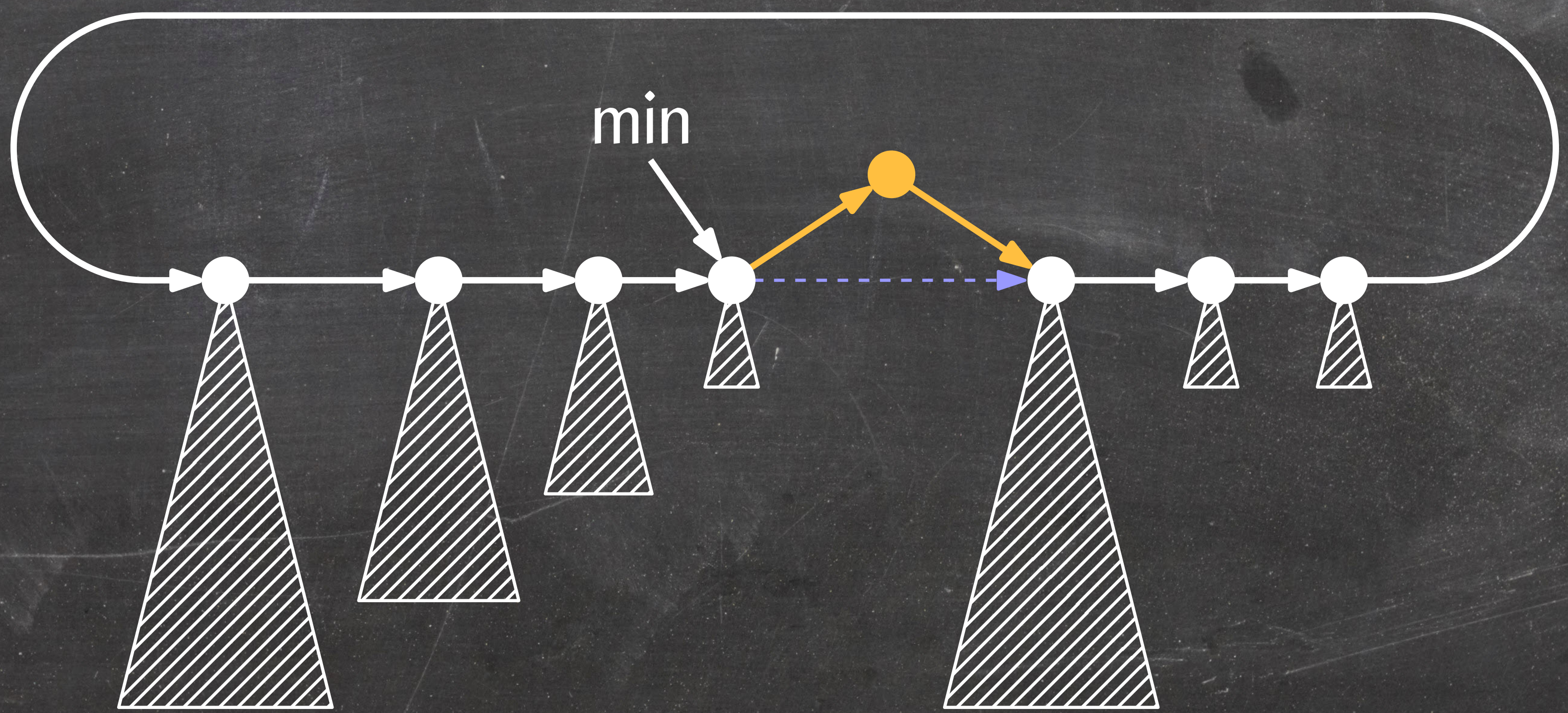

- Insert new element between min and its successor.

Insert

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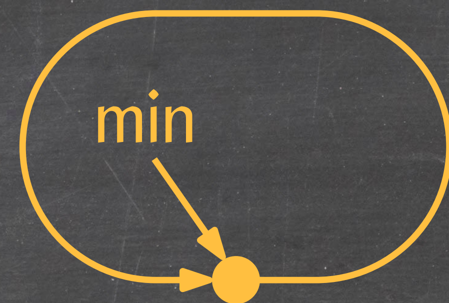

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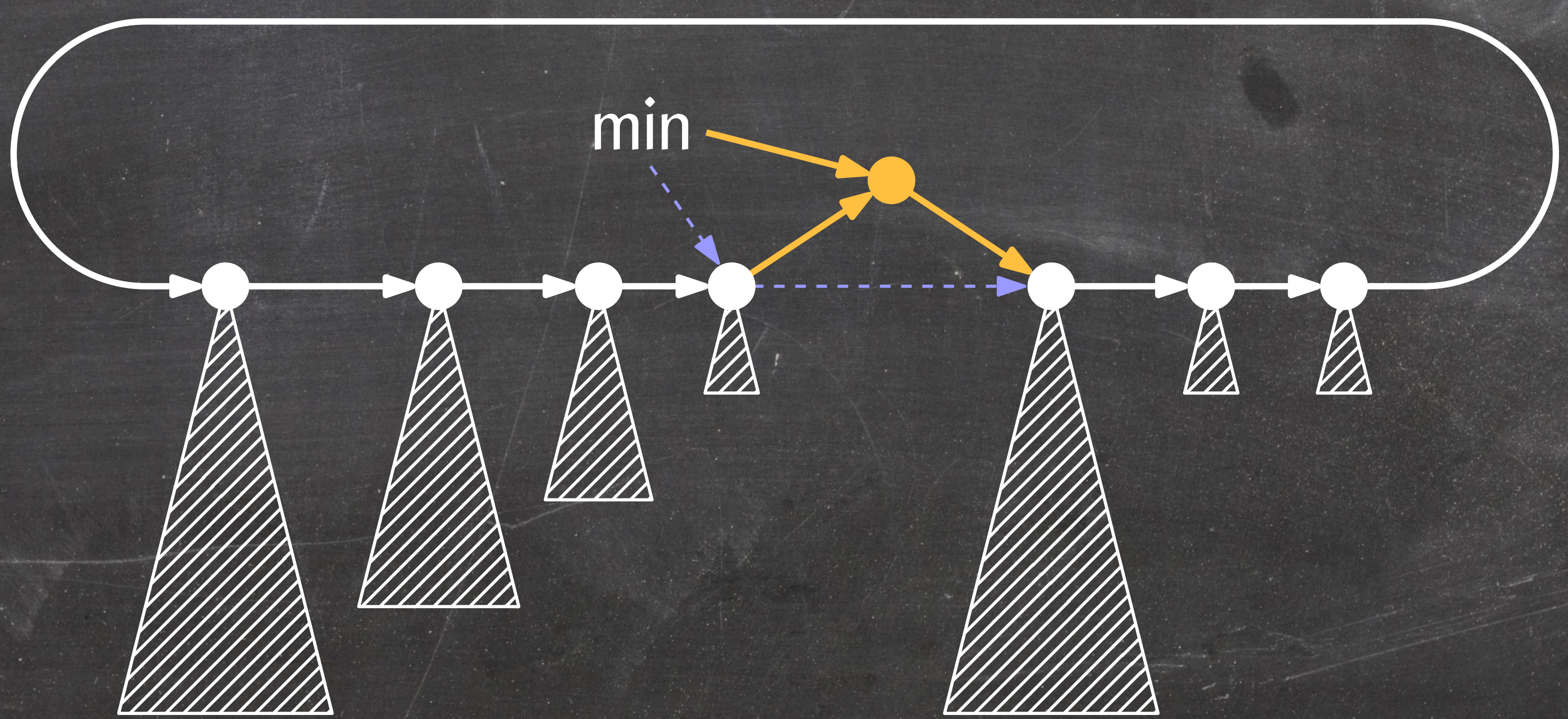
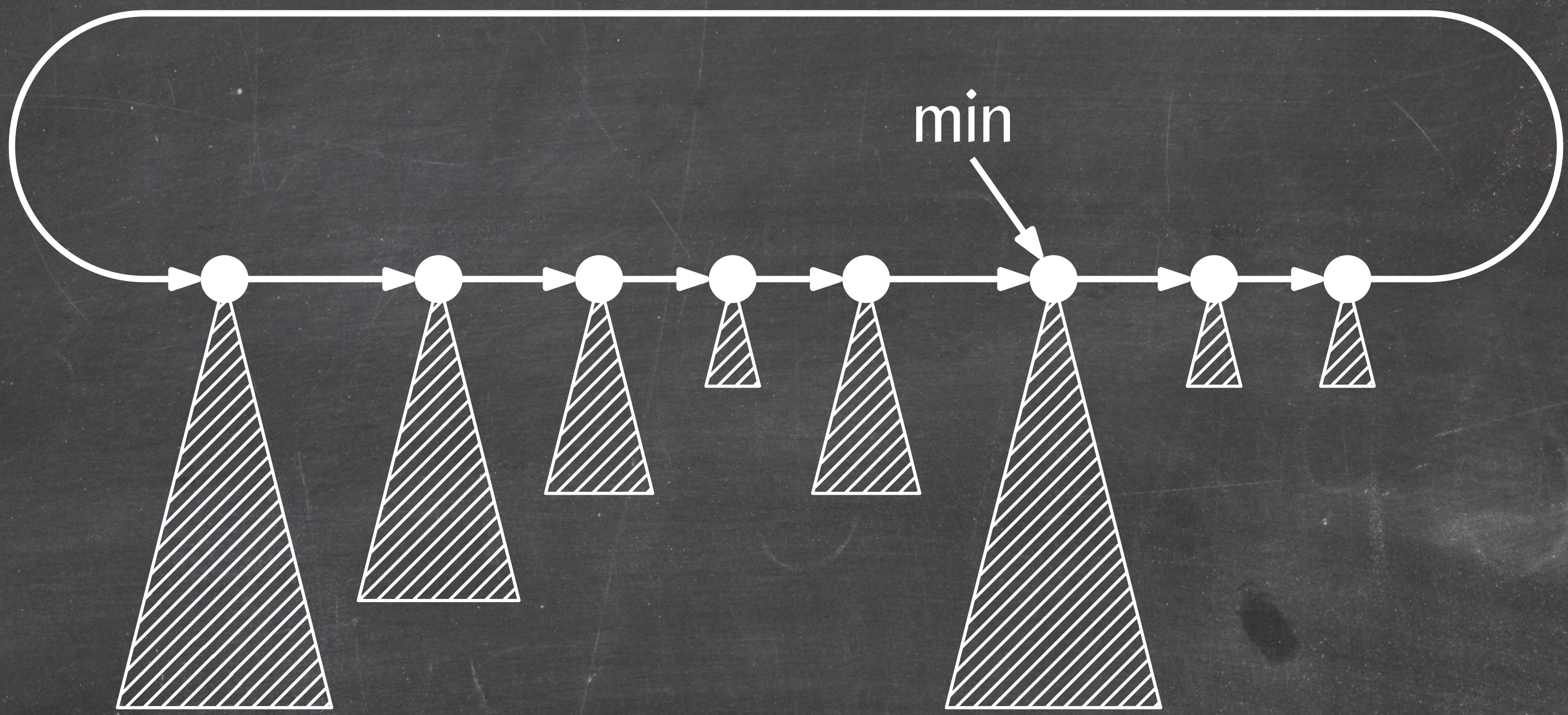


- Insert new element between min and its successor.
- Update min if the new element is the new smallest element.

DeleteMin



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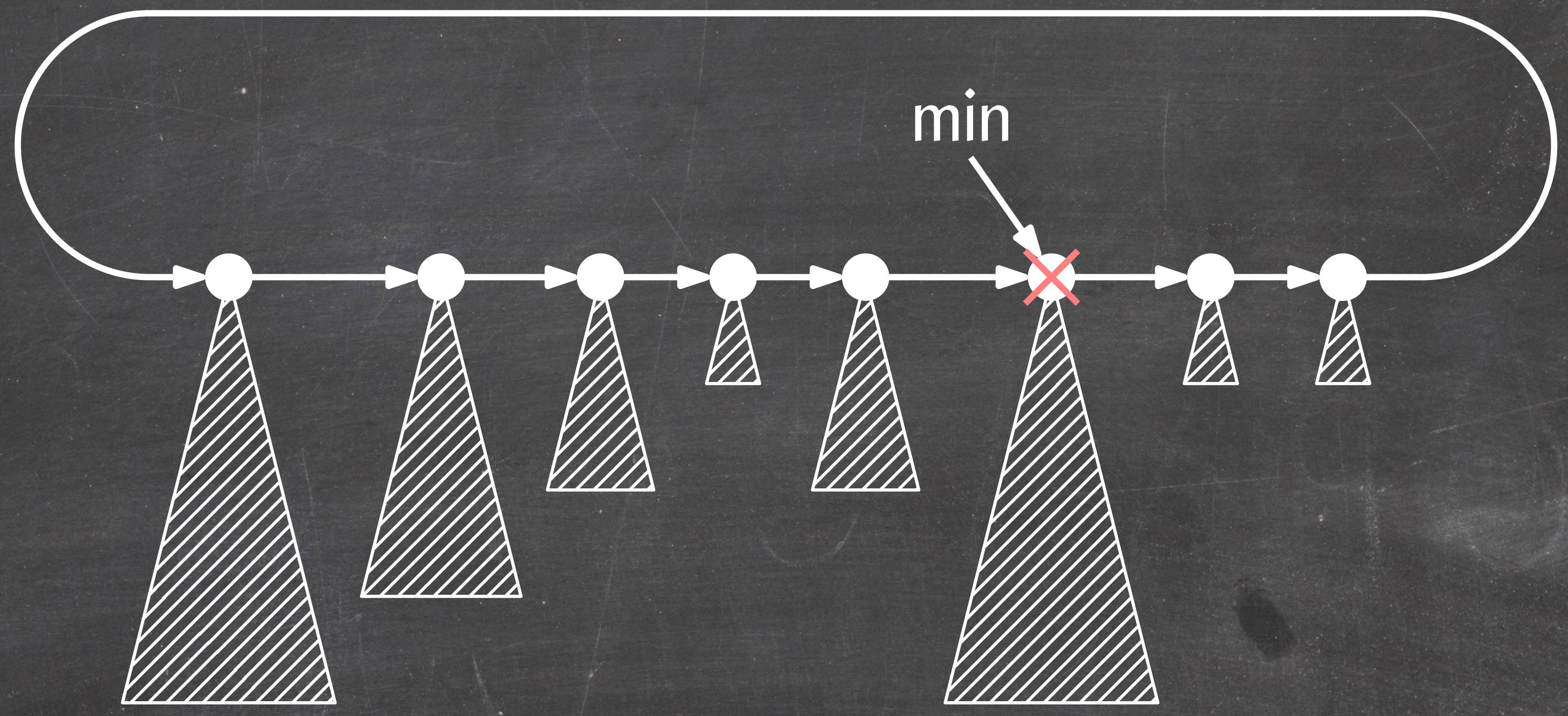

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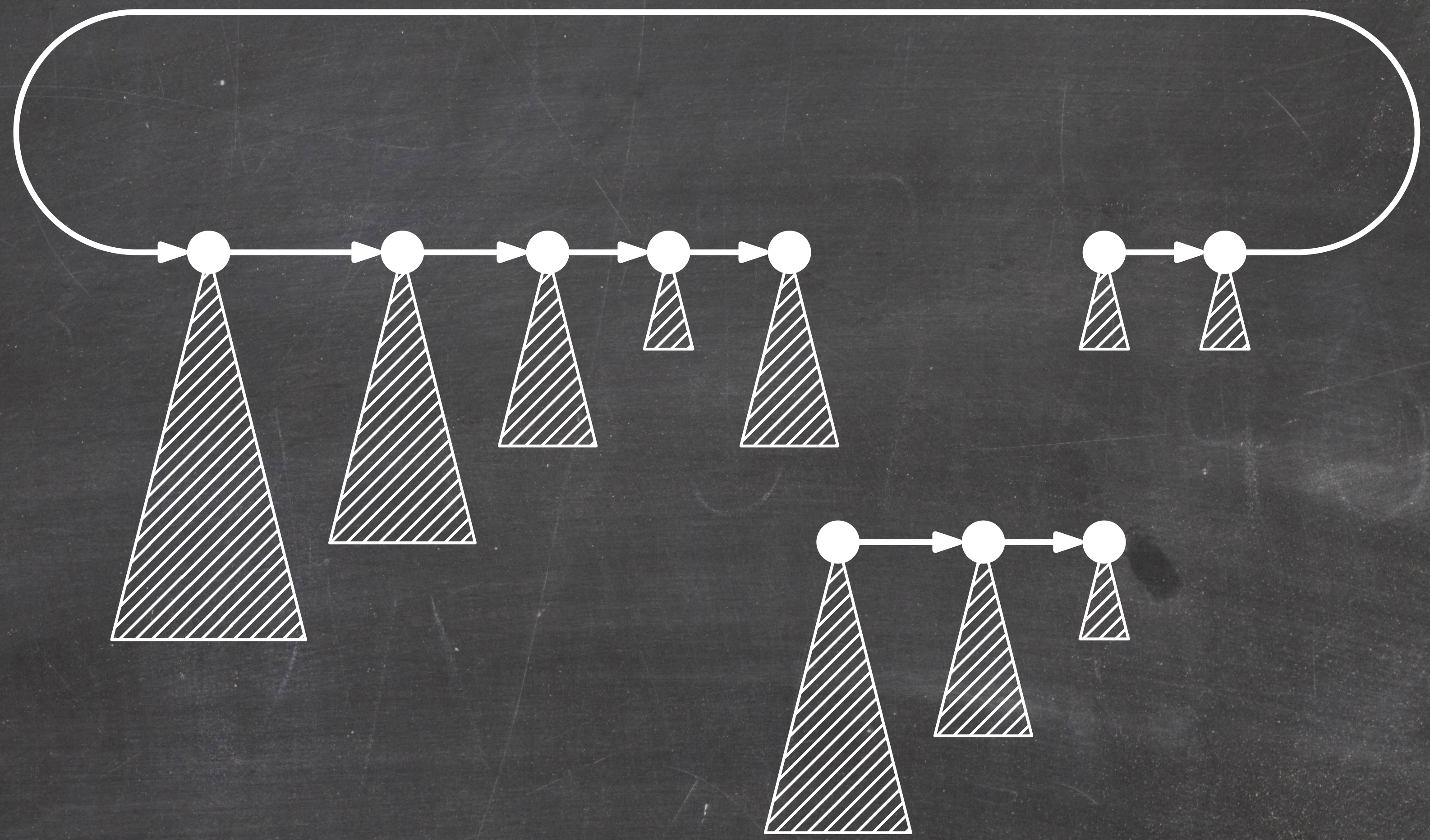

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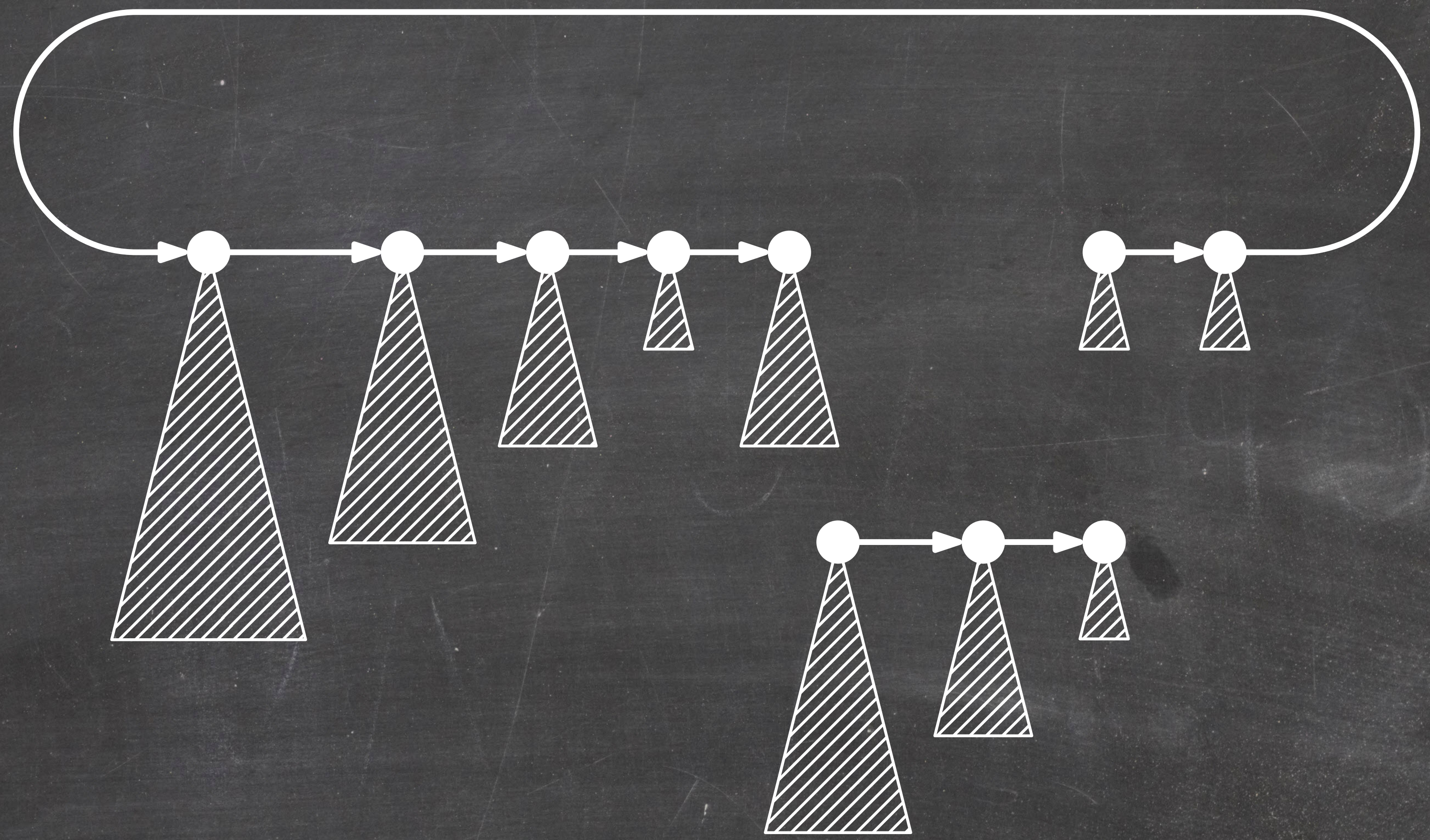
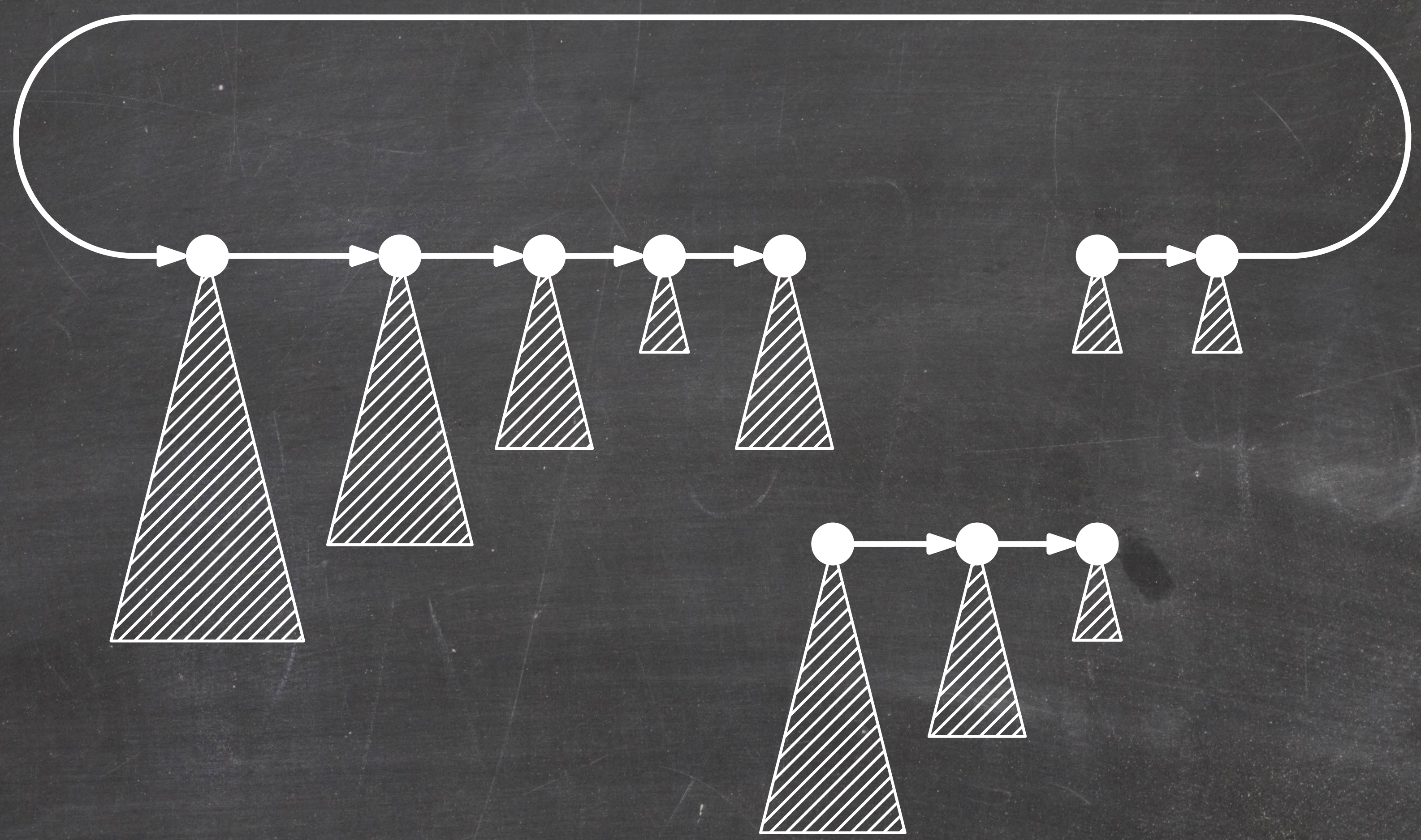


What do we do with the children?

How do we find the new minimum?

DeleteMin

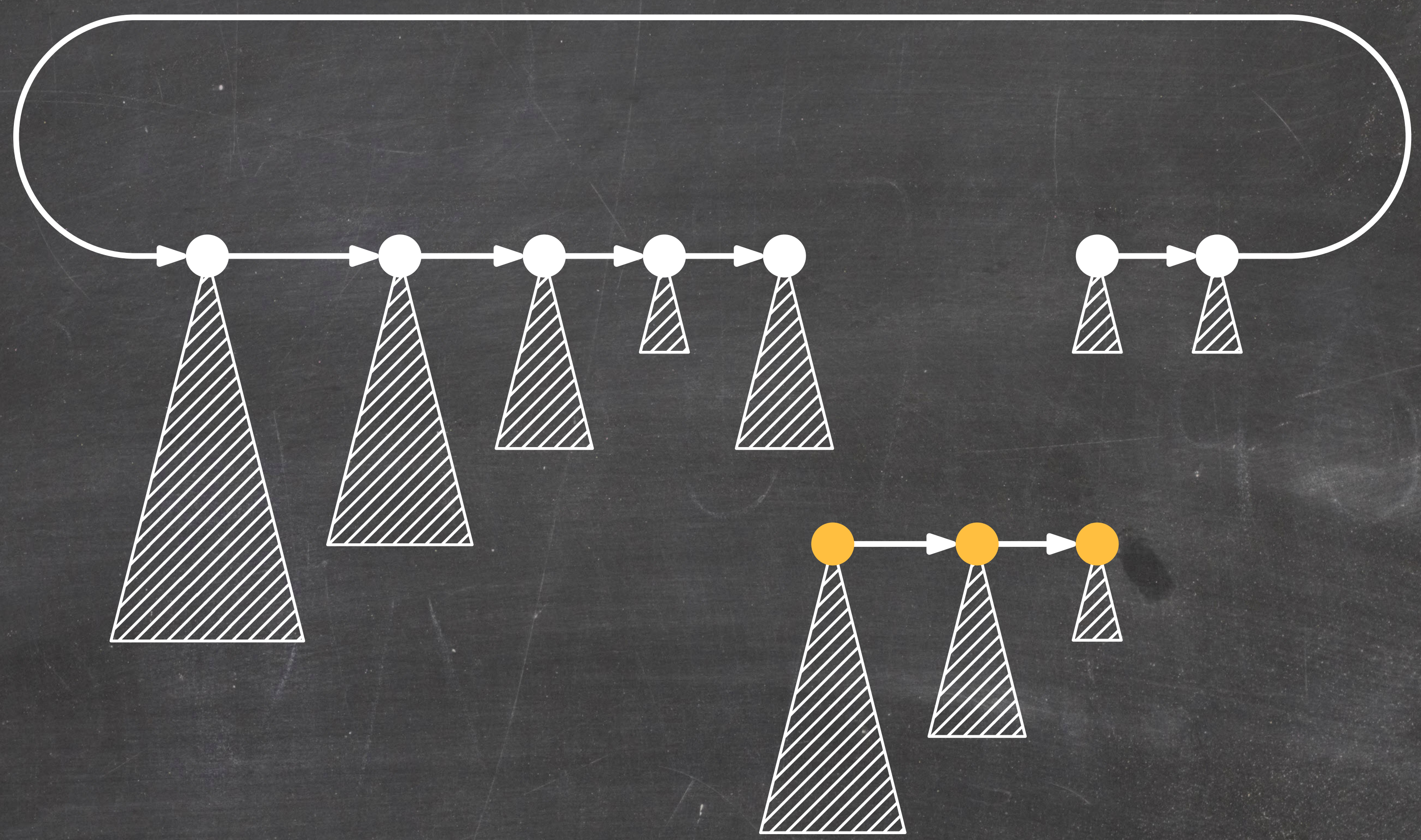


What do we do with the children?

How do we find the new minimum?

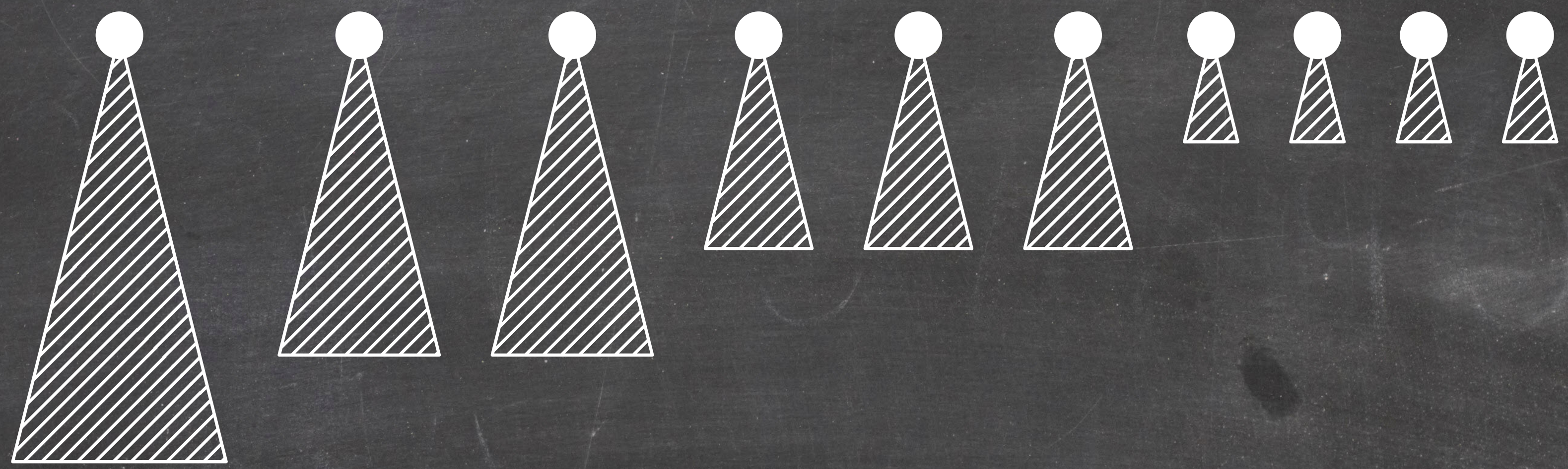
- Could be one of the children.
- Could be one of the other roots.

DeleteMin



- Ensure all former children of min are thick. How?

DeleteMin



- Ensure all former children of min are thick. How?
- Collect all roots and former children of min.

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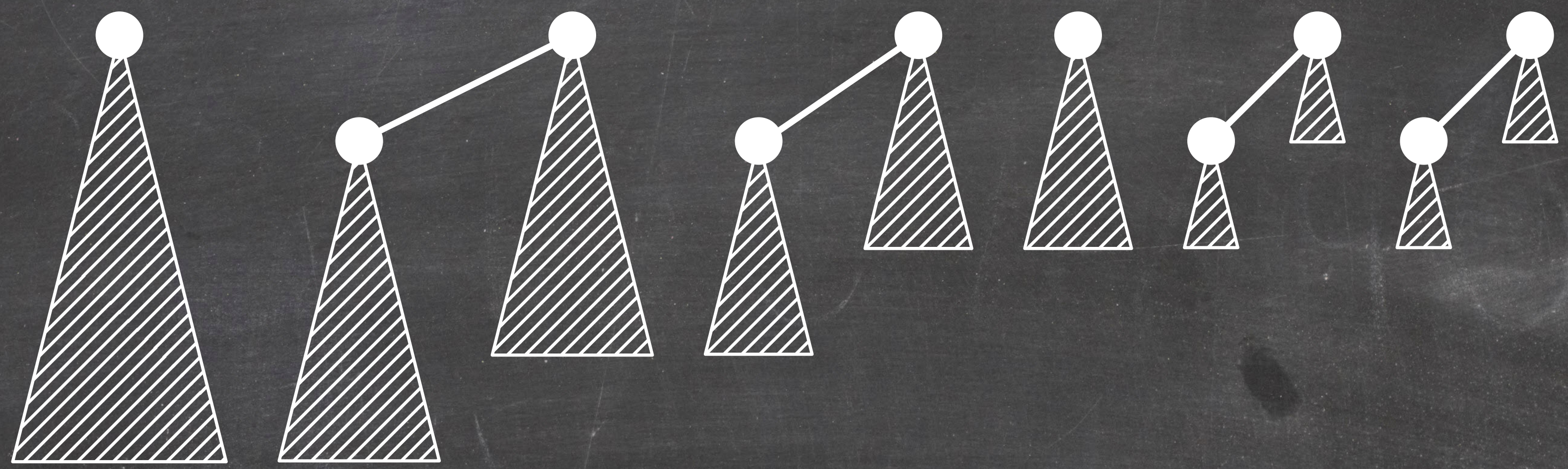

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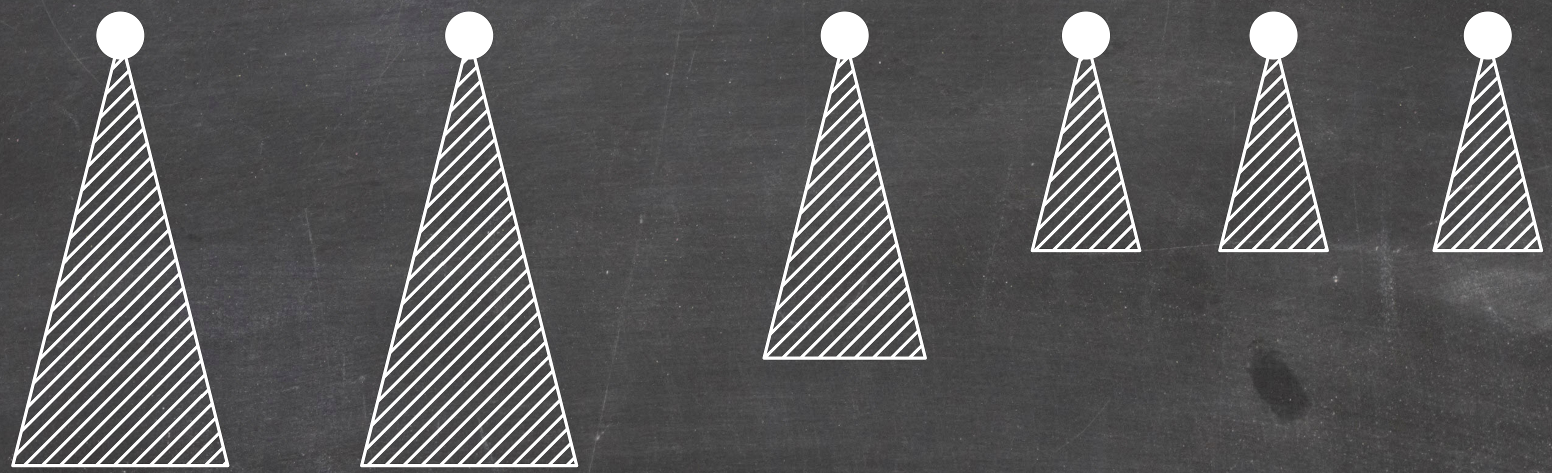

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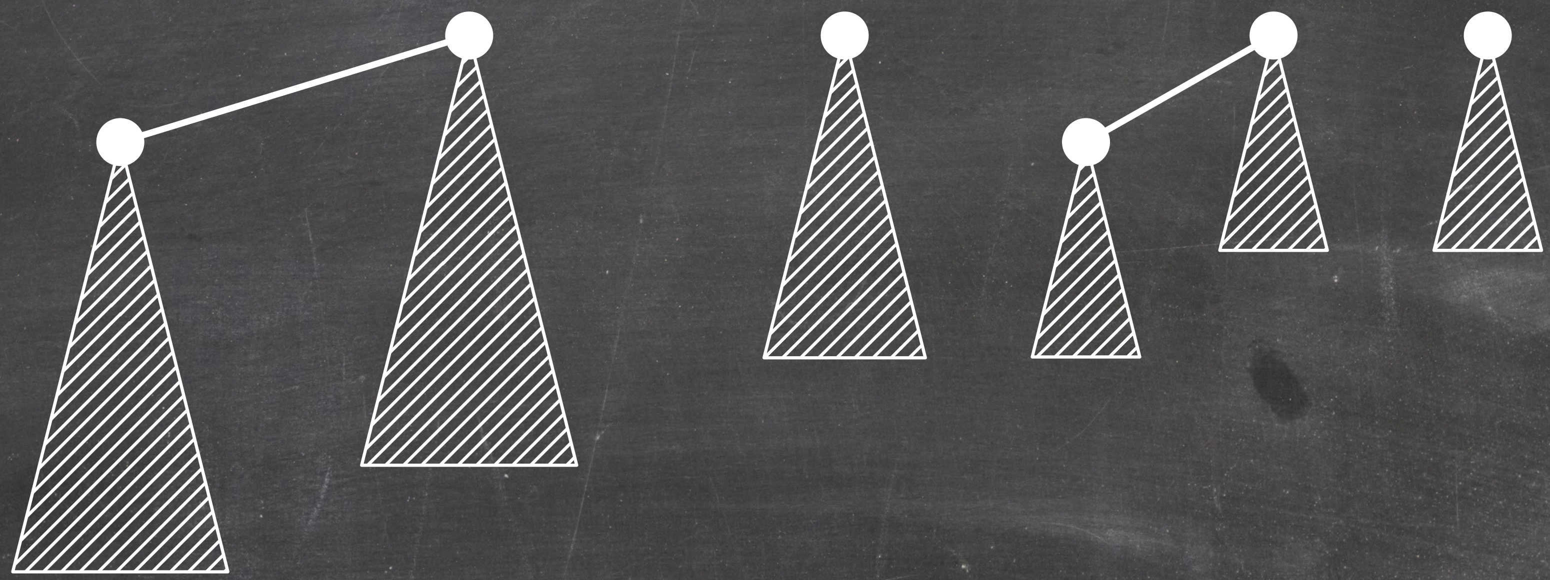

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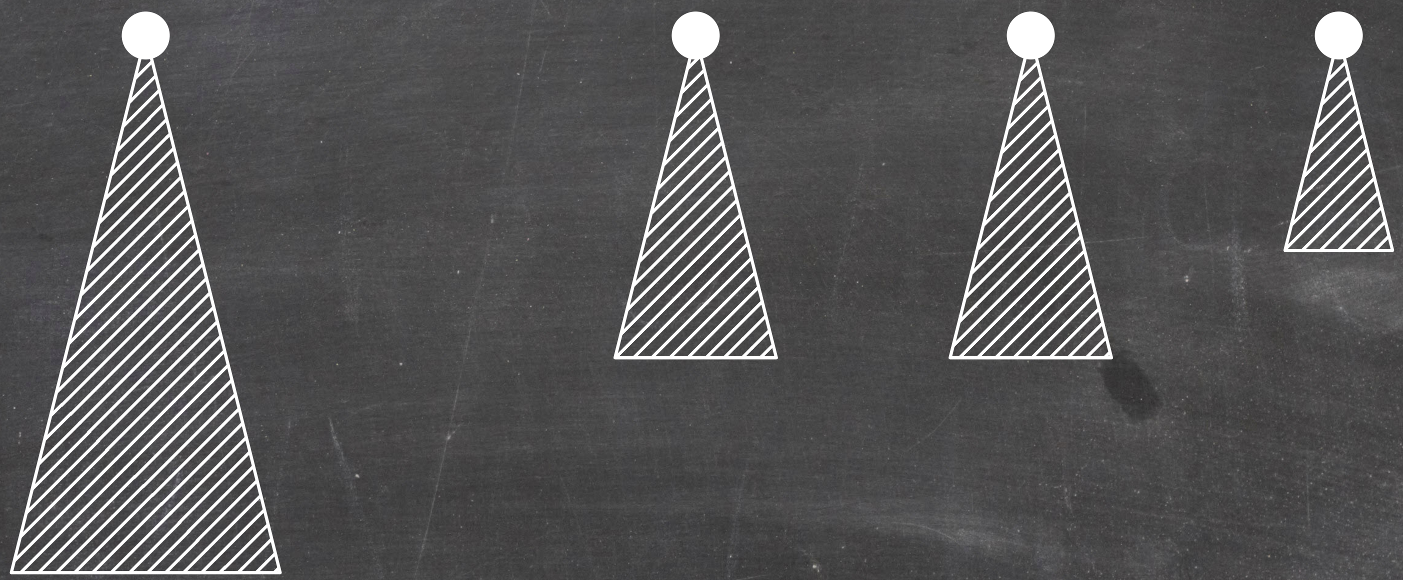

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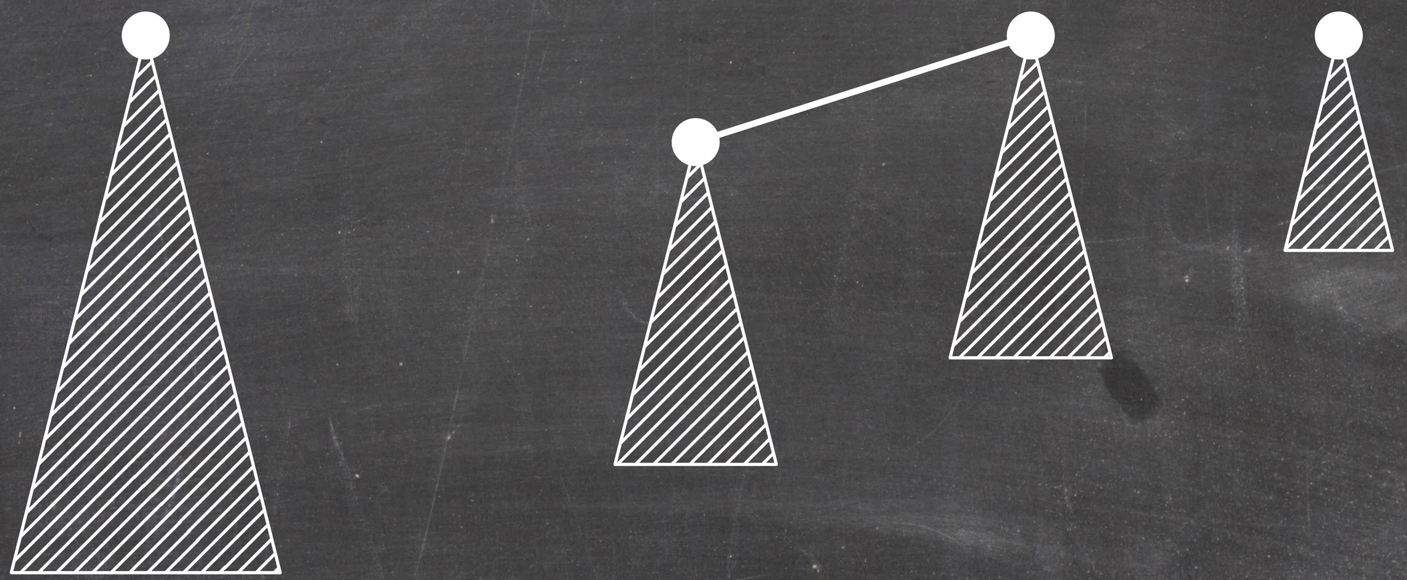

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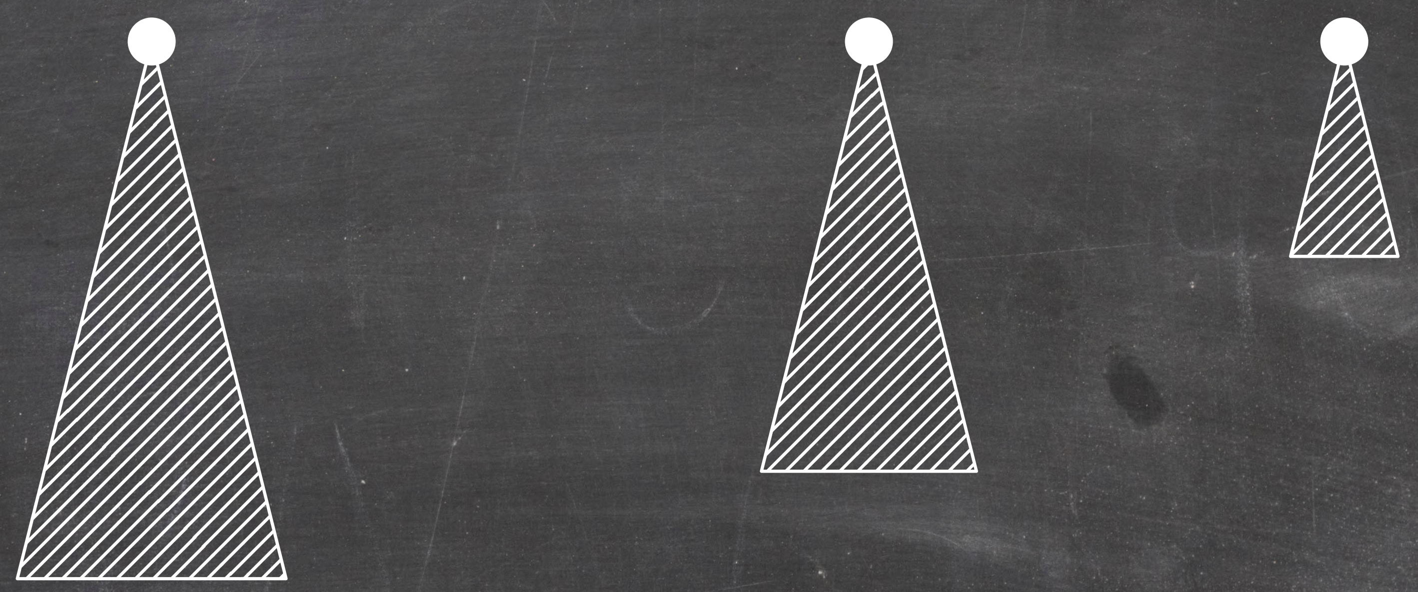

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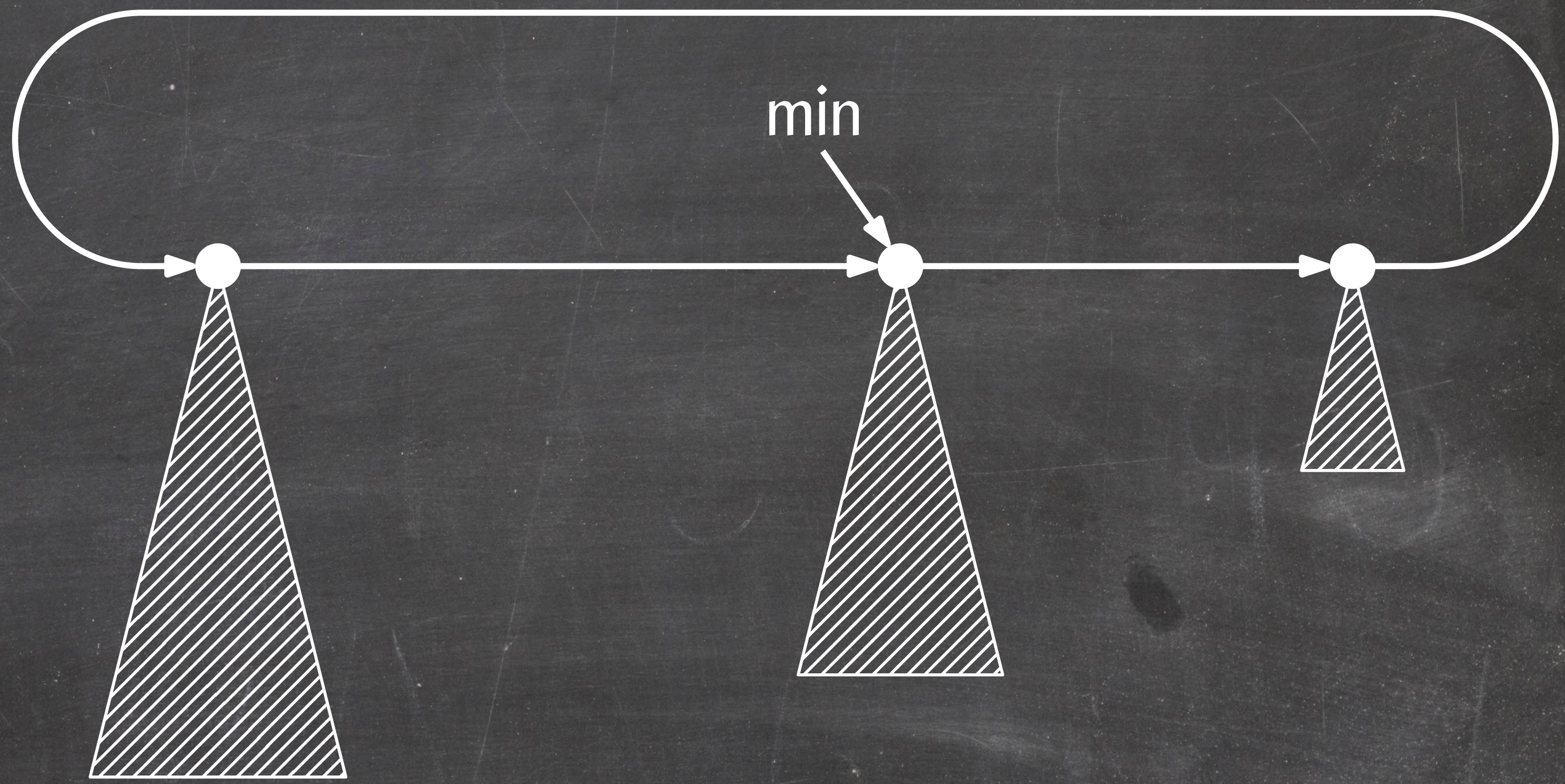

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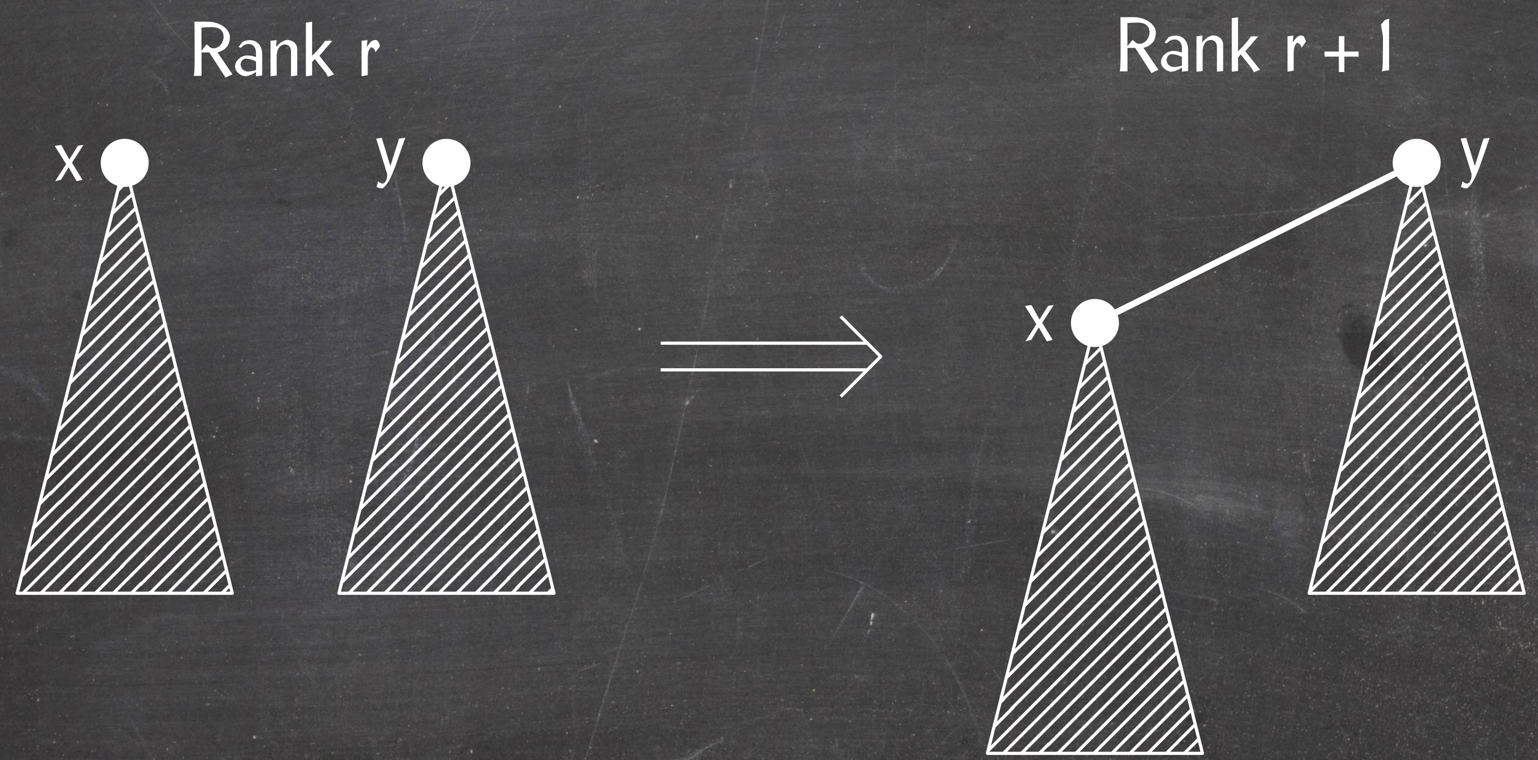
- Link trees of the same rank until at most one tree of each rank remains.
- Relink roots into circular list and make min point to the minimum root.

Linking

Important: Both nodes need to be thick and of the same rank.

Assume $y < x$ (swap the two trees otherwise).



This produces a valid thin tree:

y had r children of ranks $r-1, r-2, \ldots, 0$ before.

$\Rightarrow$ y has $r+1$ children of ranks $r, r-1, \ldots, 0$ after.

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Lemma: A tree whose root has rank r has at least F_r nodes, where F_r is the rth Fibonacci number.

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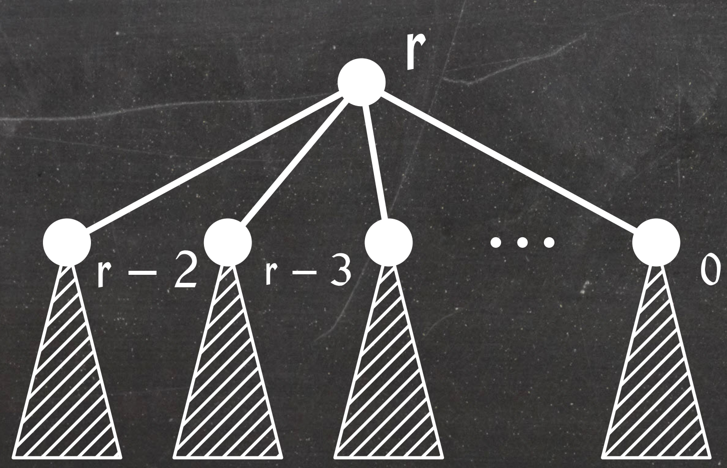

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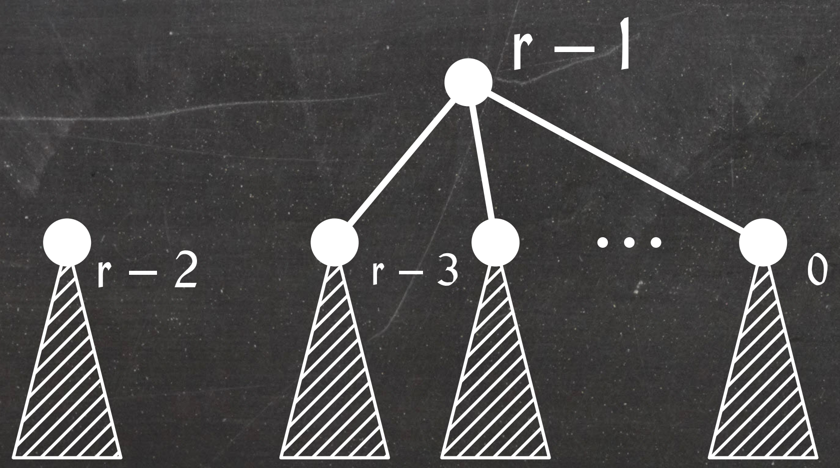

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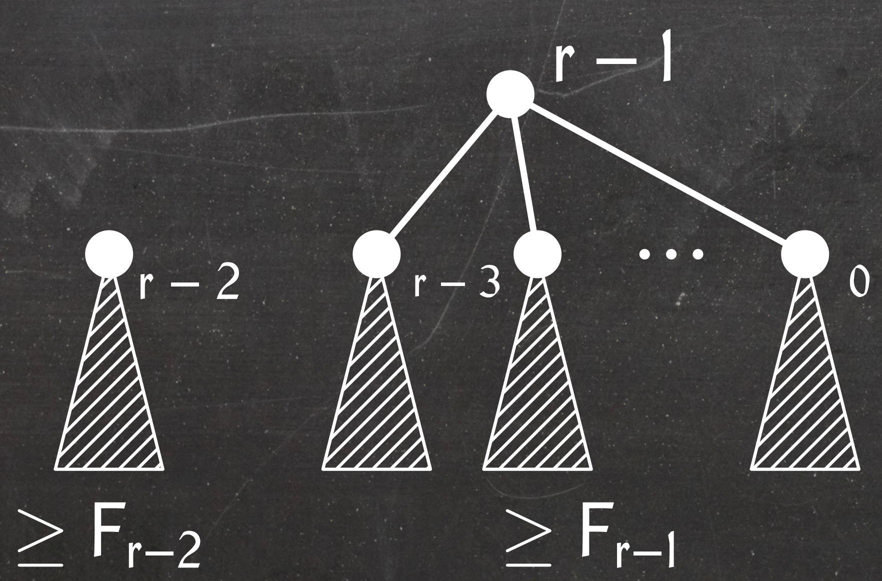

$$F_{r-1} + F_{r-2} = F_r$$

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Corollary: The maximum rank in a Thin Heap storing n elements is $\log_\phi n < 2 \lg n$.

Implementation of DeleteMin

Q.deleteMin()

```
x = Q.min
R = array of size 2 lg n with all its entries initially null.
for every root r other than Q.min
    do LinkTrees(R, r)
for every child c of Q.min
    do decrease c's rank if necessary to make it thick
       LinkTrees(R, c)
Q.min = null
for i = 0 to 2 lg n
    do if R[i] ≠ null
          then R[i].leftSibOrParent = null
               if Q.min = null
                  then Q.min = R[i]
                       Q.min.rightSib = Q.min
                  else R[i].rightSib = Q.min.rightSib
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                       if R[i].val < Q.min.val
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LinkTrees(R, x)

```
r = x.rank
while R[r] ≠ null
    do x = Link(x, R[r])
       R[r] = null
       r = r + 1
R[r] = x
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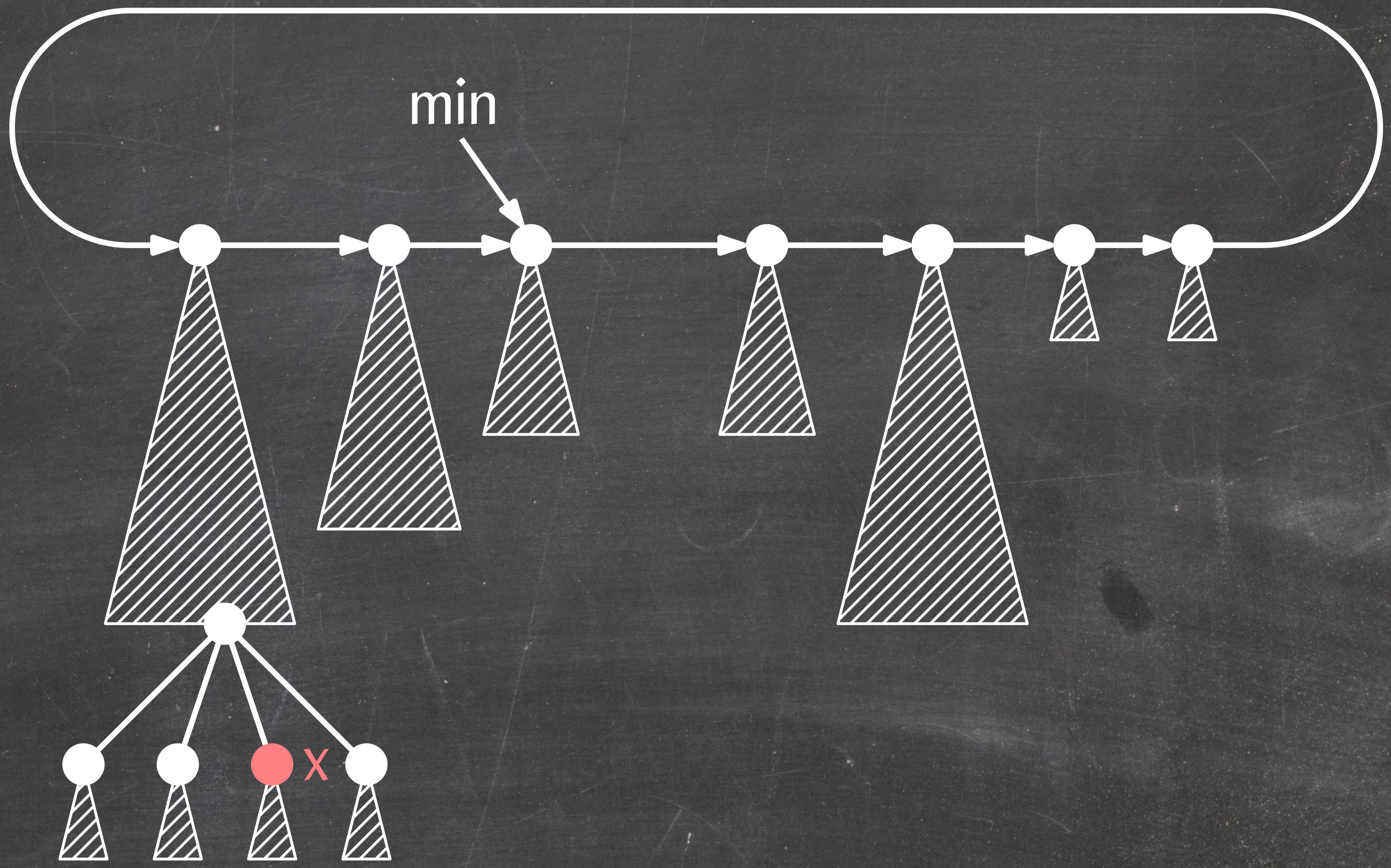
```

Collect remaining trees and form circular list.

DecreaseKey



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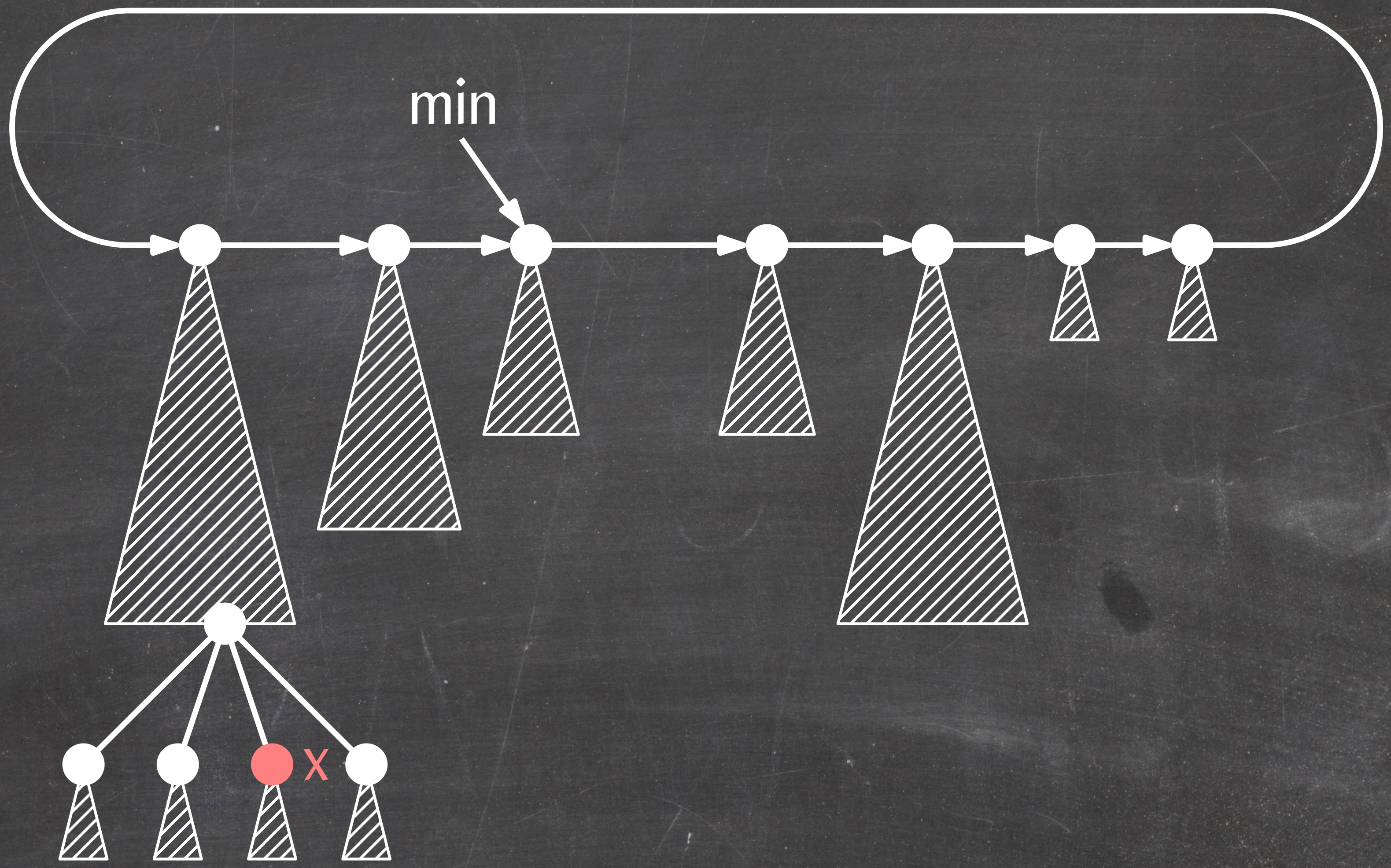
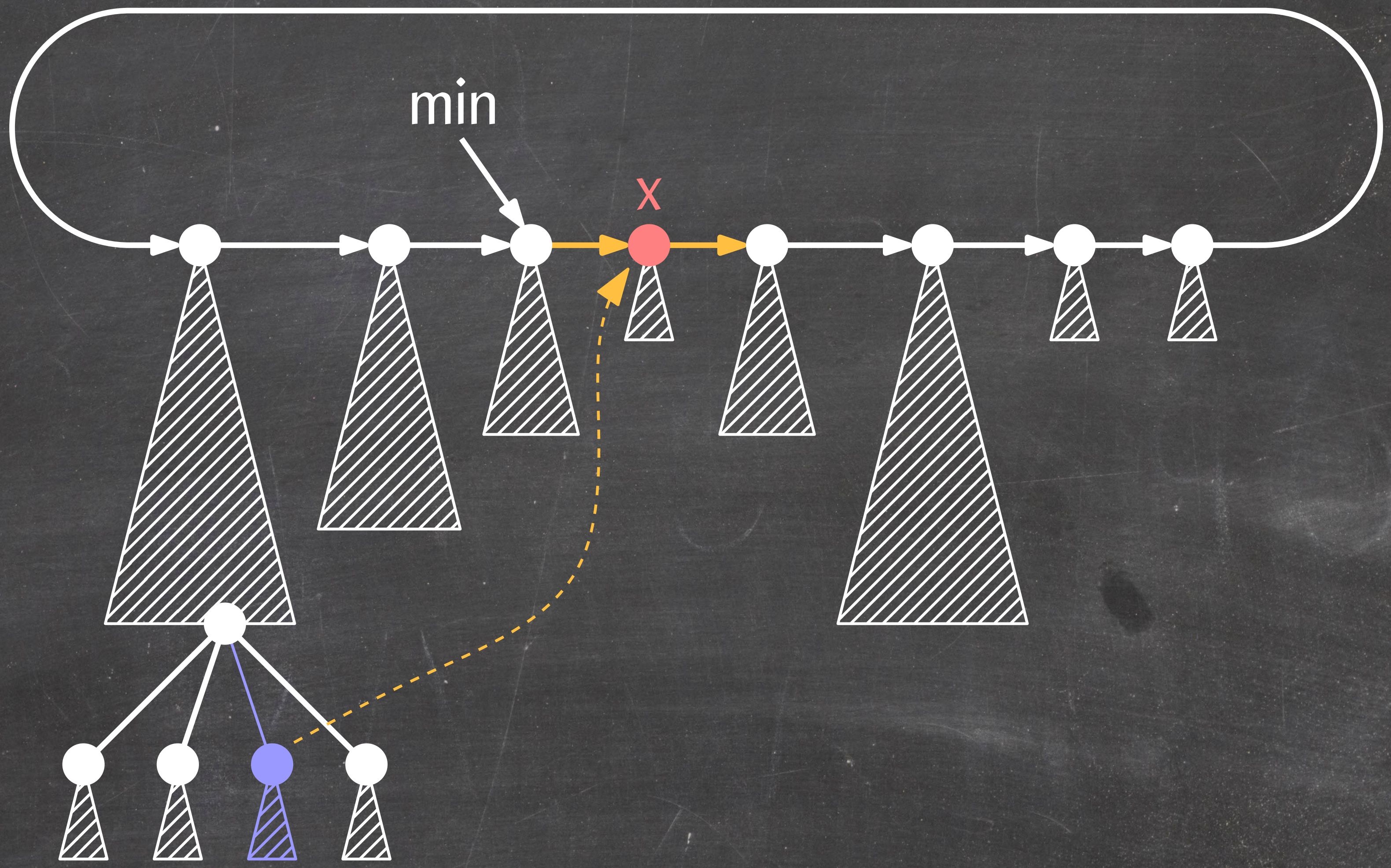


- Update x's priority

DecreaseKey



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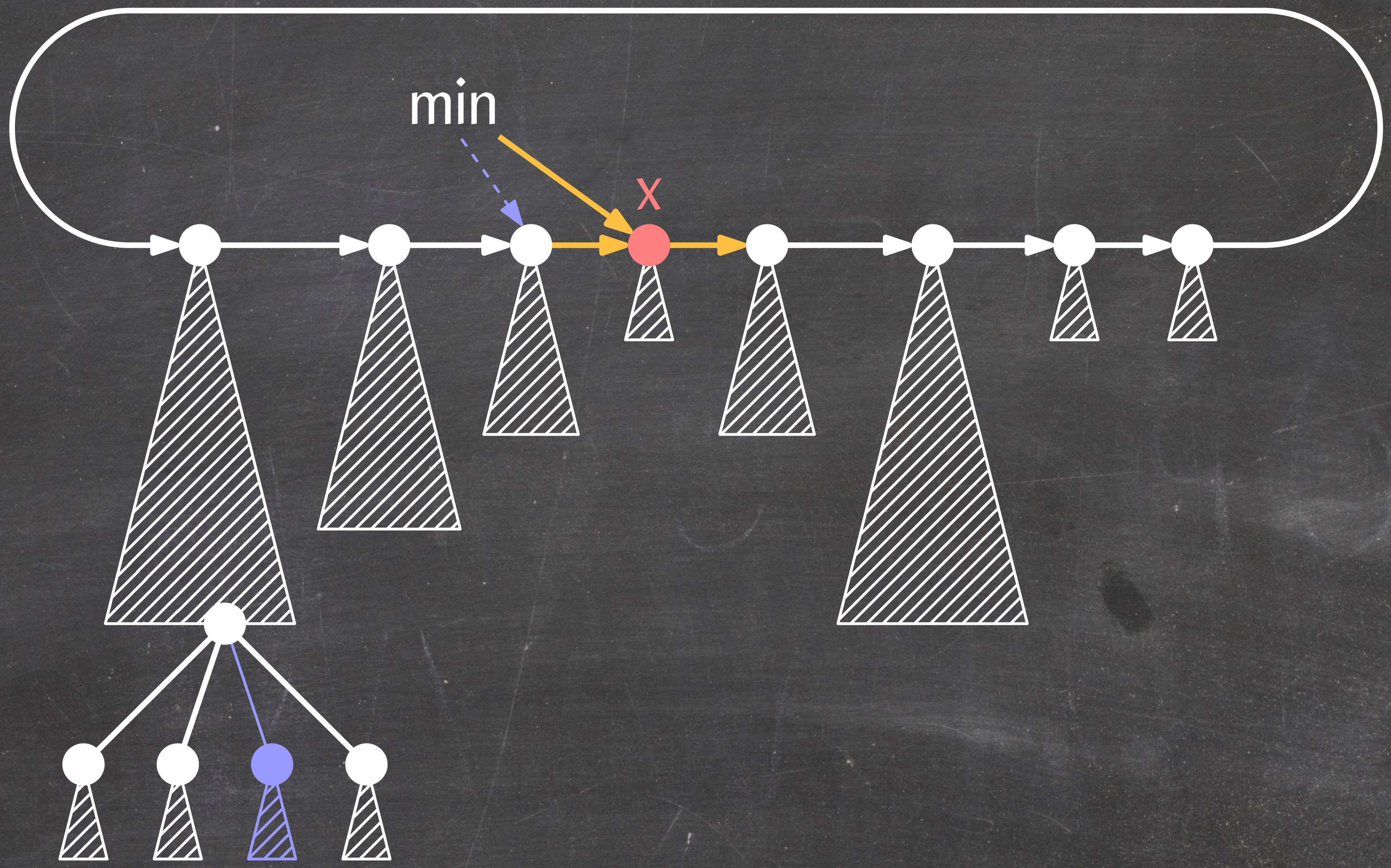

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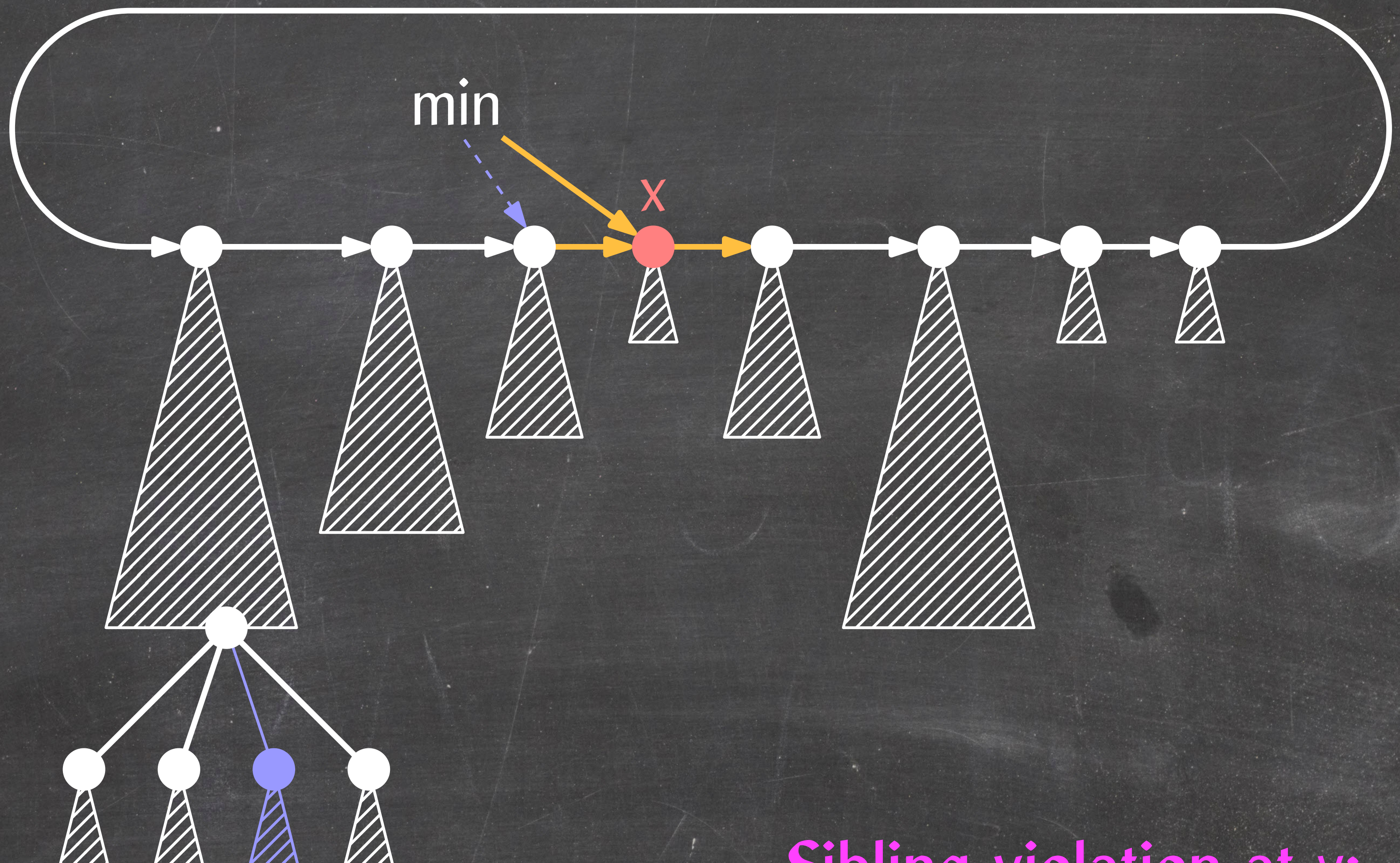

Sibling violation at y:

y.rank > 0 and y has no right sibling or y.rightSib.rank < y.rank – 1.

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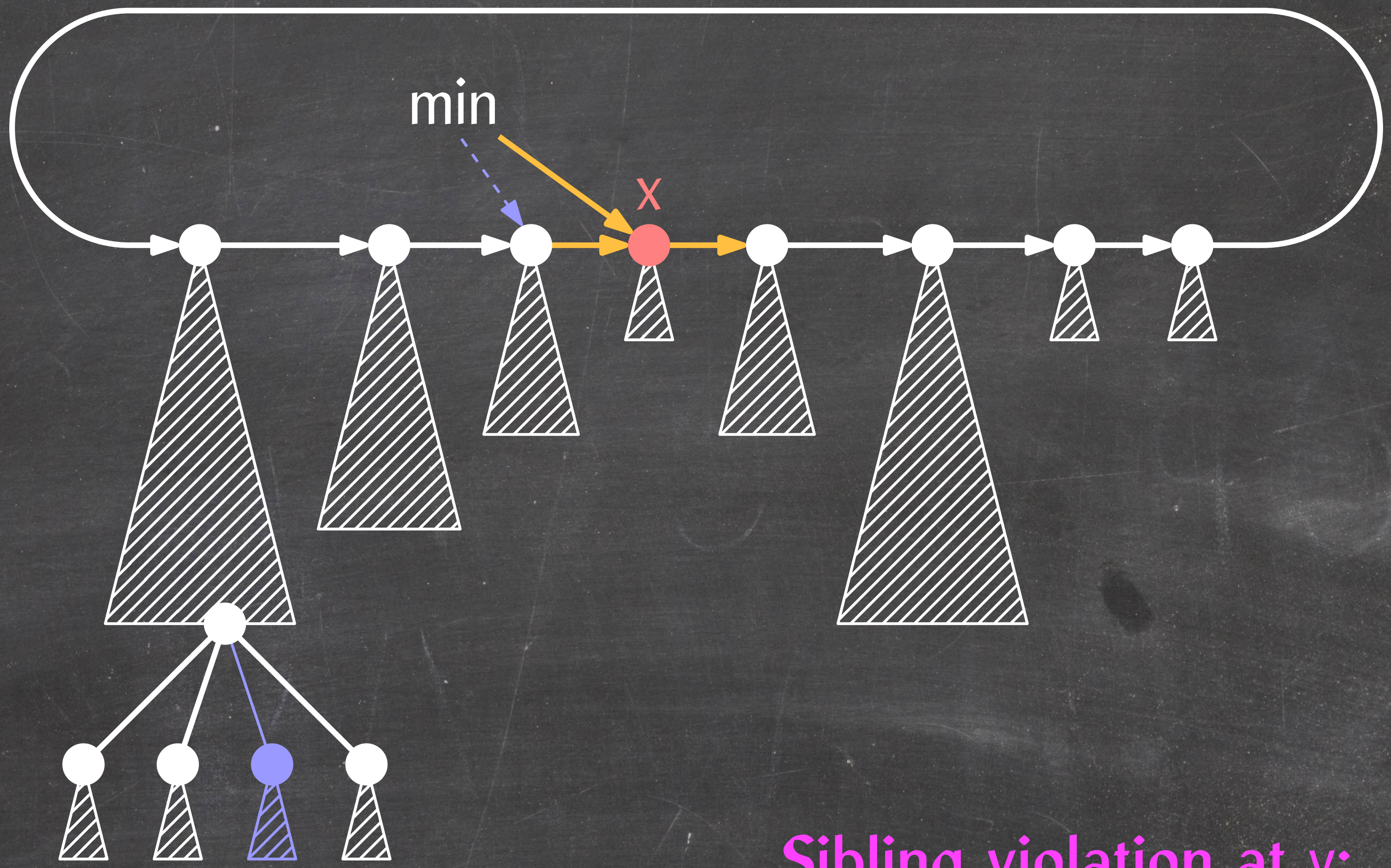

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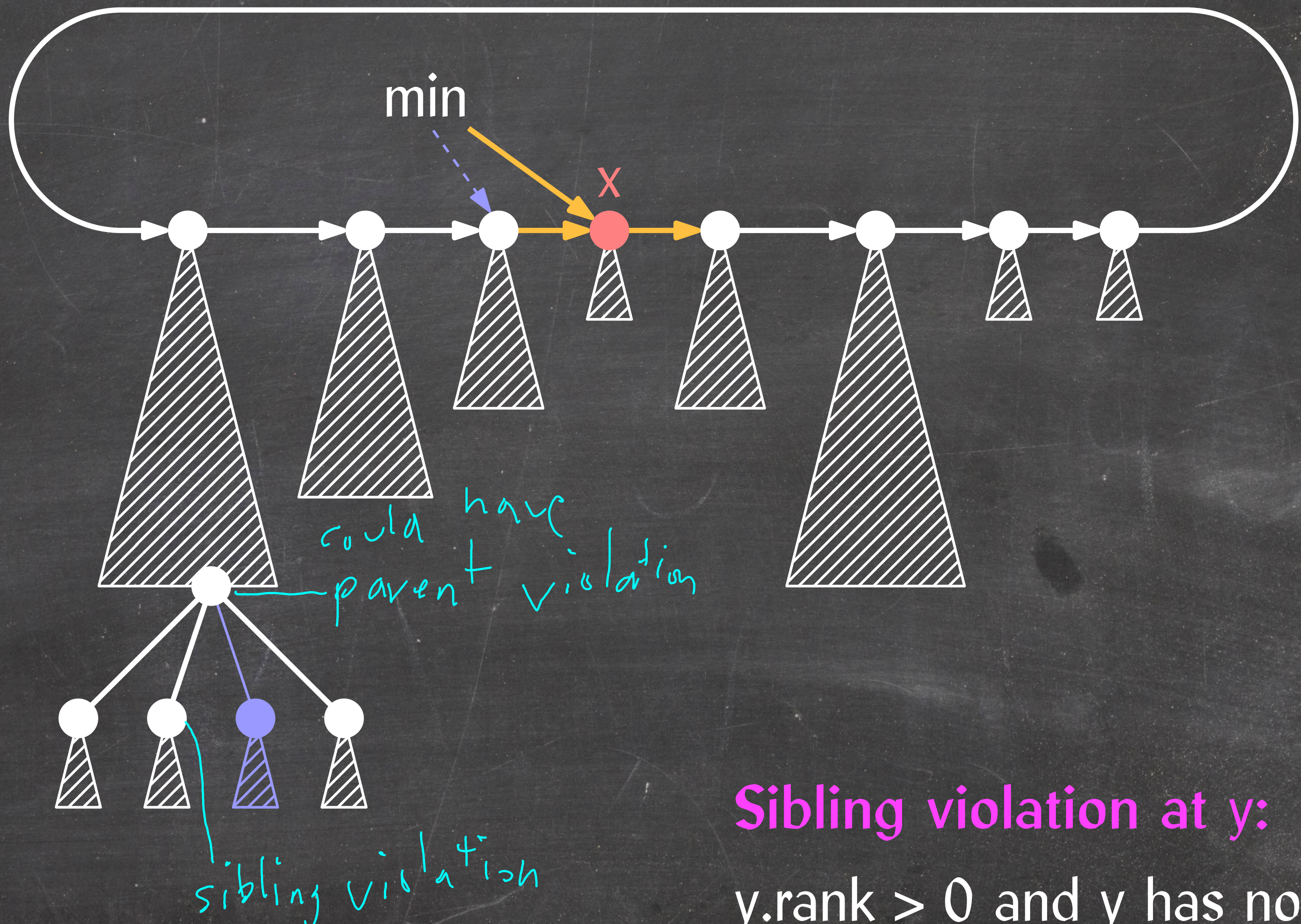

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- Make x a root
- Fix parent/sibling violations

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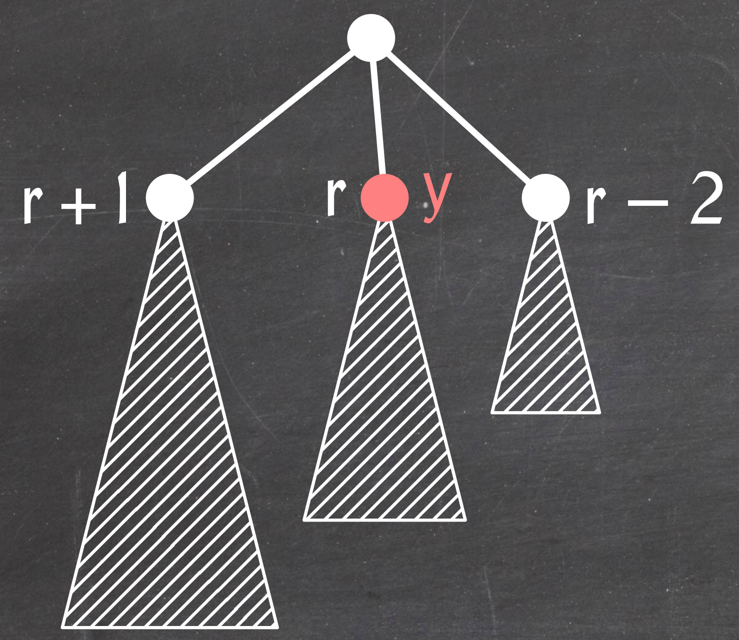

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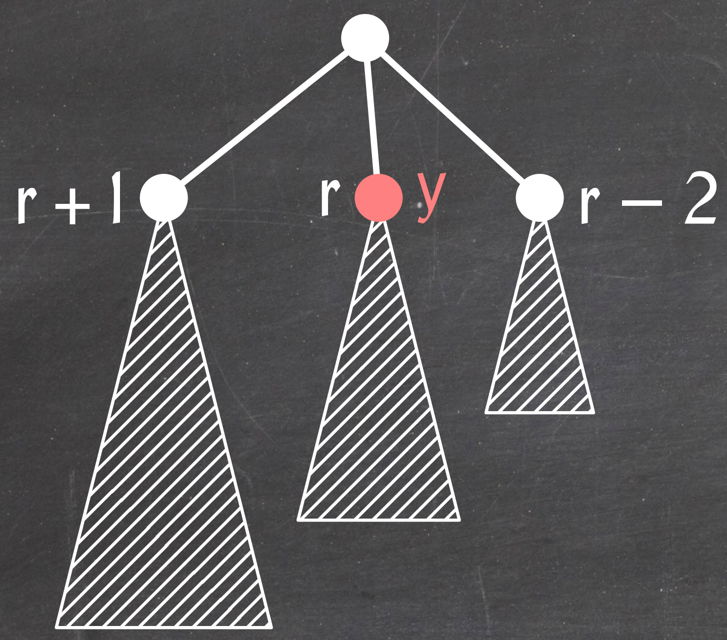

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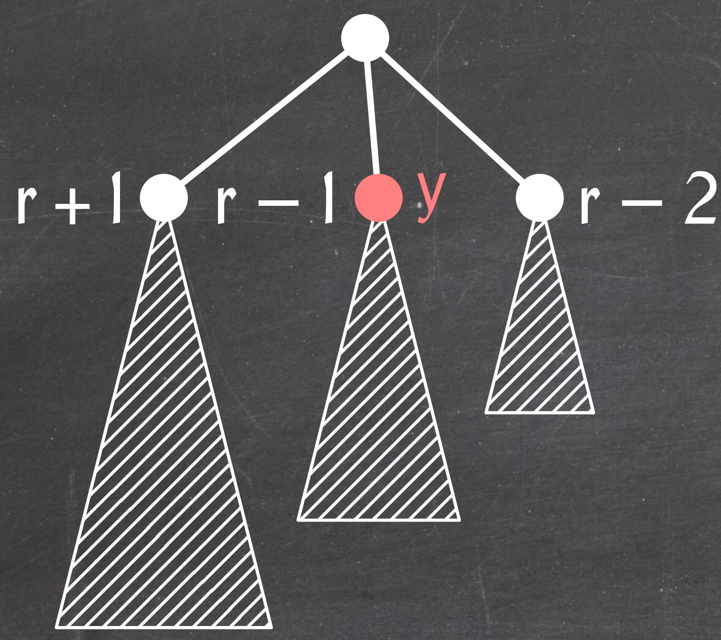

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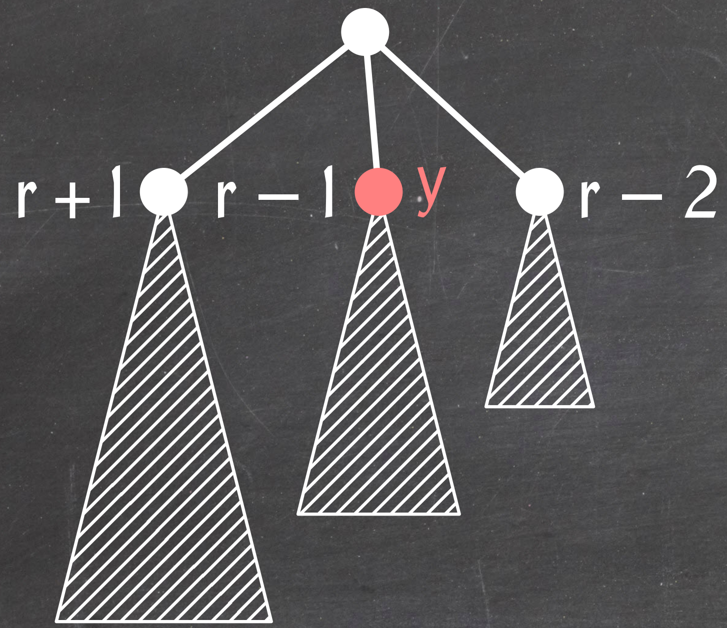

If y is thin, then

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- fix violation at y.leftSibOrParent.

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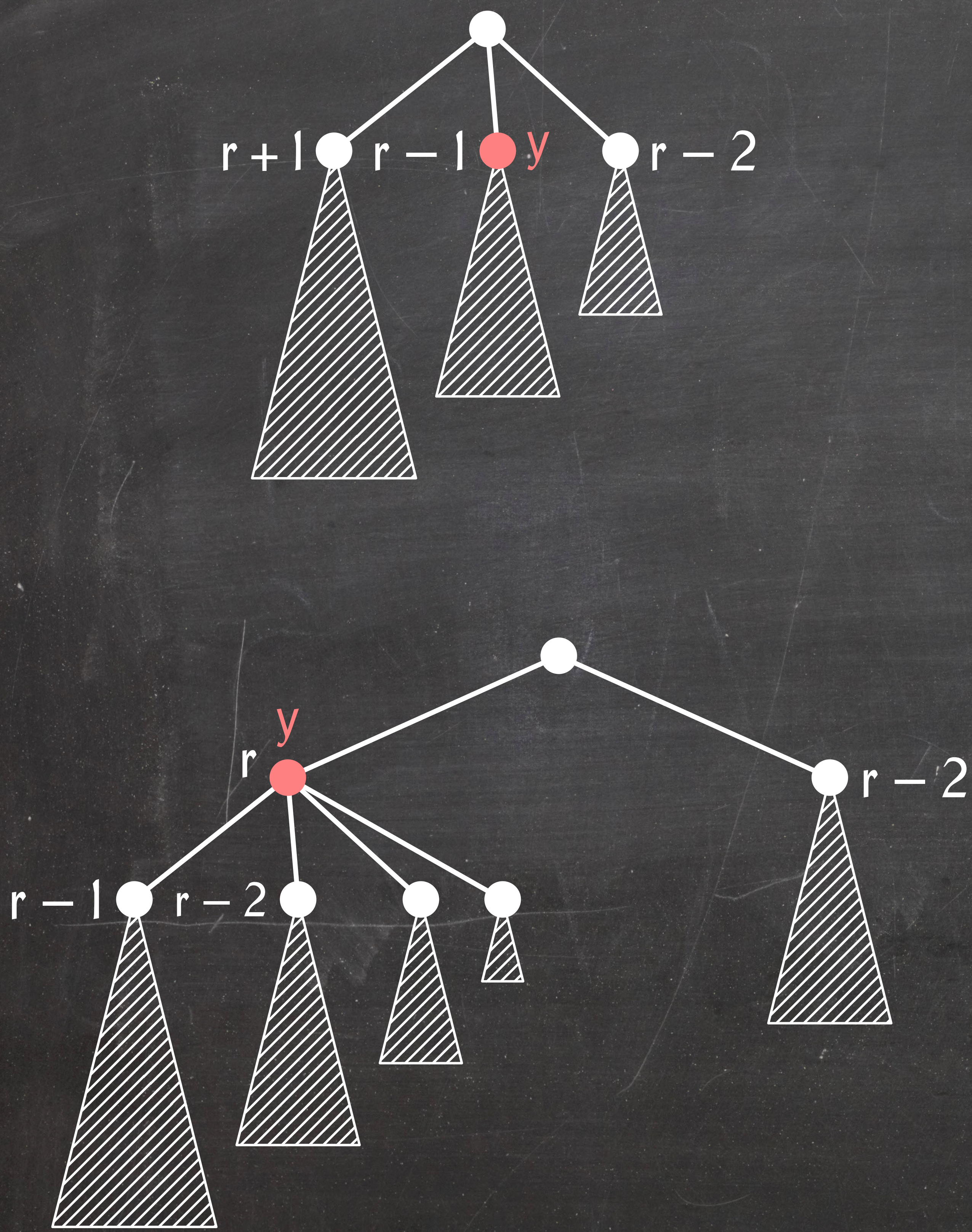

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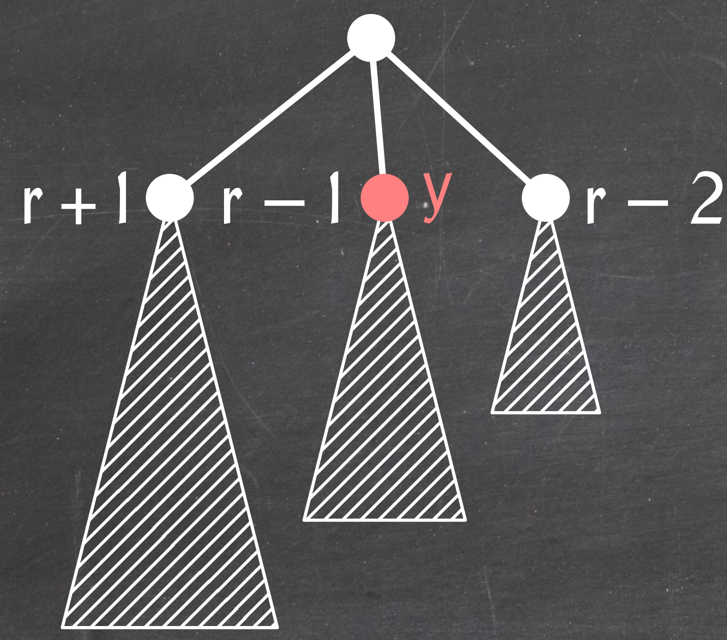

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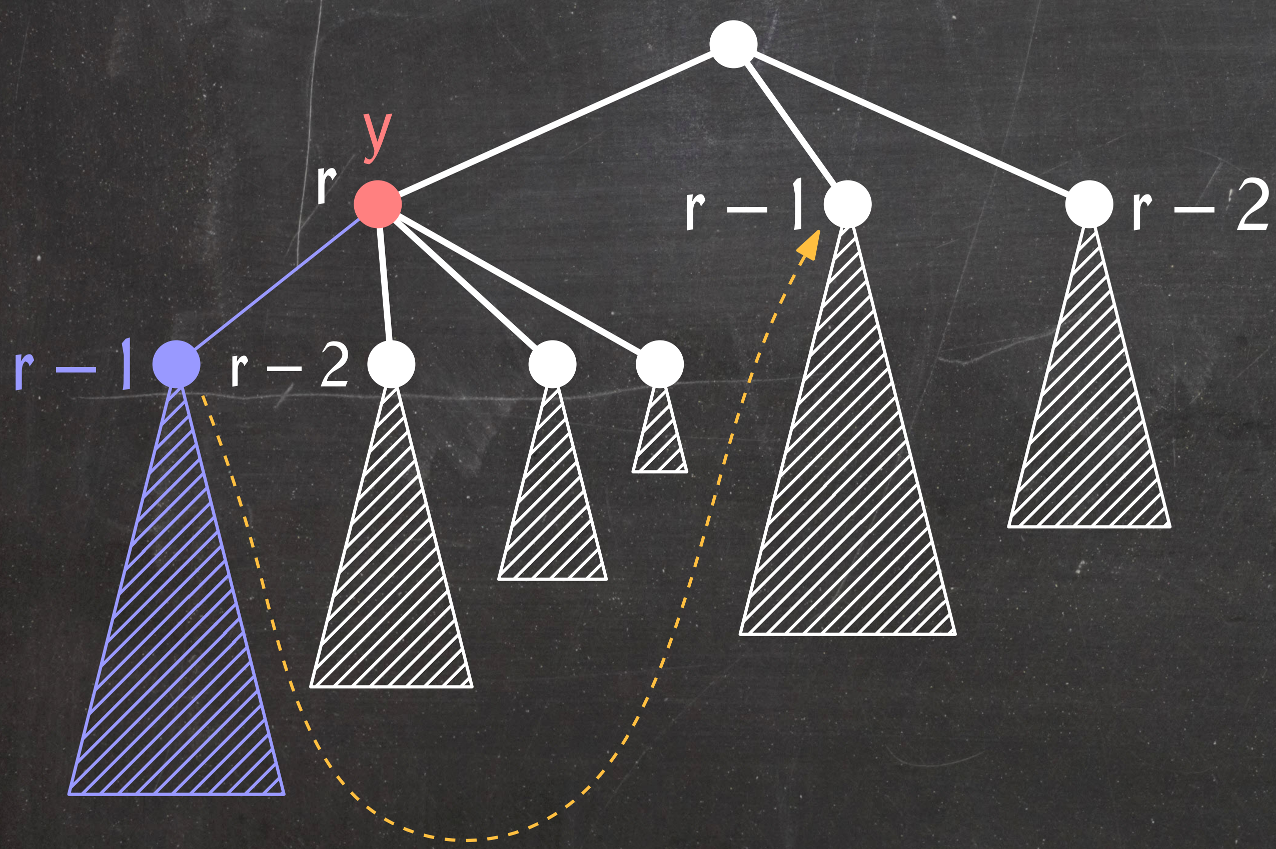
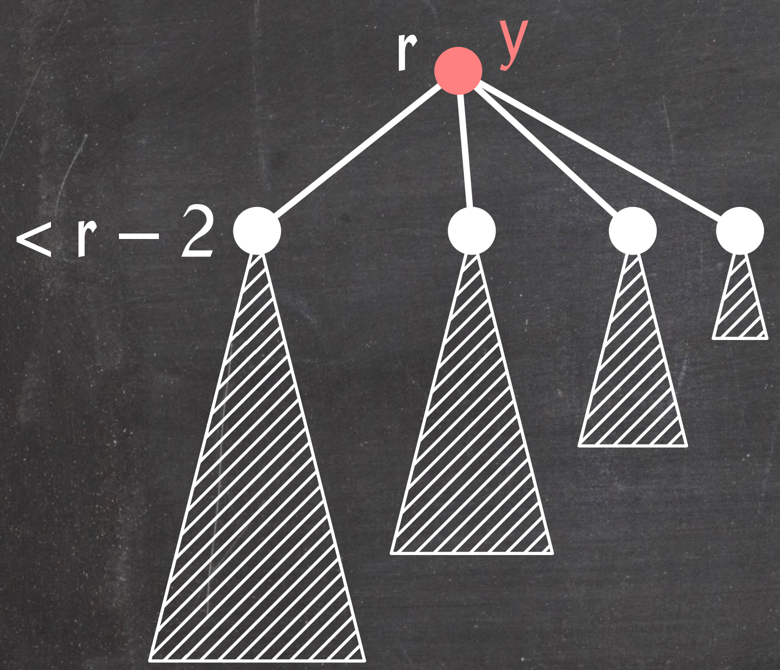


If y is thick, then make y.child y's right sibling.

=> y is thin

Parent Violation



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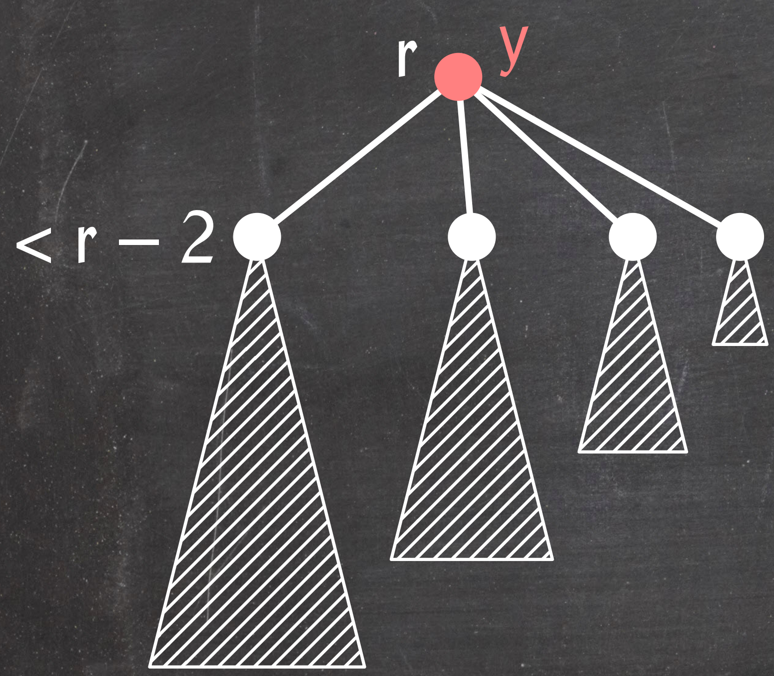

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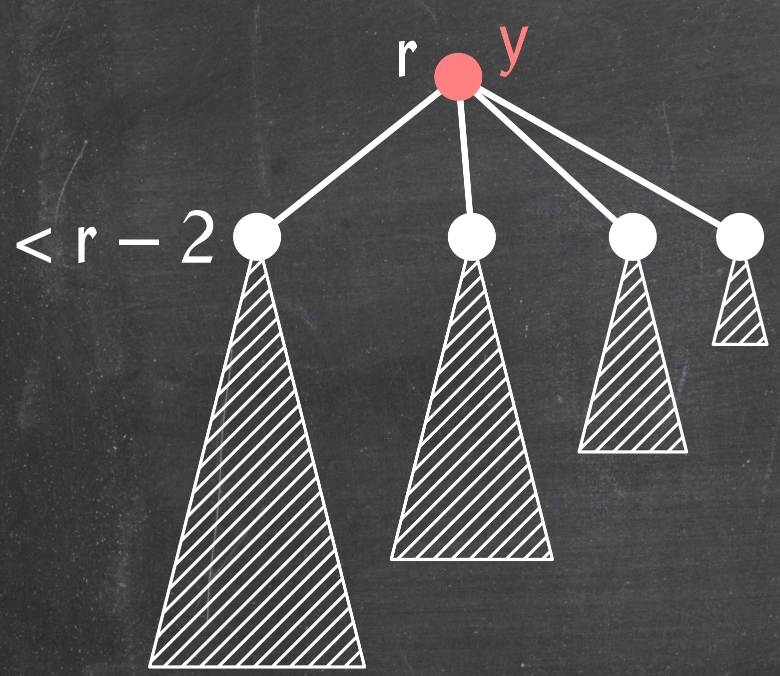

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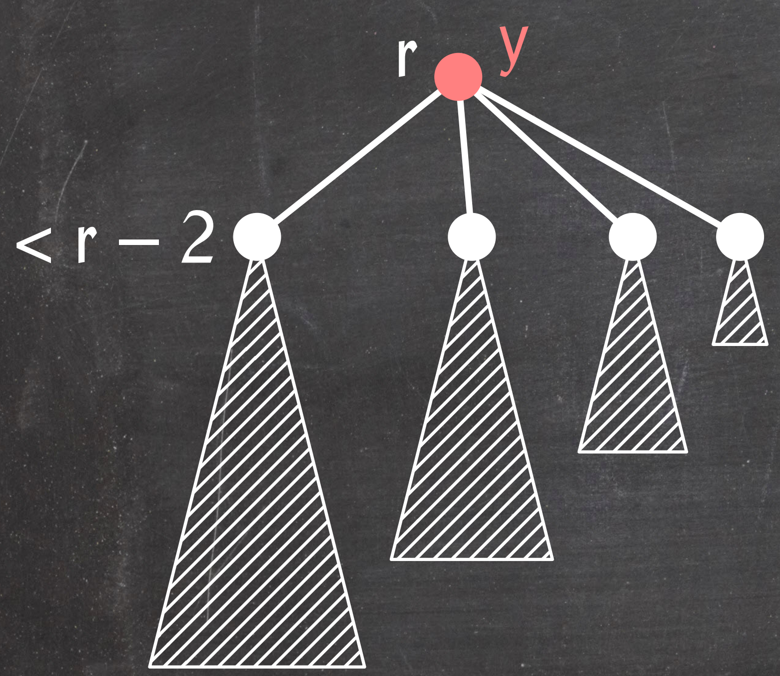

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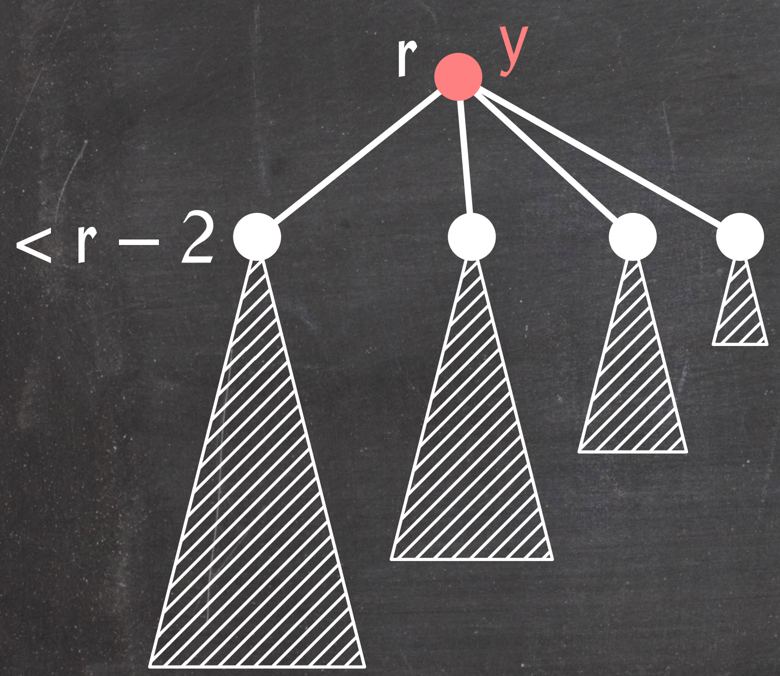

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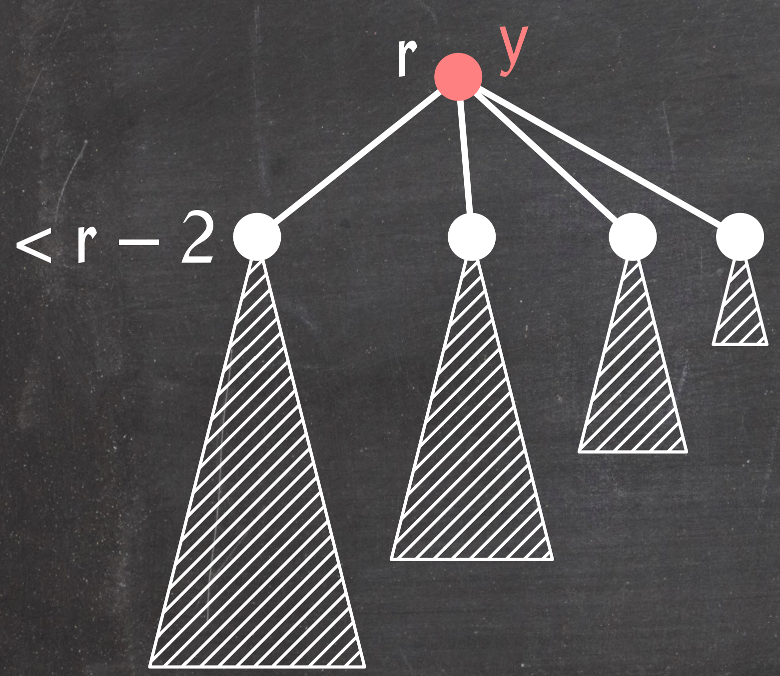

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These costs are completely fictitious but must satisfy an important condition to be useful:

$$\sum_{i=1}^{m} c_i \leq \sum_{i=1}^{m} \hat{c}_i$$

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Techniques for Proving Amortized Bounds

The most important ones are the Accounting Method and Potential Functions.

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$$\sum_{i=1}^{m} \hat{c}_i = \sum_{i=1}^{m} (c_i + \Phi_i - \Phi_{i-1}) = \sum_{i=1}^{m} c_i + \Phi_m - \Phi_0 \geq \sum_{i=1}^{m} c_i$$

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Intuition:

- The potential captures parts of the data structure that can make operations expensive.
- If operations that take long eliminate these "expensive" parts of the data structure, then there can't be many expensive operations without lots of operations that create these expensive parts.
- These operations can "pay" for the cost of the expensive operations.

Amortized Analysis: Stack with MultiPop Operation

Operations:

S.push(x)	Push element x on the stack
S.pop()	Pop the topmost element from the stack
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$$\Phi = |S|$$

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$\Rightarrow \; \hat{c} = c + \Delta\Phi = O(1 + \min(k, |S|)) - \min(k, |S|) = O(1)$

Amortized Analysis: Binary Counter

Consider a binary counter initially set to 0.

The only operation we want to support is Increment.

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↓

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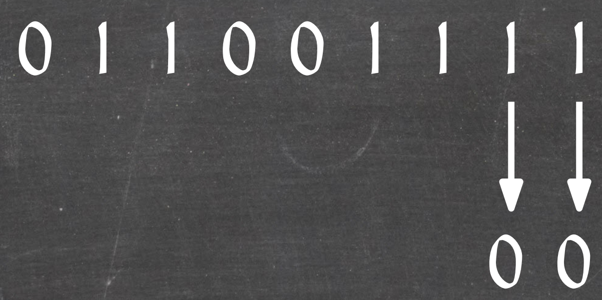

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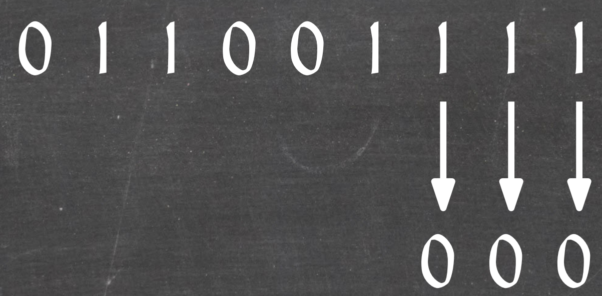

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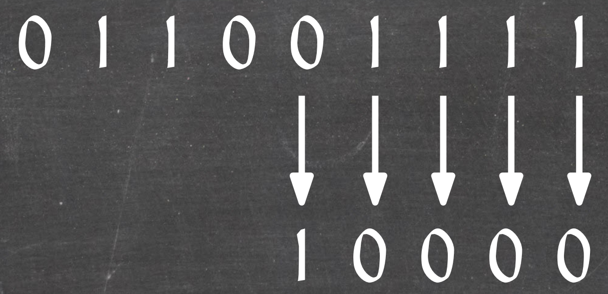

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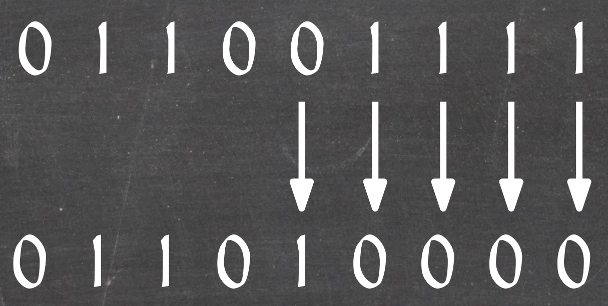

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Φ = #1s in the current counter value

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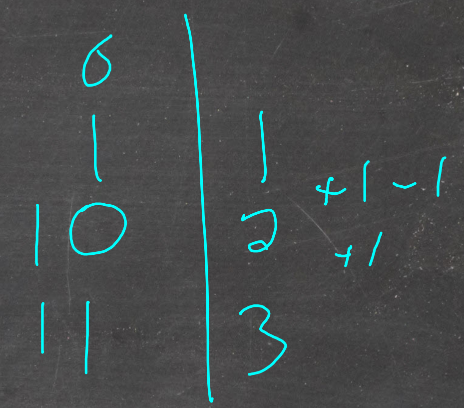

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A Potential Function for Thin Heap

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$$\Phi = 2 \cdot \text{number of thin nodes} + \text{number of roots}$$

Amortized Cost of Insert, FindMin, and Delete

Insert:

- $c \in O(1)$
- $\Delta\Phi = +1$:
  - Δ(number of roots) = +1
  - Δ(number of thin nodes) = 0

$\Rightarrow \hat{c} \in O(1)$

Amortized Cost of Insert, FindMin, and Delete

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FindMin:

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  - The heap structure doesn't change.

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Delete:

- We show that $\hat{c}(\text{DecreaseKey}) \in O(1)$.
- We show that $\hat{c}(\text{DeleteMin}) \in O(\lg n)$.

$\Rightarrow \hat{c} \in O(\lg n)$

Amortized Cost of DeleteMin

Actual cost: $O(\lg n + \text{number of roots} + \text{number of children of Q.min})$

- $O(\lg n)$ for initializing R
- $O(1)$ per addition to R
- $O(1)$ per link operation
- $O(\lg n)$ to collect final list of roots from R
- Number of additions to R = number of roots and children of Q.min
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$\Rightarrow$ $c \in O(\lg n + \text{number of roots})$

- $\Delta(\text{number of thin nodes}) \le 0$
- $\Delta(\text{number of roots}) \le 2 \lg n - \text{number of roots}$

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Amortized cost:

$\hat{c} = c + \Delta\Phi = O(\lg n + \text{number of roots}) + 2 \lg n - \text{number of roots} \in O(\lg n).$

Amortized Cost of DecreaseKey

Make affected element x a root (if it isn't already a root):

- $c \in O(1)$
- $\Delta(\text{number of roots}) \leq 1$
- $\Delta(\text{number of thin nodes}) \leq 1$:
  - x's parent becomes thin if it was thick and x is the leftmost child.

$\Rightarrow \Delta\Phi \leq 3$

$\Rightarrow \hat{c} \in O(1)$

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We prove that

- Fixing the last violation has constant amortized cost,
- Fixing all other violations has amortized cost 0!

$\Rightarrow$ The amortized cost of fixing all violations is in $O(1)$.

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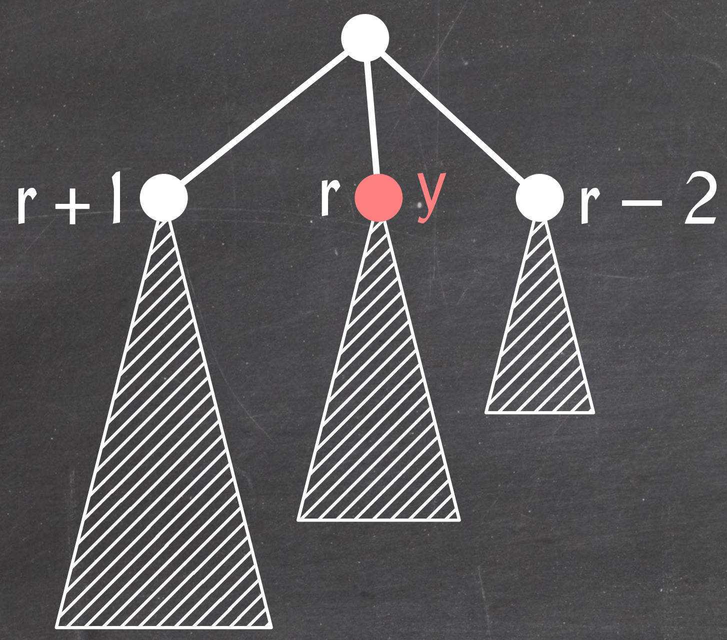
$\Rightarrow \hat{c}(\text{DecreaseKey}) \in O(1)$.

Amortized Cost of Fixing Sibling Violations



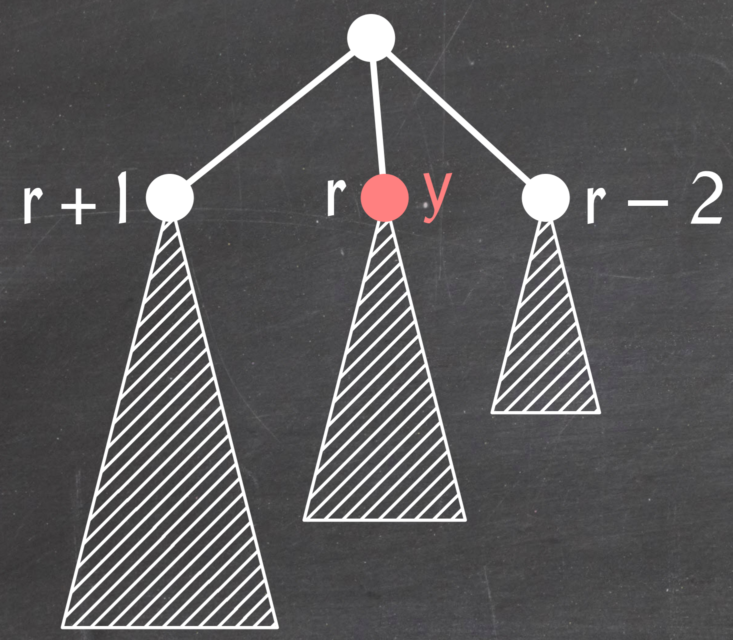
If y is thin,

- $c \in O(1)$
- Δ(number of thin nodes) = -1
- Δ(number of roots) = 0

$\Rightarrow \Delta\Phi = -2$

$\Rightarrow \hat{c} = 0$

Amortized Cost of Fixing Sibling Violations



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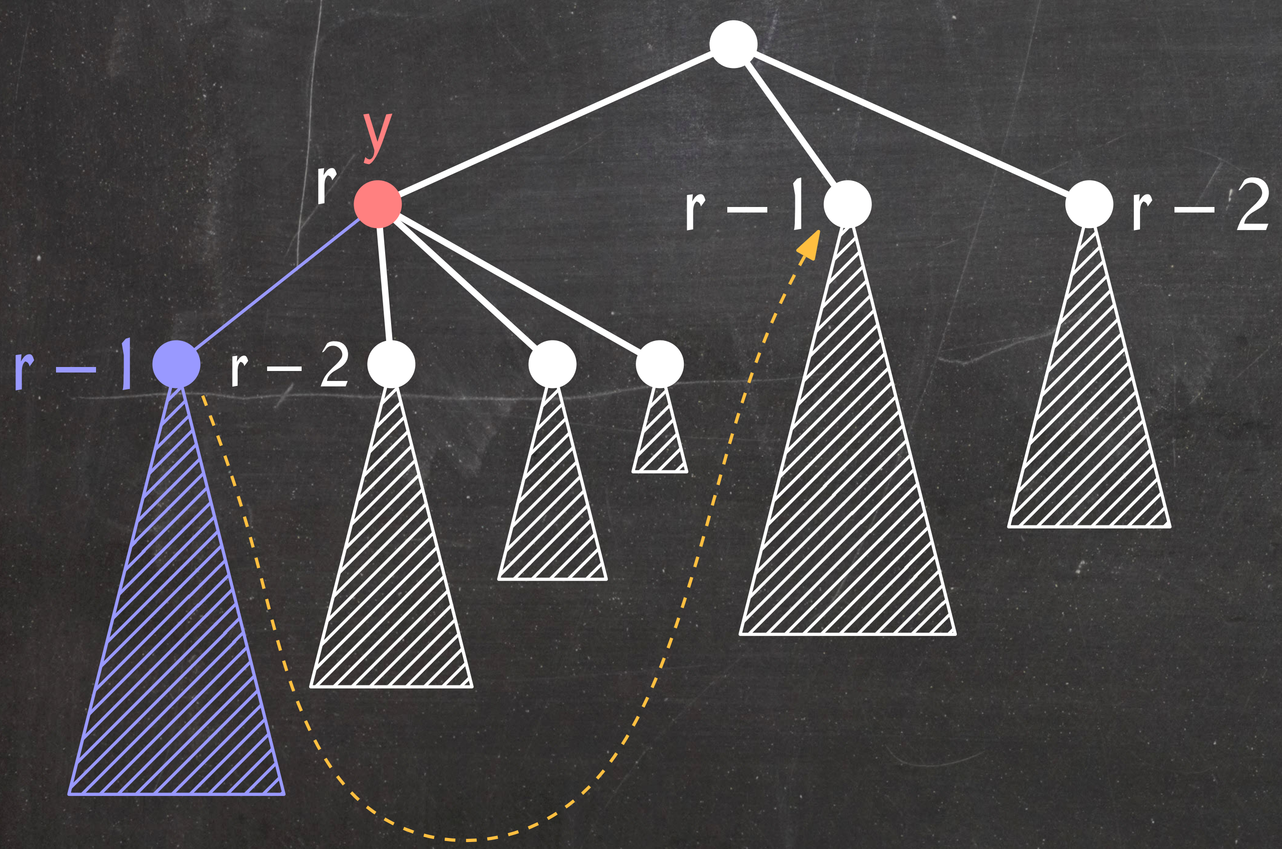


If y is thick,

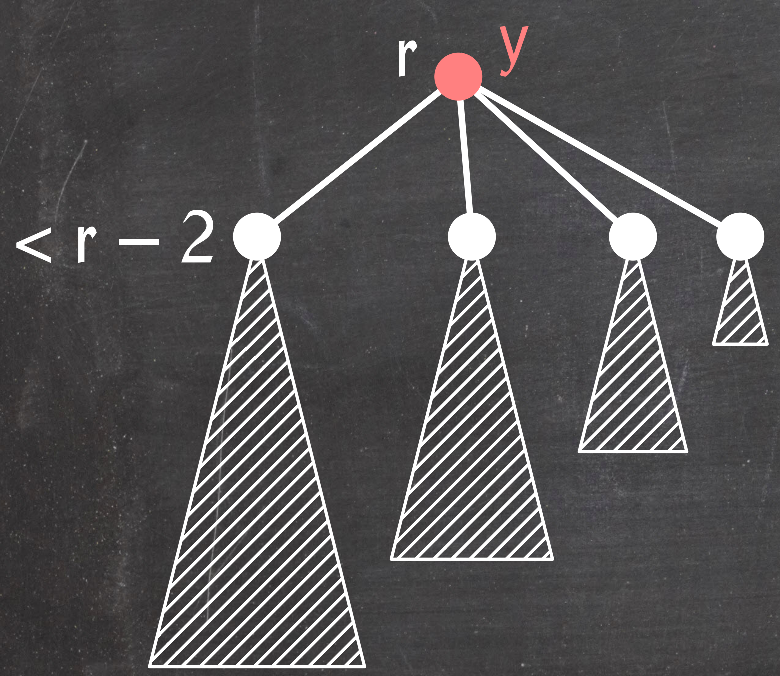
- $c \in O(1)$
- Δ(number of thin nodes) = $+1$
- Δ(number of roots) = 0

$\Rightarrow \Delta\Phi = +2$

$\Rightarrow \hat{c} \in O(1)$

After this, we're done!

Amortized Cost of Fixing Parent Violations



If y is a root, then

- $c \in O(1)$
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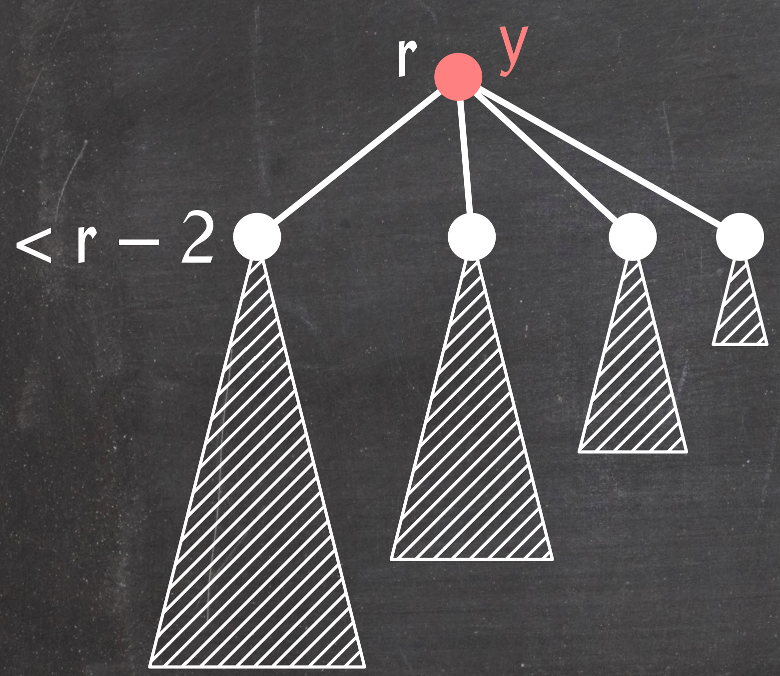
- Δ(number of thin nodes) = −1

$\Rightarrow \Delta\Phi = -2$

$\Rightarrow \hat{c} = 0$

Amortized Cost of Fixing Parent Violations



If y is a root, then

- $c \in O(1)$
- $\Delta(\text{number of roots}) = 0$
- $\Delta(\text{number of thin nodes}) = -1$

$\Rightarrow \Delta\Phi = -2$

$\Rightarrow \hat{c} = 0$

If y is not a root and is not the leftmost child of its parent, then

- $c \in O(1)$
- $\Delta(\text{number of roots}) = +1$
- $\Delta(\text{number of thin nodes}) = -1$

$\Rightarrow \Delta\Phi = -1$

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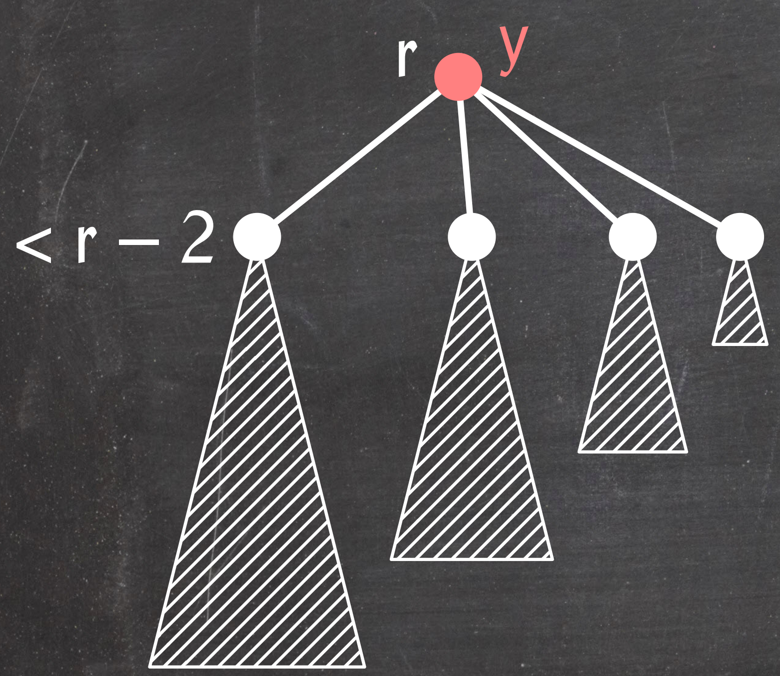

If y is not a root and is the leftmost child of its parent, and its parent is thin, then

- $c \in O(1)$
- Δ(number of roots) = +1
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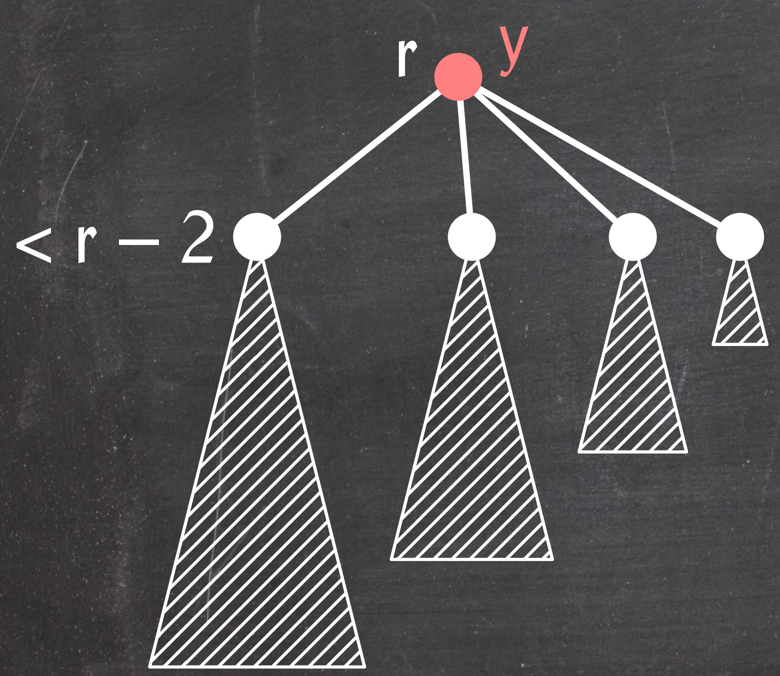

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If y is not a root and is the leftmost child of its parent, and its parent is thick, then

- $c \in O(1)$
- $\Delta(\text{number of roots}) = +1$
- $\Delta(\text{number of thin nodes}) = 0$

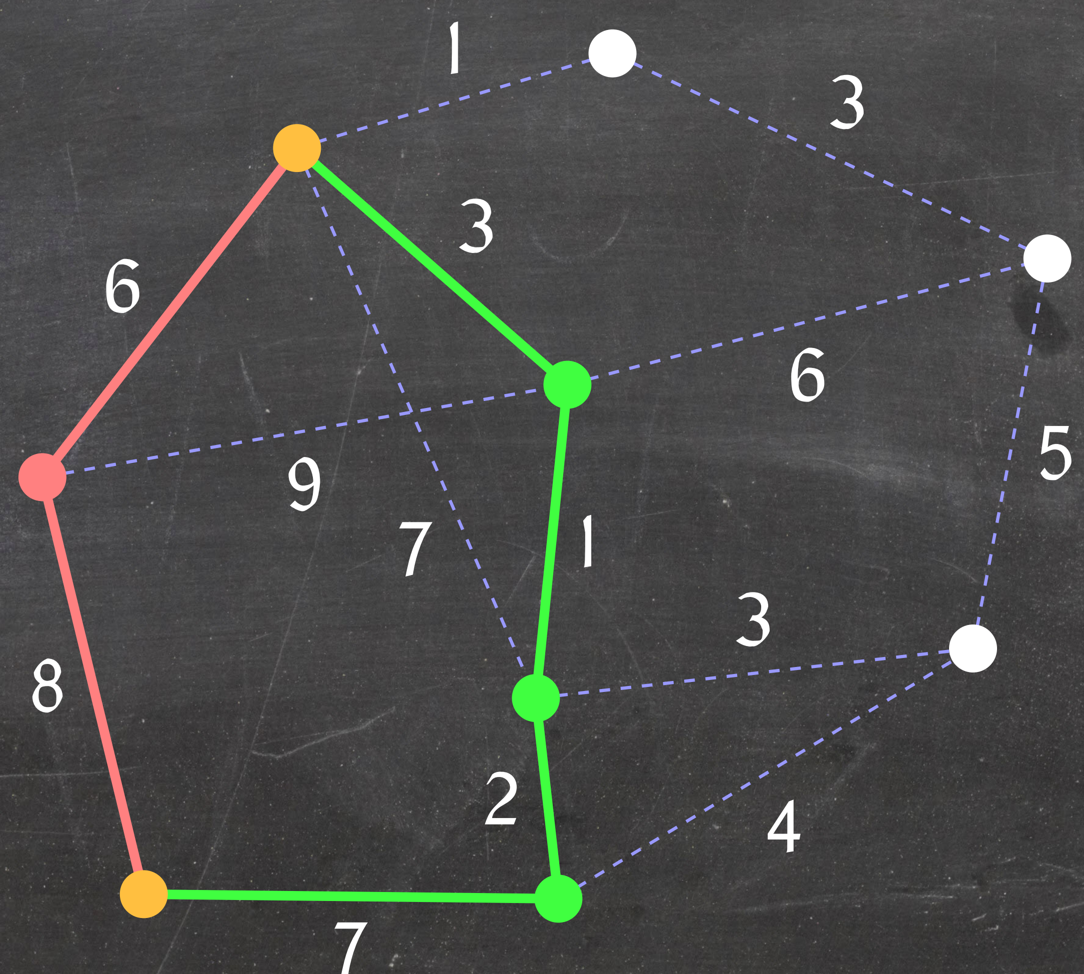
$\Rightarrow \Delta\Phi = +1$

$\Rightarrow \hat{c} \in O(1)$

After this, we're done!

Shortest Path

Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a shortest path from u to v is a path from u to v with minimum total edge weight among all paths from u to v.



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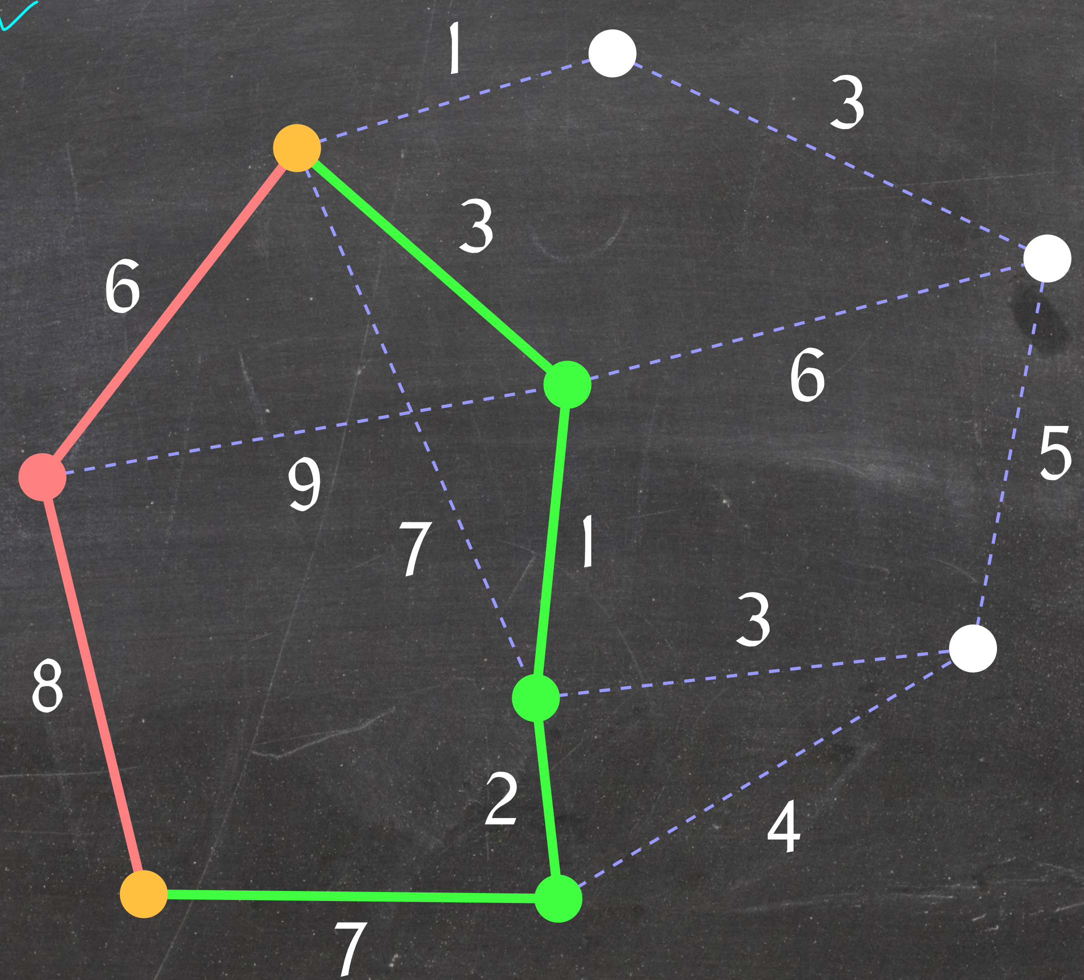

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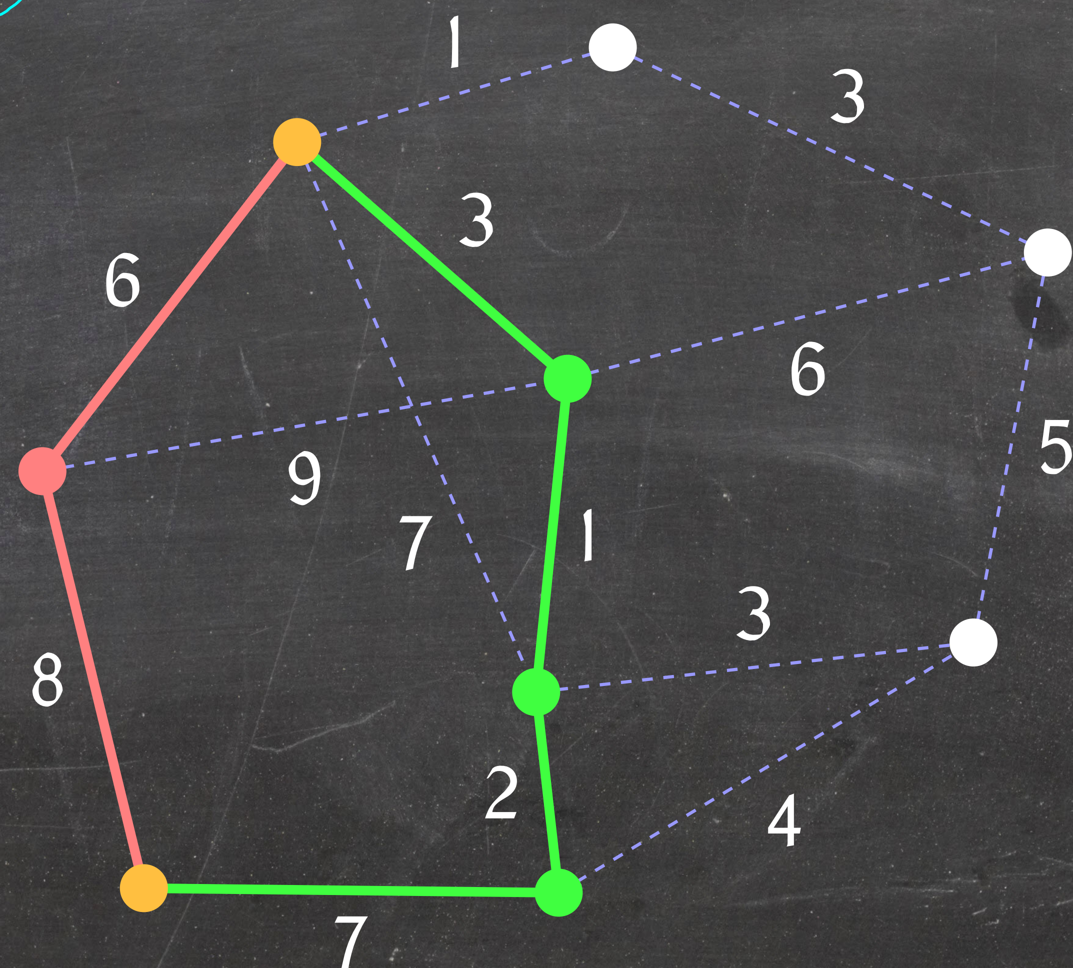

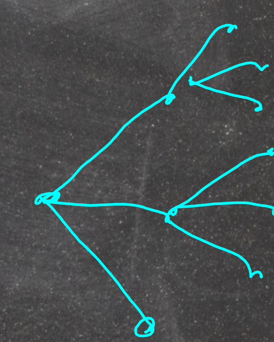

This is well-defined only if there is no negative cycle (cycle with negative total edge weight) that has a vertex on a path from u to v.

Optimal Substructure of Shortest Paths

For a path P and two vertices u and w in P, let P[u, w] be the subpath of P from u to w.

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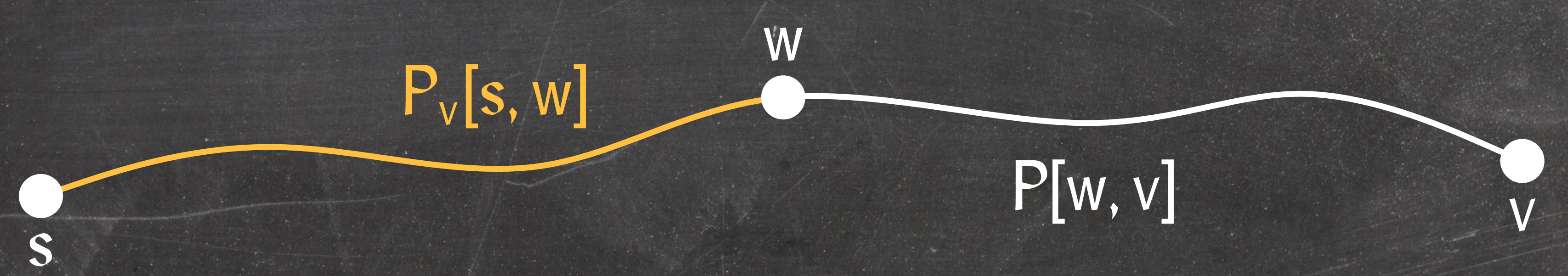

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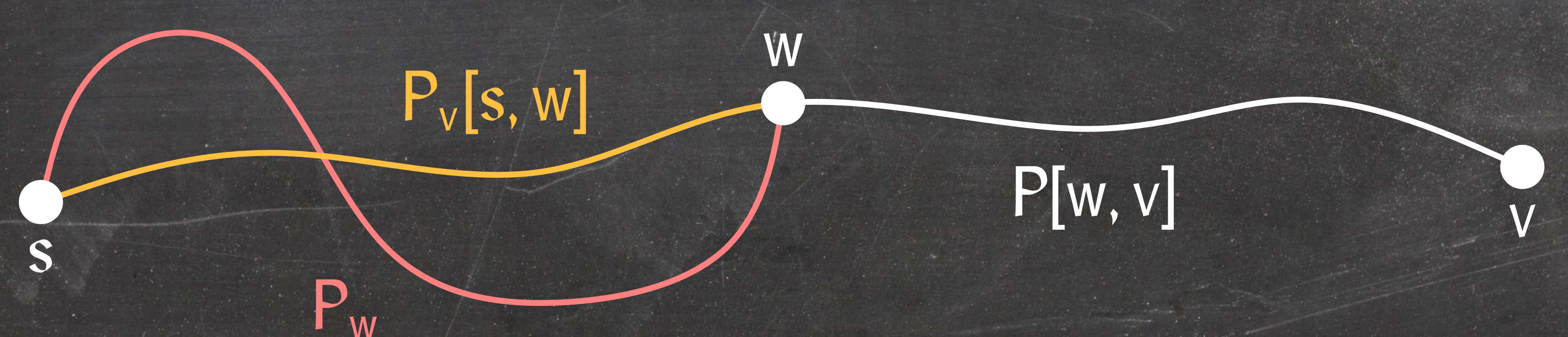

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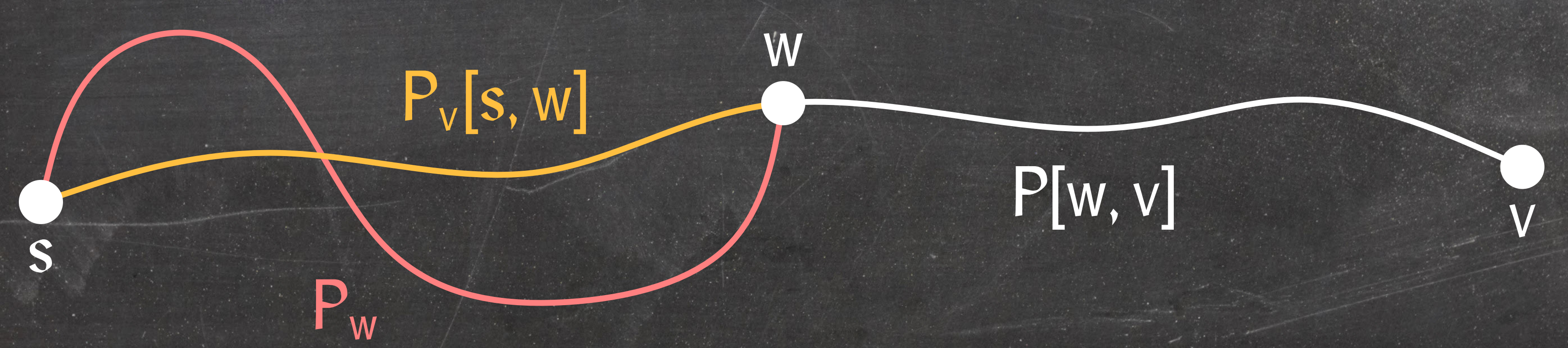

Then $w(P_w \circ P_v[w, v]) < w(P_v[s, w] \circ P_v[w, v]) = w(P_v)$, a contradiction because P_v is a shortest path from s to v.

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For a vertex $s \in G$, let $R(s)$ be the set of vertices reachable from s: for every vertex $v \in R(s)$, there exists a path from s to v.

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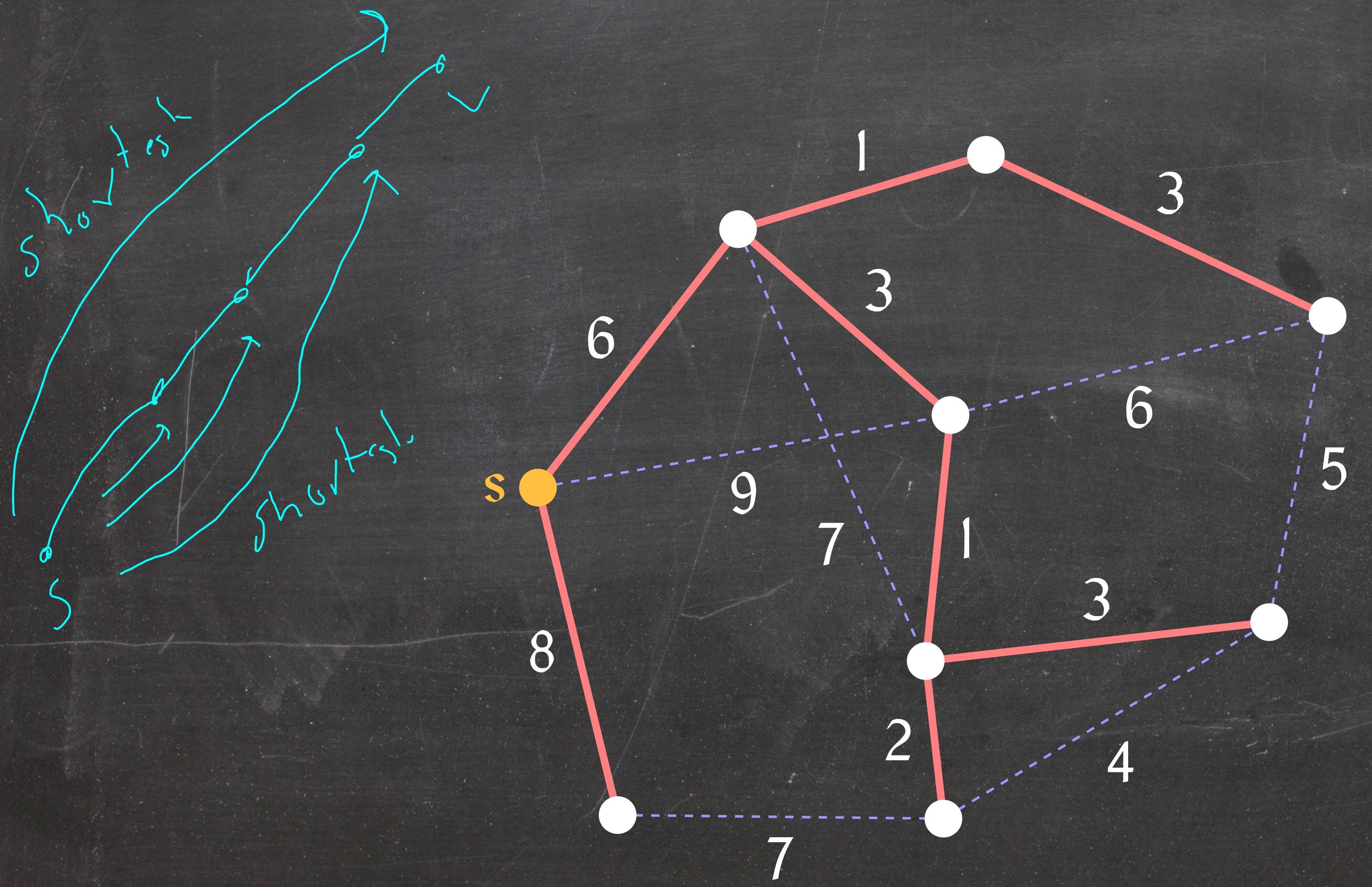

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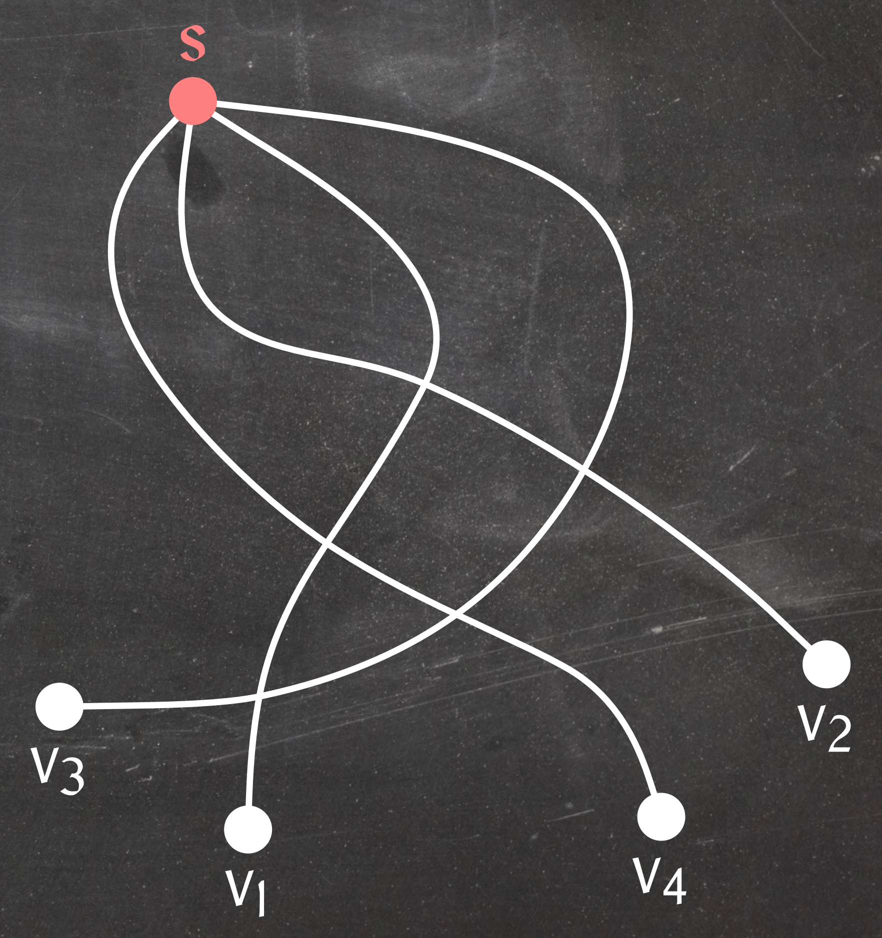

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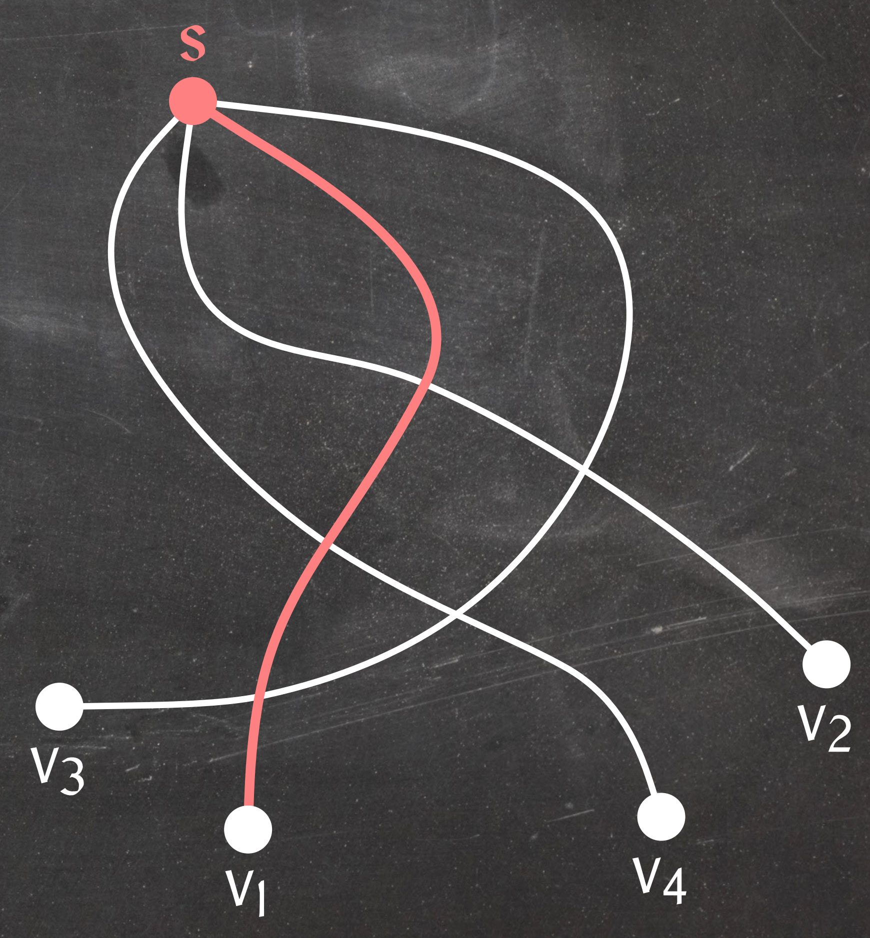

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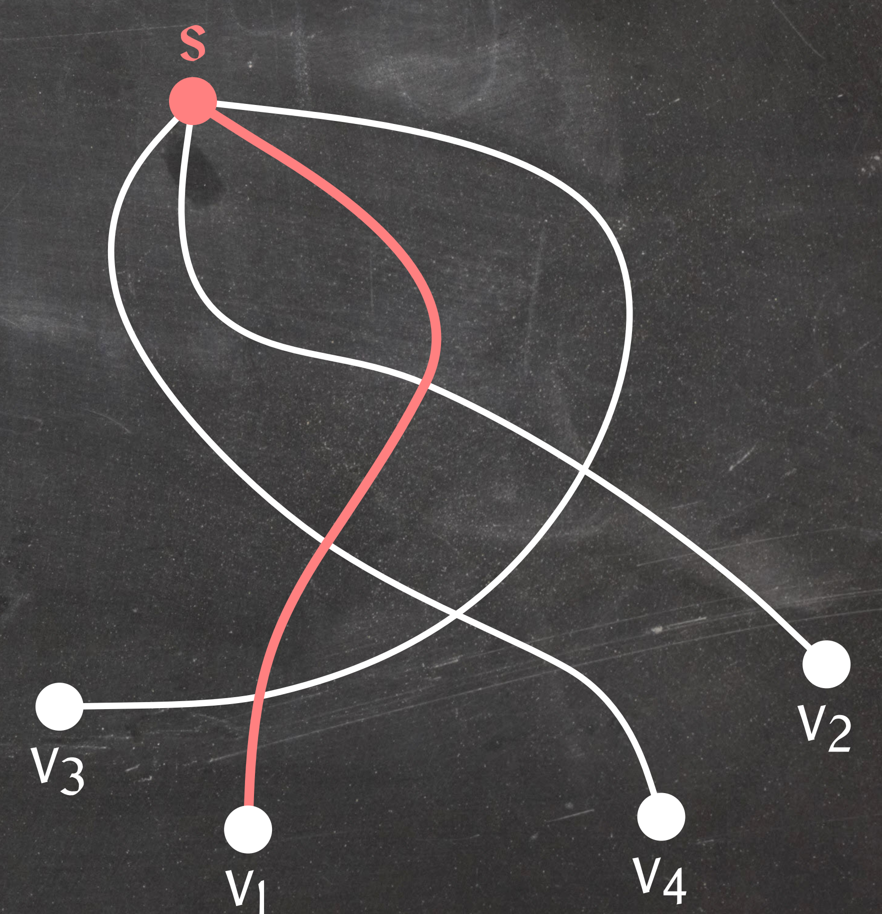

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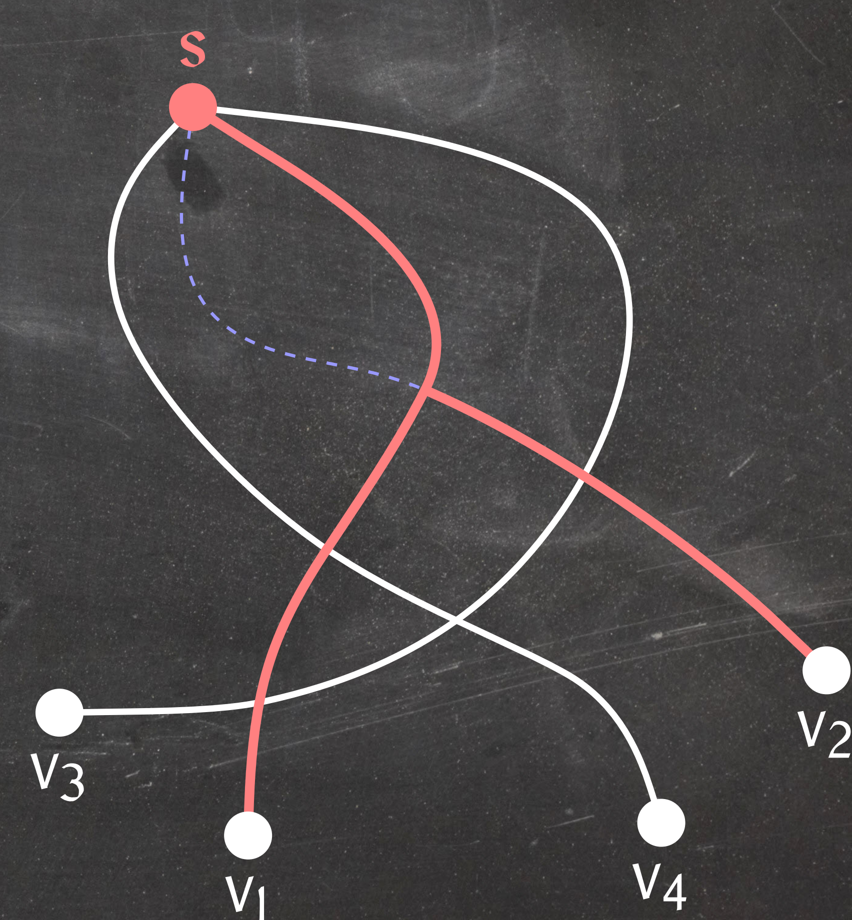

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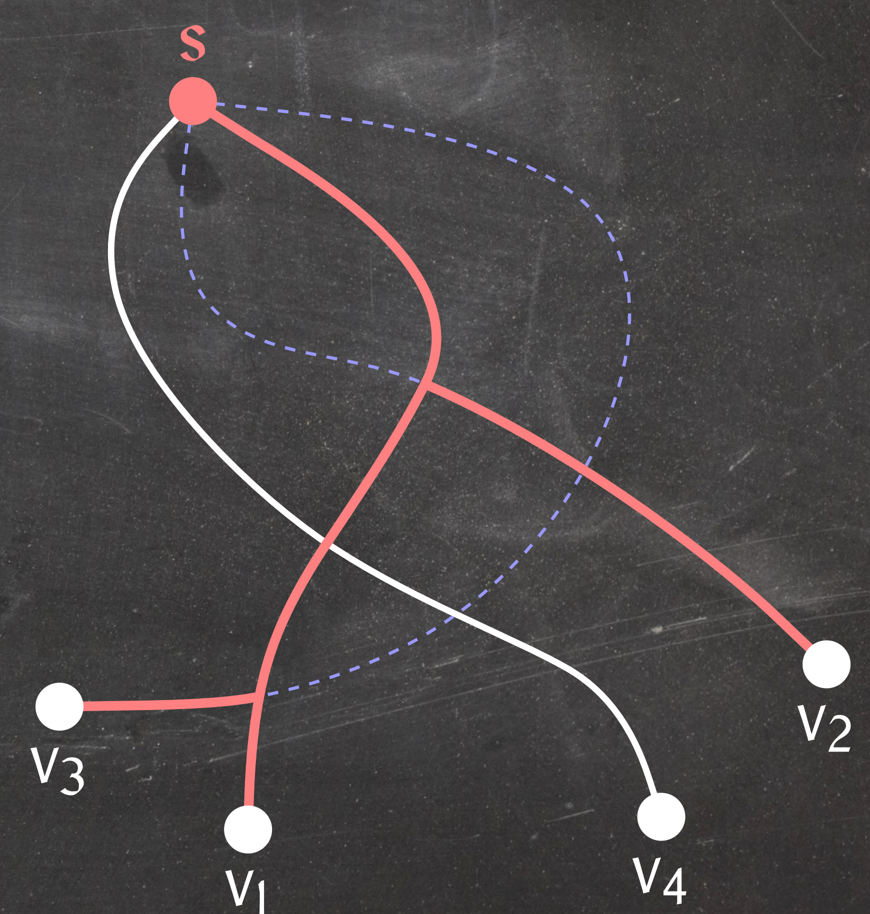

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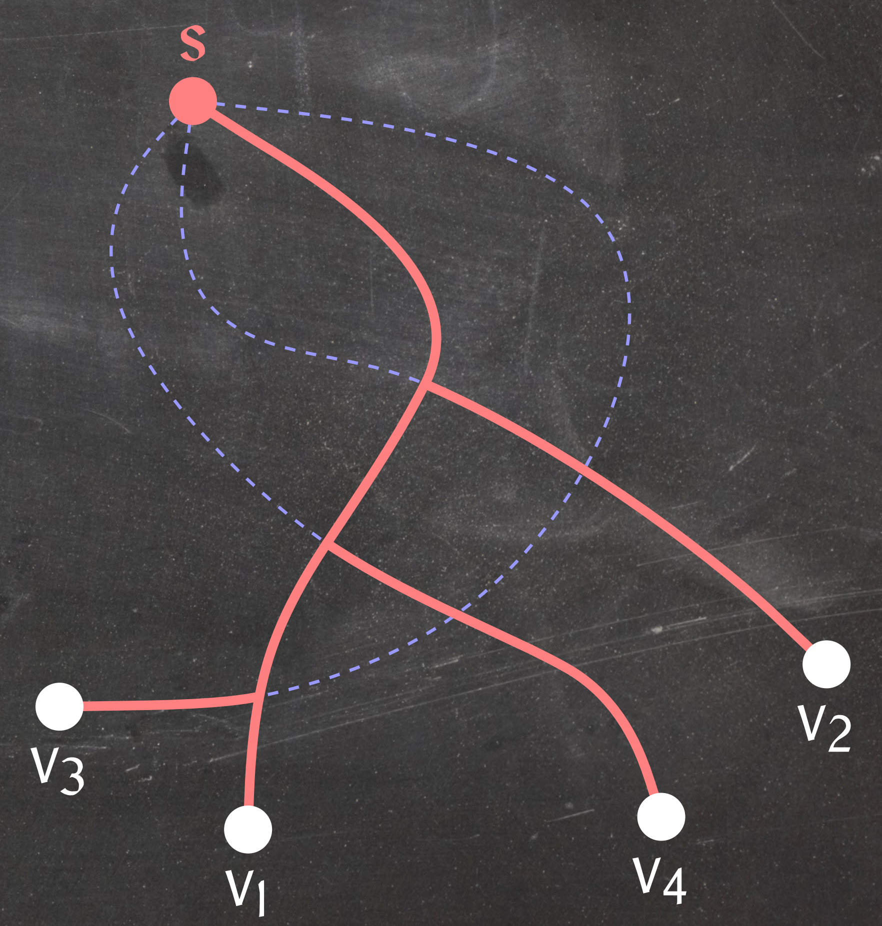

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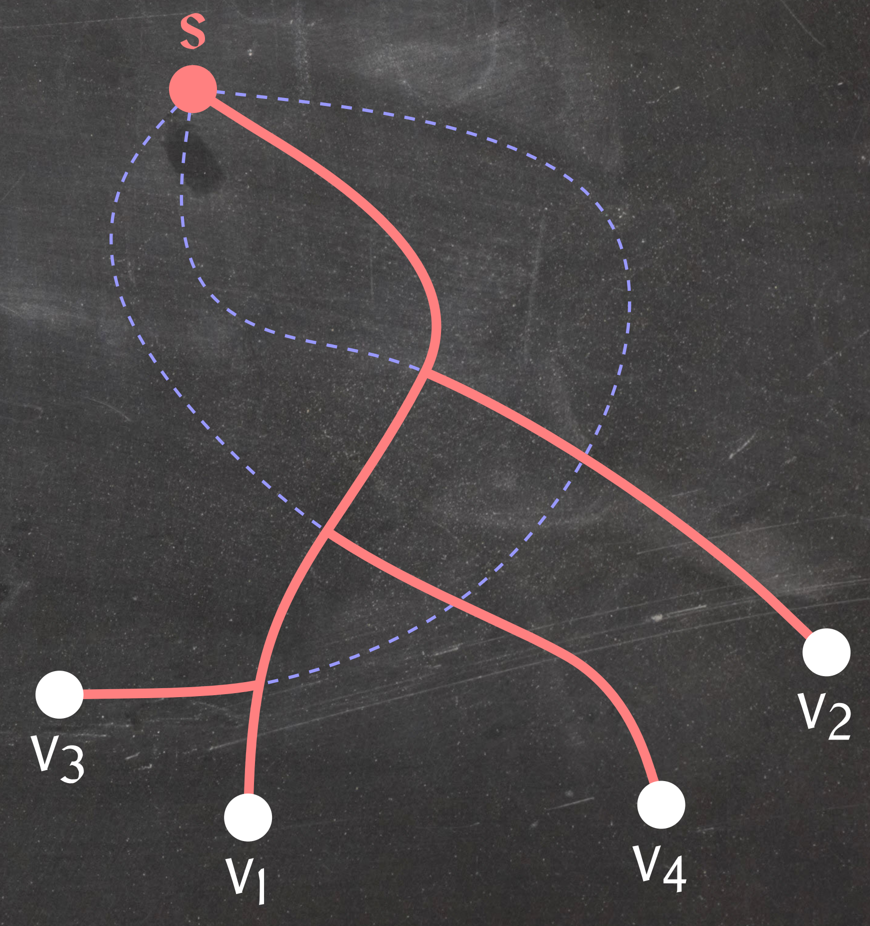

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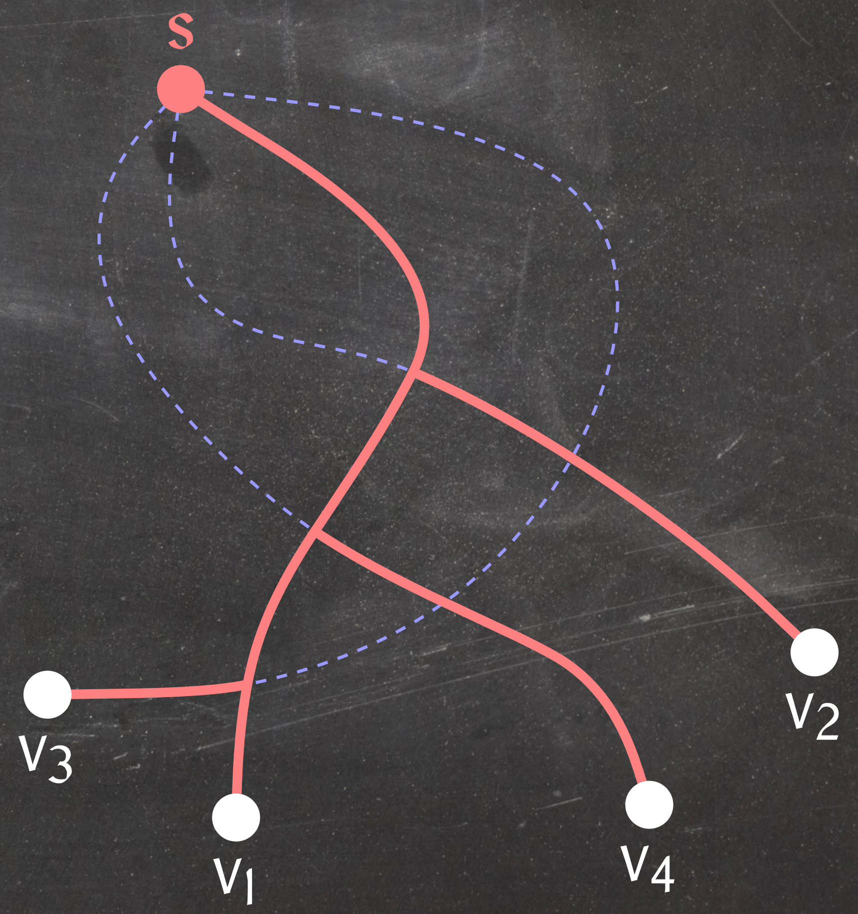
T_t is a tree:

- T_1 is a tree.
- T_i is obtained by adding a path to T_{i-1} that shares only one vertex with T_{i-1}.
- To create a cycle, the added path would have to share two vertices with T_{i-1}.



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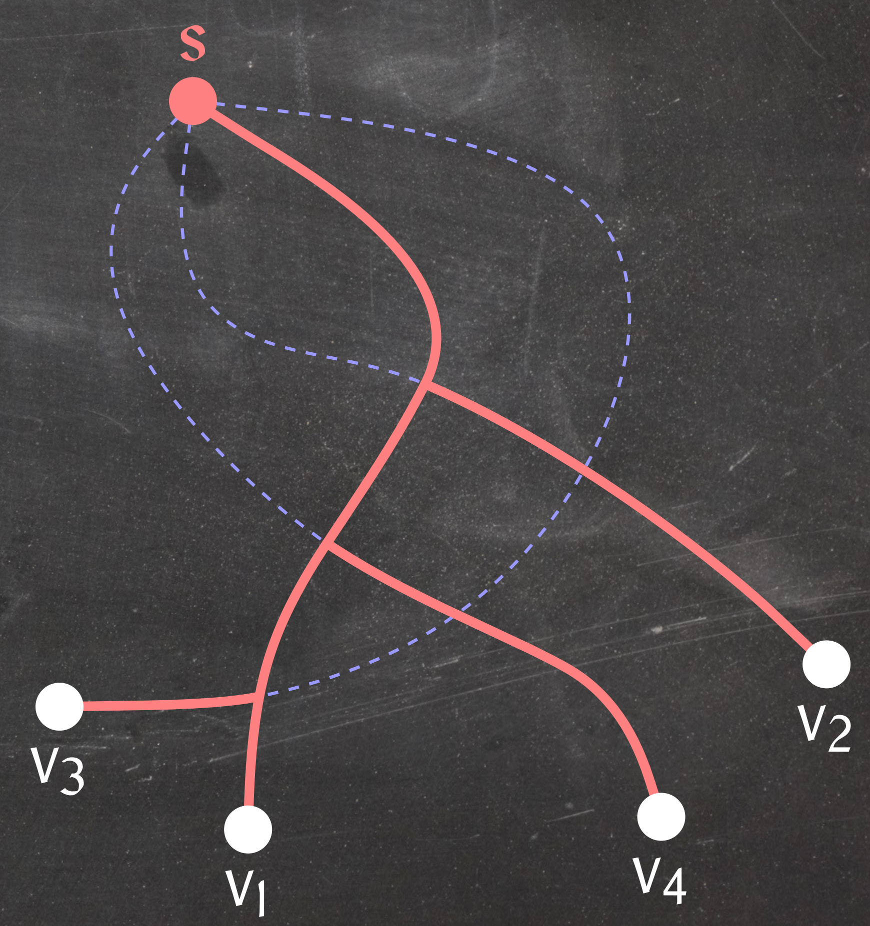

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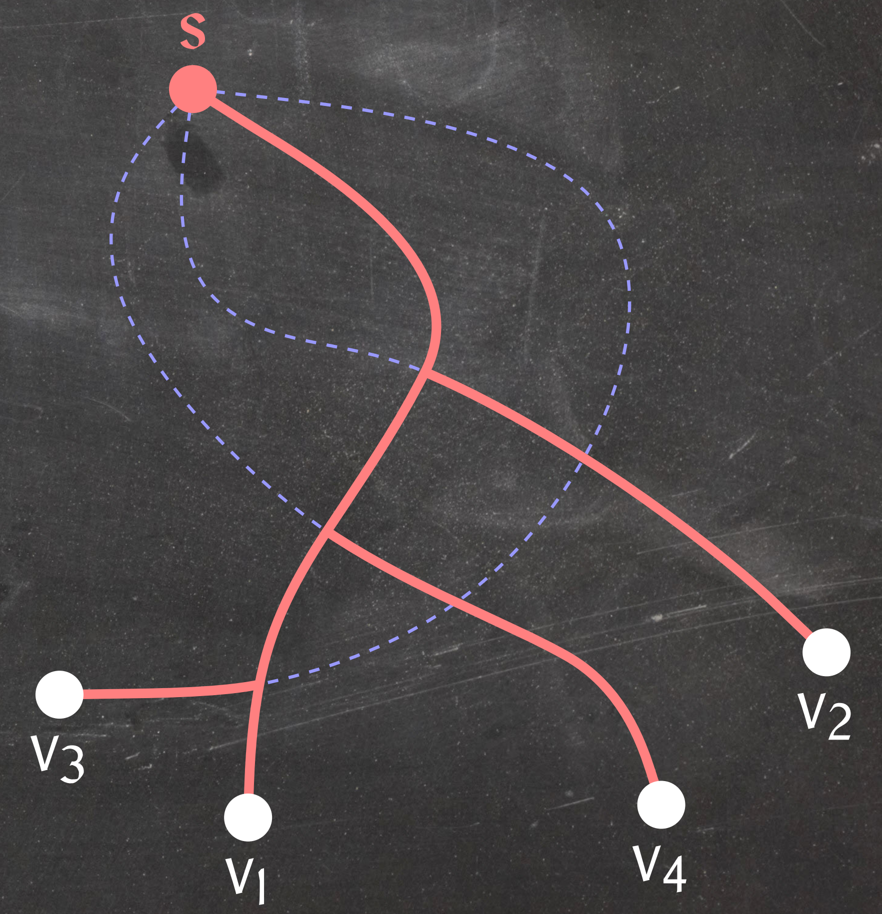
Prove by induction on i that $T_i[s, v]$ is a shortest path from s to v, for all $v \in T_i$.



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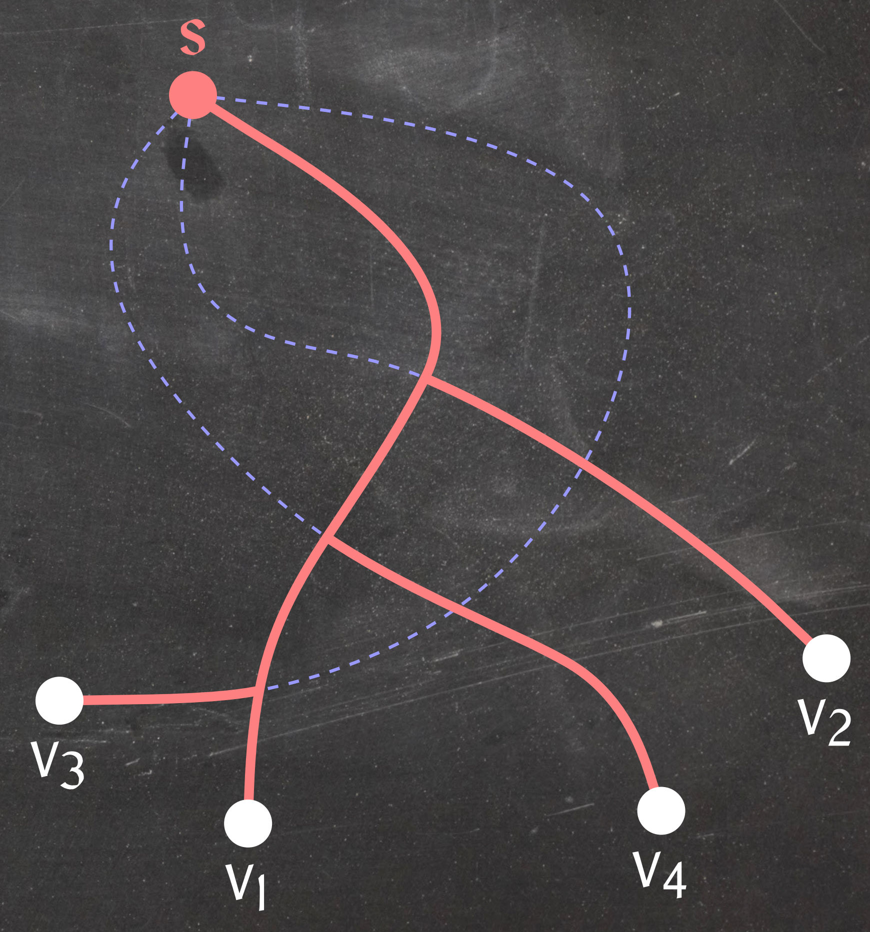

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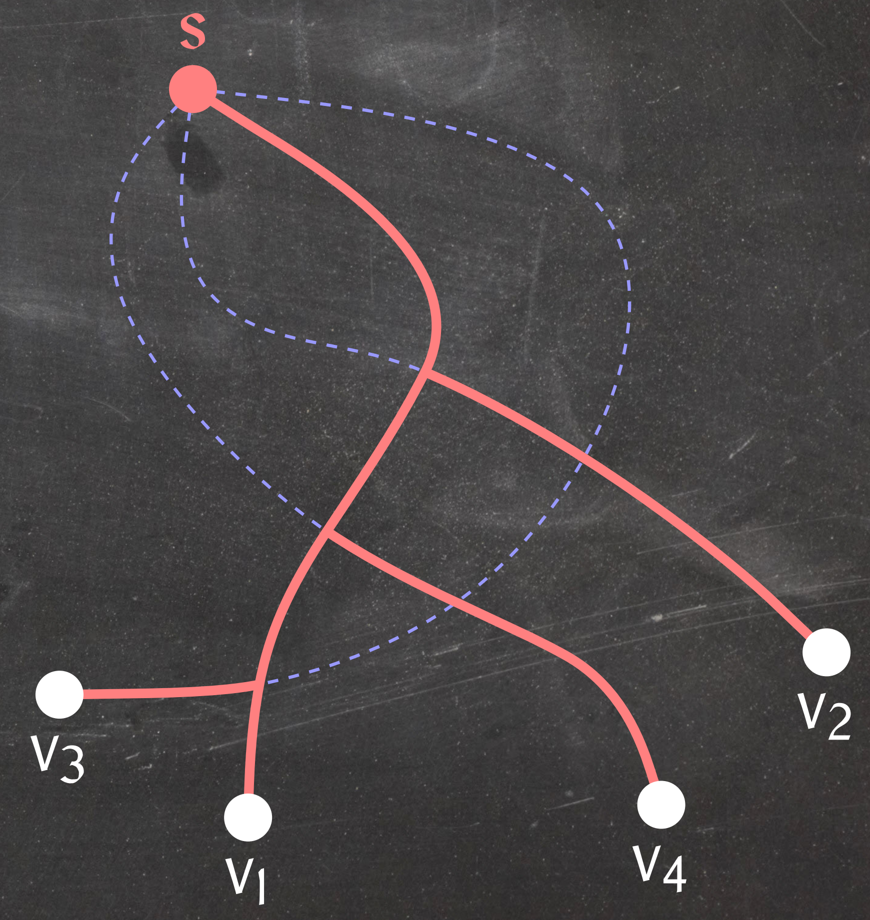

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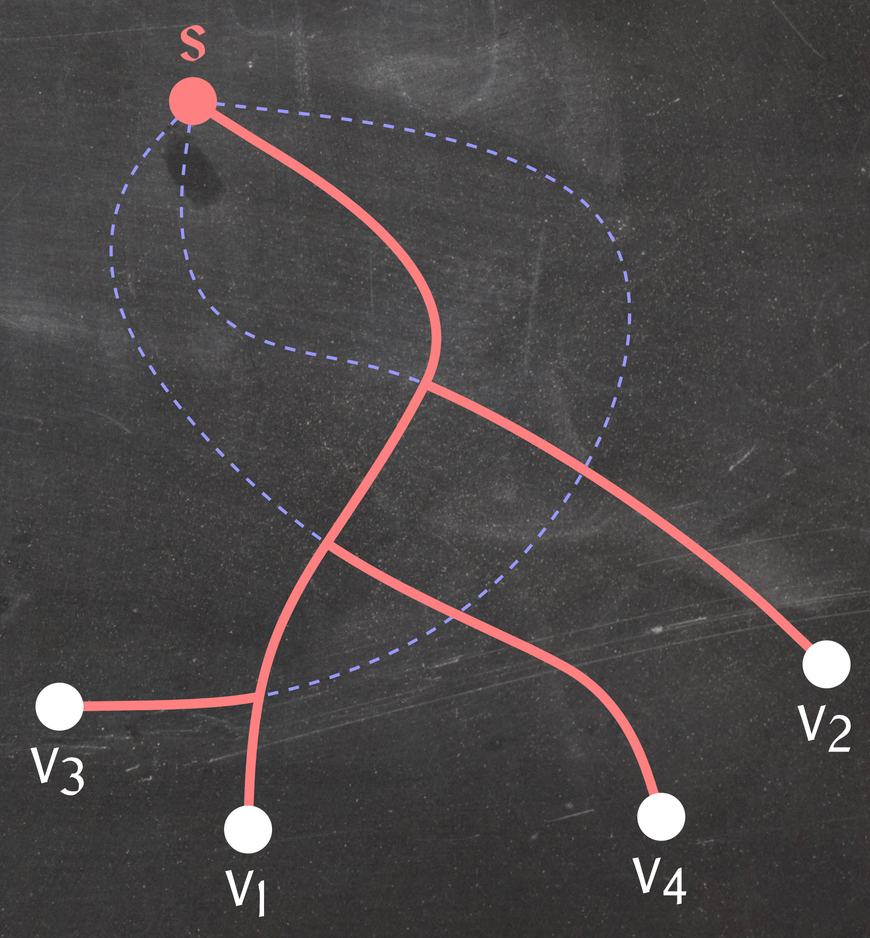

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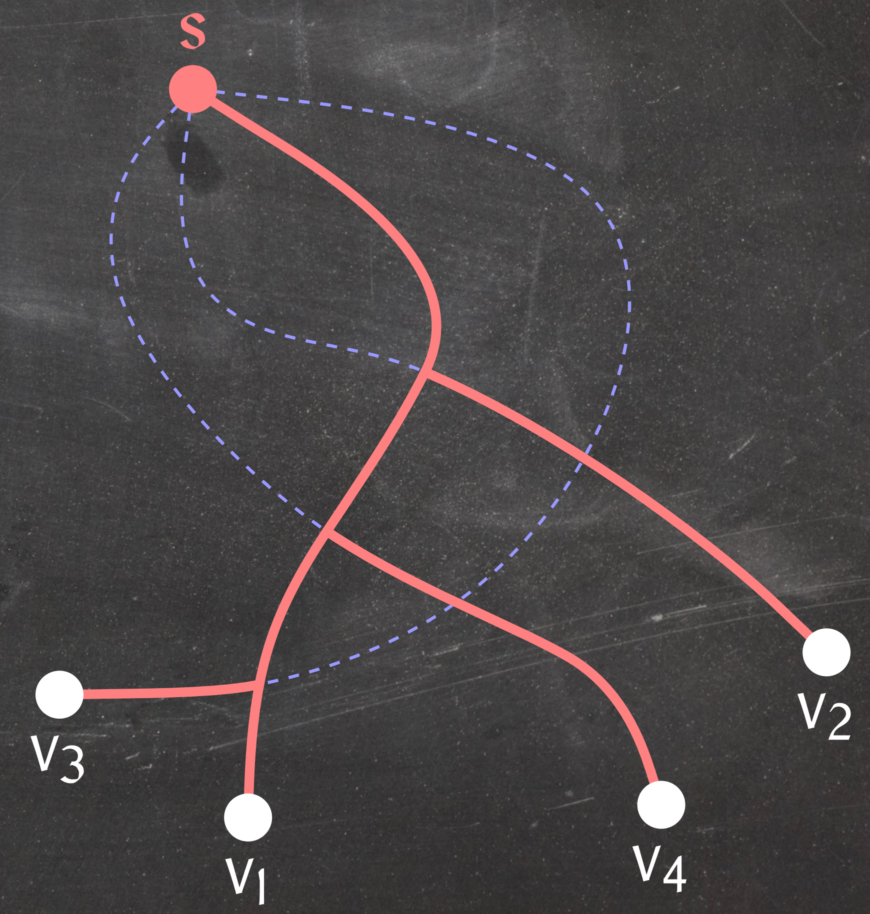
Since P'_{v_i} is a shortest path from s to v_i, so is P_{v_i}.



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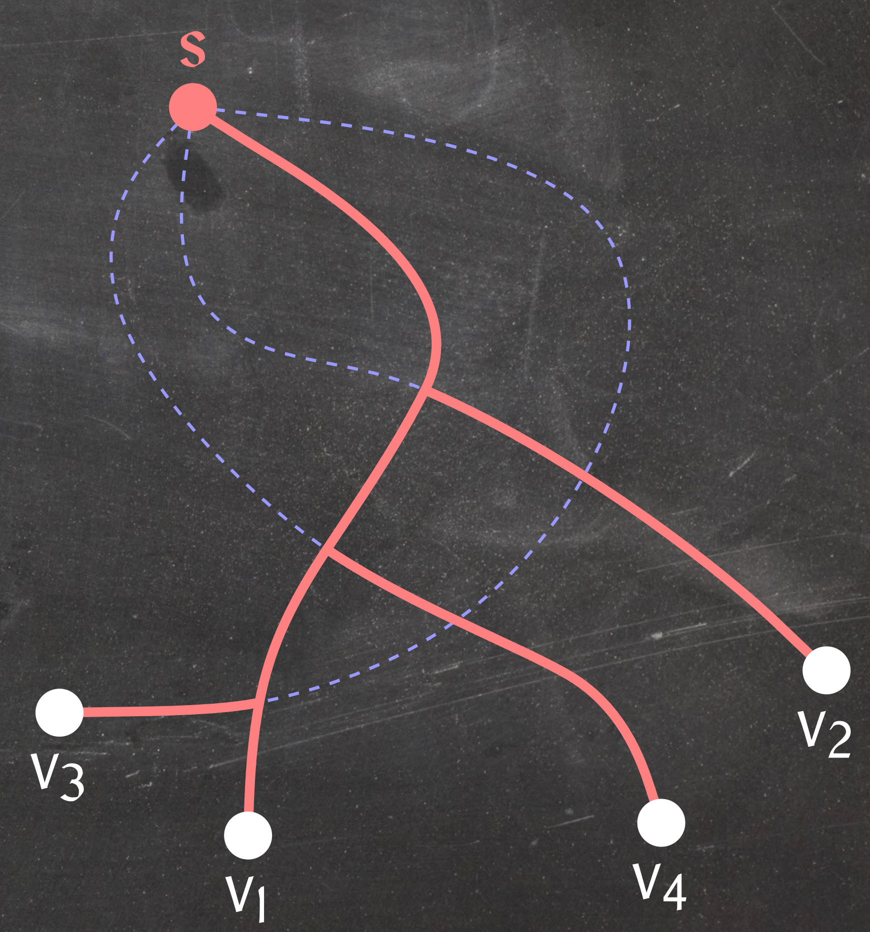
Thus, $w(T_{i-1}[s, w]) \leq w(P'_{v_i}[s, w])$ and therefore $w(P_{v_i}) = w(T_{i-1}[s, w]) + w(P'_{v_i}[w, v_i]) \leq w(P'_{v_i})$.

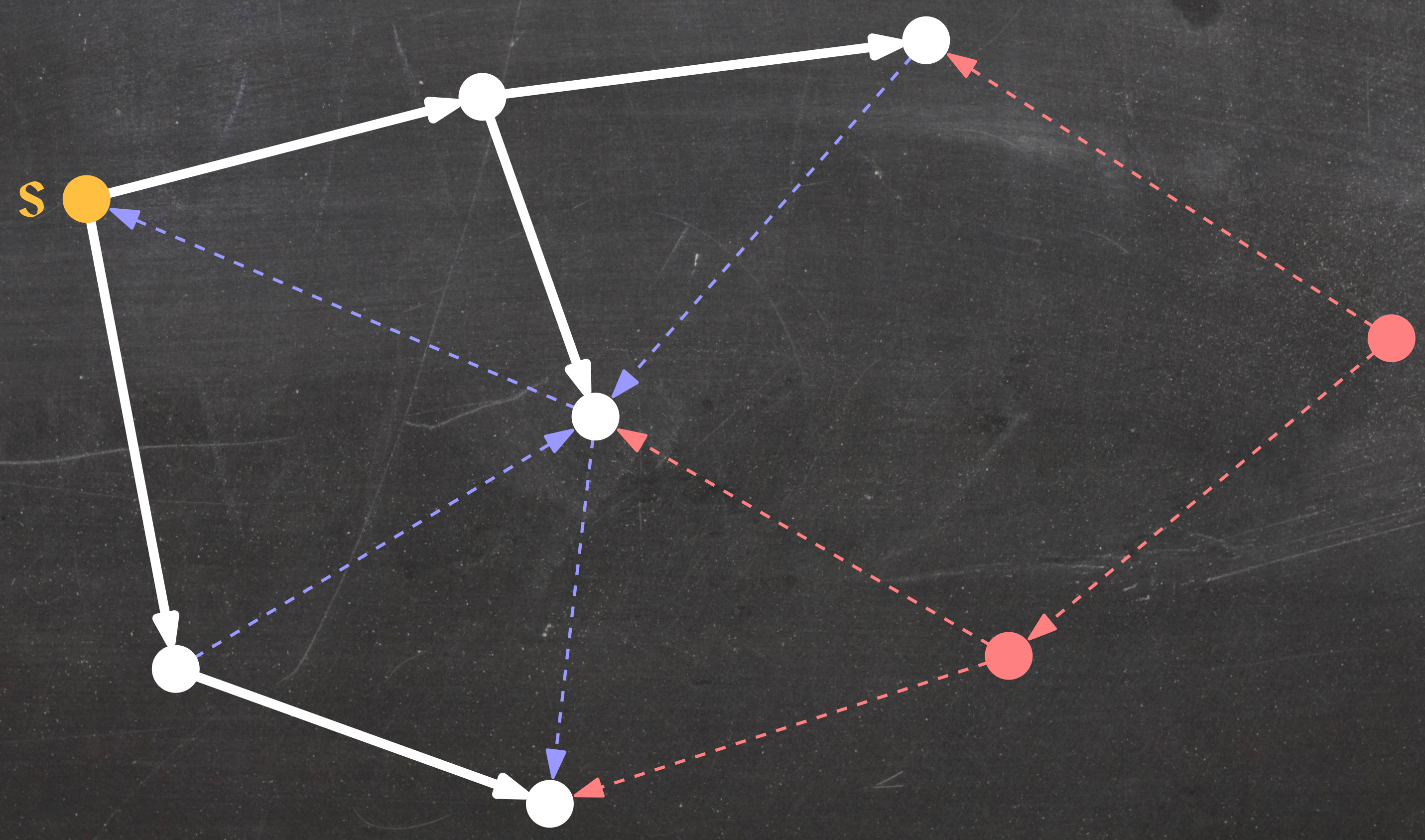
Since P'_{v_i} is a shortest path from s to v_i, so is P_{v_i}.

By optimal substructure $P_{v_i}[s, v]$ is a shortest path from s to v, for all $v \in P_{v_i}$.



A Characterization of Shortest Path Trees

An out-tree of s is a spanning tree T of G[R(s)] = (R(s), E[R(s)]), where E[R(s)] = {(v, w) ∈ E | v, w ∈ R(s)}, such that there exists a path from s to v in T, for all v ∈ R(s).



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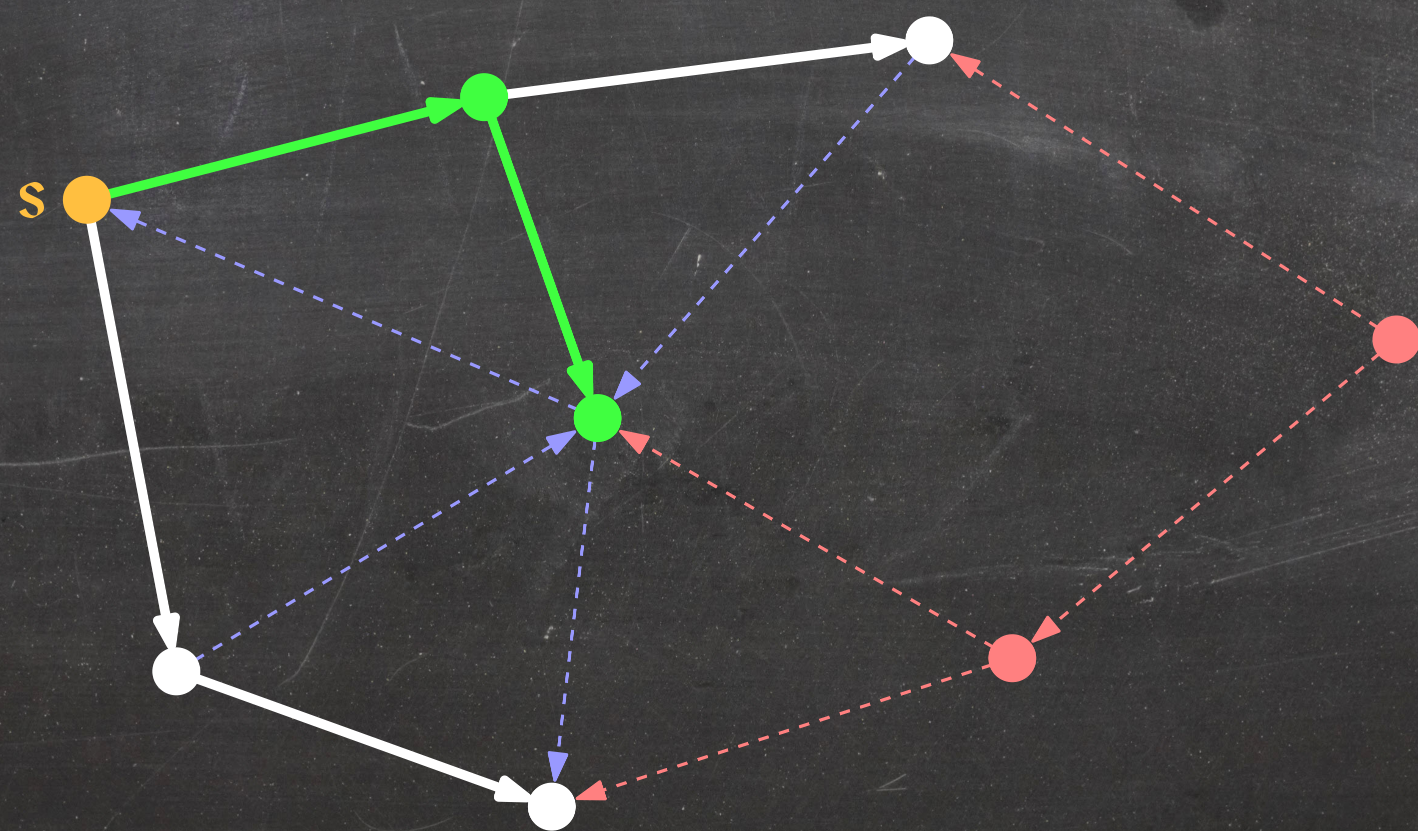

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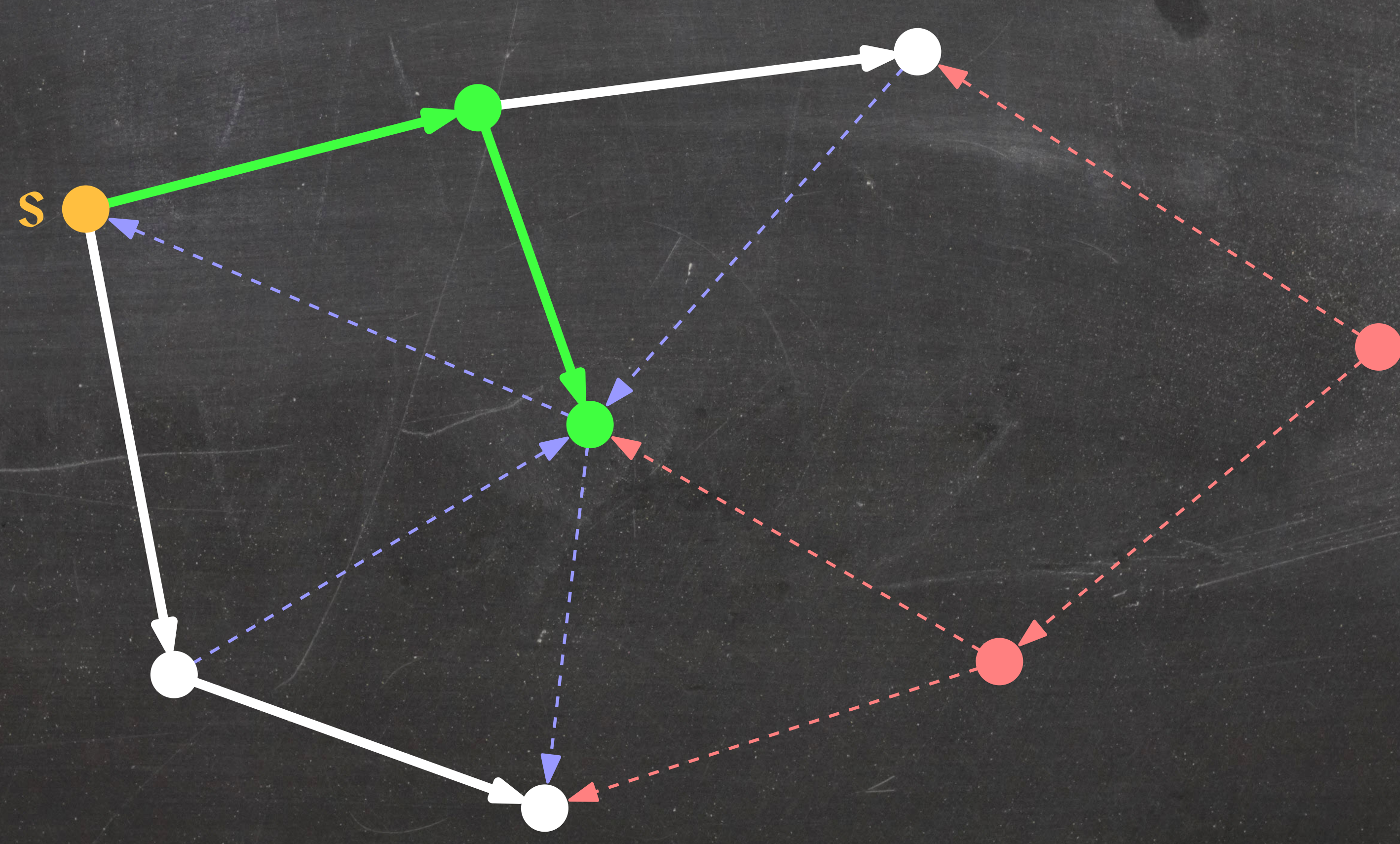

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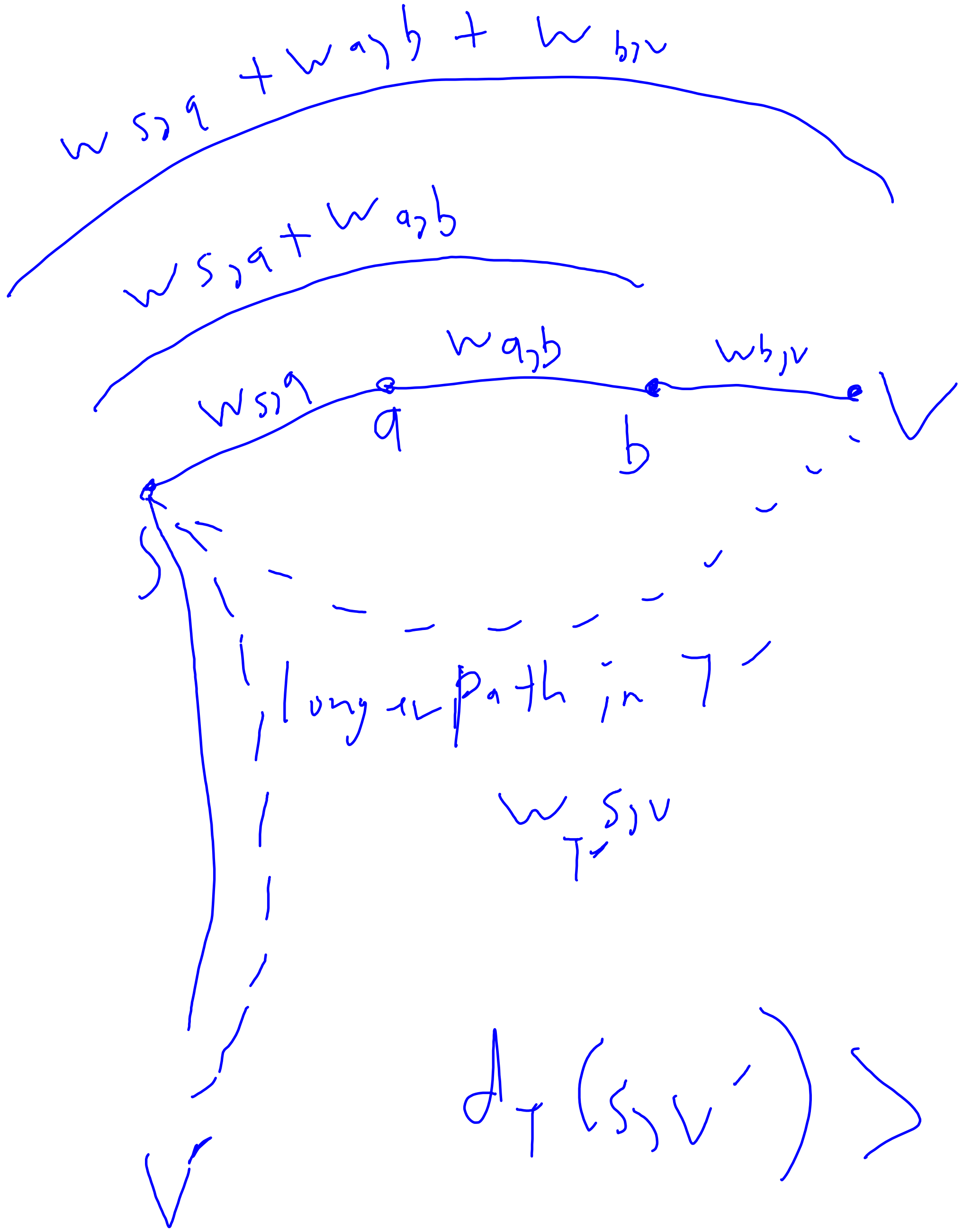
$d_T(s,w)$



$D_T = D_{T'}$

$D_T = D_{T'} - d_T(s,v) + w_{T', s,v} + \dots$

$d_T(s,v) < w_{T'}(s,v)$

weight substracting from another path

$d_T(s,v') > d_{T'}(s,v')$

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$\Rightarrow$ T' is a shortest path tree.

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Build a shortest-path tree by starting with s and adding vertices in R(s) one by one.

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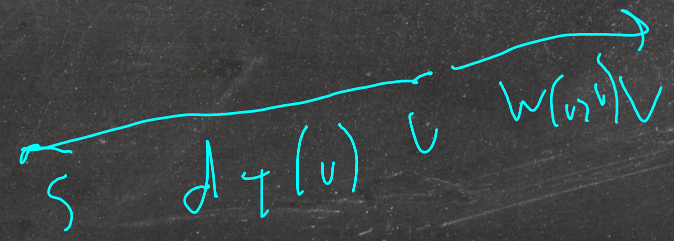
Dijkstra(G, s)

```
T = ({s}, ∅)
while some vertex in T has an out-neighbour not in T
    do choose an edge (u, v) such that
         • u ∈ T,
         • v ∉ T, and
         • d_T(u) + w(u, v) is minimized.
       add v and (u, v) to T
return T
```



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T = (V, ∅)
mark every vertex of G as unexplored
set d(v) = +∞ and e(v) = nil for every vertex v ∈ G
mark s as explored and set d(v) = 0
Q = an empty priority queue
for every edge (s, v) incident to s
    do Q.insert(v, w(s, v))
       d(v) = w(s, v)
       e(v) = (s, v)
while not Q.isEmpty()
    do u = Q.deleteMin()
       mark u as explored
       add e(u) to T
       for every edge (u, v) incident to u
           do if v is unexplored and (v ∉ Q or d(u) + w(u, v) < d(v))
                 then d(v) = d(u) + w(u, v)
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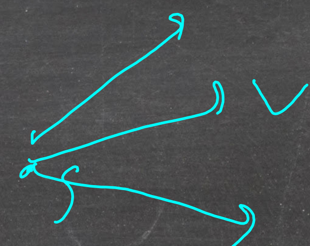
d(s) = 0



insert v into Q with weight w(s,v)

d(v) = w(s,v)
best known way to v

e(v) = (s,v)
best candidate edge to v

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$\Rightarrow$ Dijkstra's algorithm takes $O(n \lg n + m)$ time.

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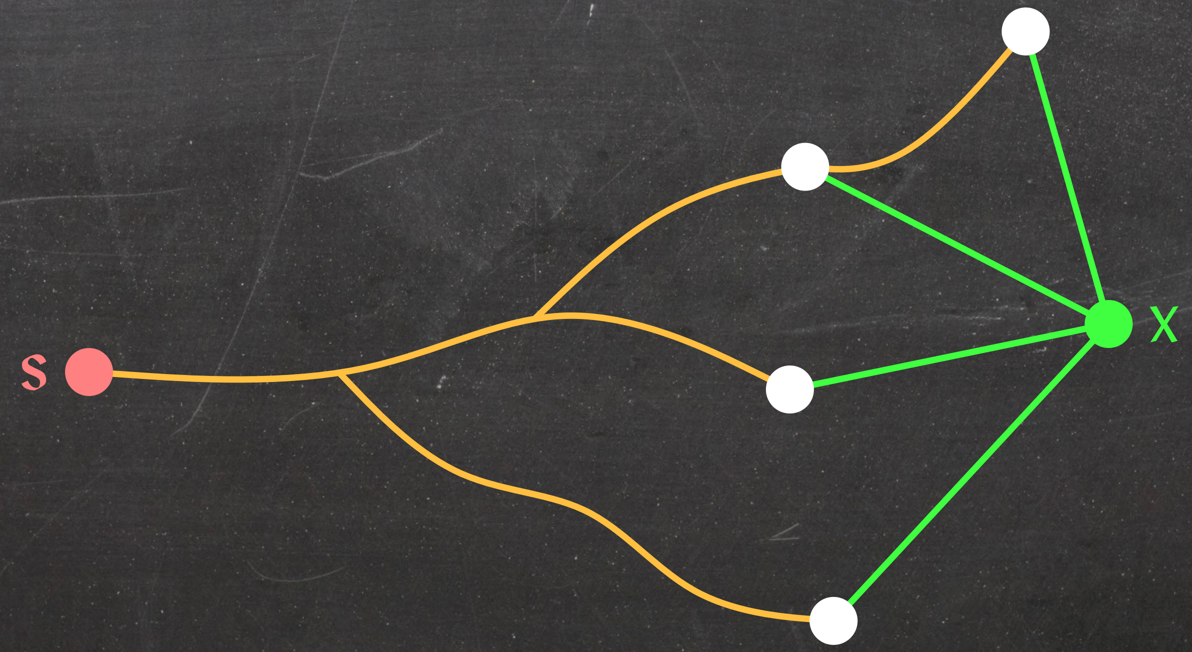
For every vertex $x \notin T$, we have

$$d(x) = \min_{\substack{(u,x) \in E \\ u \in T}} d(u) + w(u, x) = \min_{\substack{(u,x) \in E \\ u \in T}} \mathrm{dist}(s, u) + w(u, x).$$



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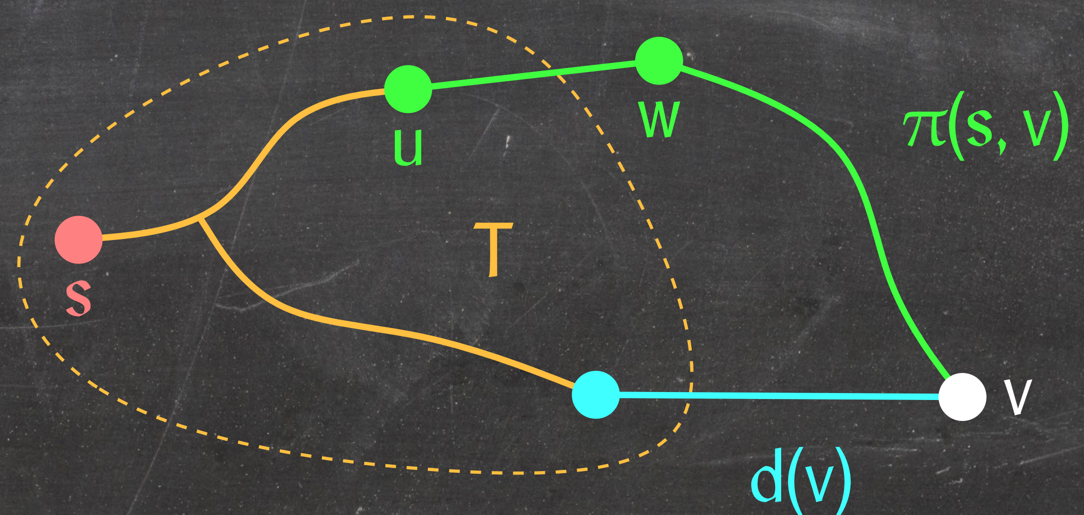
The shortest path $\pi(s, v)$ from s to v must include a vertex $w \notin T$ whose predecessor u in $\pi(s, v)$ belongs to T.



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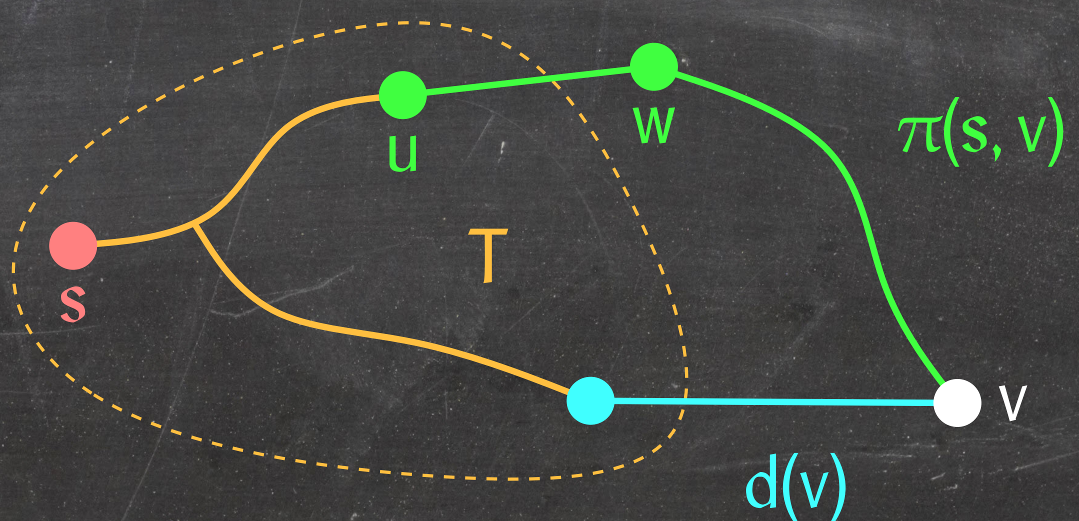

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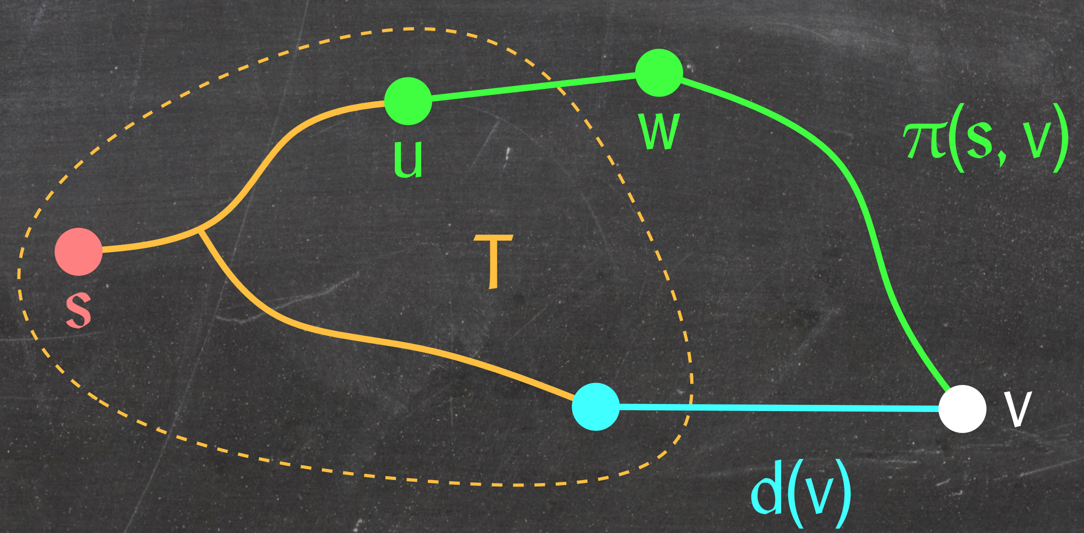

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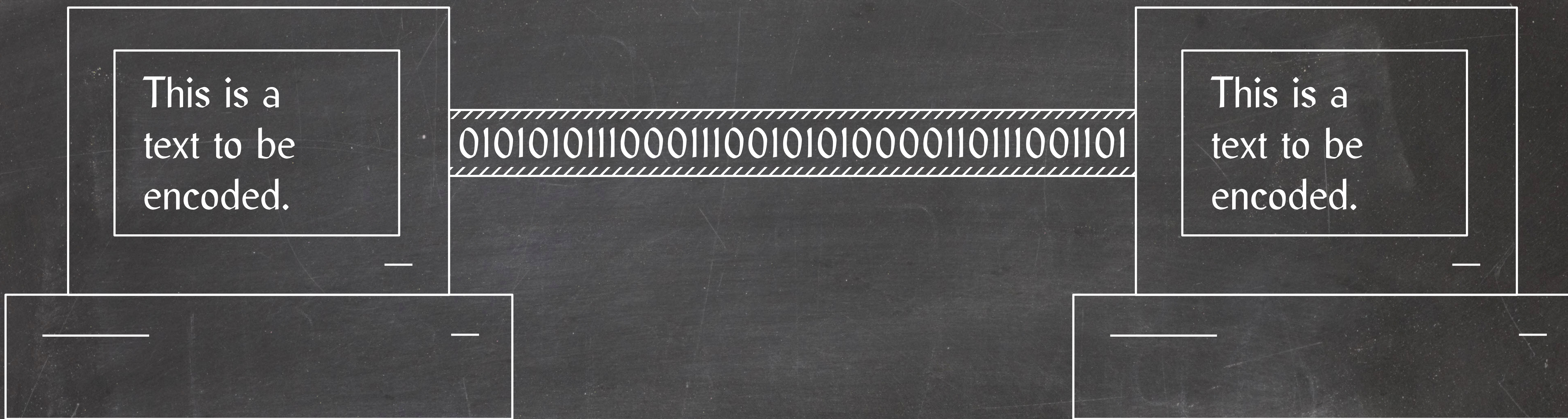
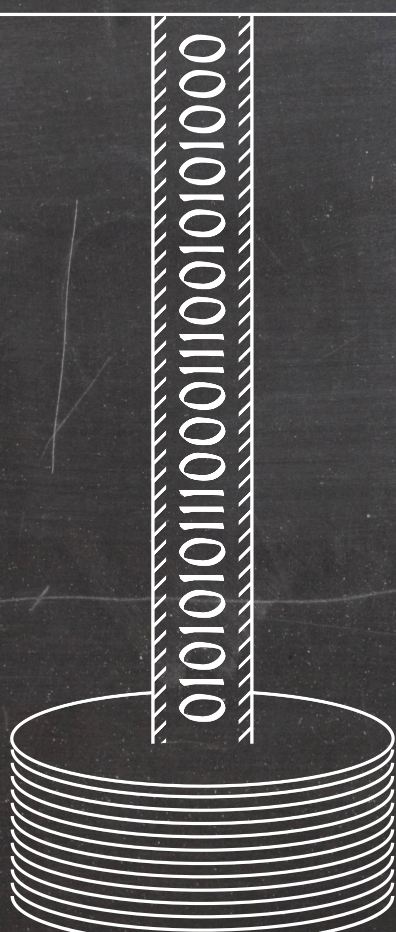
$\Rightarrow$ v is not the next vertex we add to T, a contradiction.

Minimum Length Codes





Goal:

- Encode a given text using as few bits as possible:
  - Limit amount of disk space required to store the text.
  - Send the text over a potentially slow network.
  - ...

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C_1(prefix-free) = 011 100 000 001 010 101 110 001 100 000 000 (33 bits)

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C_3	0	1	00	01	10	11	000

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	e	f	i	p	r	x	-
C_1	000	001	010	011	100	101	110
C_2	00	010	0110	0111	10	110	111
C_3	0	1	00	01	10	11	000

C_1(prefix-free) = 011 100 000 001 010 101 110 001 100 000 000 (33 bits)

C_2(prefix-free) = 0111 10 00 010 0110 110 111 010 10 00 00 (30 bits)

C_3(prefix-free) = 01 10 0 1 00 11 000 1 10 0 0 (18 bits)

A code $C(\cdot)$ is prefix-free if there are no two characters x and y such that $C(x)$ is a prefix of $C(y)$.

Codes That Can Be Decoded

A code is a mapping $C(\cdot)$ that maps every character x to a bit string $C(x)$, called the encoding of x.

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Non-prefix-free codes cannot always be decoded uniquely!

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$C(T)$

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$\Rightarrow$ $C(\langle x_1, x_2, \ldots, x_{i-1} \rangle) = C(\langle y_1, y_2, \ldots, y_{i-1} \rangle)$ and
$C(\langle x_i, x_{i+1}, \ldots, x_m \rangle) = C(\langle y_i, y_{i+1}, \ldots, y_n \rangle)$.

$C(T)$	$C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$	$C(x_i)$	$C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$
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$$\Rightarrow \quad C(\langle x_1, x_2, \ldots, x_{i-1} \rangle) = C(\langle y_1, y_2, \ldots, y_{i-1} \rangle) \text{ and}$$
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Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

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Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

Since both $C(x_i)$ and $C(y_i)$ are prefixes of $C(\langle x_i, x_{i+1}, \ldots, x_m \rangle)$, $C(x_i)$ must be a prefix of $C(y_i)$, a contradiction.

$C(T)$	$C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$	$C(x_i)$	$C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$
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Prefix Codes and Binary Trees

Observation: Every prefix-free code C(·) can be represented as a binary tree $\mathcal{T}_C$ whose leaves correspond to the letters in the alphabet.

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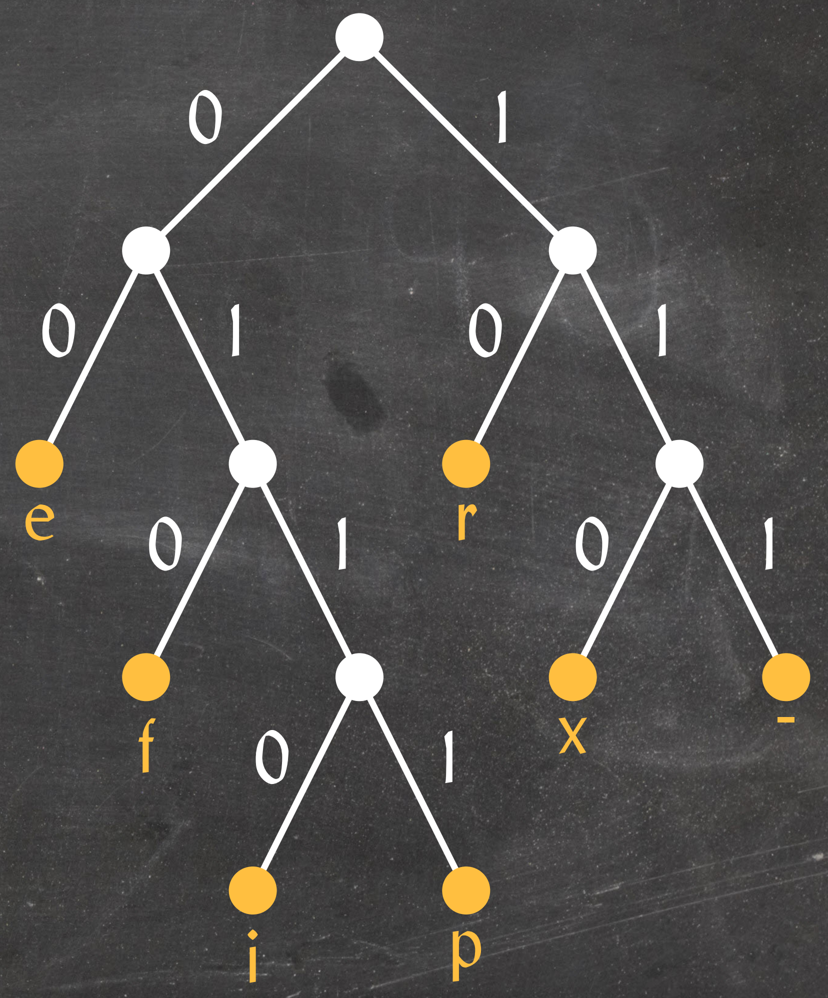

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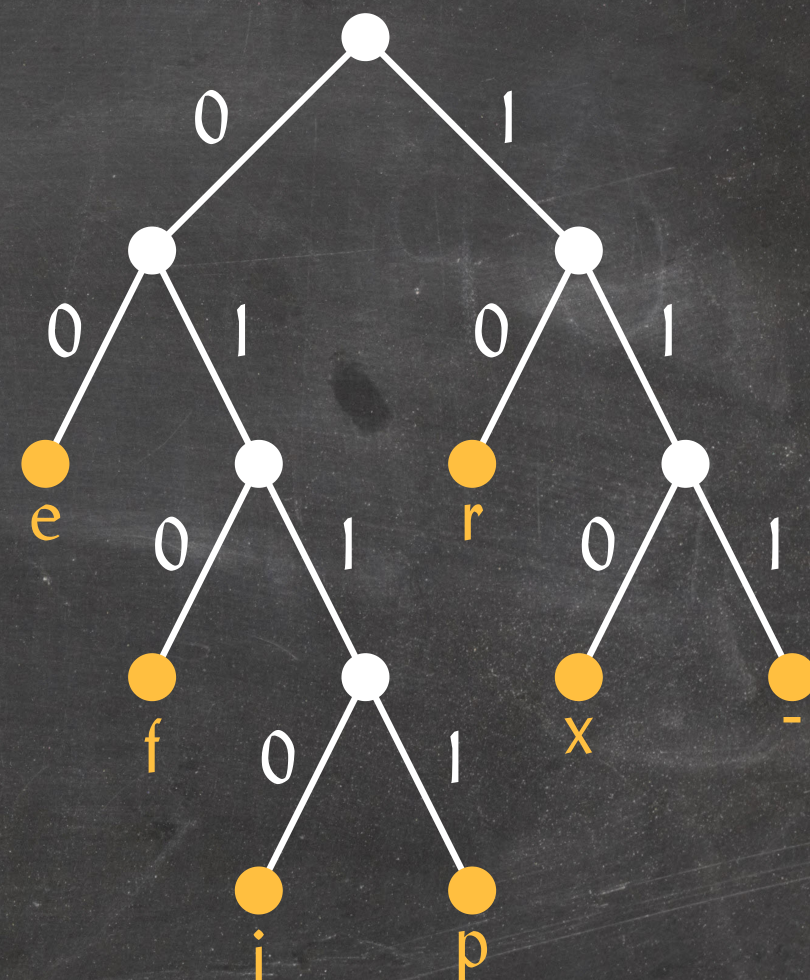

The depth of character x in $\mathcal{T}_C$ is the number of bits $|C(x)|$ used to encode x using $C(\cdot)$.

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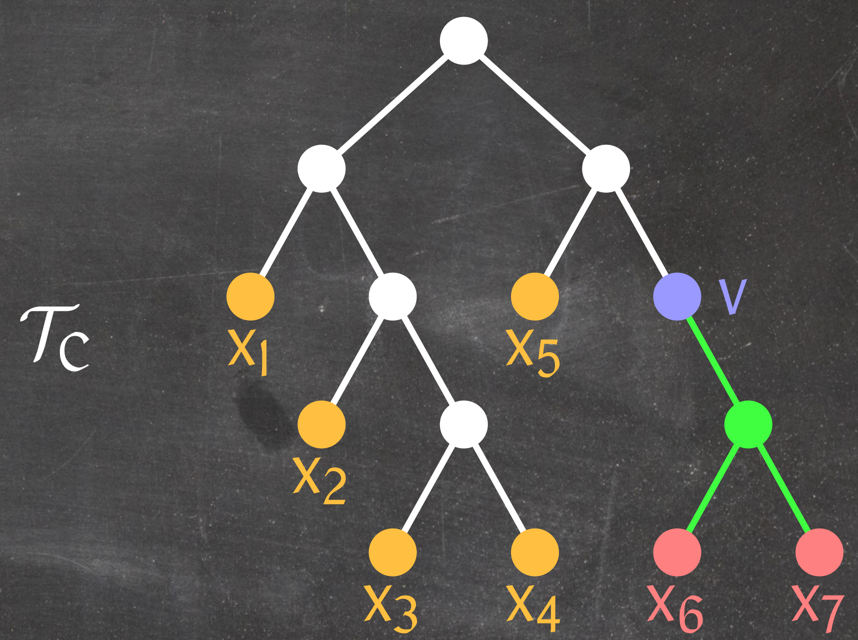

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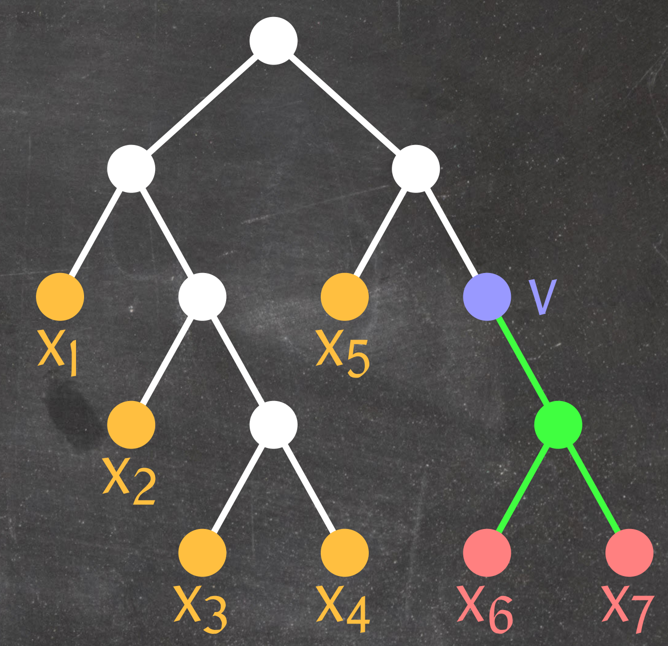
Choose C(·) so that $\mathcal{T}_C$ has as few internal nodes with only one child as possible among all optimal prefix-free codes for T.

$\mathcal{T}_C$



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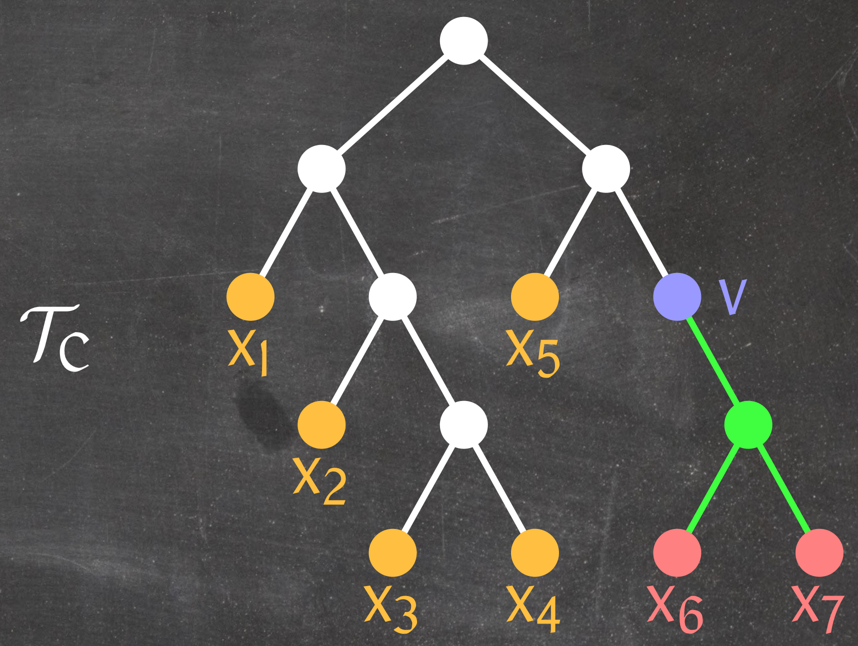
If $\mathcal{T}_C$ has no internal node with only one child, the lemma holds.



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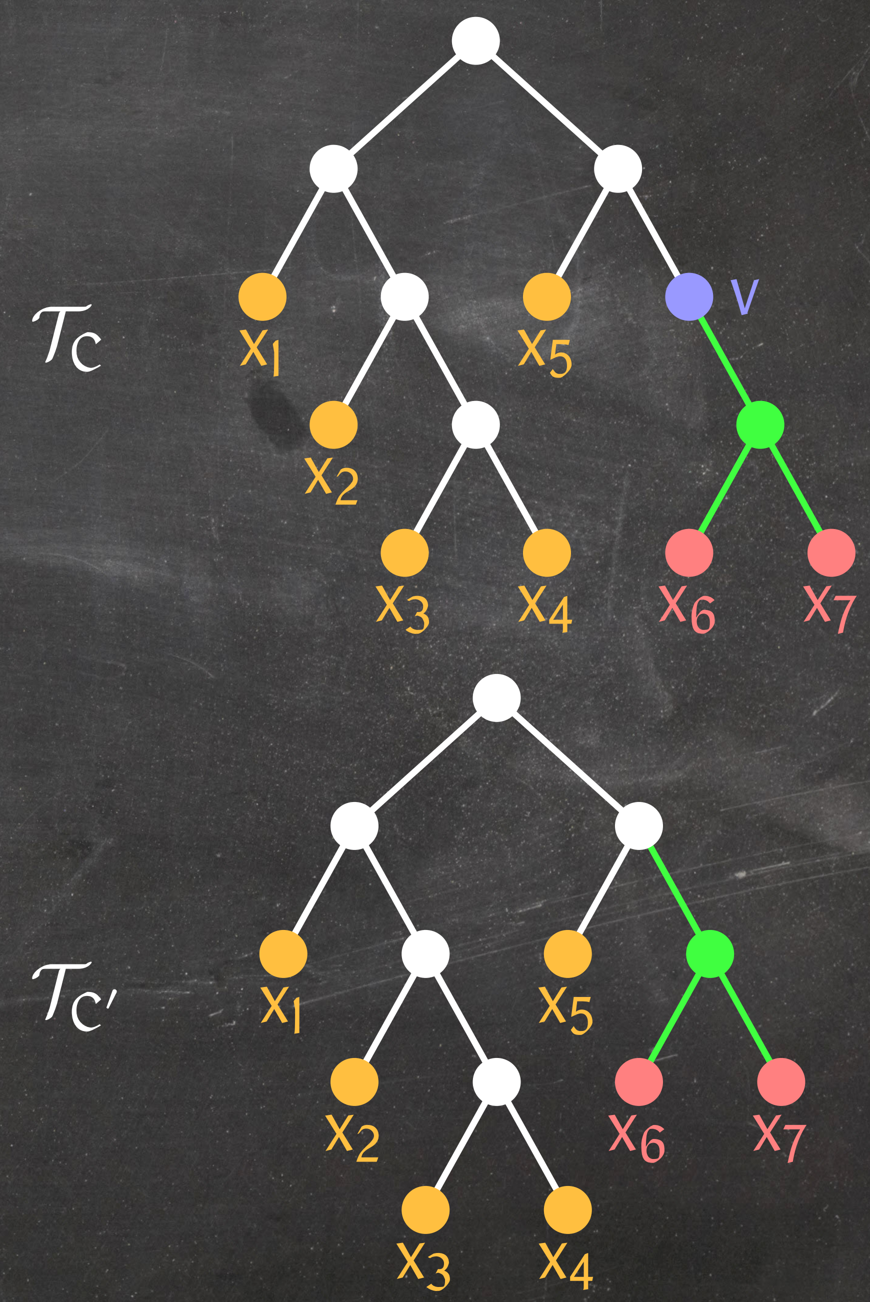
Otherwise, choose an internal node v with only one child w and contract the edge (v, w).



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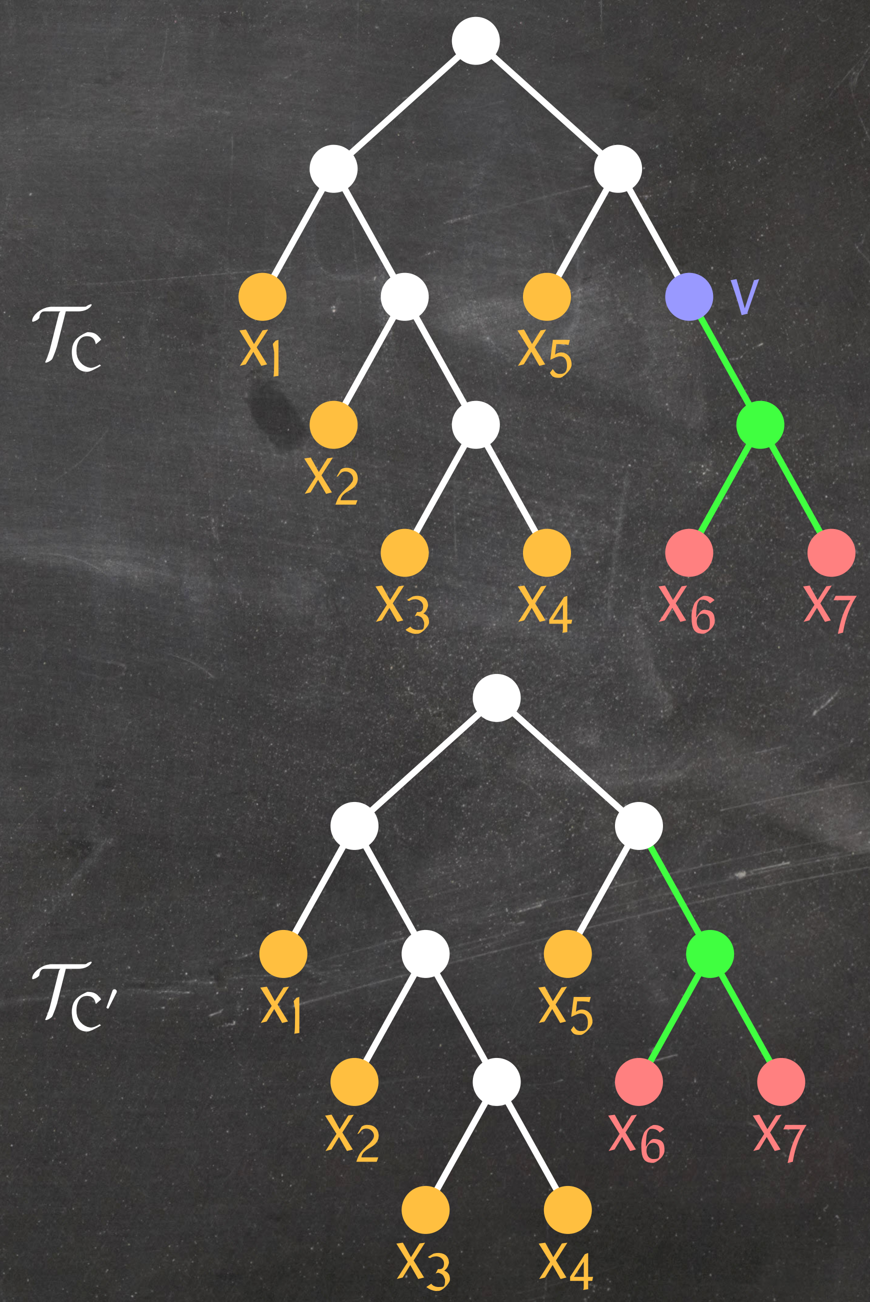
The resulting tree $\mathcal{T}_{C'}$ has one less internal node with only one child and represents a prefix-free code $C'(\cdot)$ with the property that $|C'(x)| \le |C(x)|$ for every character x.



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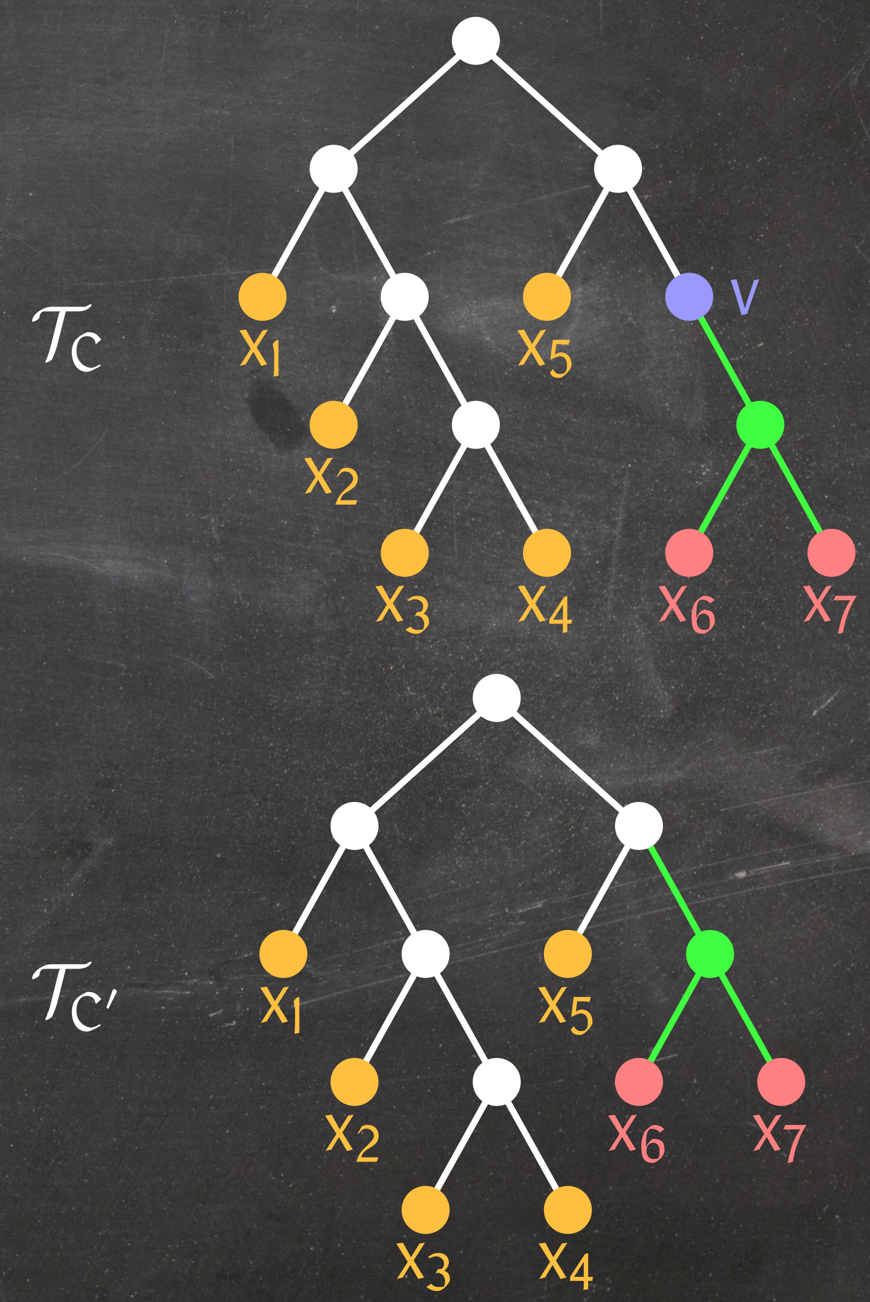
$\Rightarrow$ $|C'(T)| \le |C(T)|$, contradicting the choice of C.



A Greedy Choice for Optimal Prefix Codes

We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.

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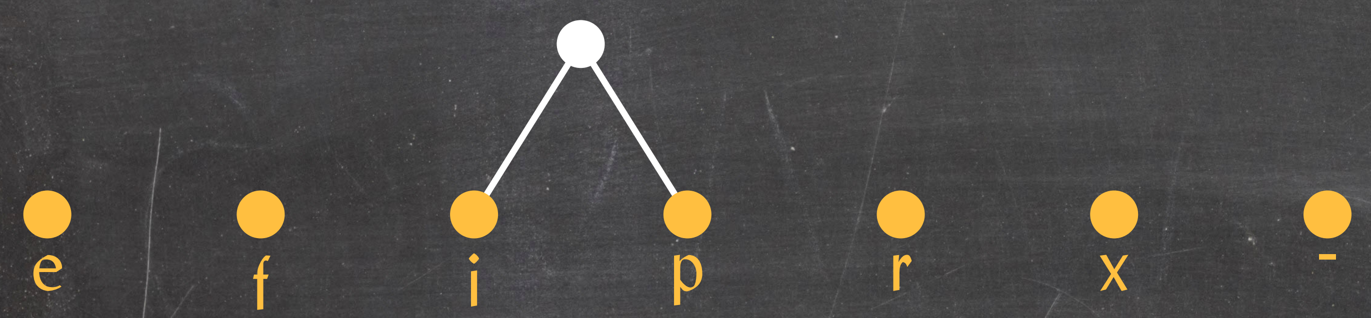

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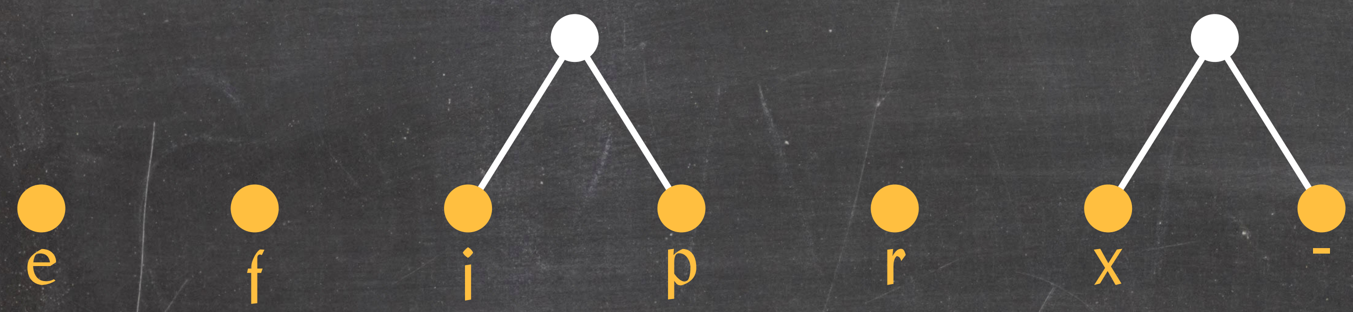

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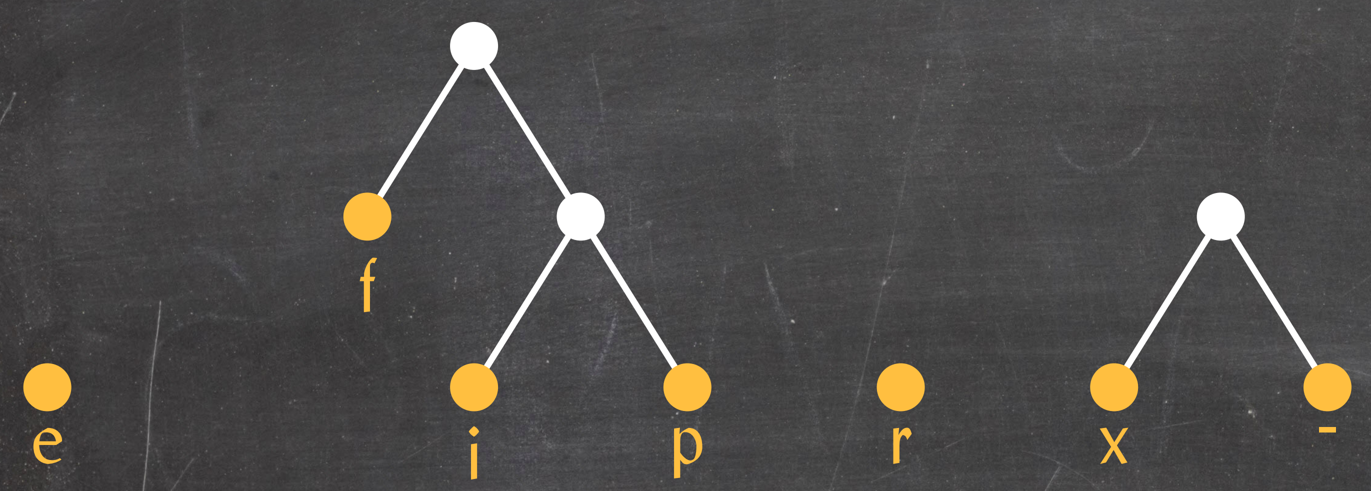

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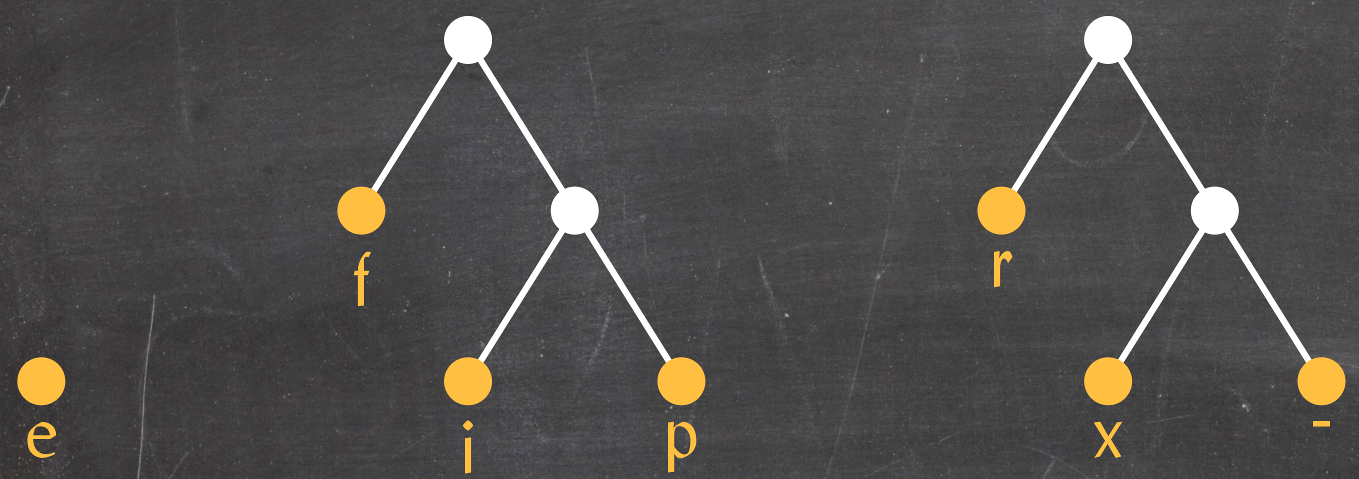

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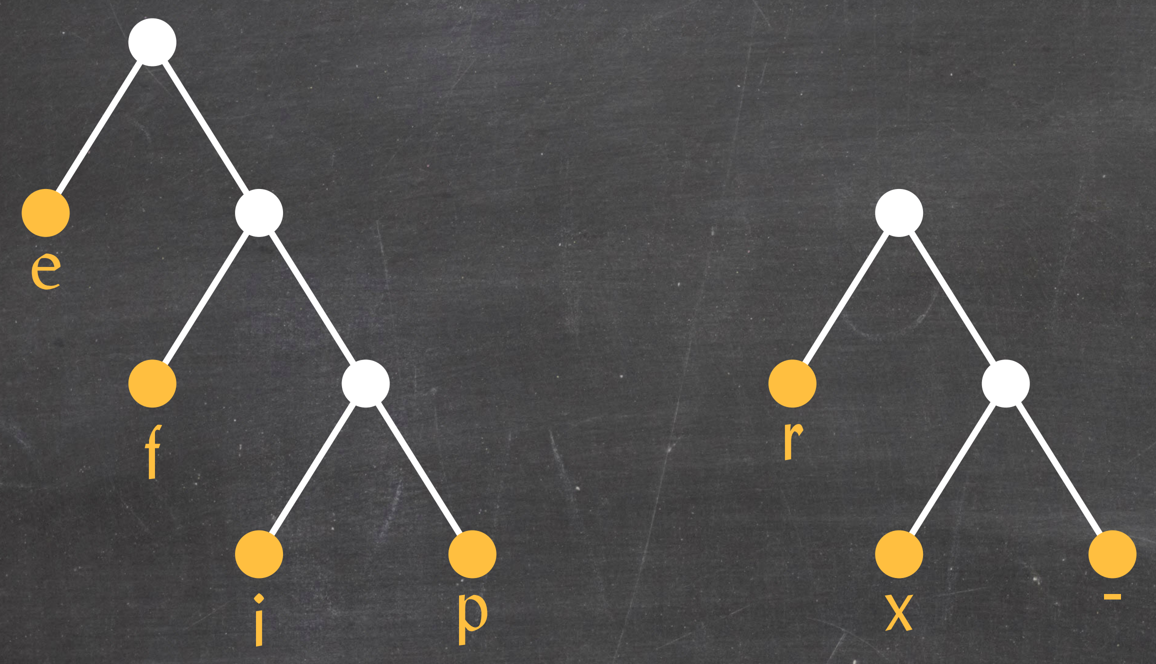

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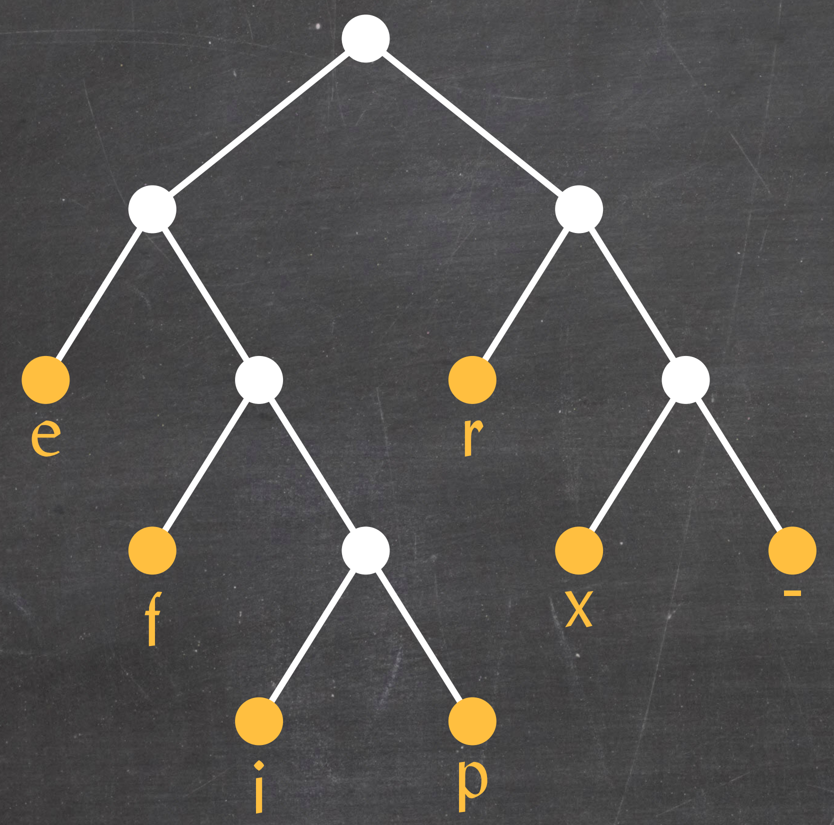

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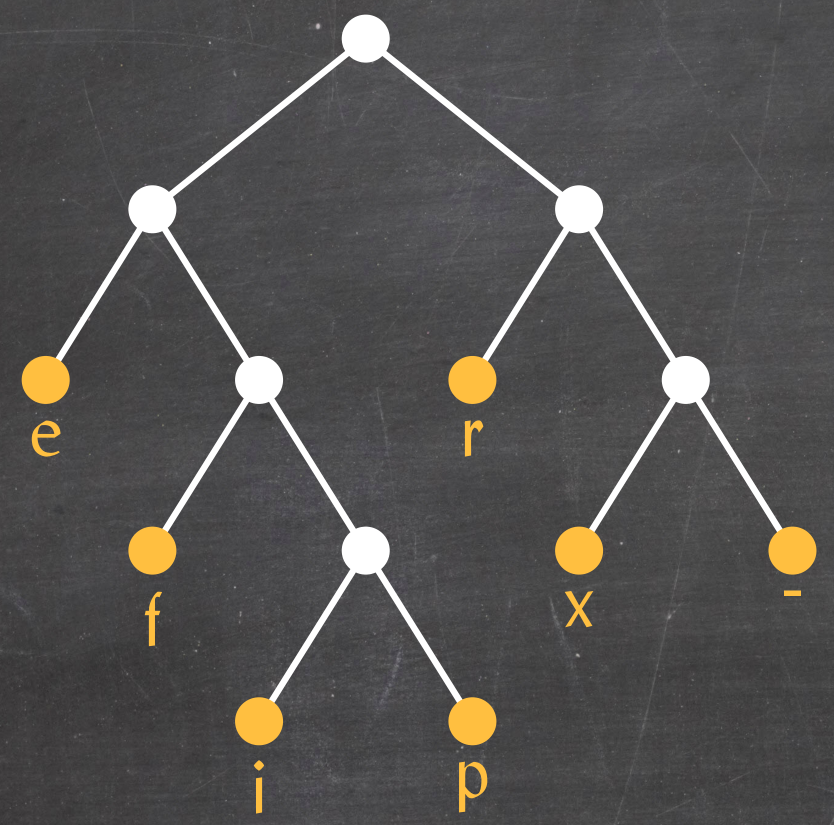

The length of the encoding of T is $|C(T)| = \sum_x f_T(x)|C(x)|$, where $f_T(x)$ is the frequency of x in T.

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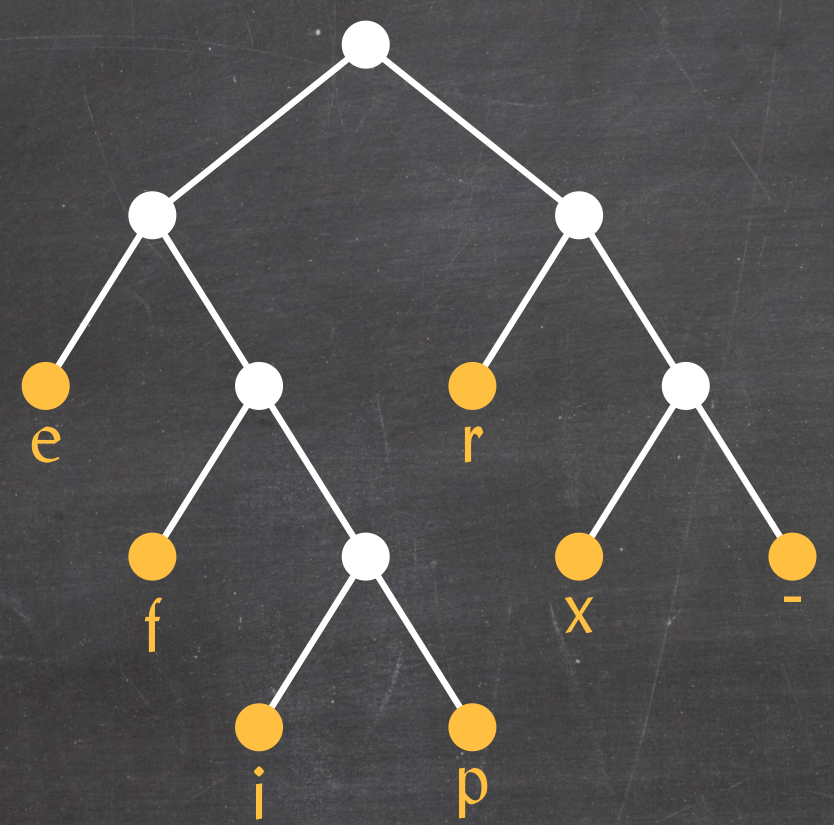

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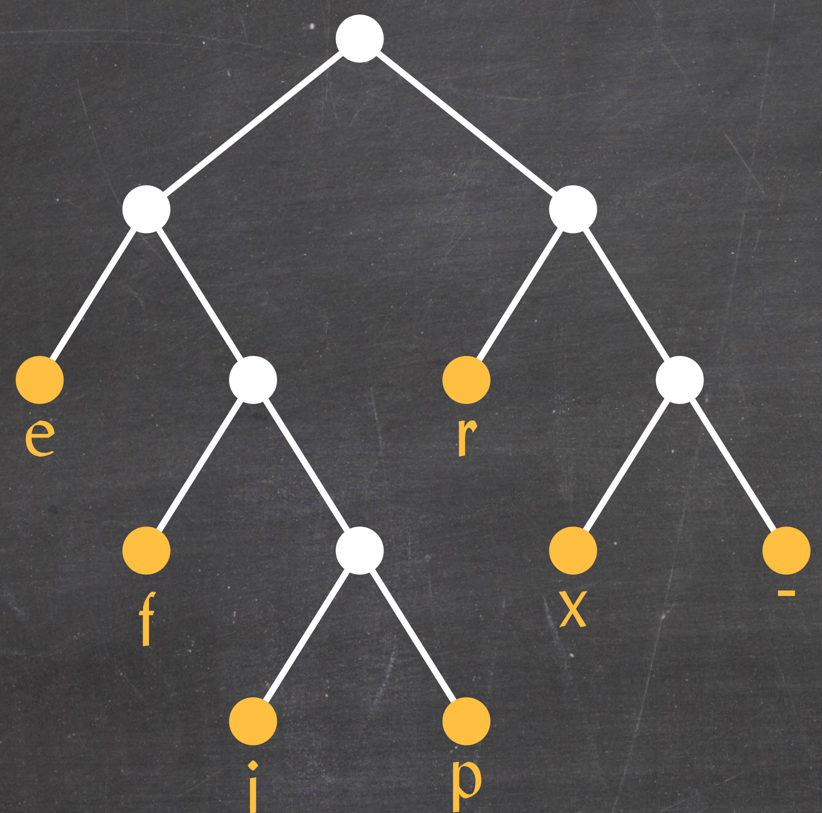

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$\Rightarrow$ By choosing the two roots with minimum total frequency of their descendent leaves, we minimize the increase in $|C(T)|$.

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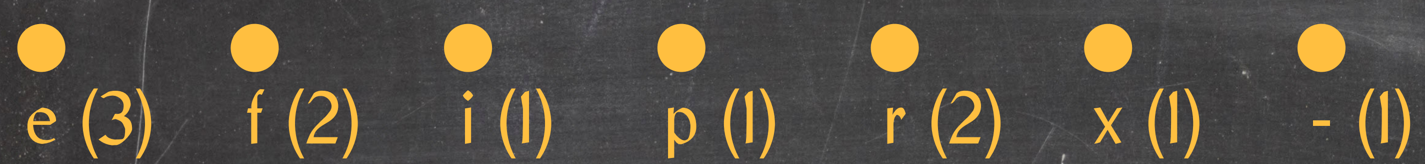
$f_T(x)$	3	2	1	1	2	1	1



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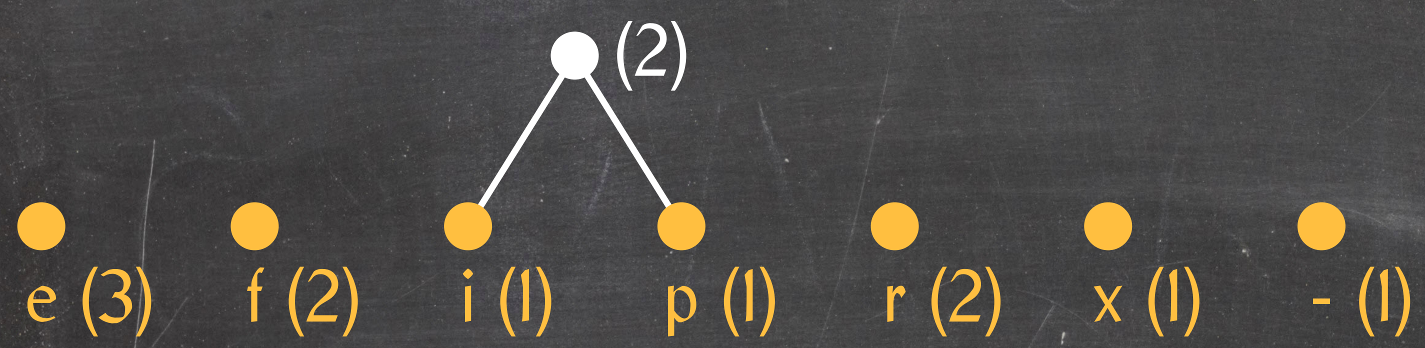

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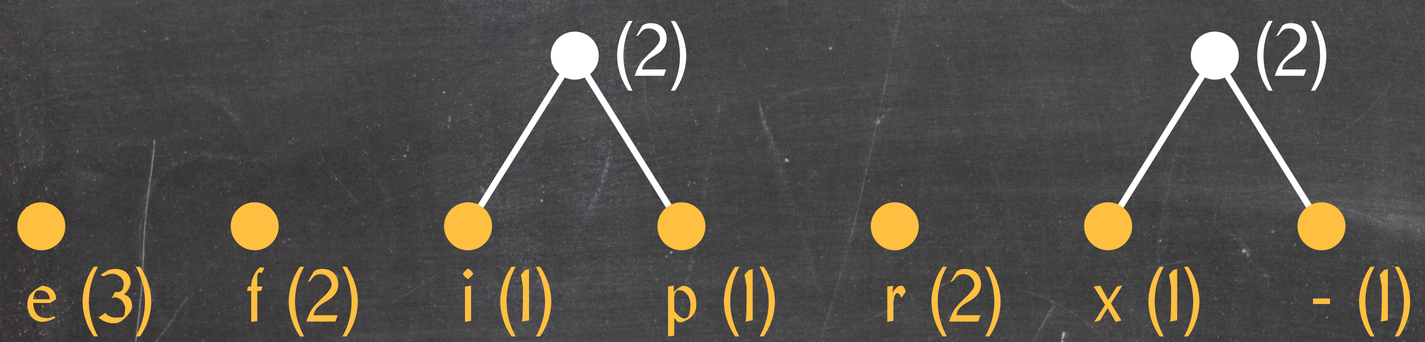

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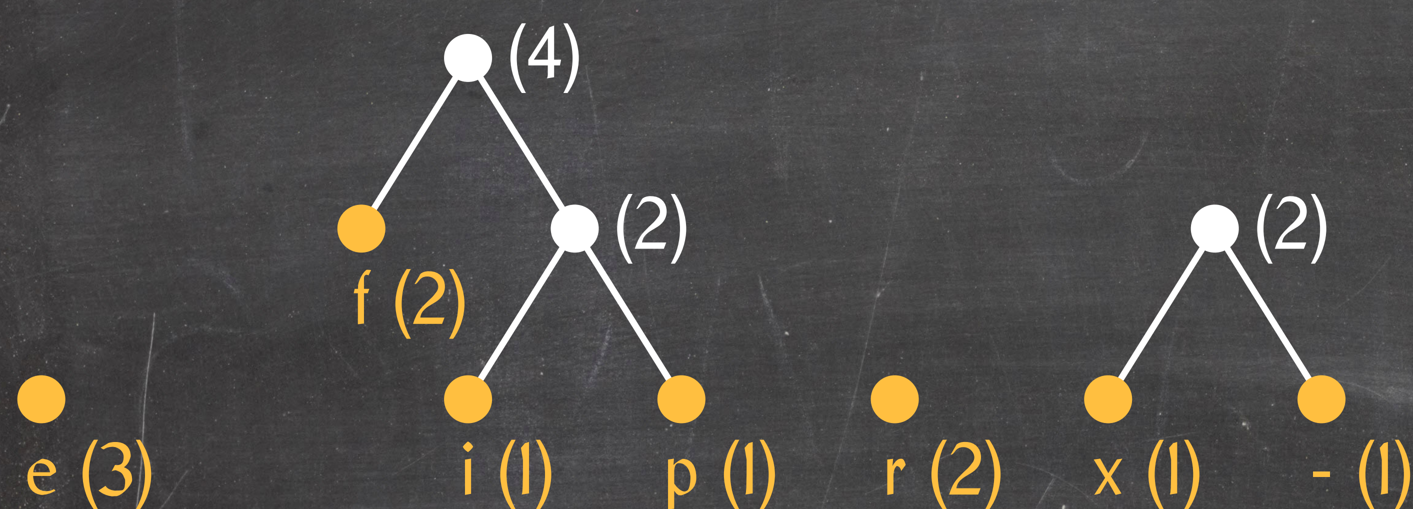

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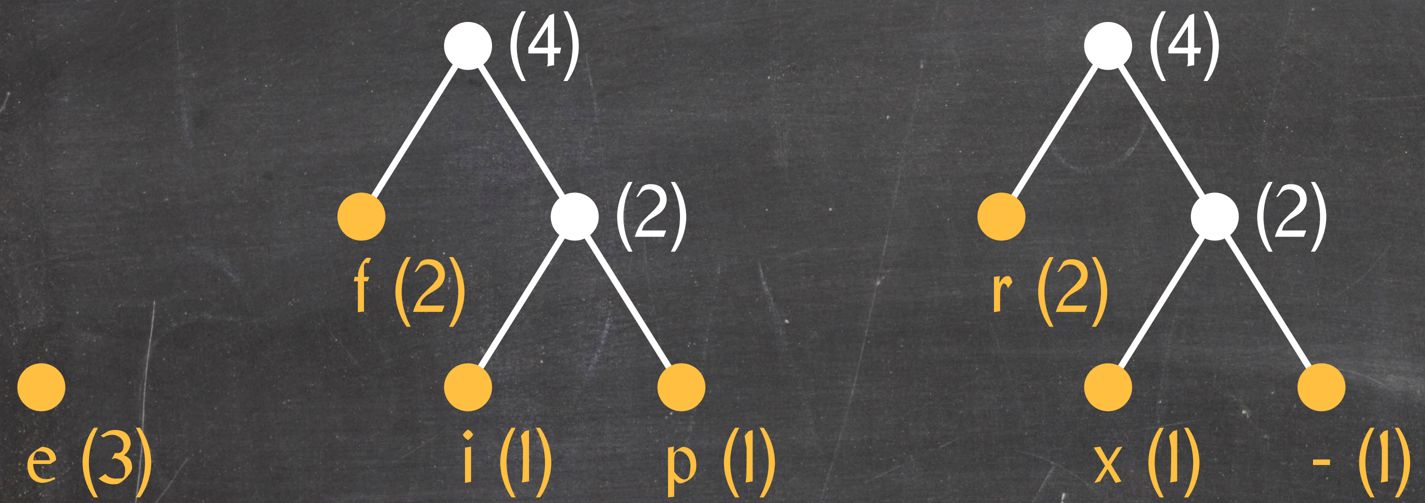

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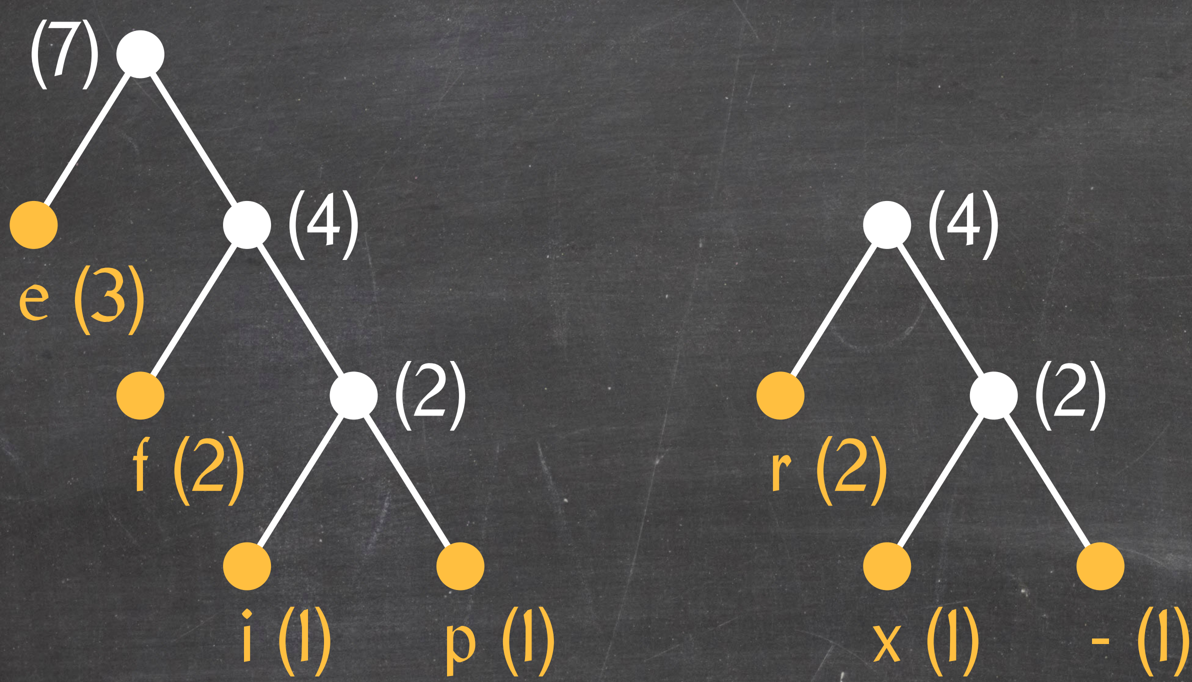

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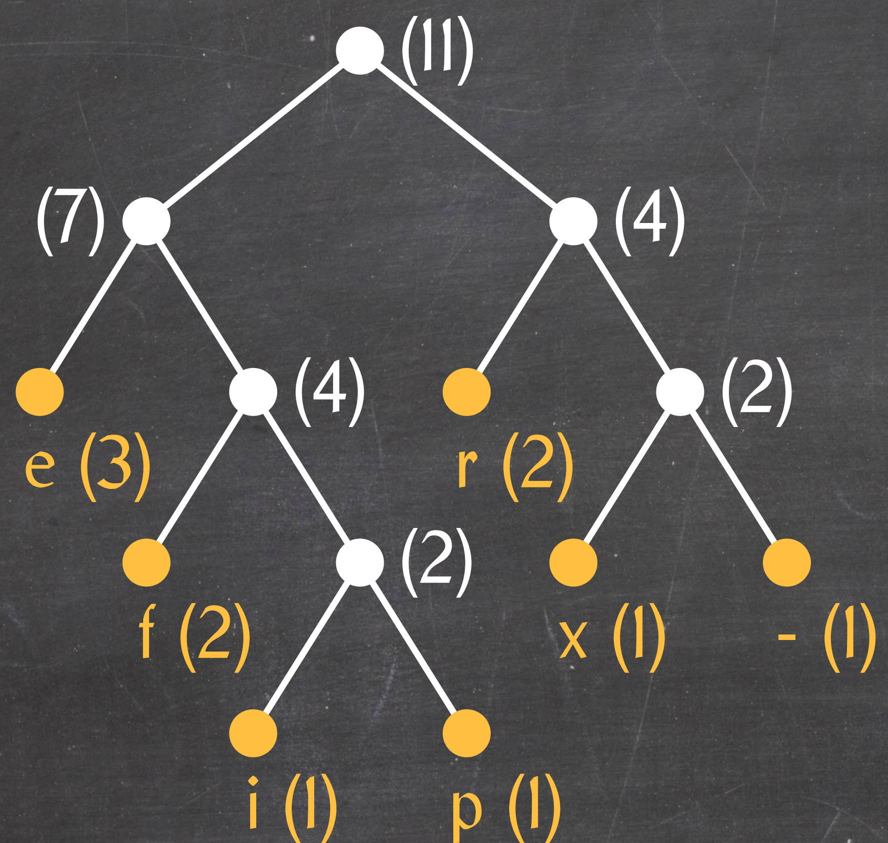

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Huffman's Algorithm

Huffman(T)

```
determine the set A of characters that occur in T and their frequencies
Q = an empty priority queue
for every character x ∈ A
    do create a node v associated with x and define f(v) = f(x)
       Q.insert(v, f(v))
while |Q| > 1
    do v = Q.deleteMin()
       w = Q.deleteMin()
       u = a new node with frequency f(u) = f(v) + f(w)
       make v and w children of u
       Q.insert(u, f(u))
return Q.deleteMin()
```

Lemma: Huffman's algorithm runs in $O(m \lg n)$ time, where $m = |T|$ and n is the size of the alphabet.

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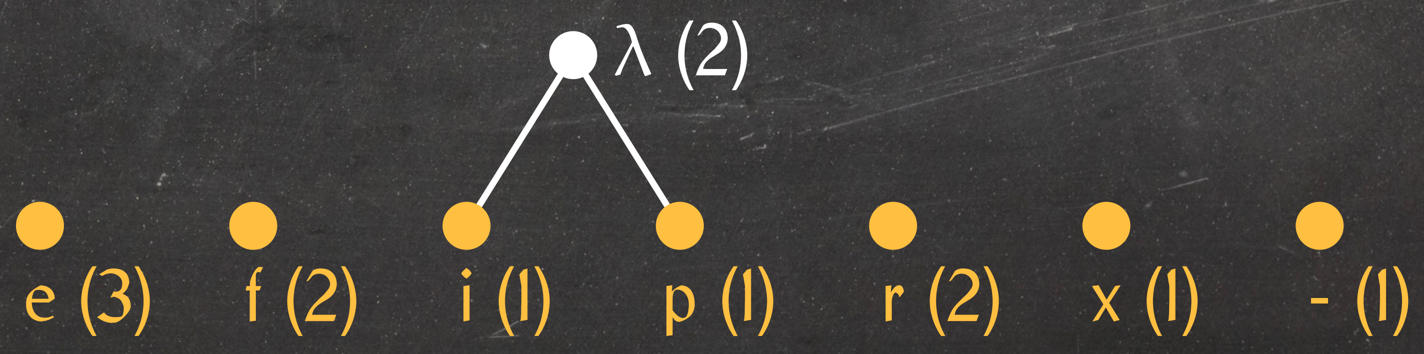

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"prefix-free"

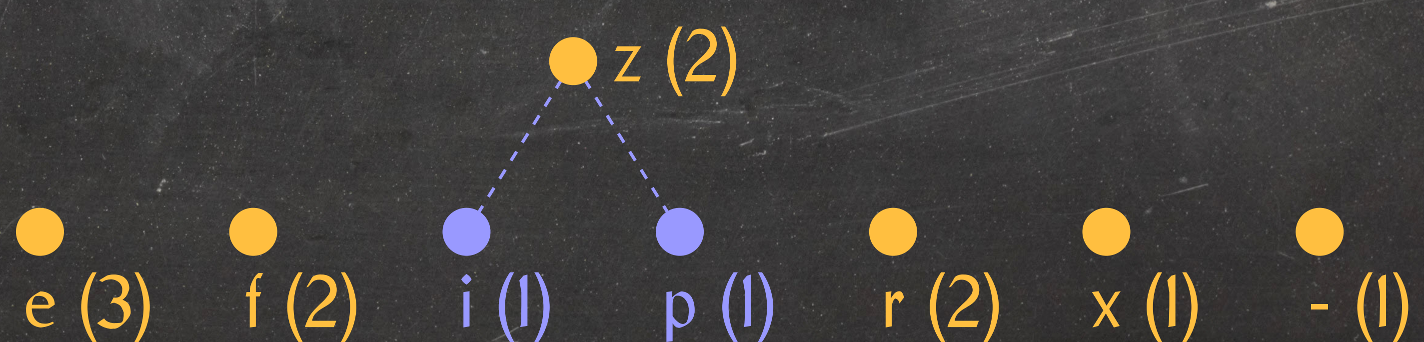
⇓

"zrefzx-free"

Inductive step: n > 2.

Consider the first two characters a and b that are joined under a common parent z with frequency $f(z) = f(a) + f(b)$.

Replacing a and b with z in T produces a new text T' over an alphabet of size $n - 1$ where z has frequency $f(z)$.



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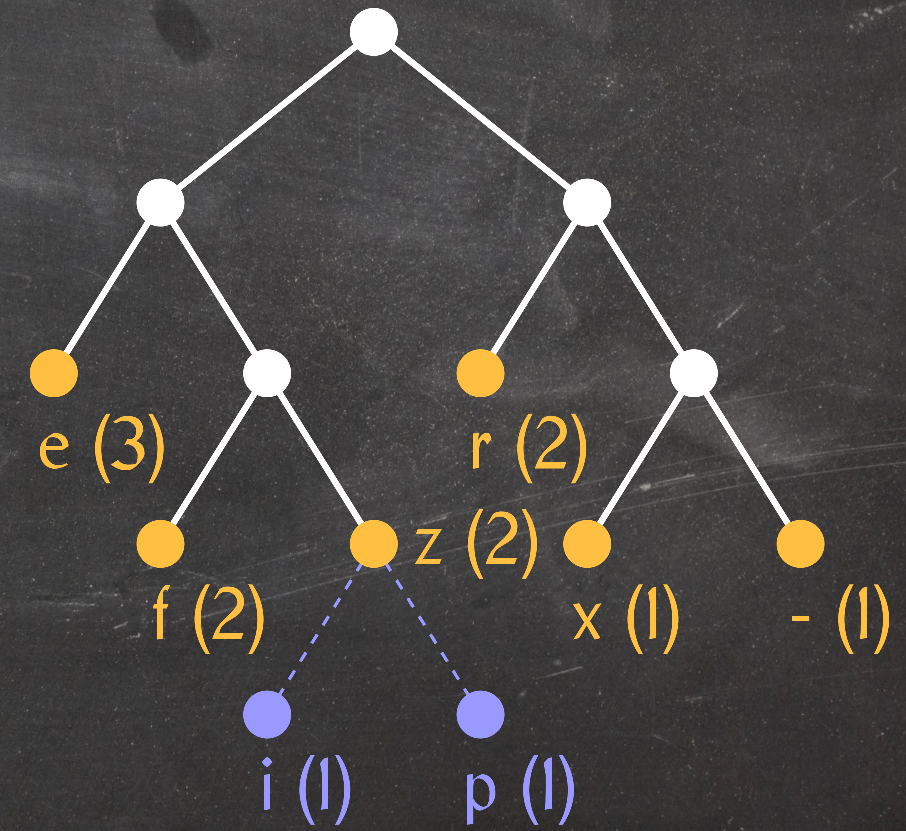
After joining a and b under z, Huffman's algorithm behaves exactly as if it was run on T′.

"prefix-free"

$\Downarrow$

"zrefzx-free"



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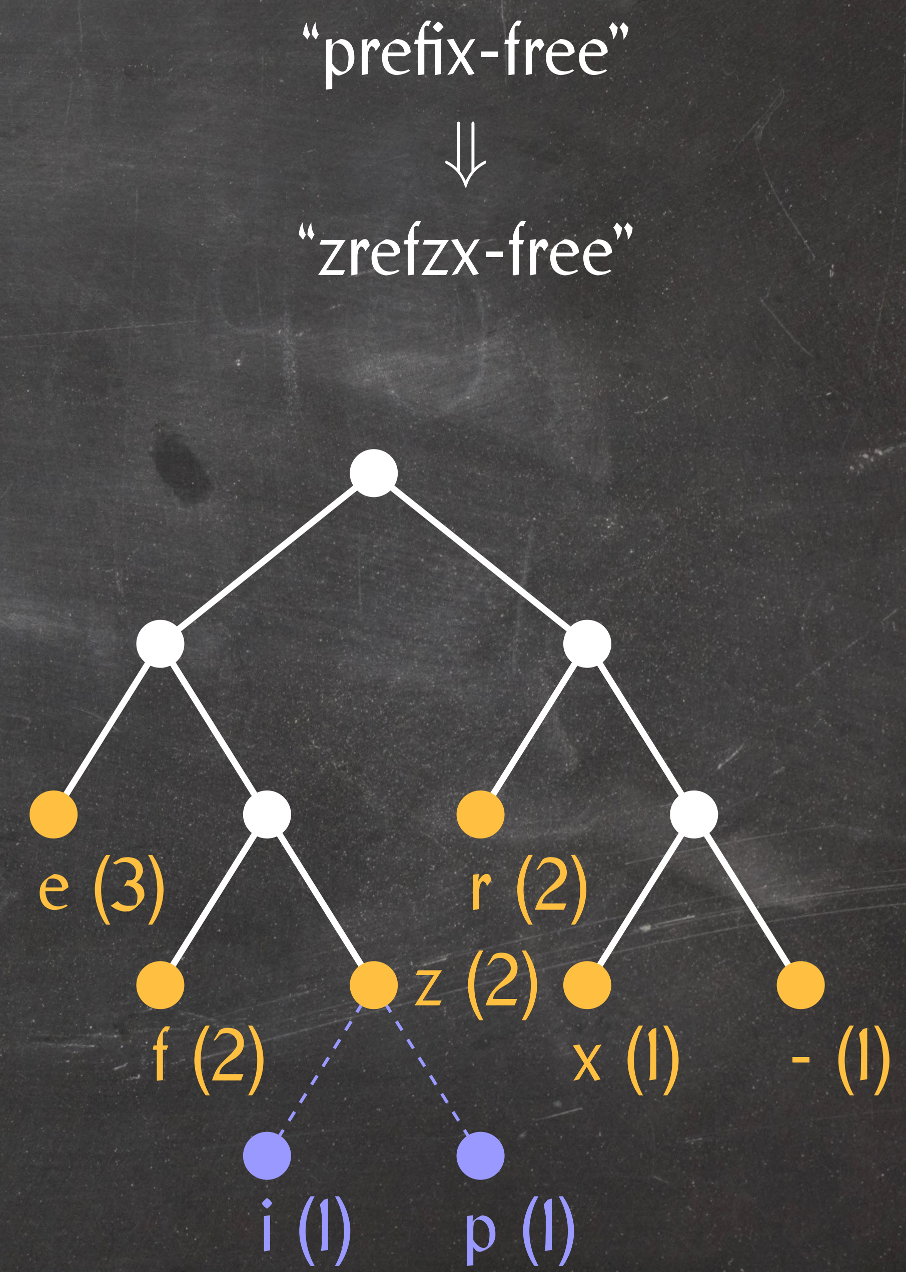
By induction, it produces an optimal code $C'(\cdot)$ for T'.



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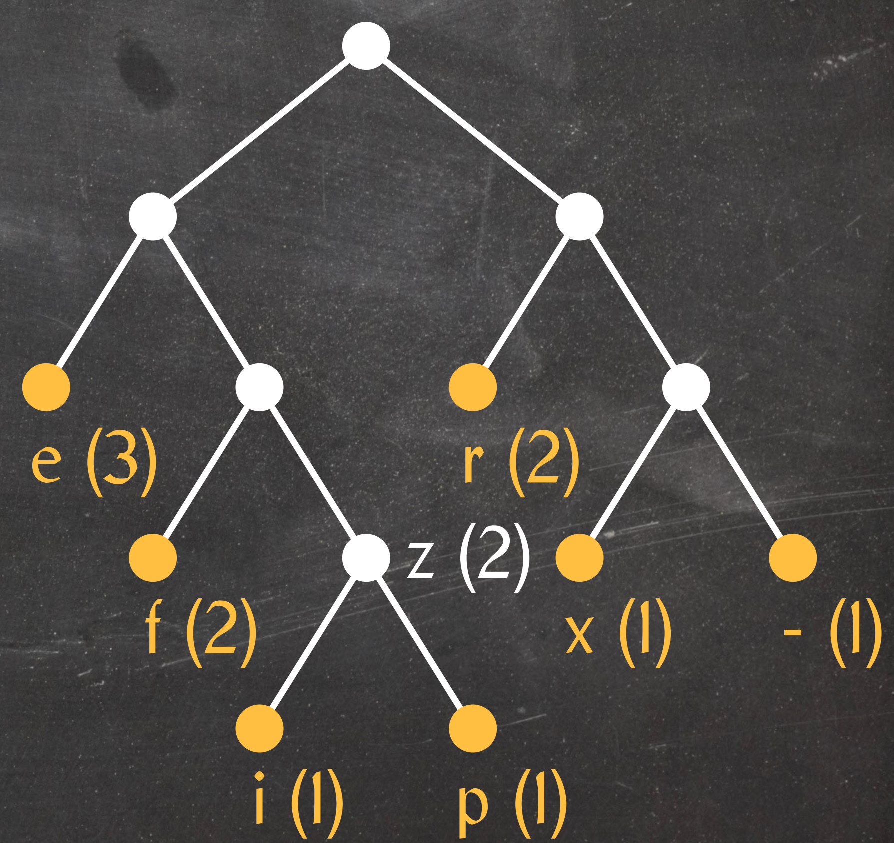
"prefix-free"



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$$C''(x) = \begin{cases} C^*(x) & x \neq z \\ \sigma & x = z \text{ and } C^*(a) = \sigma 0 \end{cases}$$

"prefix-free"

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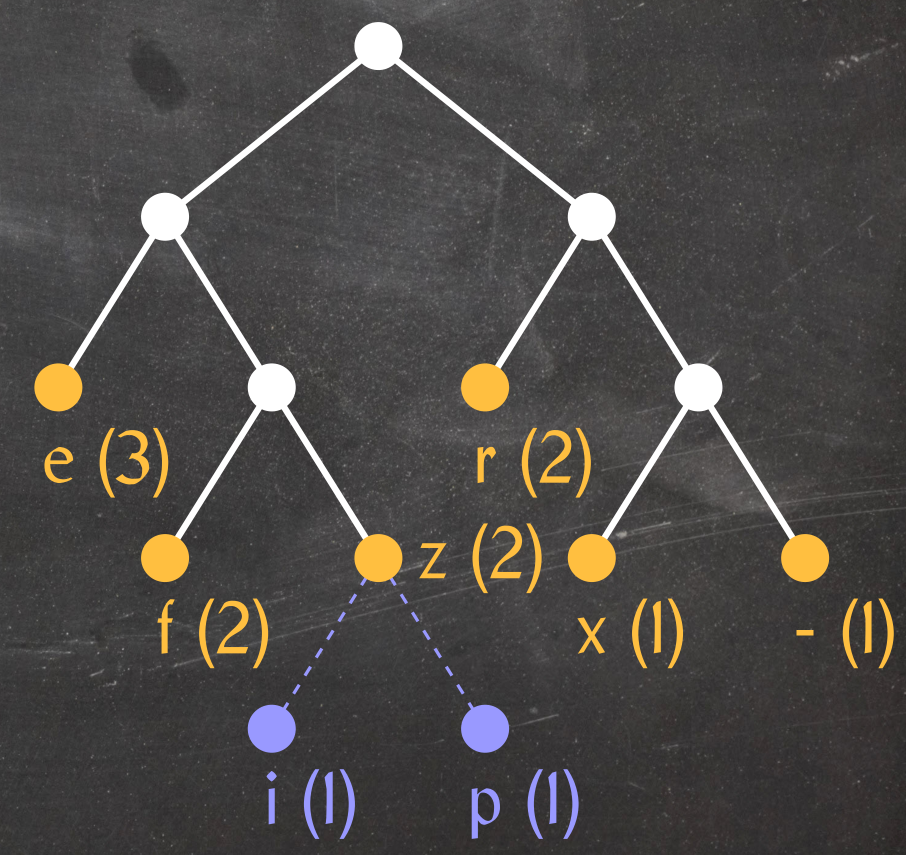

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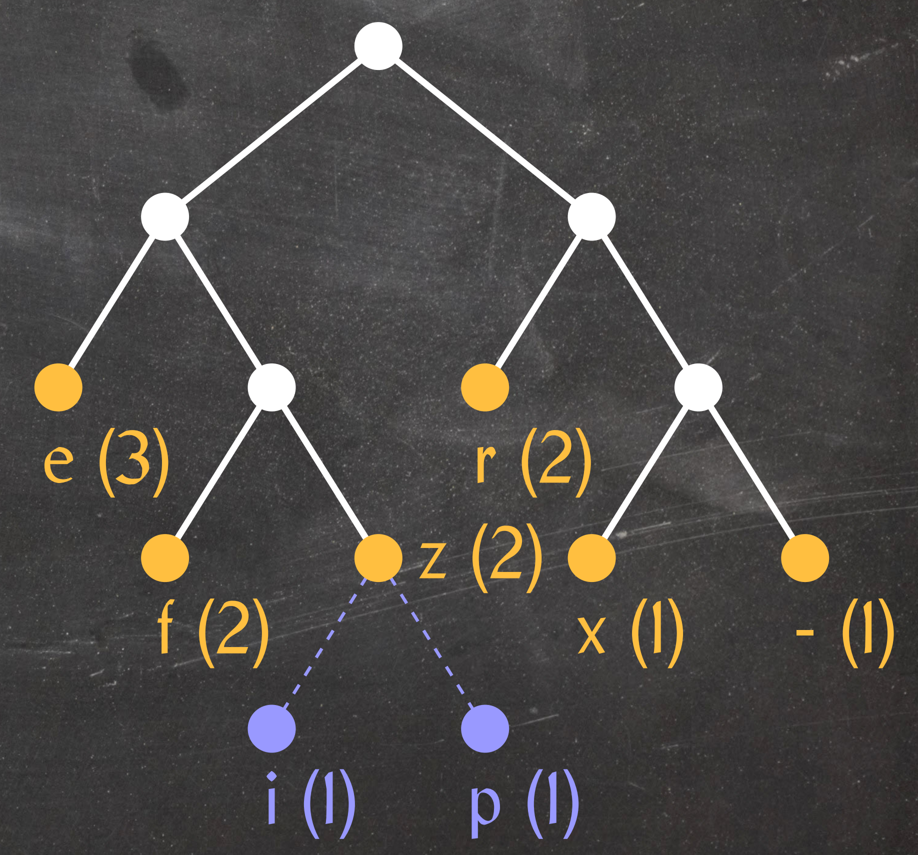

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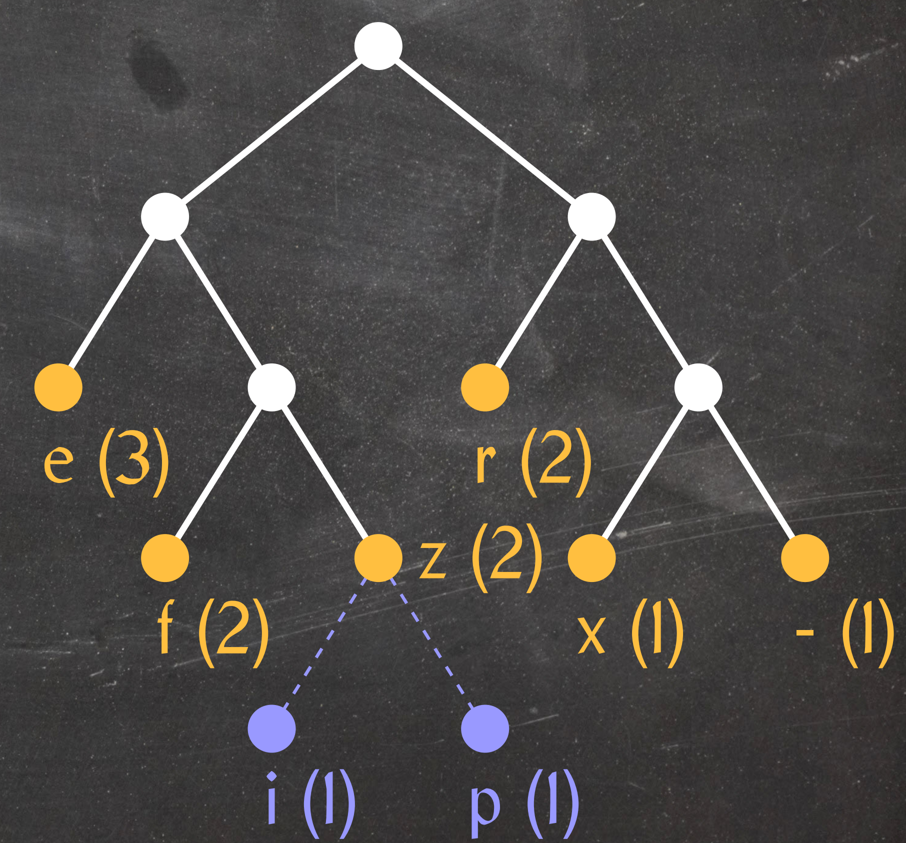

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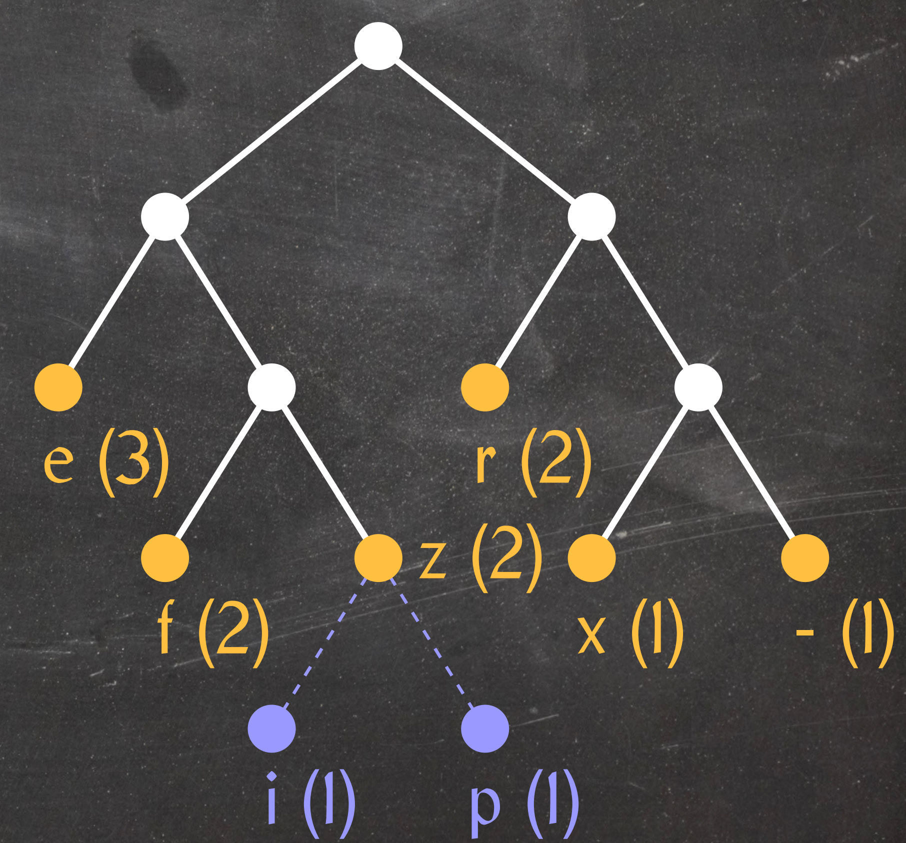

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$|C(T)| = |C'(T')| + f(z)$ and $|C^*(T)| = |C''(T')| + f(z)$.

$\Rightarrow$ $|C''(T')| < |C'(T')|$, a contradiction because $C'(\cdot)$ is optimal for T'.

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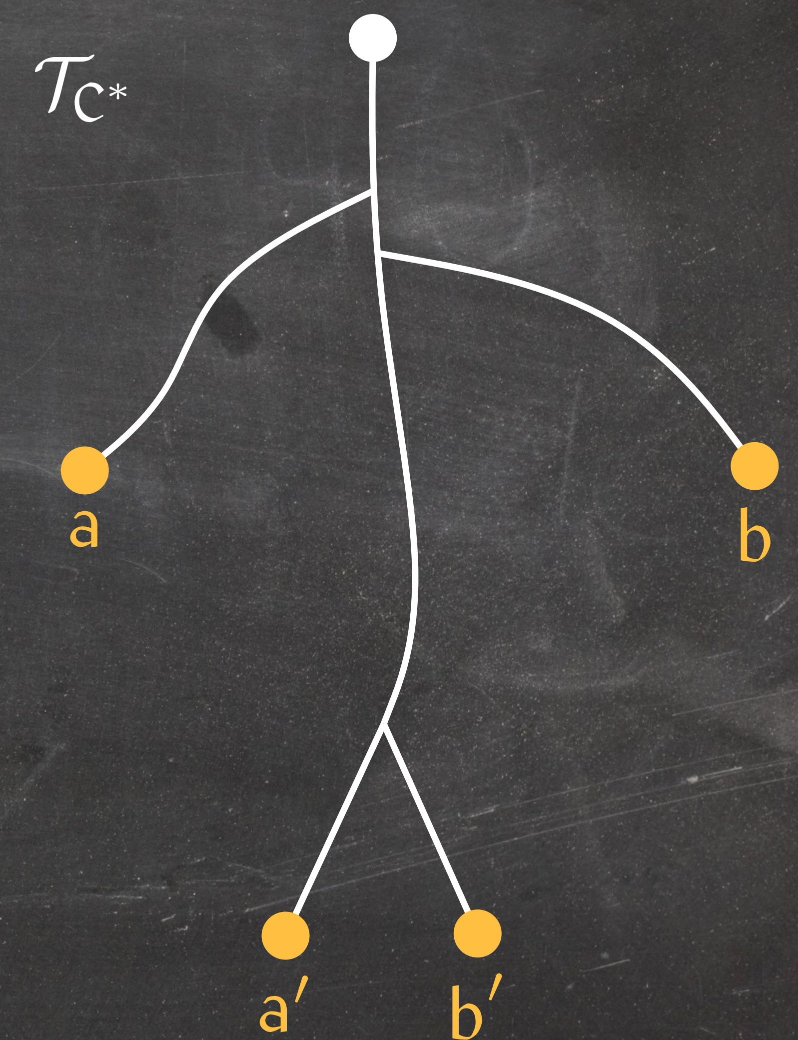
The sibling b' of the deepest leaf a' in $\mathcal{T}_{C^*}$ is also a leaf.



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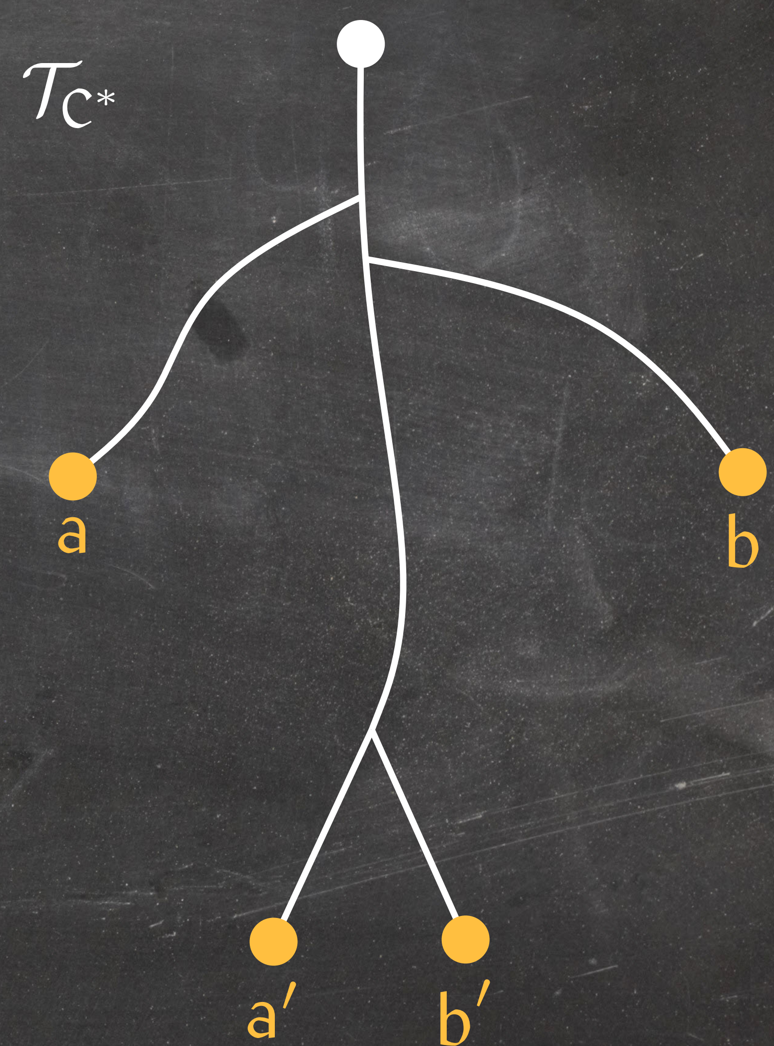

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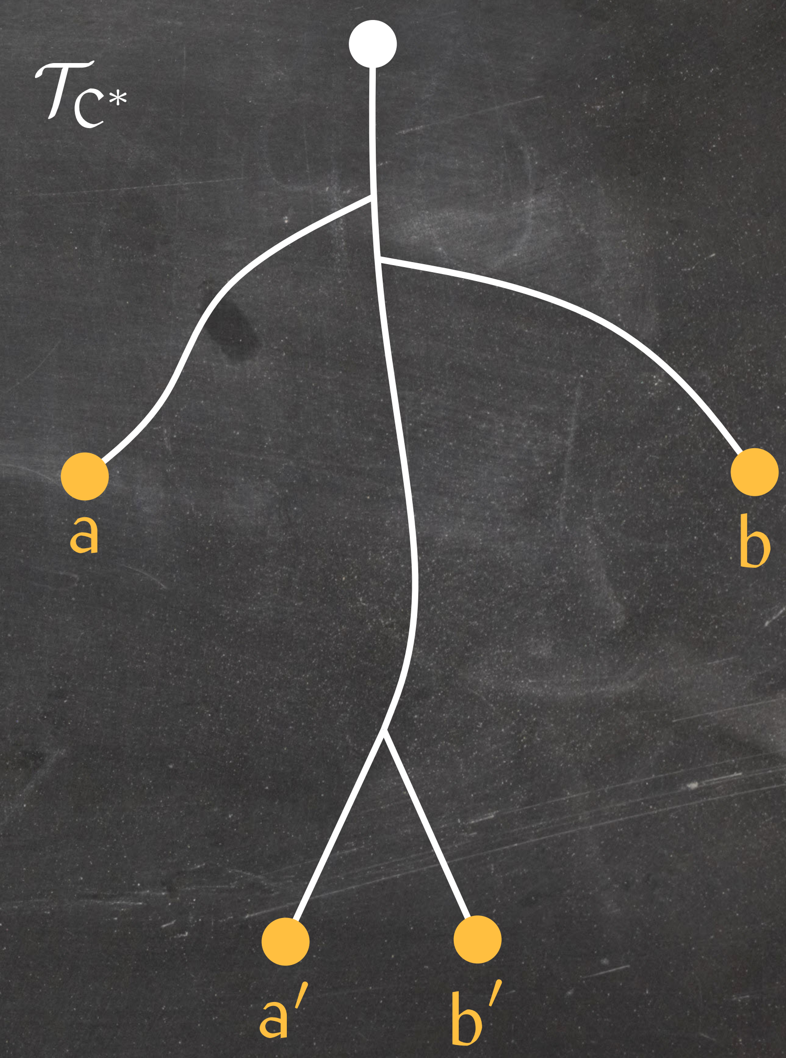

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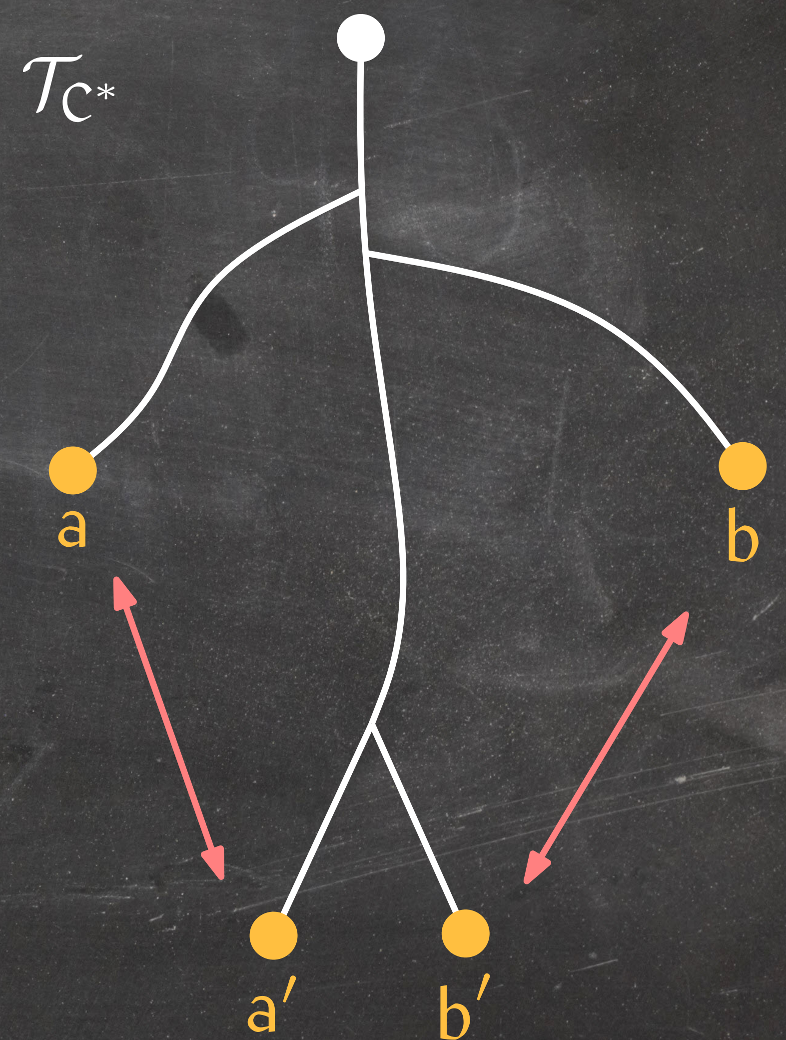

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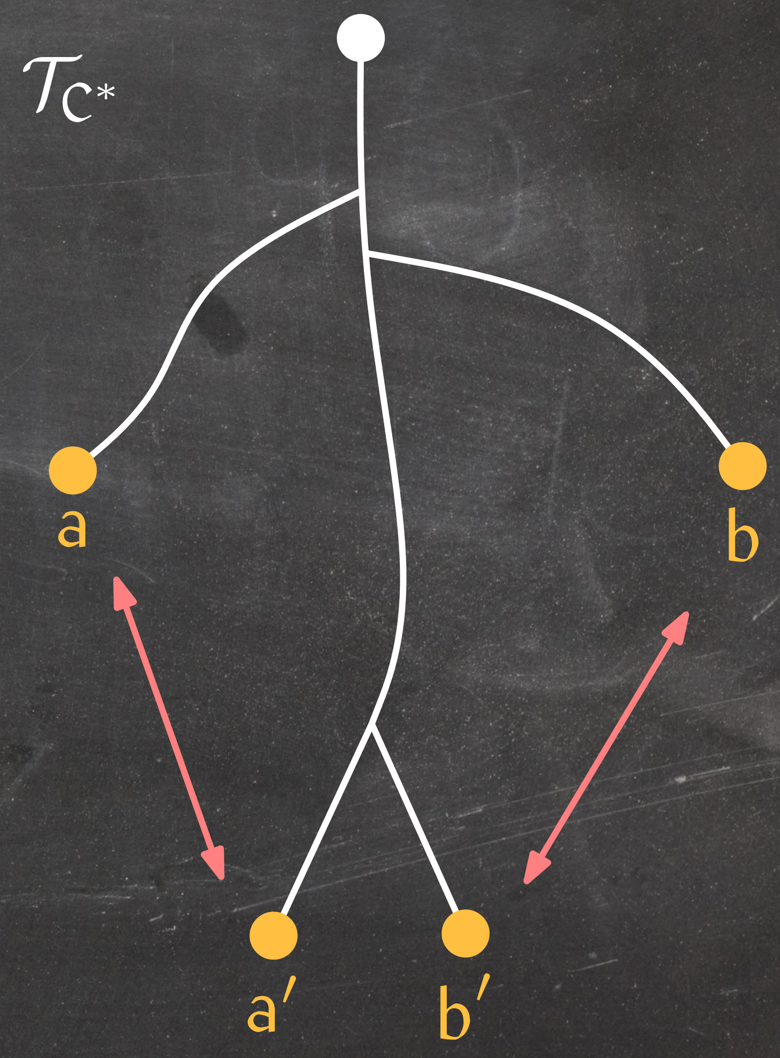
We prove that $|C(T)| \le |C^*(T)|$, that is, $C(\cdot)$ is an optimal prefix-free code for T.



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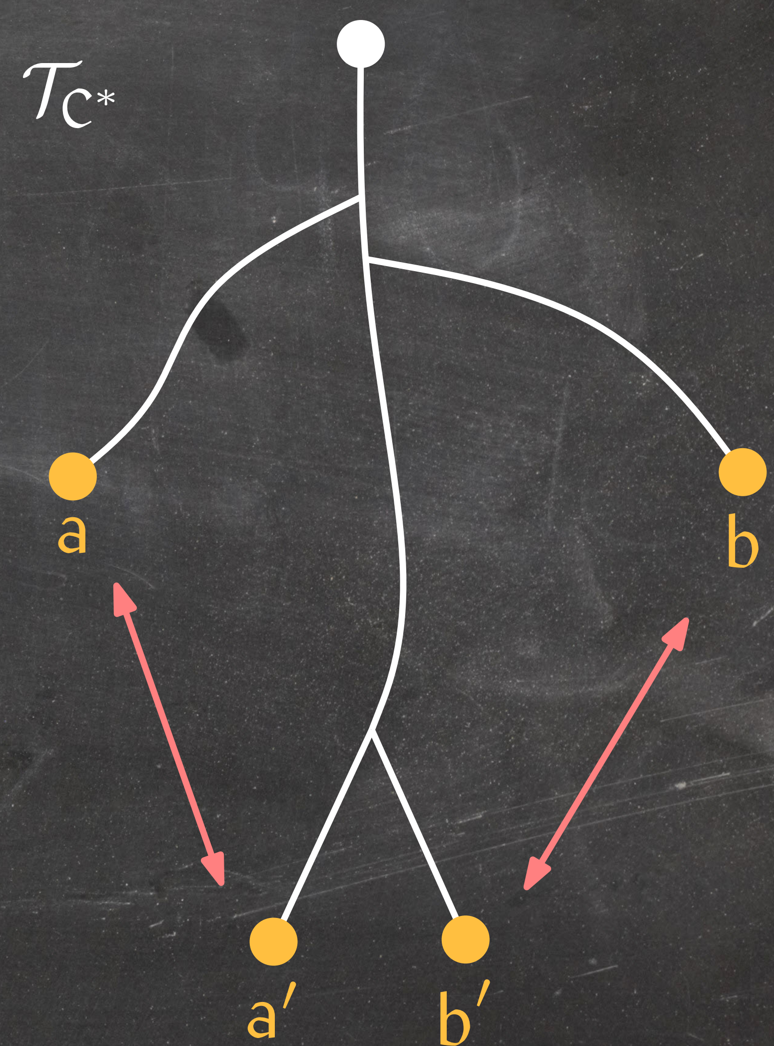
Since a and b are siblings in $\mathcal{T}_C$, this proves the claim.



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$$|C(T)| - |C^*(T)| = f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')|$$

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$$\begin{aligned}
|C(T)| - |C^*(T)| &= f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\
&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
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&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= (f(a) - f(a'))\,(|C^*(a')| - |C^*(a)|) + (f(b) - f(b'))\,(|C^*(b')| - |C^*(b)|)
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&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= \underbrace{(f(a) - f(a'))}_{\le 0}\underbrace{(|C^*(a')| - |C^*(a)|)}_{\ge 0} + \underbrace{(f(b) - f(b'))}_{\le 0}\underbrace{(|C^*(b')| - |C^*(b)|)}_{\ge 0}
\end{aligned}$$

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&\le 0
\end{aligned}$$

Summary

Greedy algorithms make natural local choices in their search for a globally optimal solution.

Many good heuristics are greedy:

- Simple
- Work well in practice

Proof that a greedy algorithm finds an optimal solution:

- Induction
- Exchange argument

Useful data structures:

- Union-find data structure
- Thin Heap

Analysis of a sequence of data structure operations:

- Amortized analysis
- Potential functions