Graph Algorithms

Textbook Reading

Chapter 22

Overview

Design principle:

- Learn the structure of the graph by systematic exploration.

Proof technique:

- Proof by contradiction

Problems:

- Connected components
- Bipartiteness testing
- Topological sorting
- Strongly connected components

Graphs, Vertices, and Edges

A graph is an ordered pair $G = (V, E)$.

- V is the set of vertices of G.
- E is the set of edges of G.
- The elements of E are pairs of vertices (v, w).

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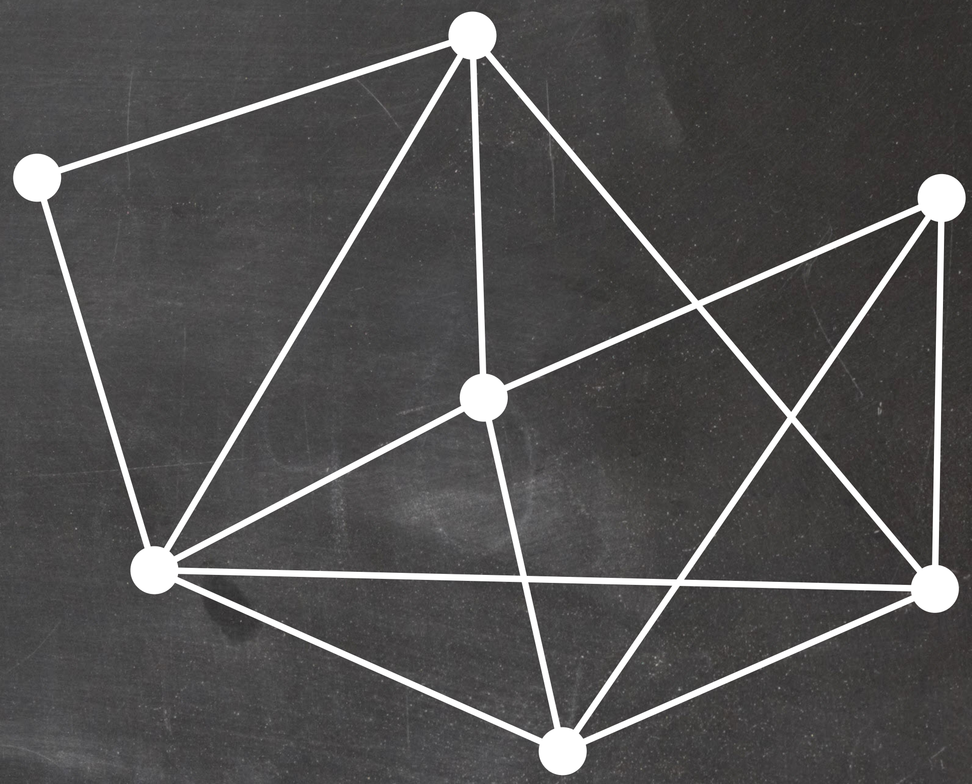

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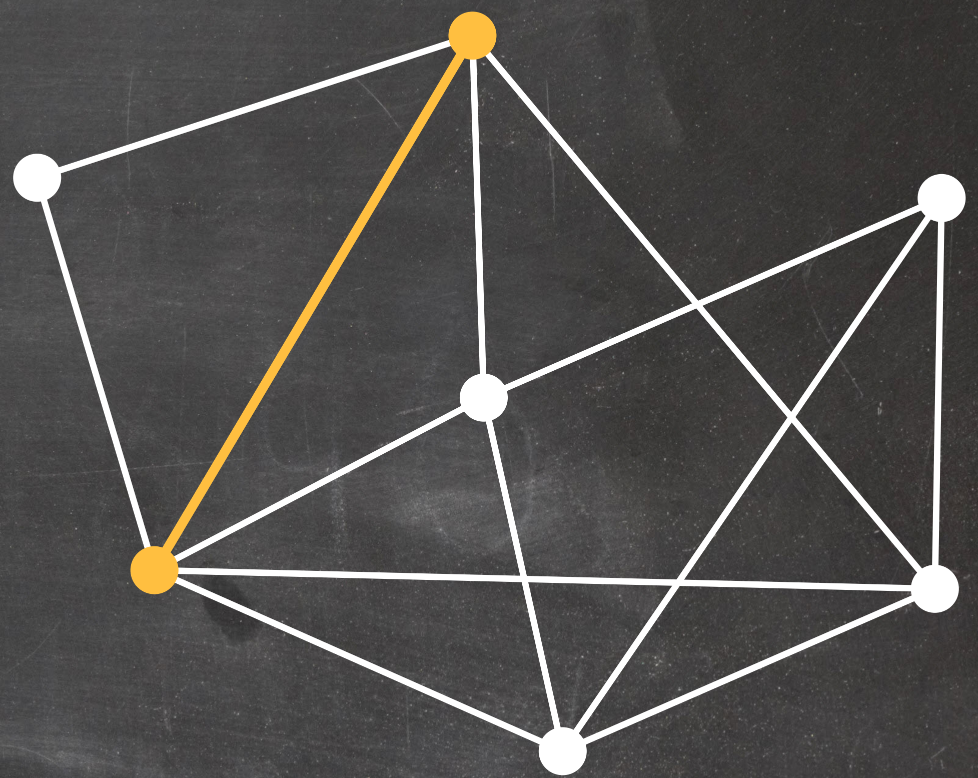

The endpoints of an edge (v, w) are v and w.

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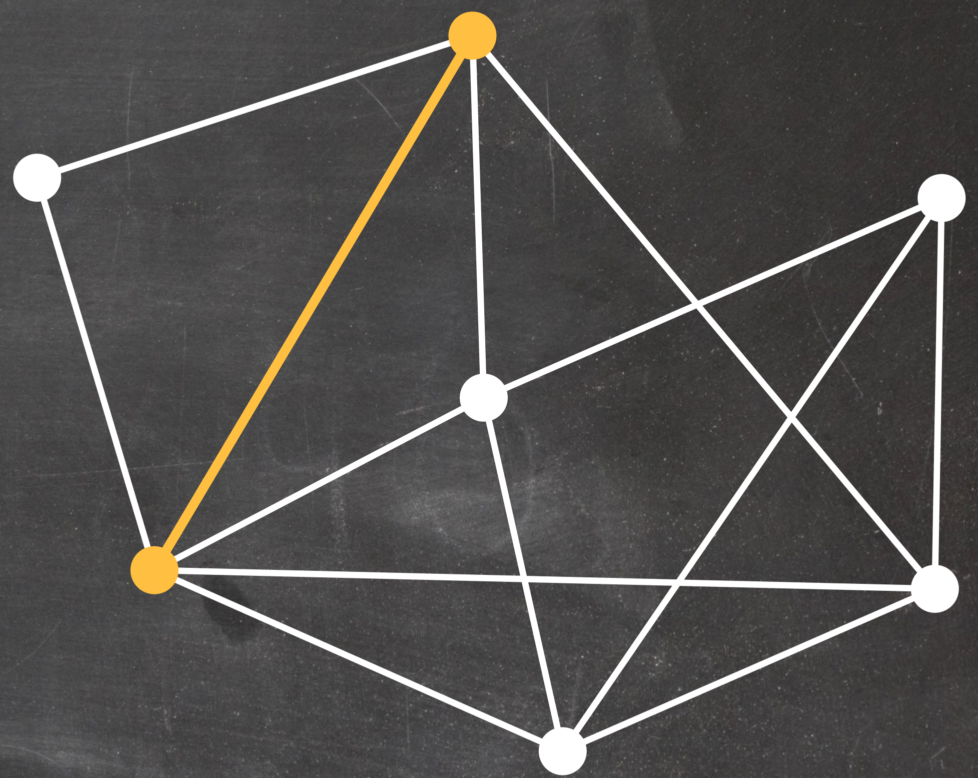

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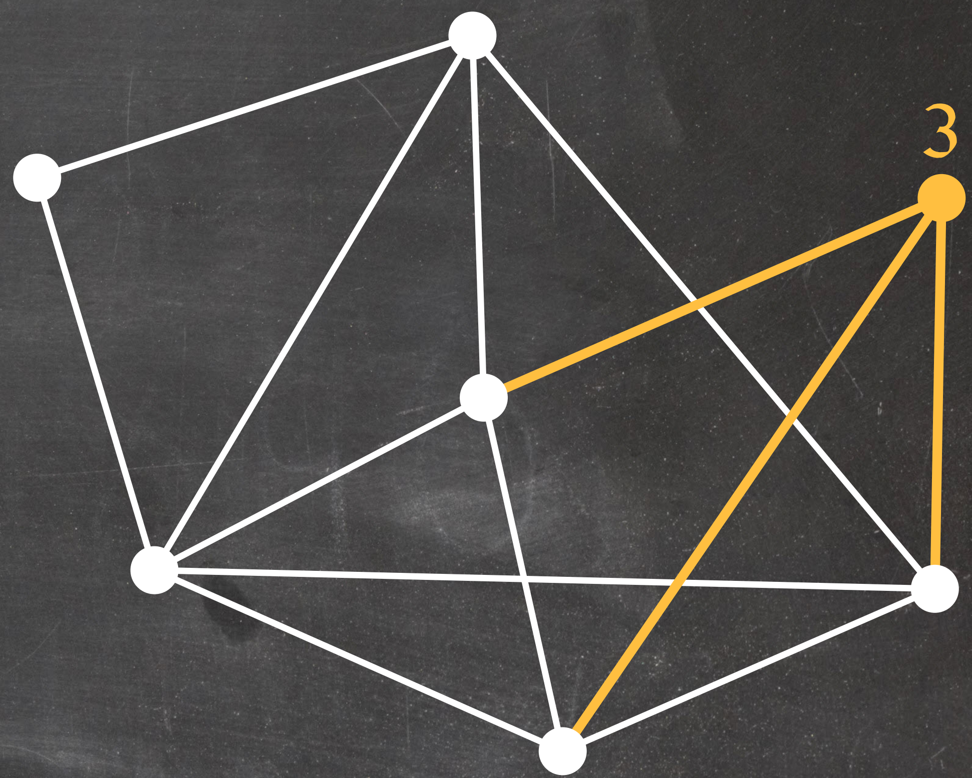

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The degree of a vertex is the number of its incident edges.

Undirected and Directed Graphs

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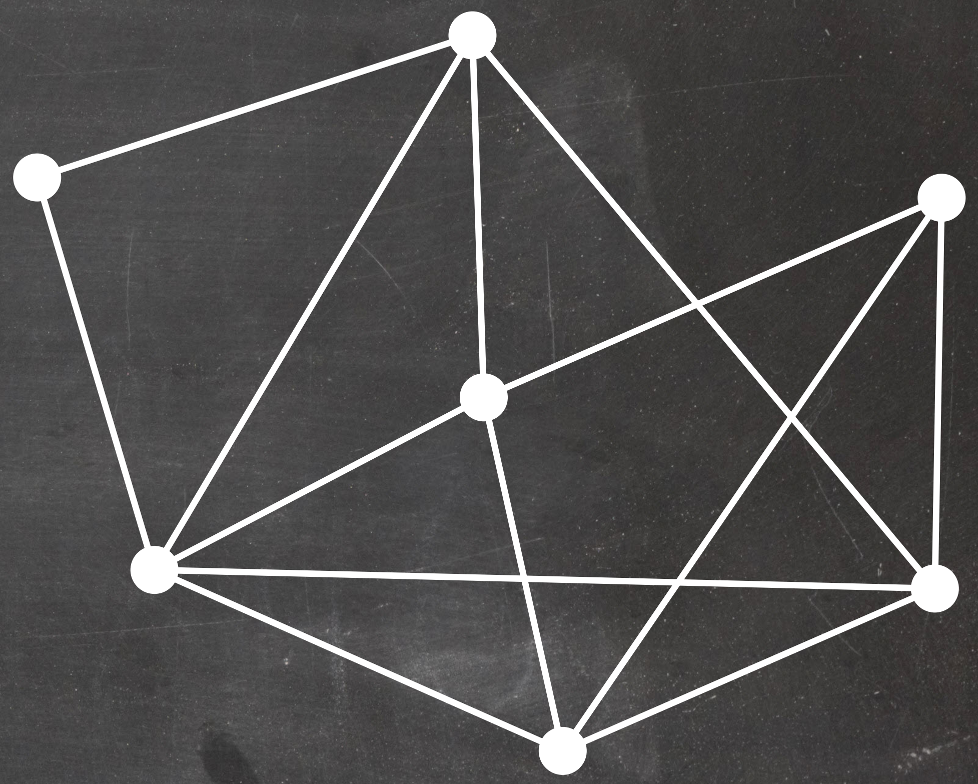

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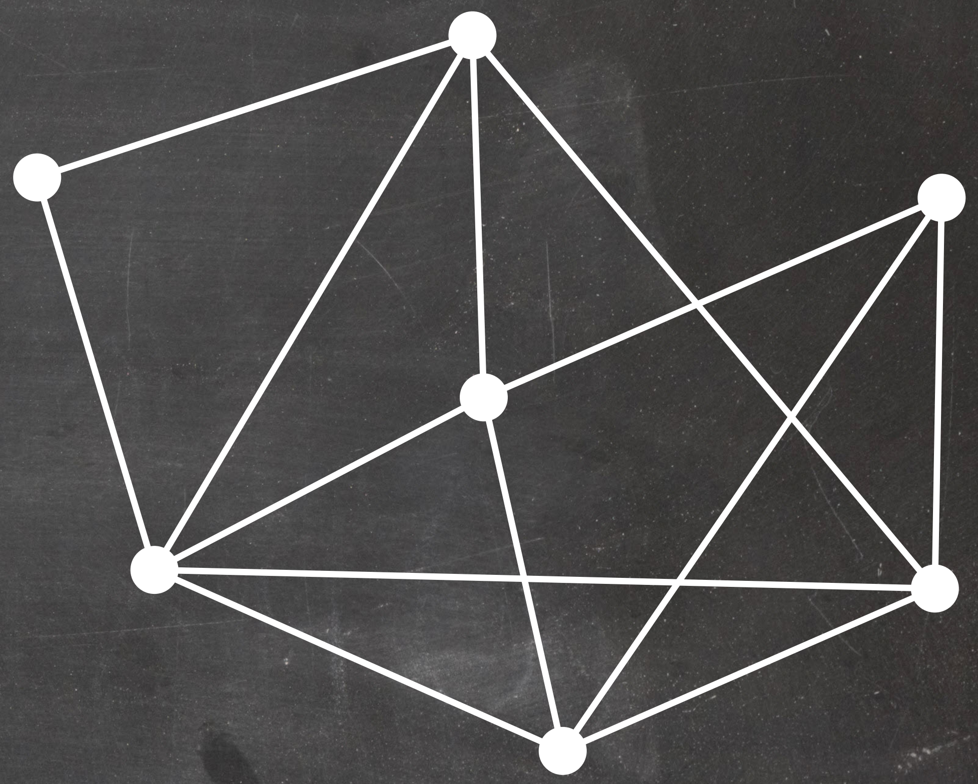

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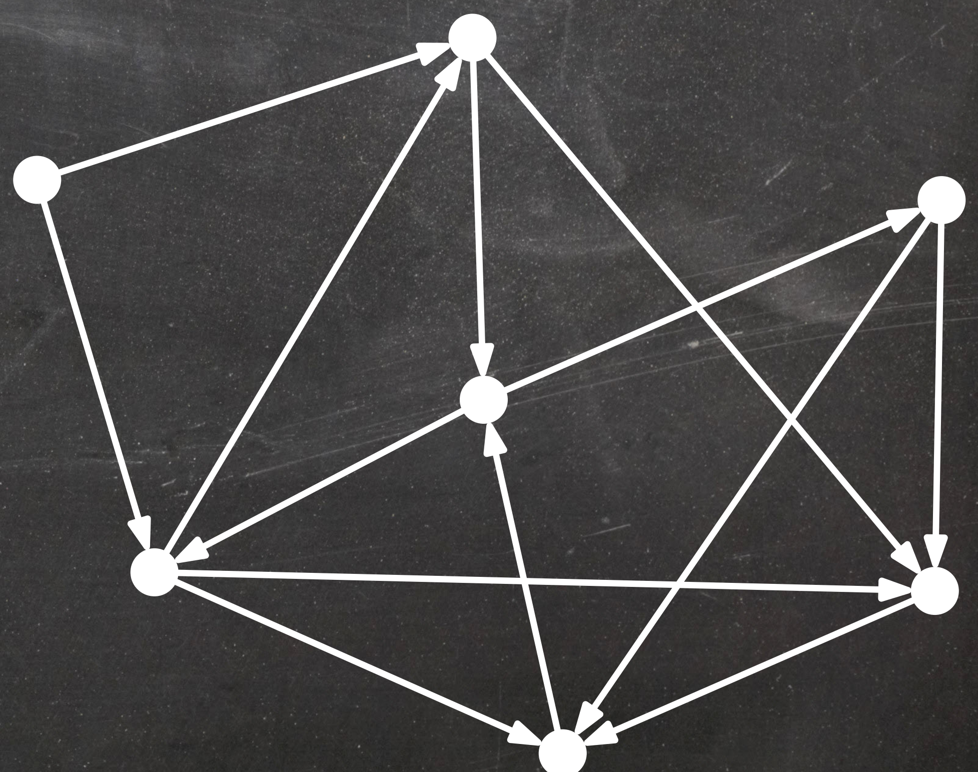

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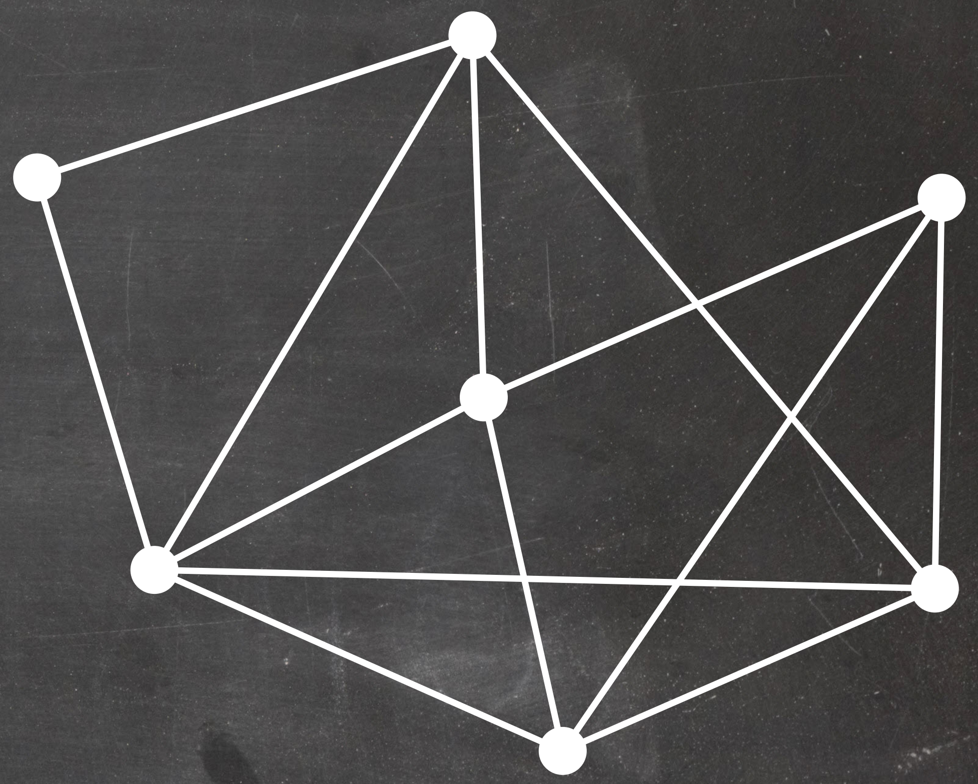

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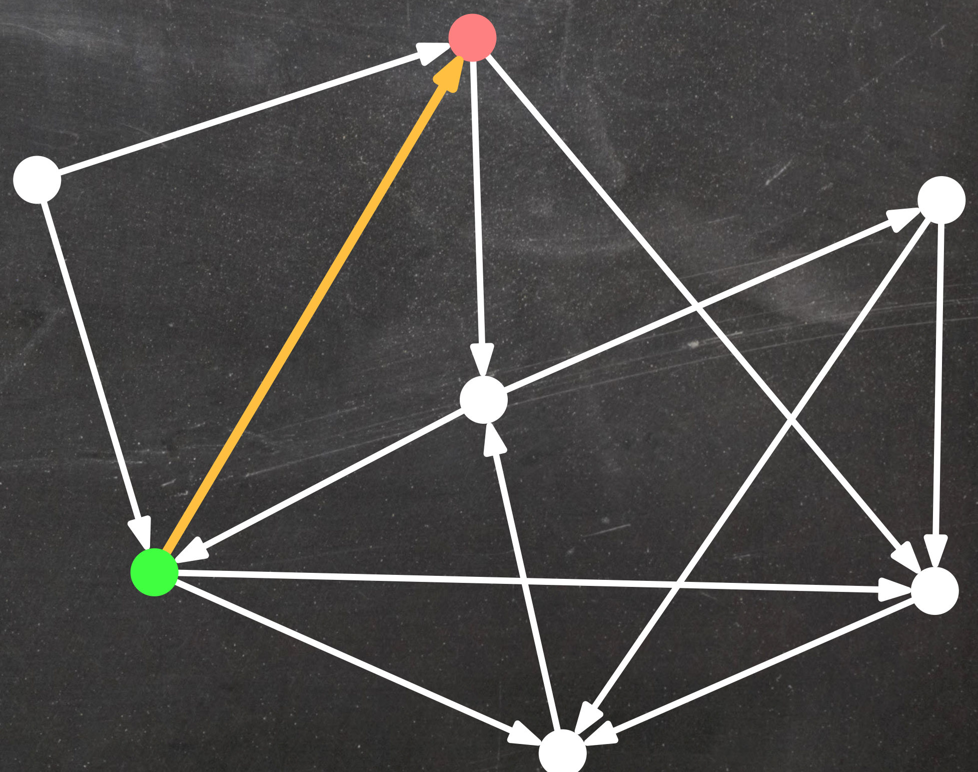
A directed edge (v, w) is an out-edge of v and an in-edge of w.



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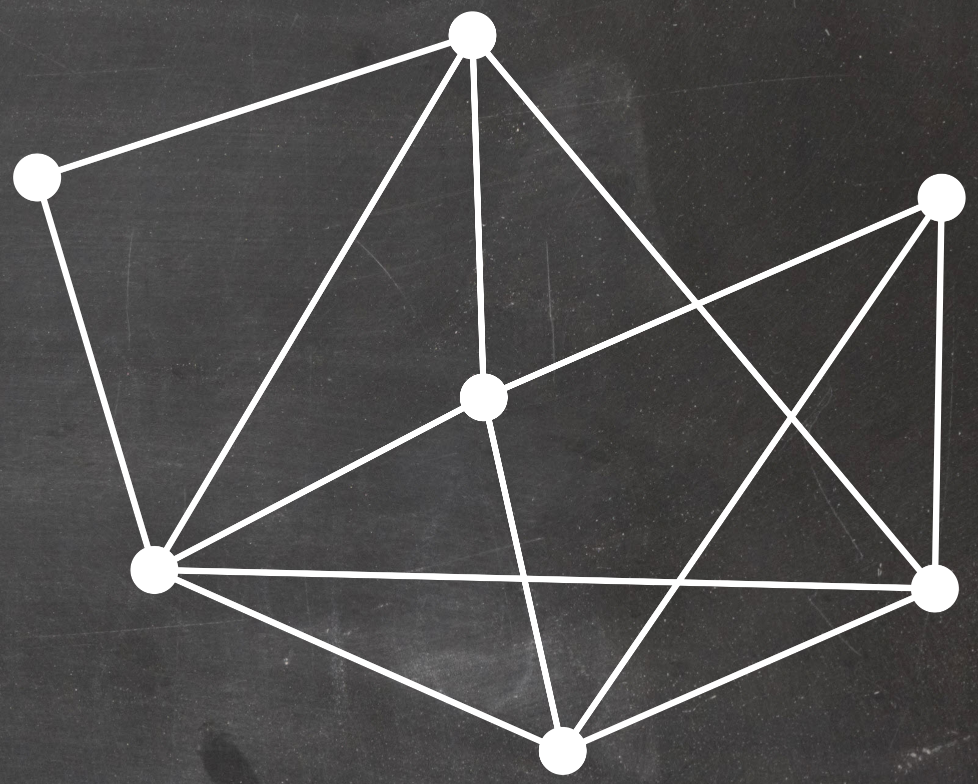

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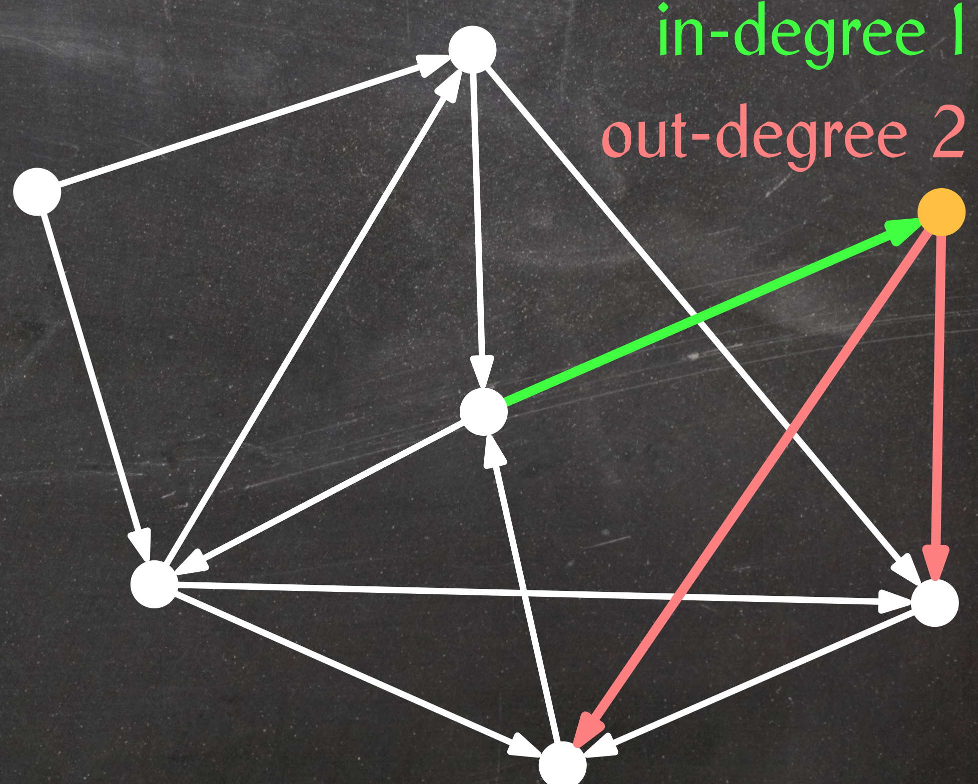

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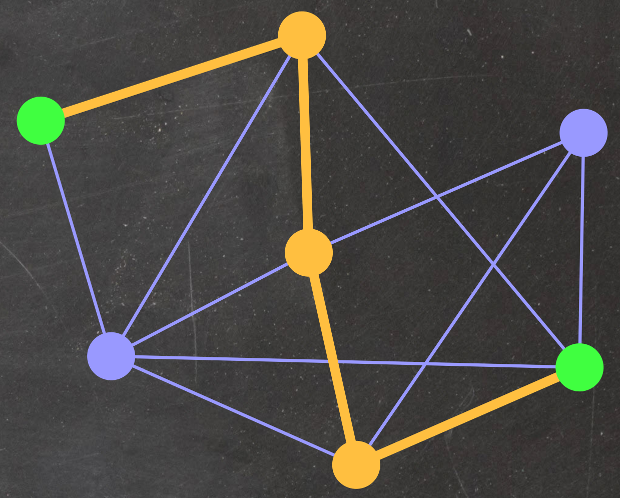
The in-degree and out-degree of a vertex are the numbers of its in-edges and out-edges, respectively.

Paths and Cycles

A path from a vertex s to a vertex t is a sequence of vertices $\langle x_0, x_1, \ldots, x_k \rangle$ such that

- $x_0 = s$,
- $x_k = t$, and
- for all $1 \le i \le k$, (x_{i-1}, x_i) is an edge of G.



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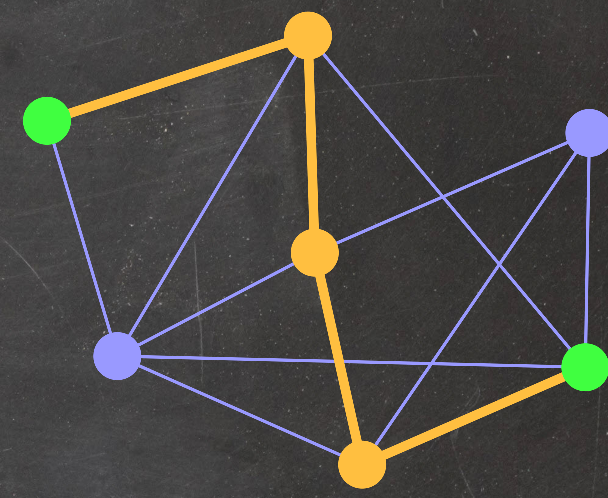
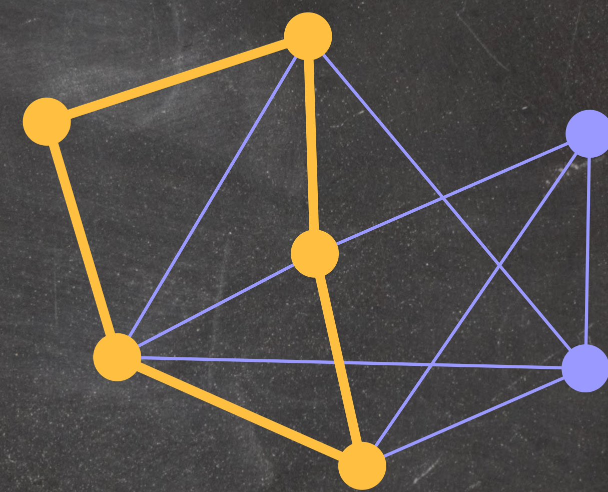


A cycle is a path from a vertex x back to itself.



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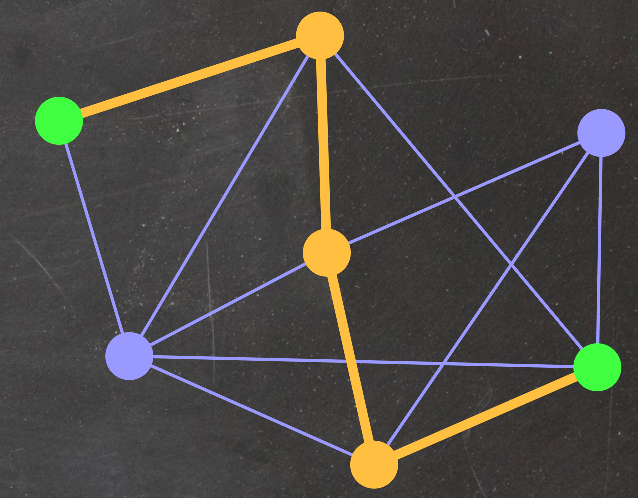

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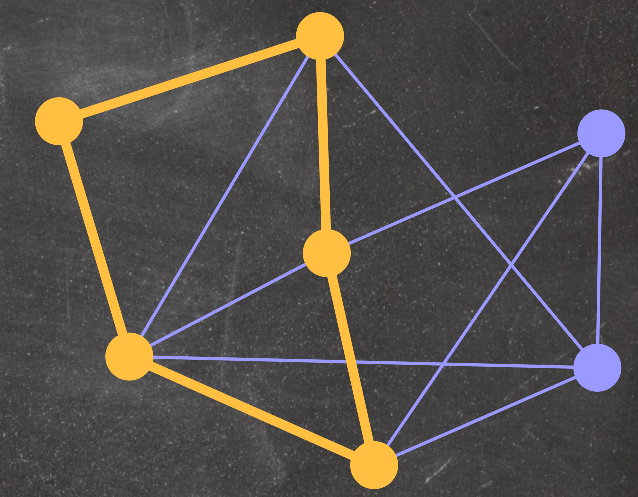
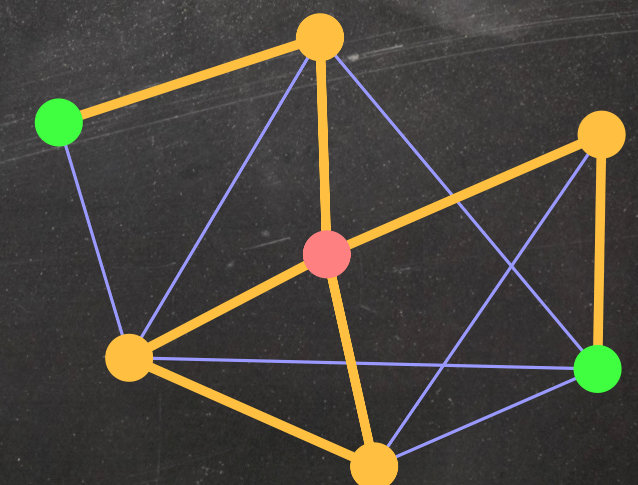
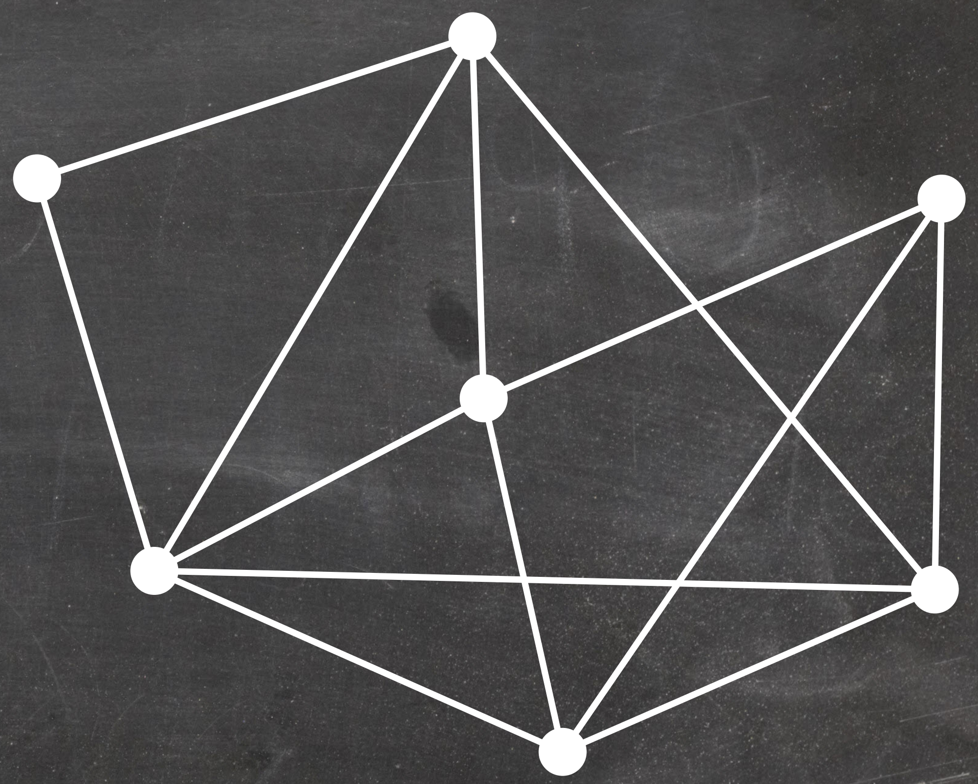


A path or cycle is simple if it contains every vertex of G at most once.



Connected Graphs, Trees, and Forests

A graph is connected if there exists a path between every pair of vertices.



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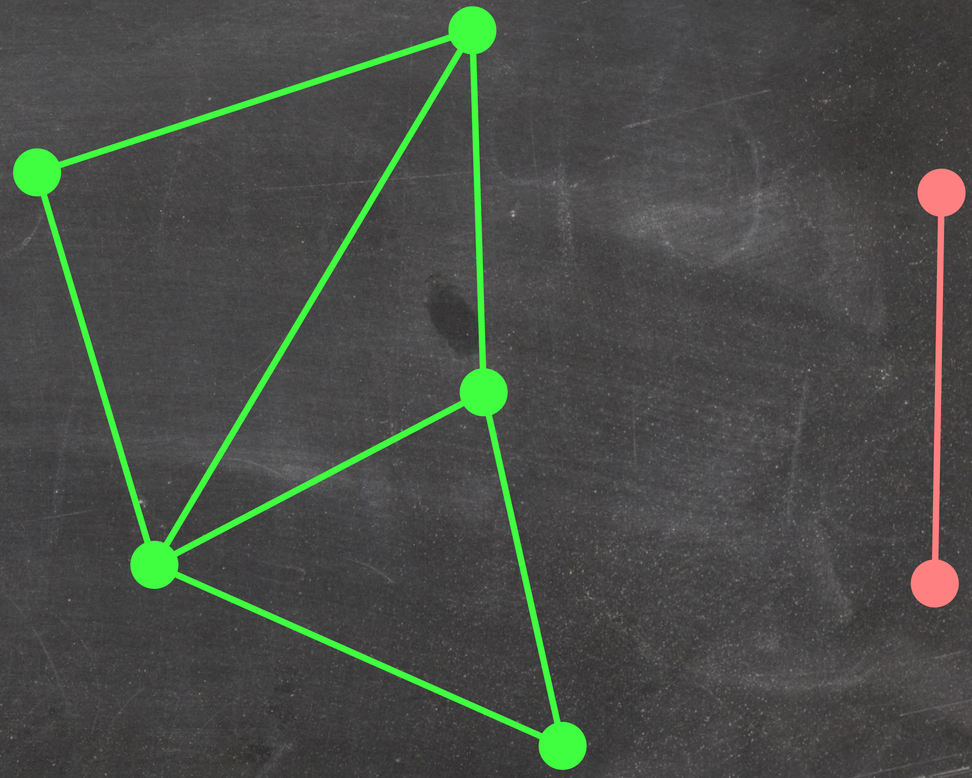

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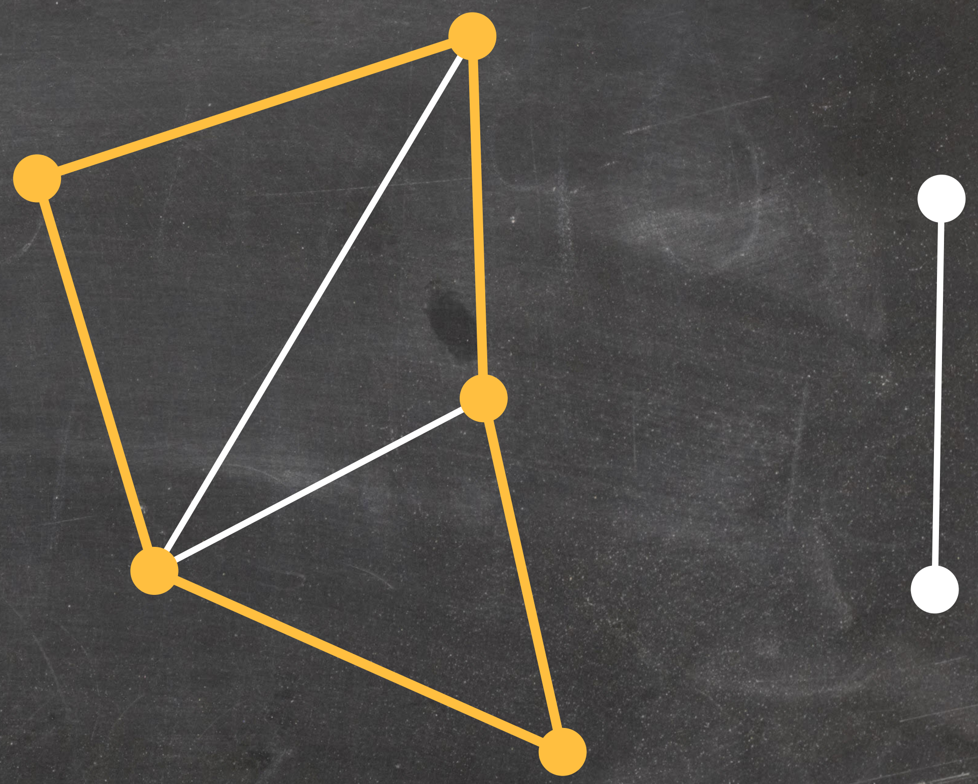
A forest is a graph without cycles.



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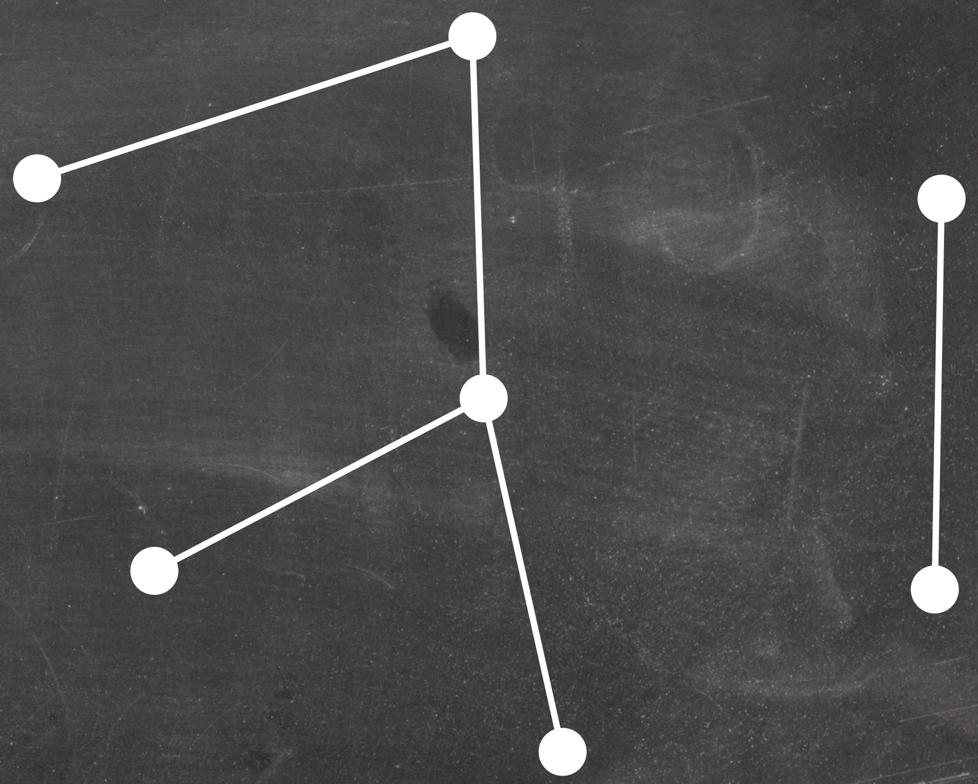

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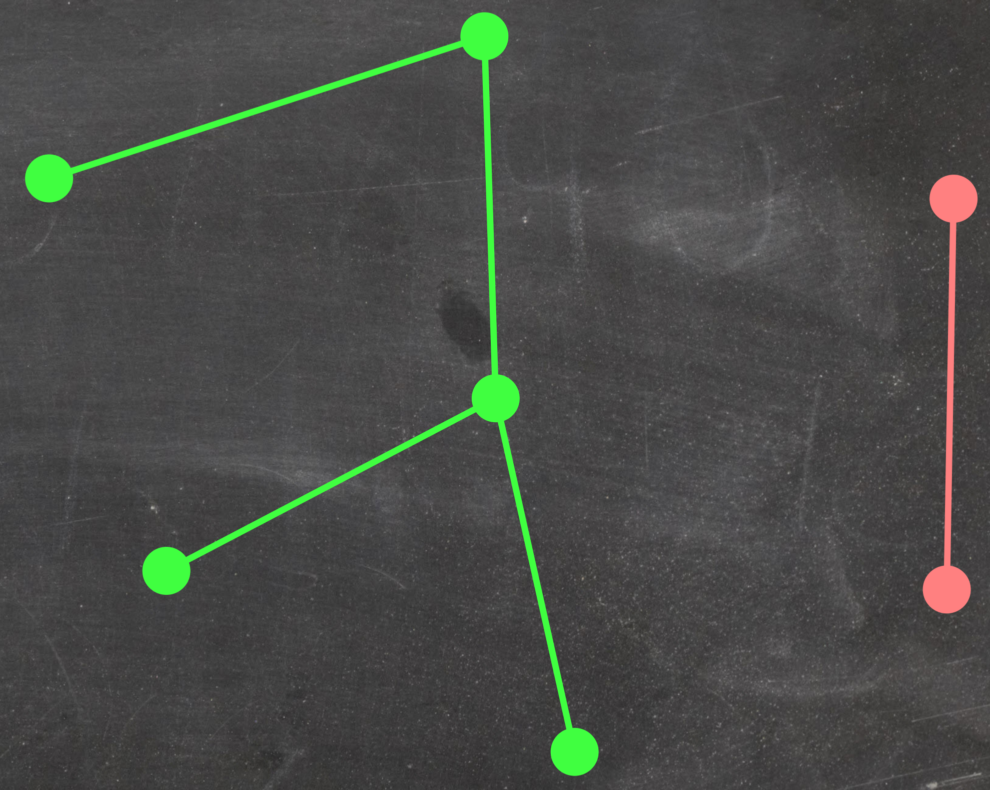

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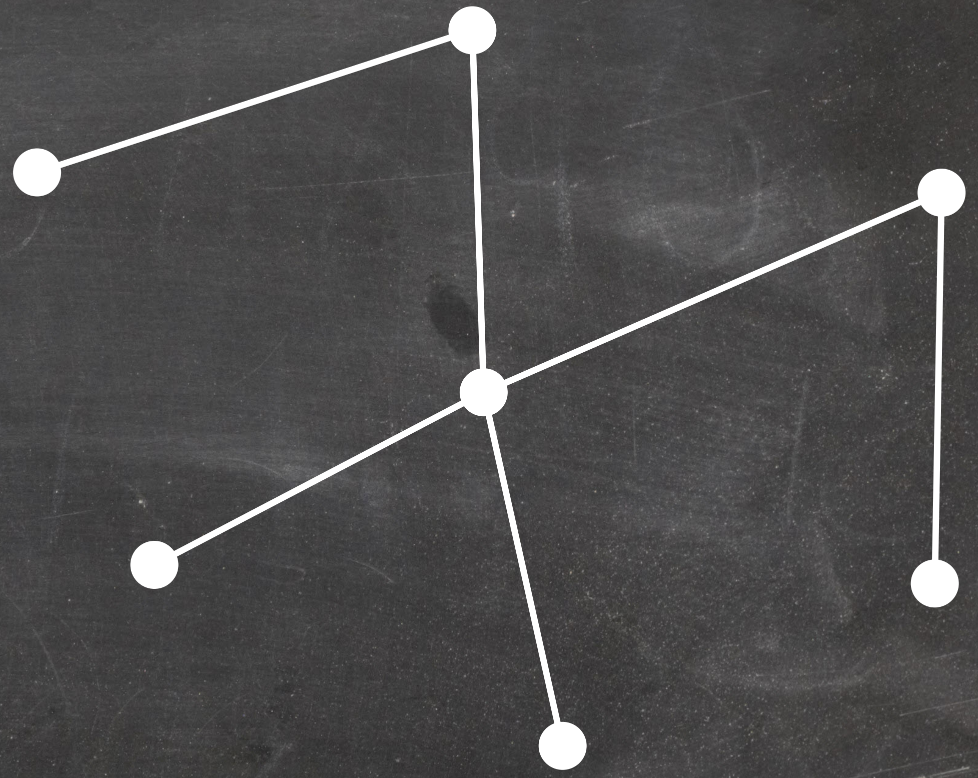
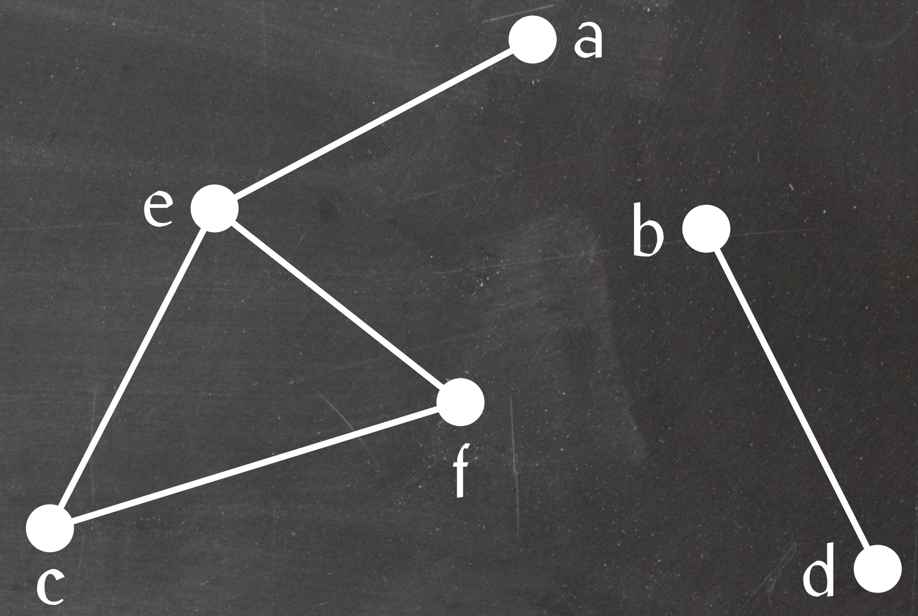
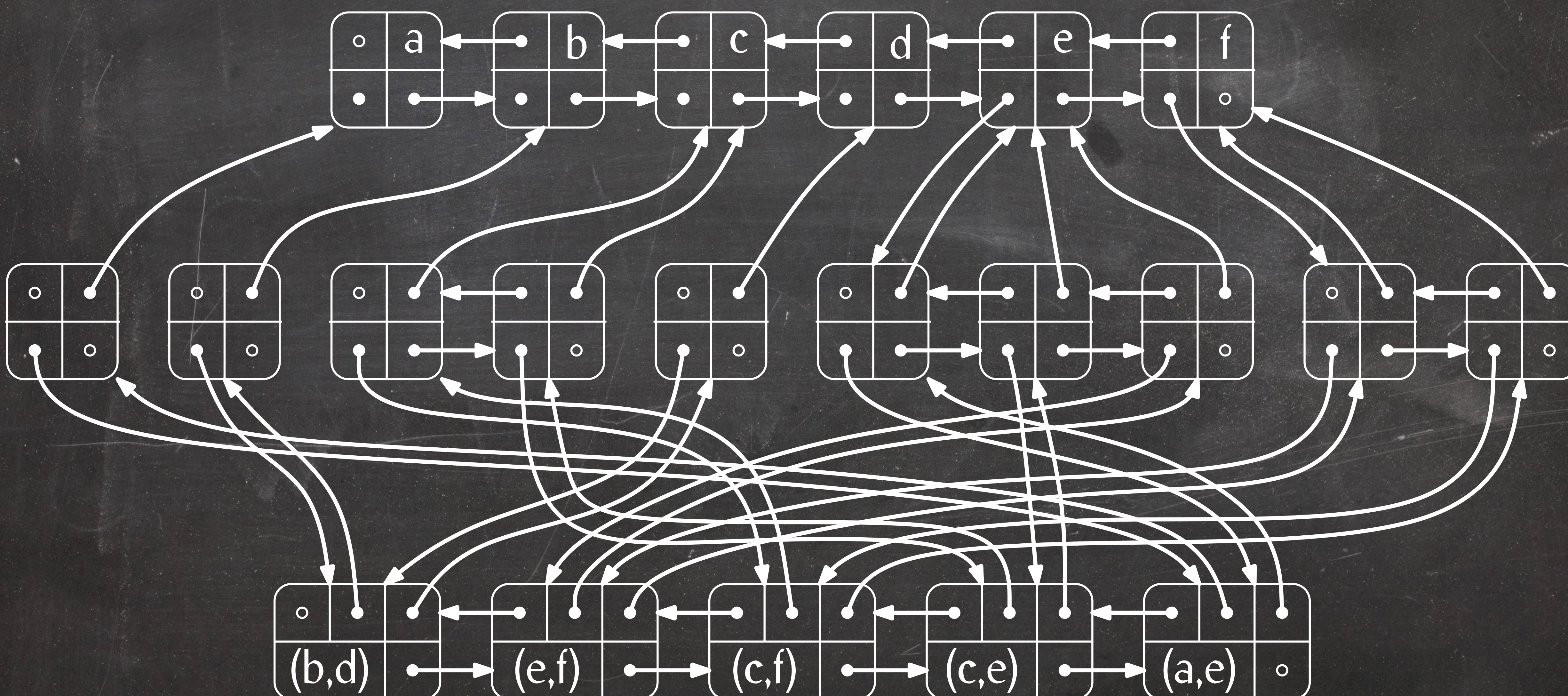


Adjacency List Representation



- Doubly-linked list of vertices
- Doubly-linked list of edges
- One doubly-linked adjacency list per vertex
- Pointers from adjacency list entries to vertices
- Cross-pointers between edges and adjacency list entries



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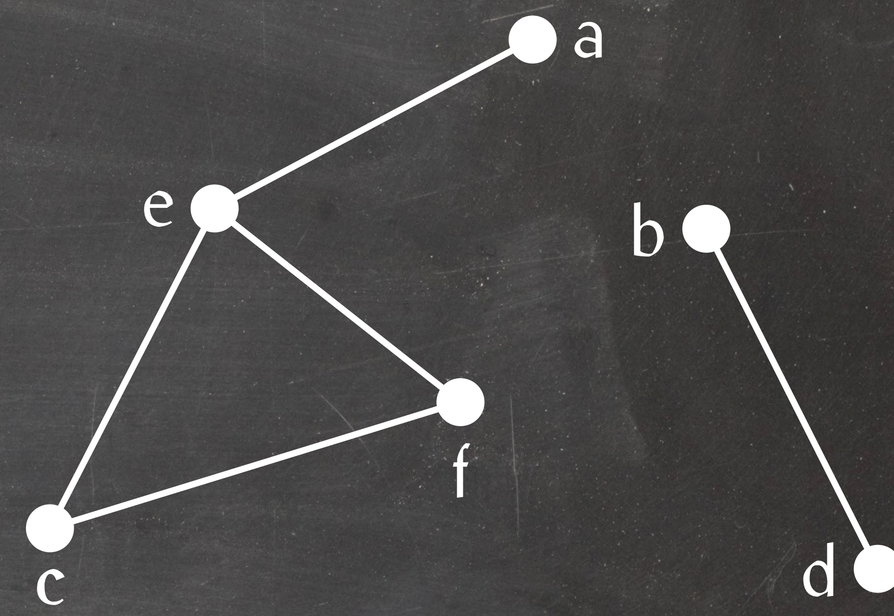

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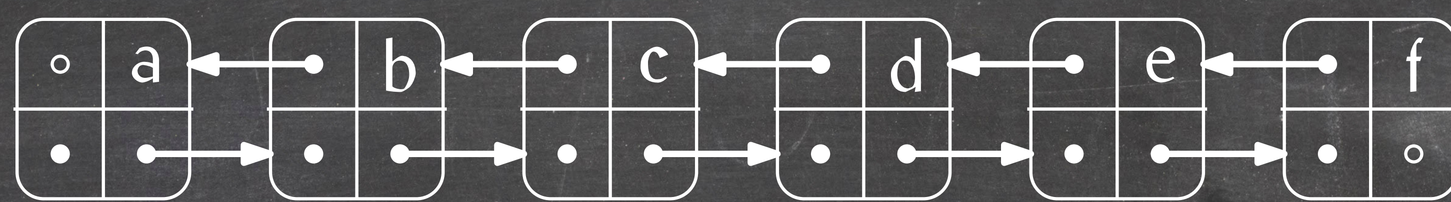

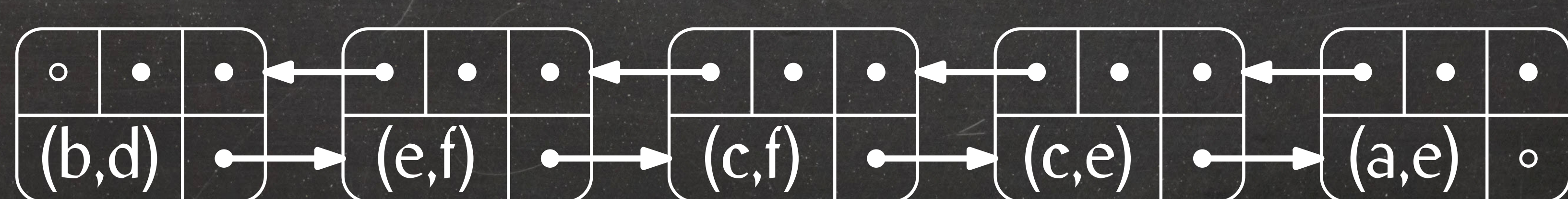

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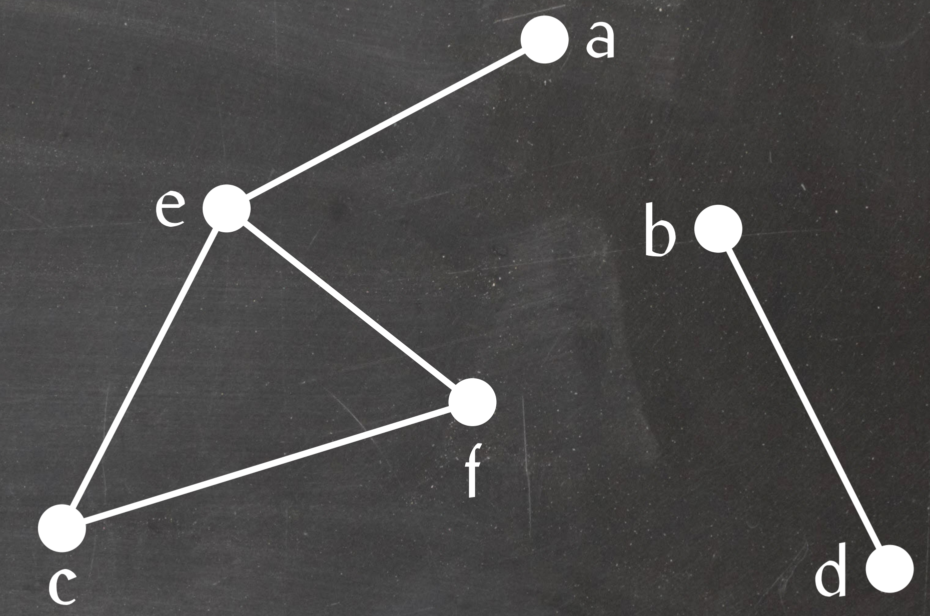

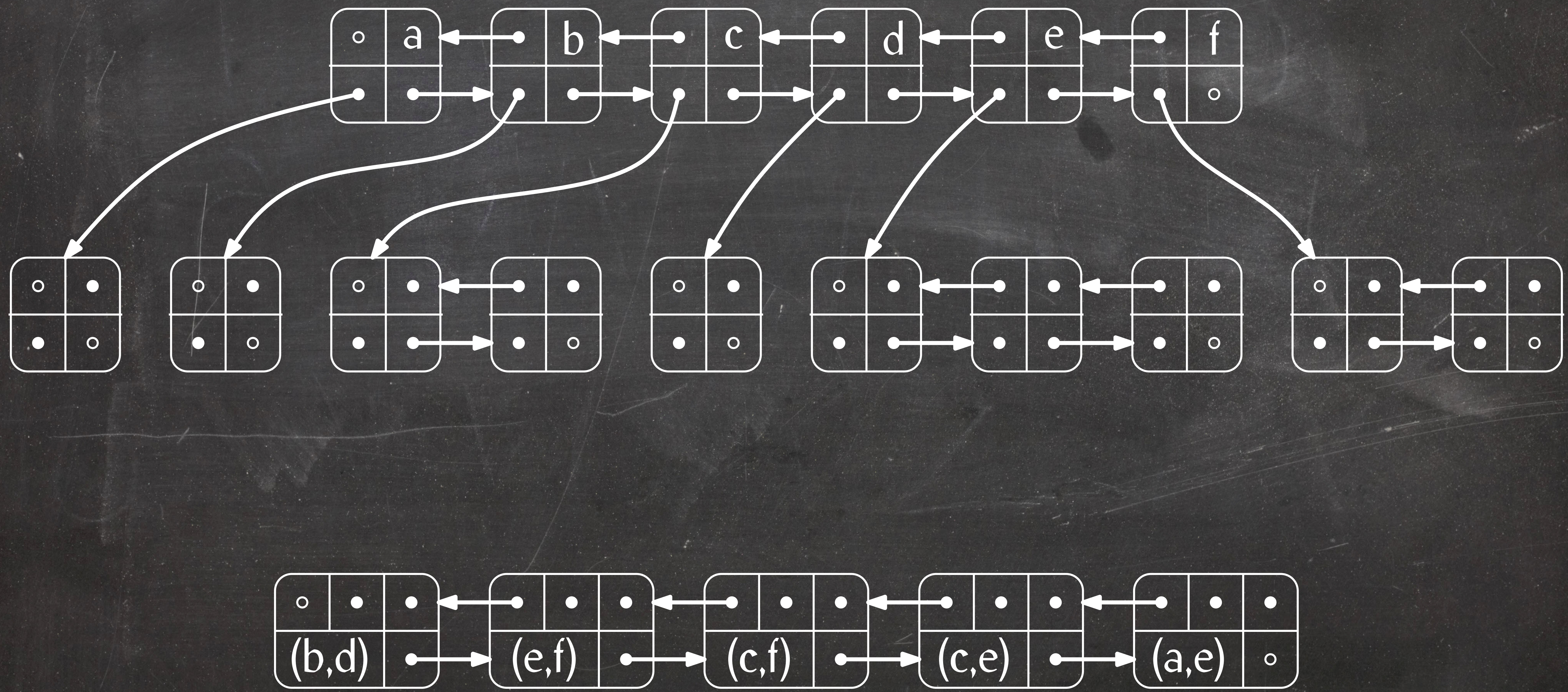

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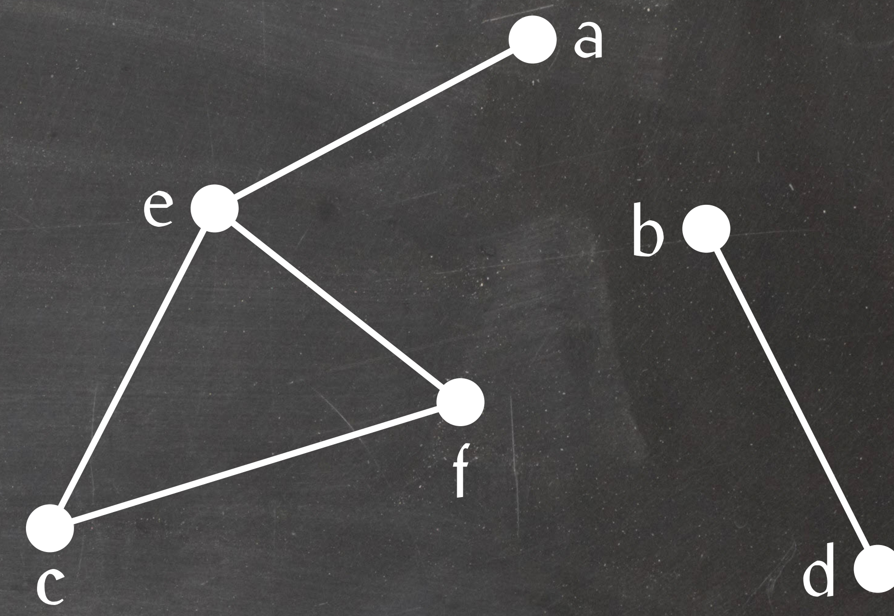
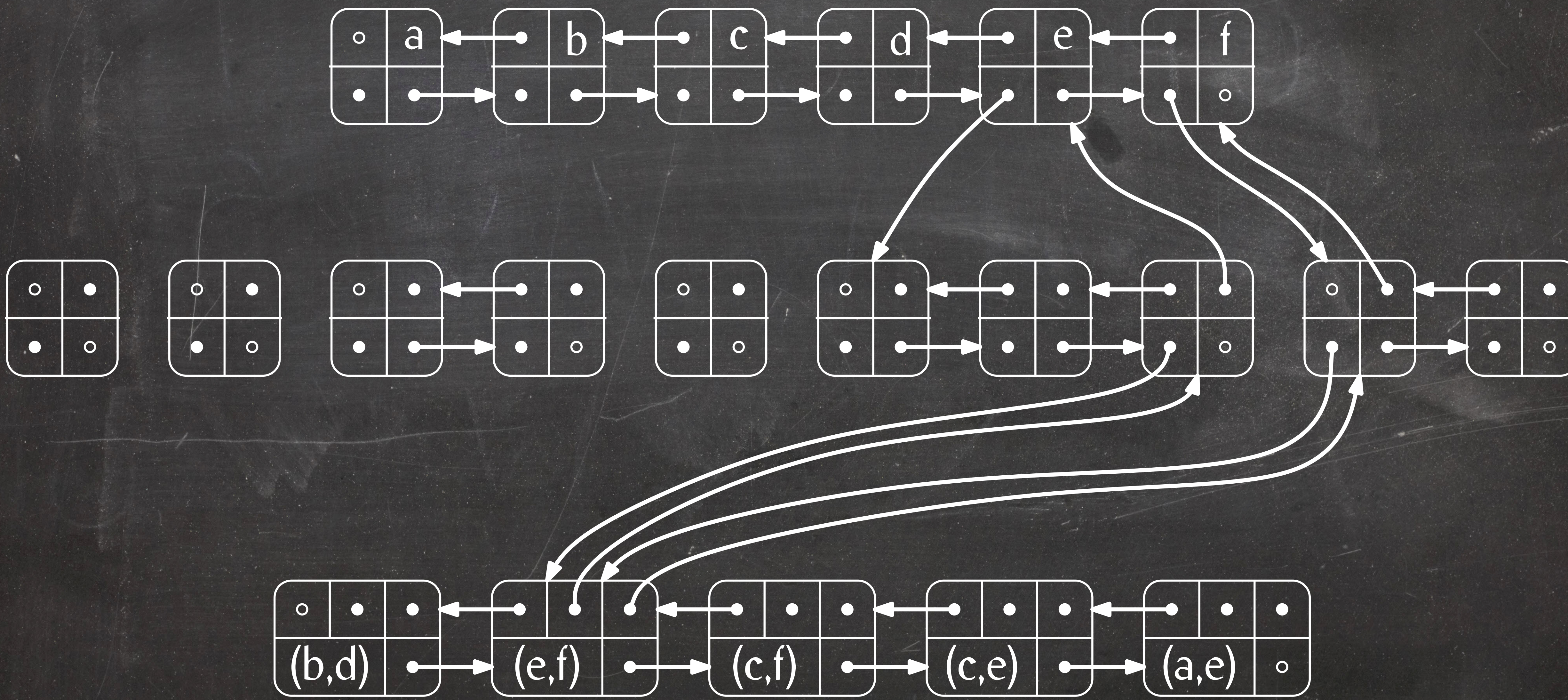


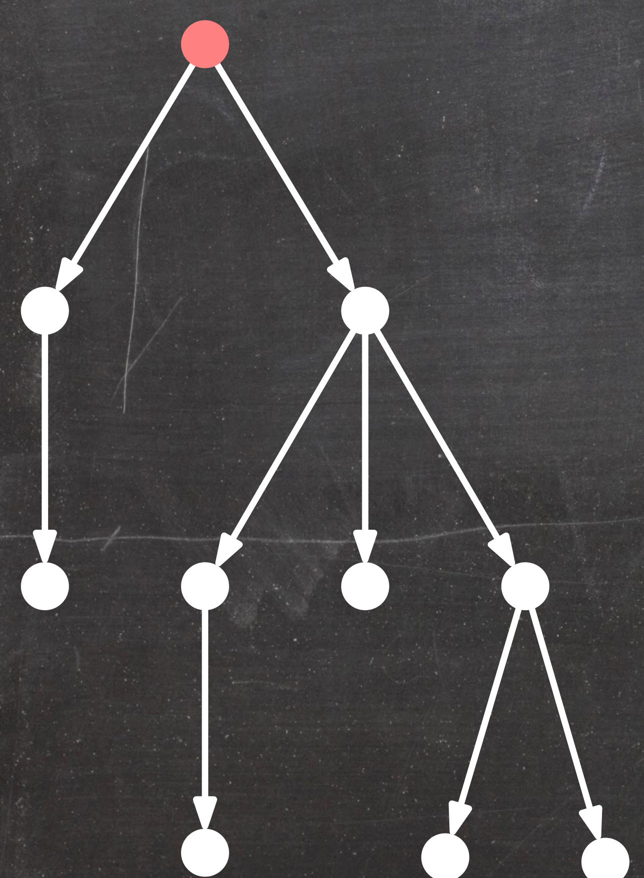


Representing Rooted Trees

A rooted tree T

- is a tree,
- is a directed graph,
- has one of its vertices, r, designated as a root.

There exists a path from r to every vertex in T.



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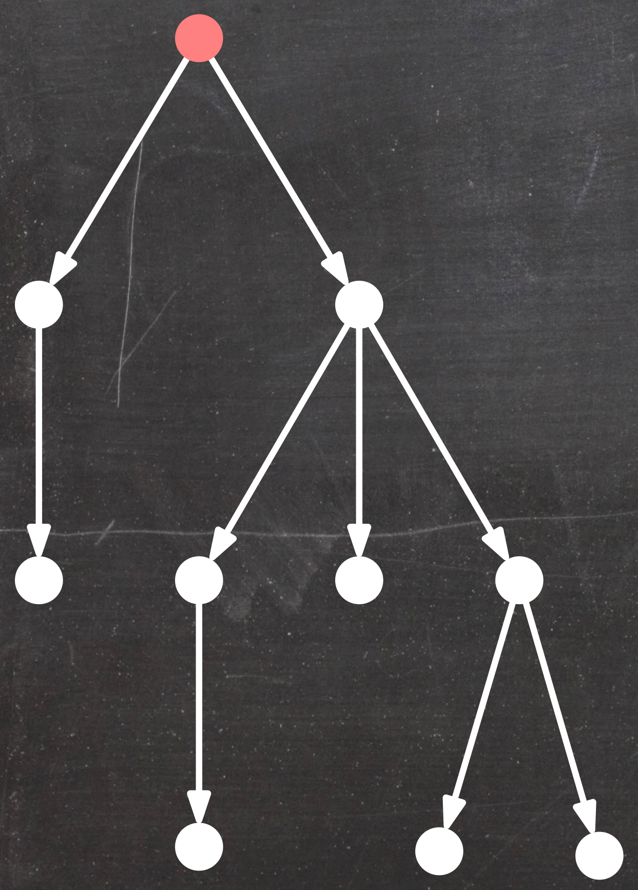


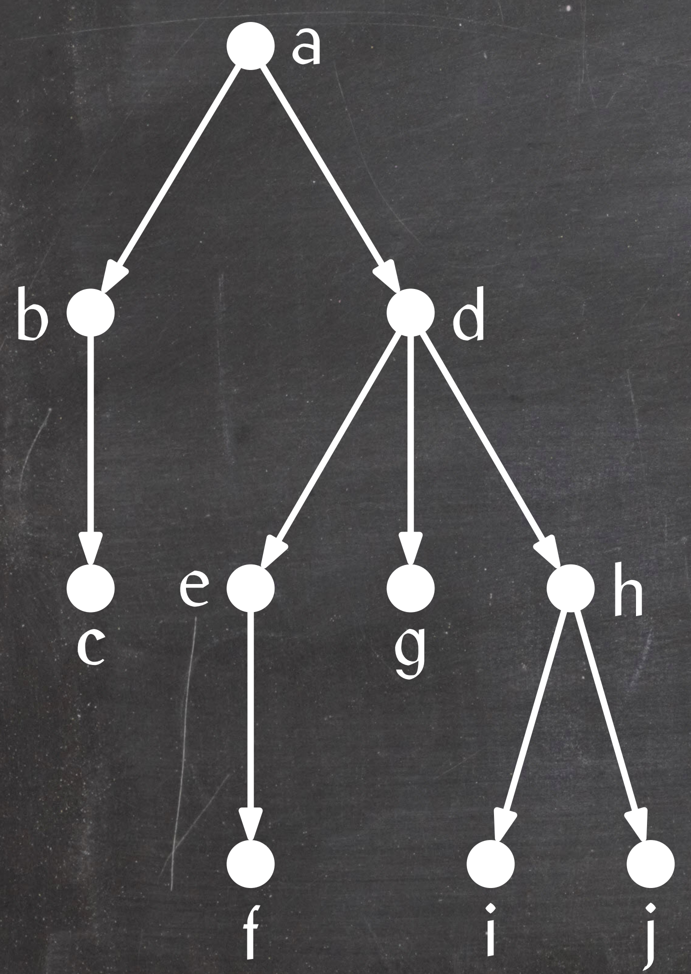
Representation:

Tree = root

Every node stores

- an arbitrary key
- a (doubly-linked) list of its children.

Standard Tree Orderings



Preorder:

- Every vertex appears before its children.
- Every vertex appears before its right sibling.
- The vertices in each subtree appear consecutively.

$\Rightarrow$ [a, b, c, d, e, f, g, h, i, j]

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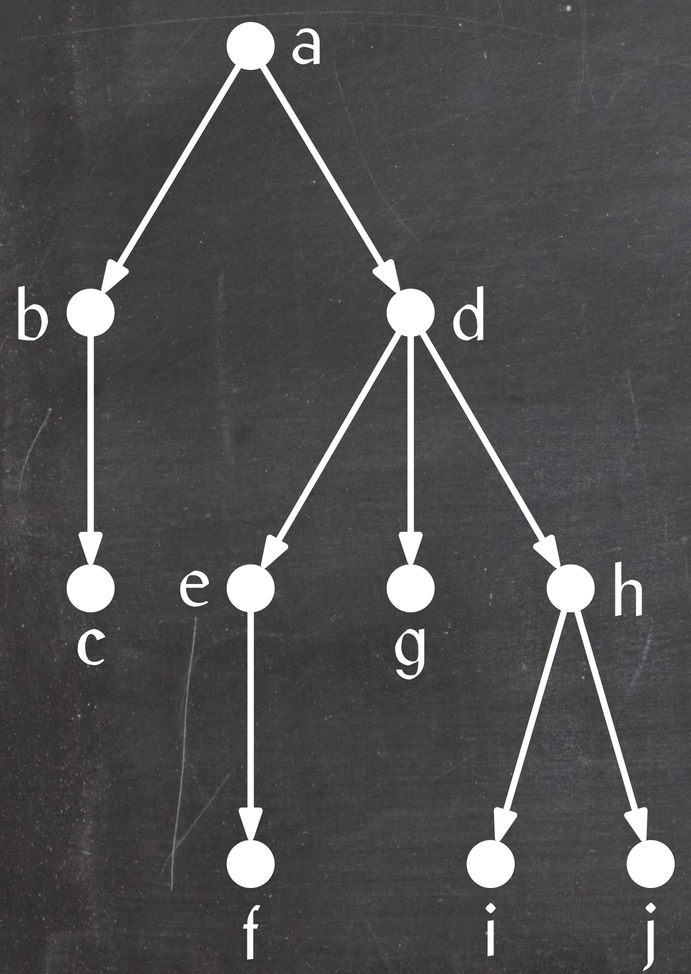

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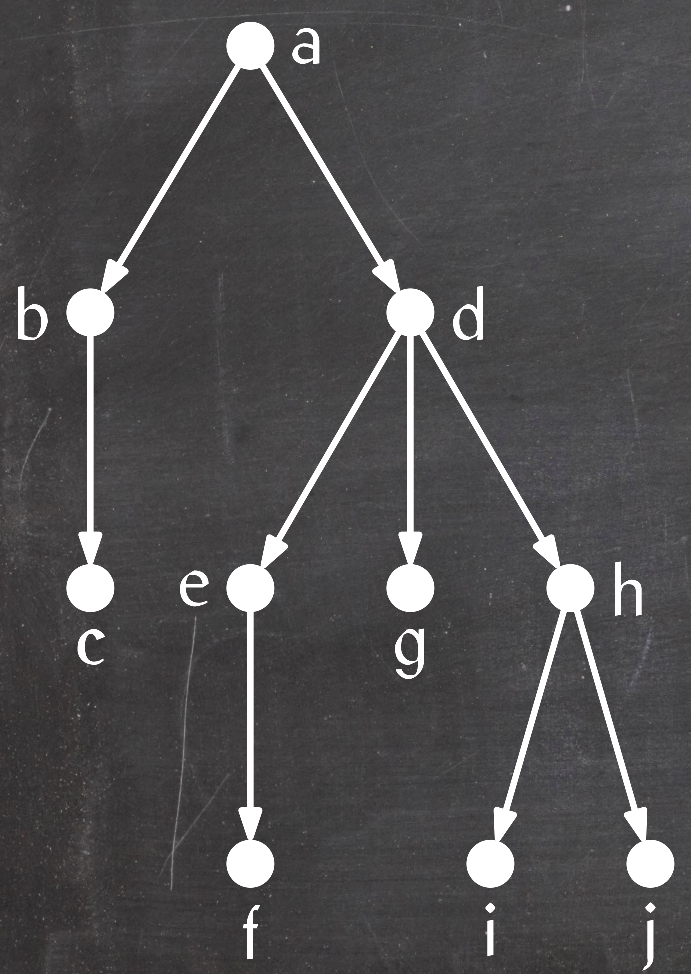

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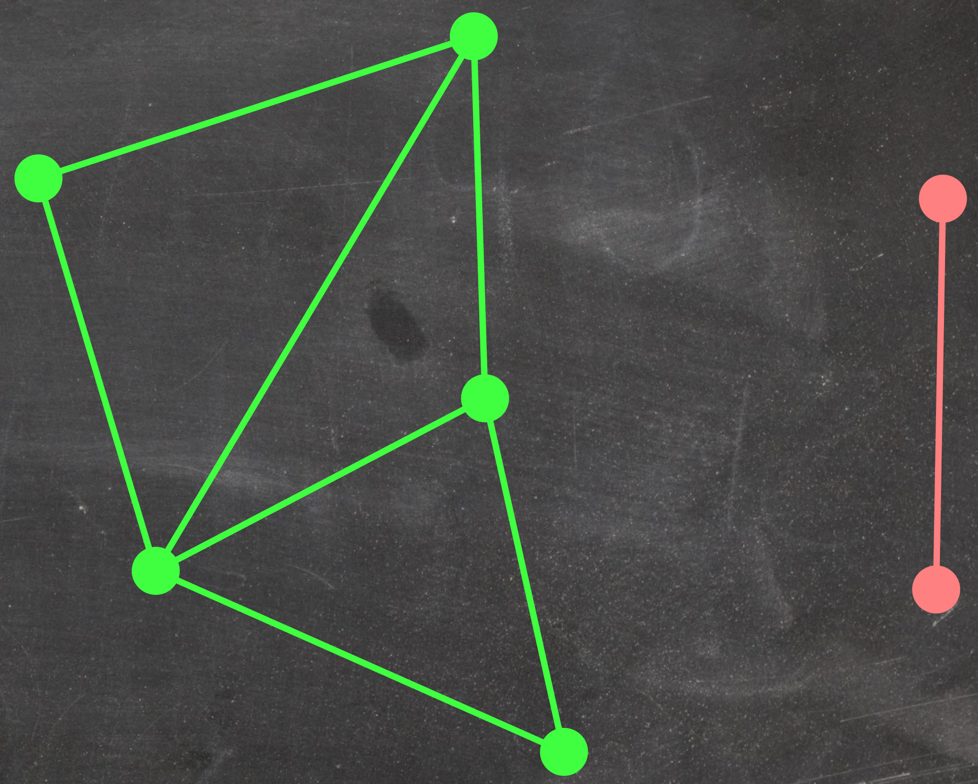
Lemma: It takes linear time to arrange the vertices of a forest in preorder or postorder.

Connected Components and Spanning Forests

The connected components of a graph G are its maximal connected subgraphs.



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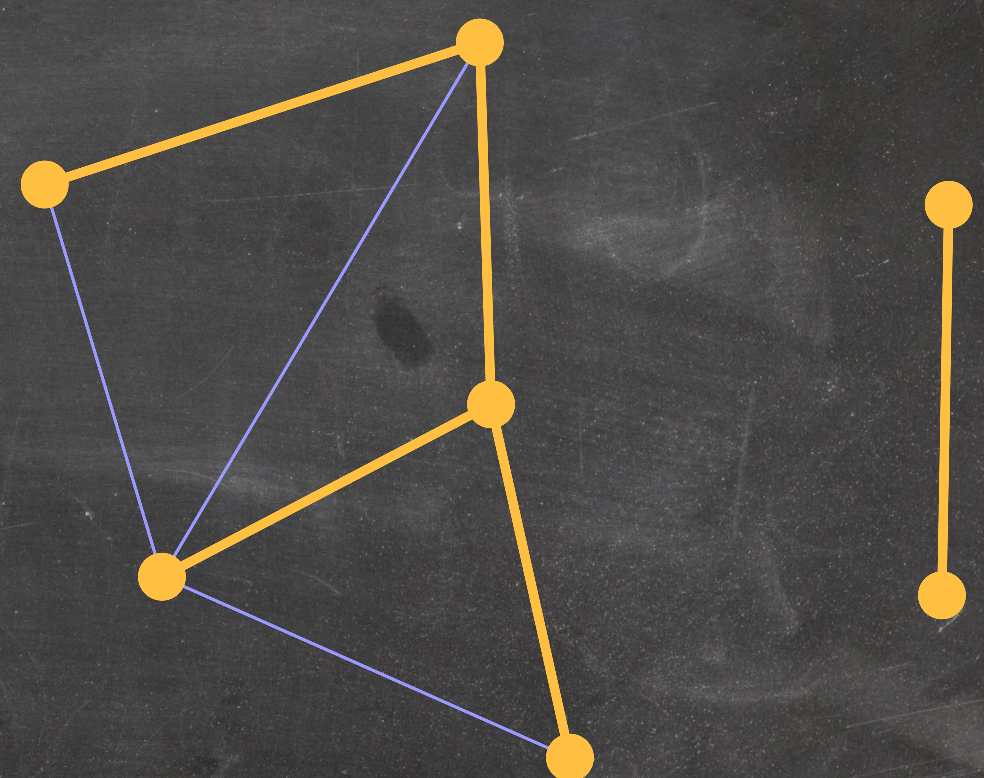
A spanning forest of a graph G is a subgraph $F \subseteq G$ with the same number of connected components and which is a forest.



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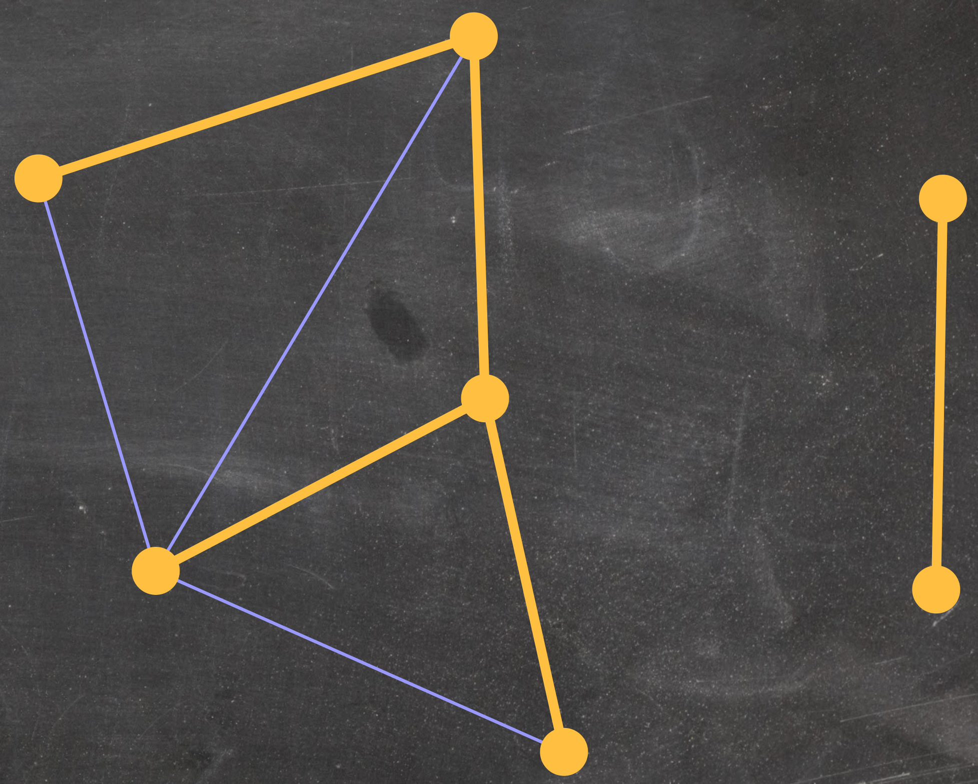

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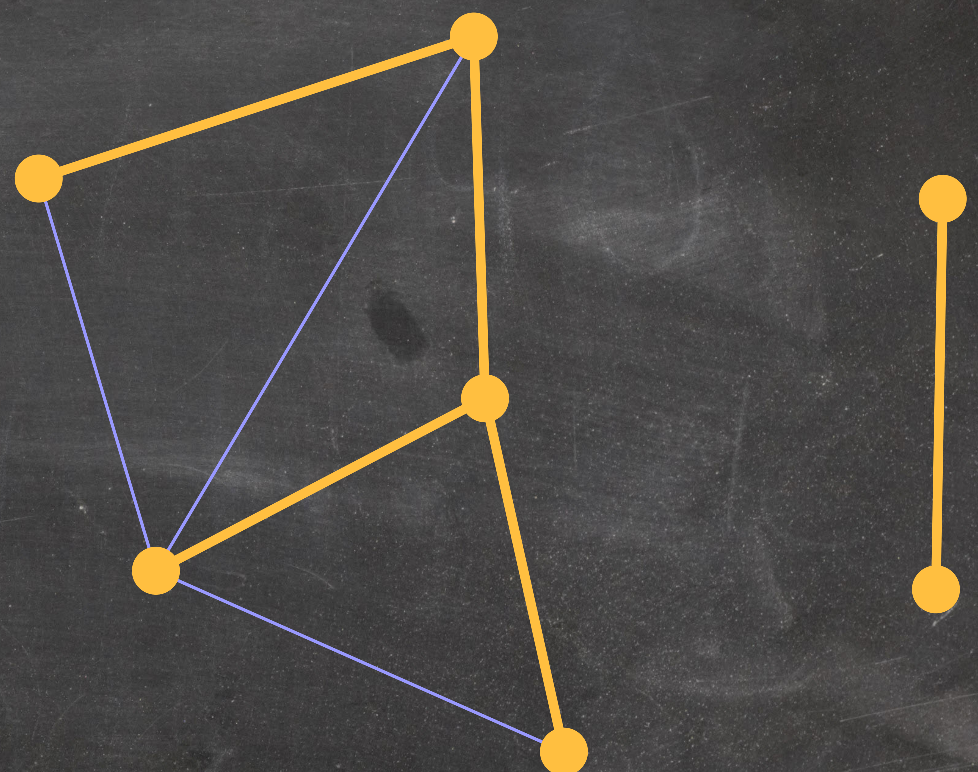
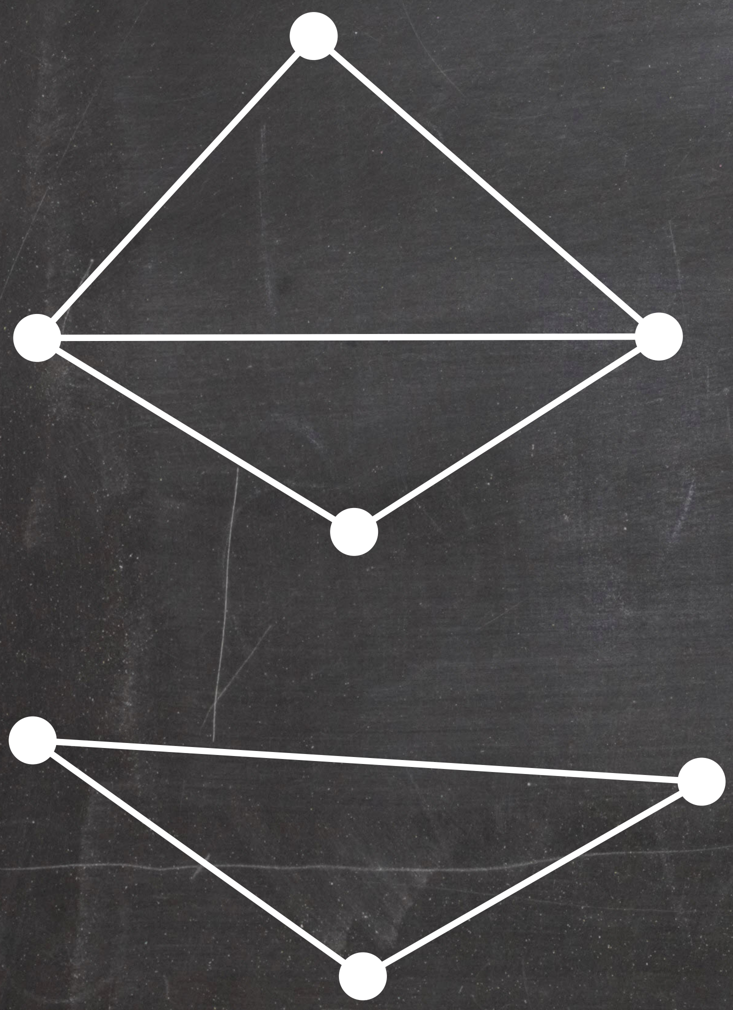
Representation: List of rooted trees



Graph Traversal

We use graph traversal to build a spanning forest of G.



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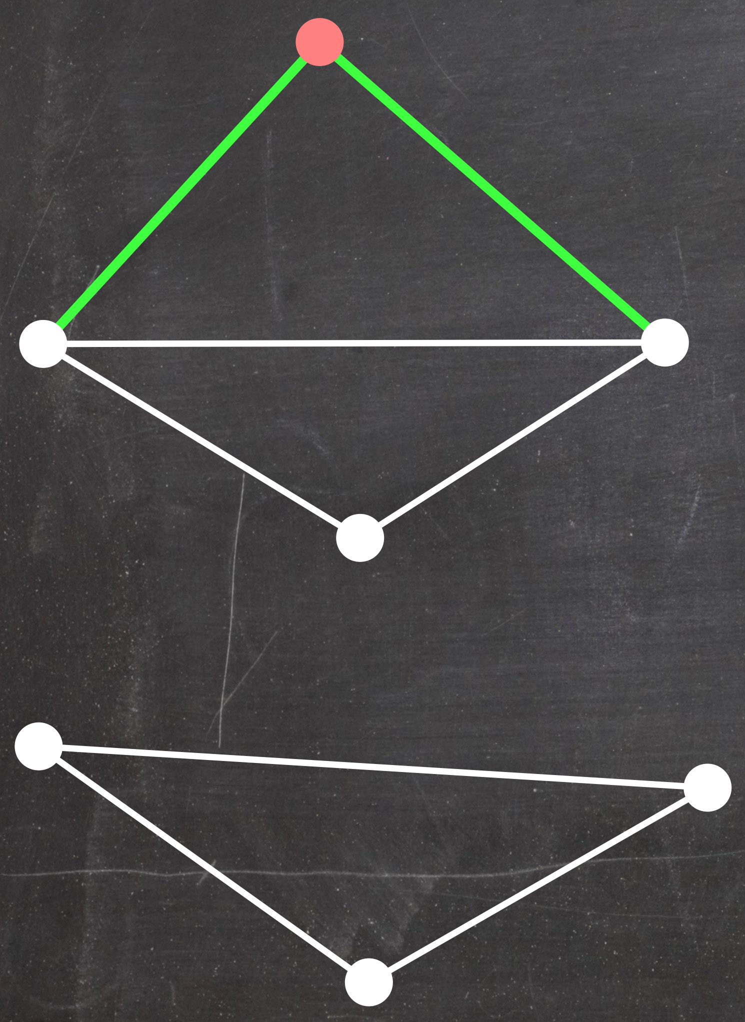

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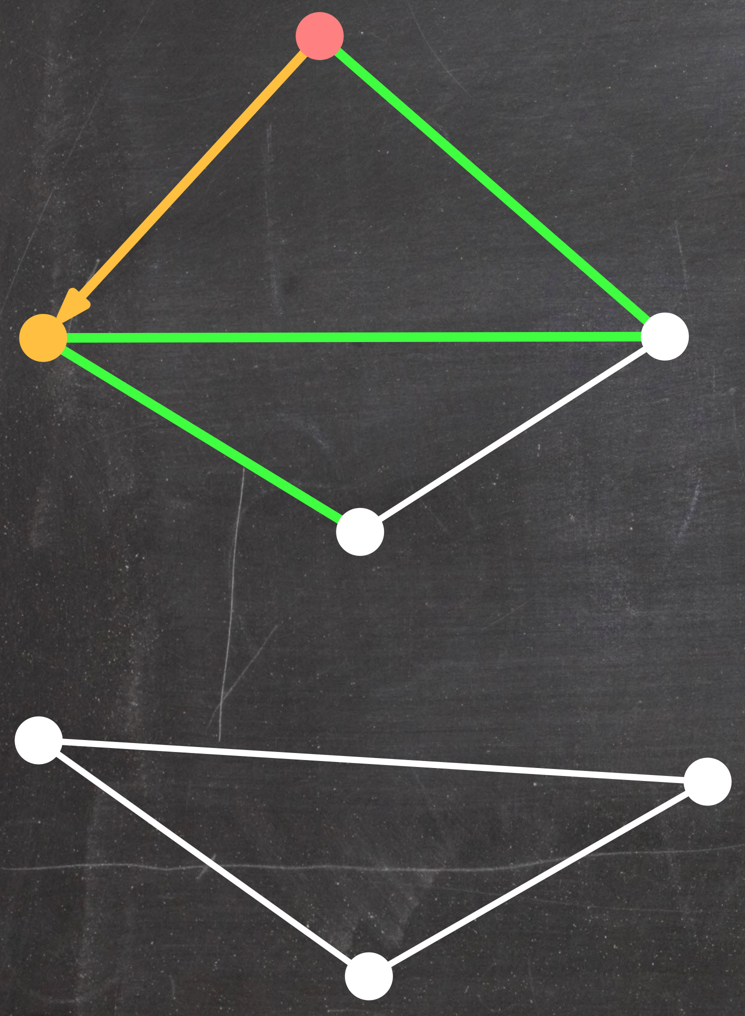

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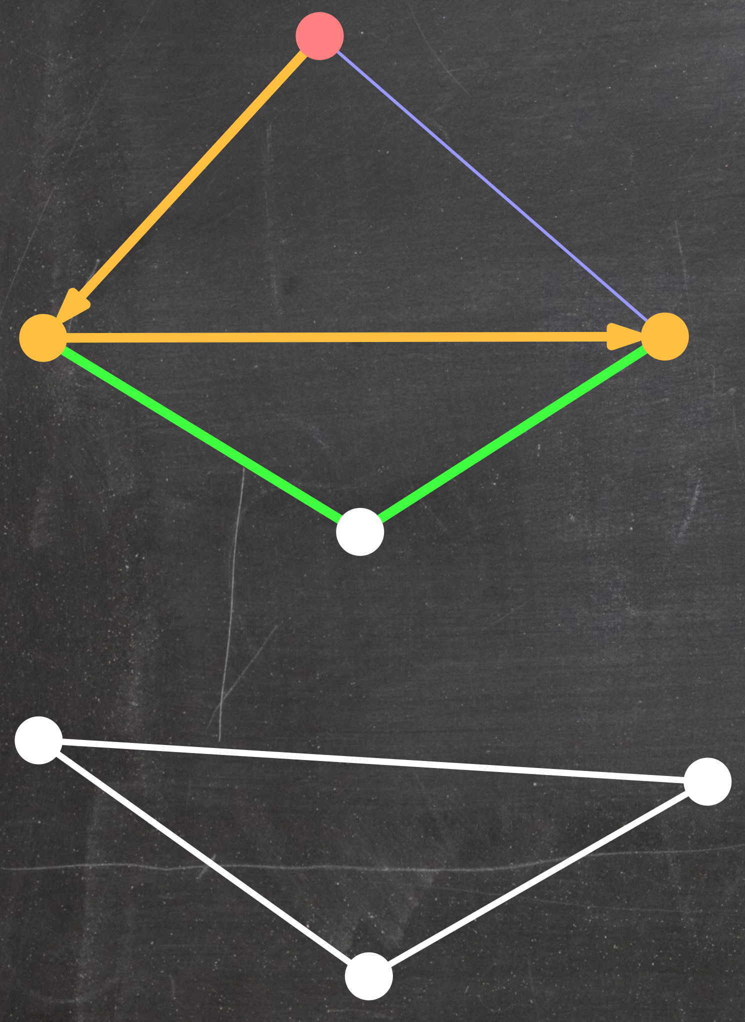

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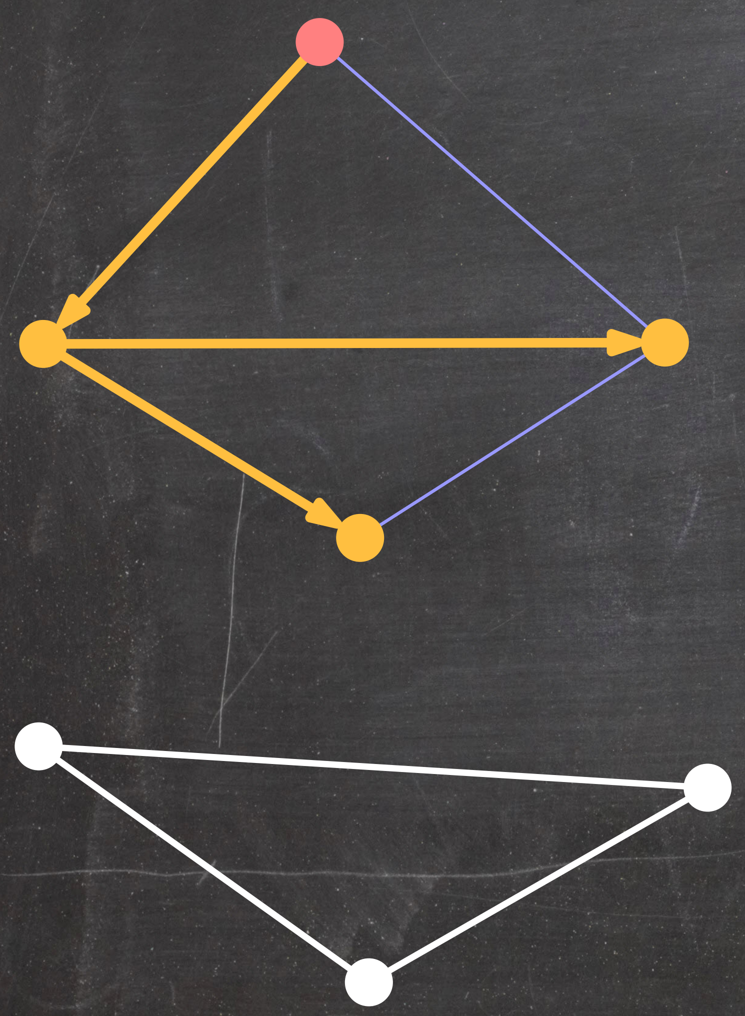

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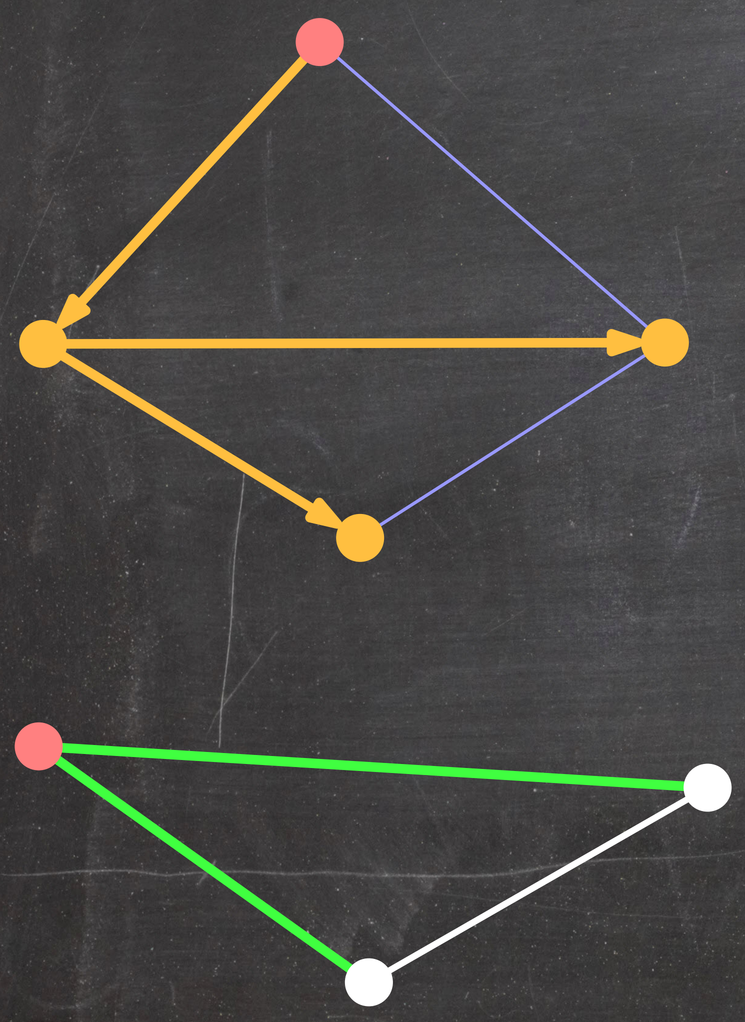

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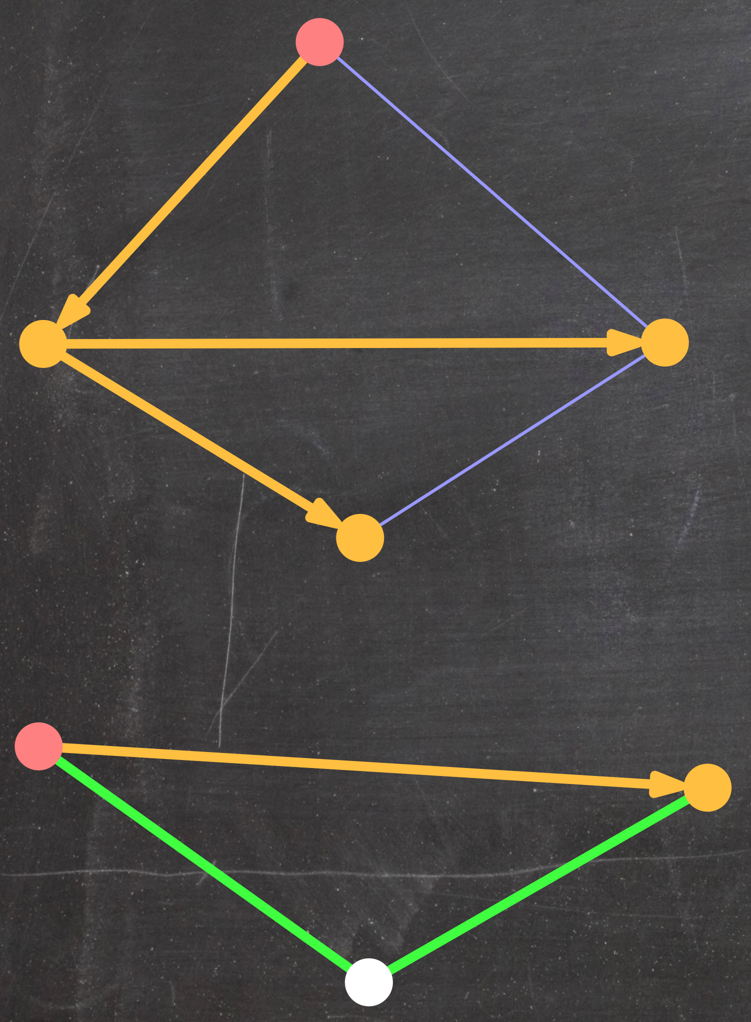

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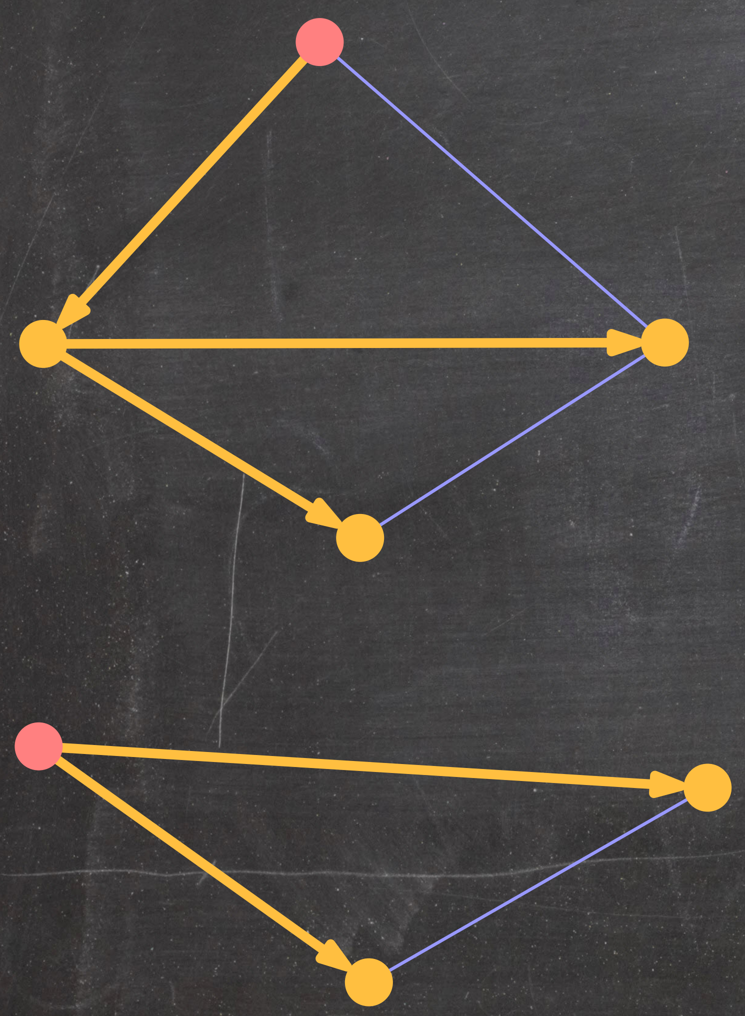

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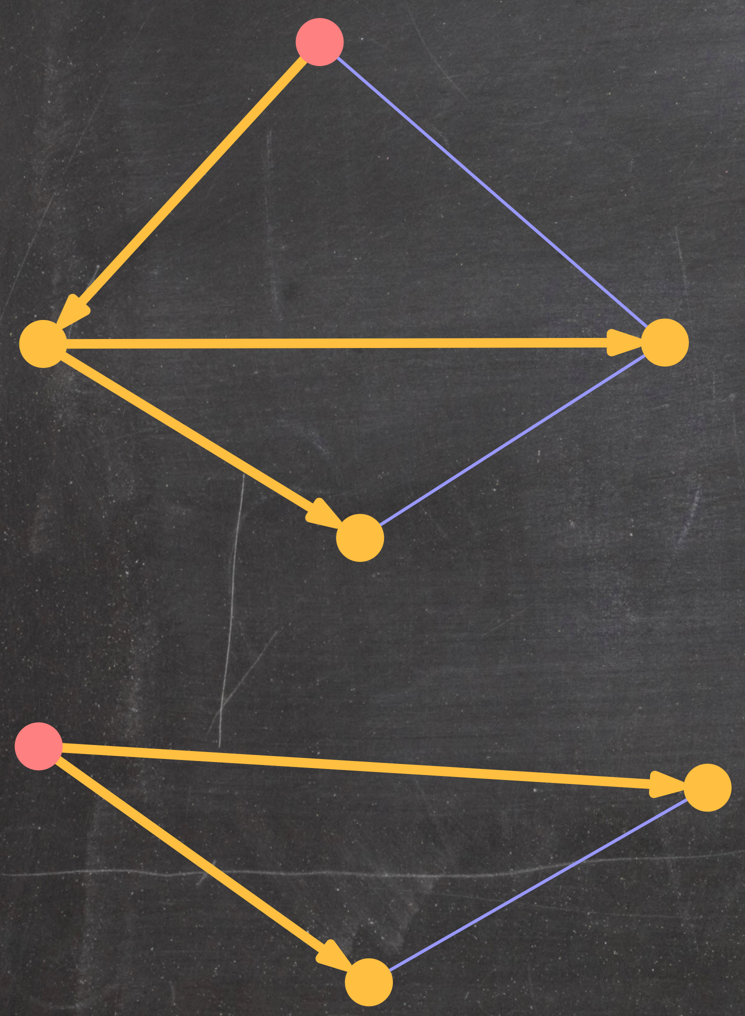

Different traversal strategies lead to different spanning forests:

- Breadth-first search
- Depth-first search
- Prim's algorithm for computing minimum spanning trees
- Dijkstra's algorithm for computing shortest paths

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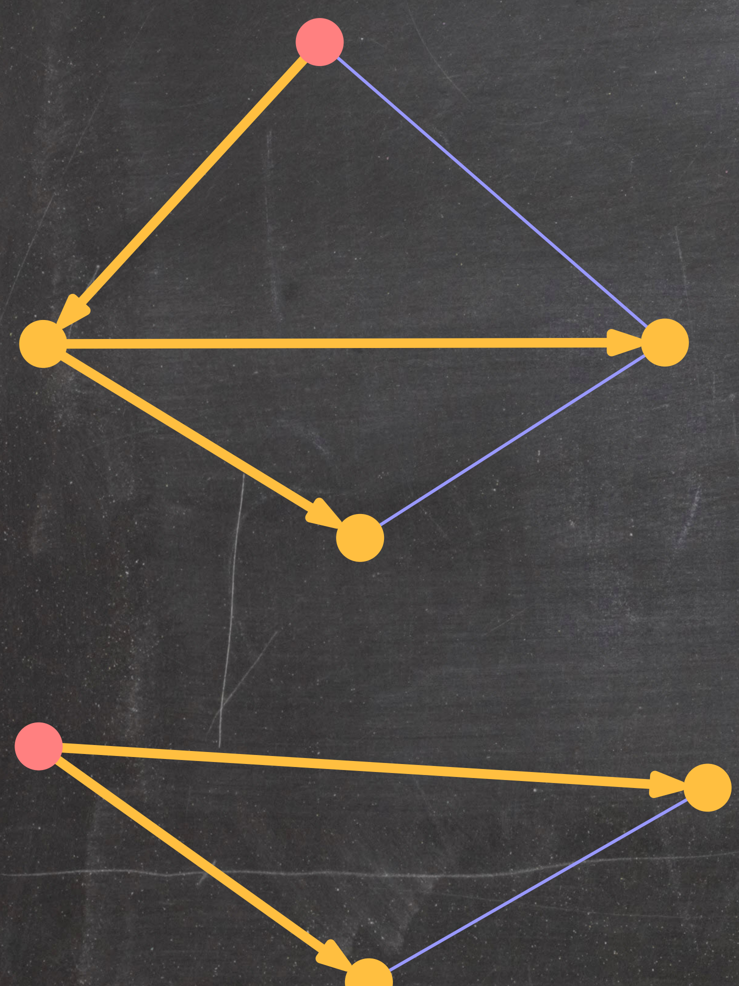
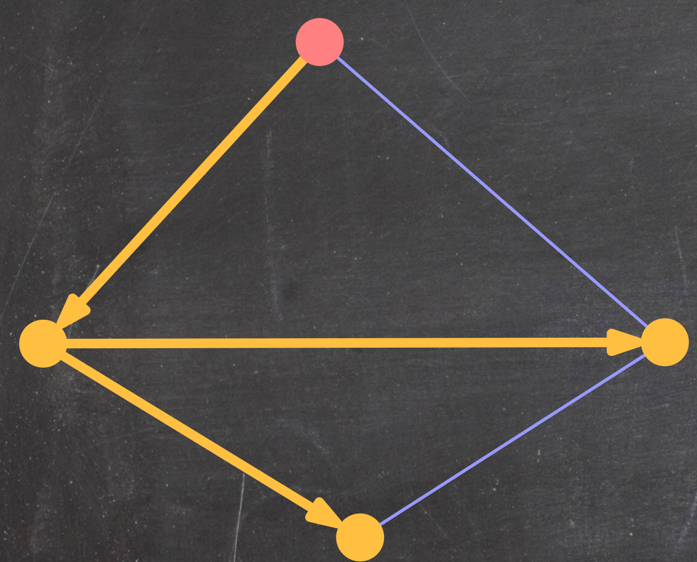


TraverseGraph(G)

```
Mark every vertex of G as unexplored
F = [ ]
for every vertex u ∈ G
    do if not u.explored
            then F.append(TraverseFromVertex(G, u))
return F
```

Graph Traversal



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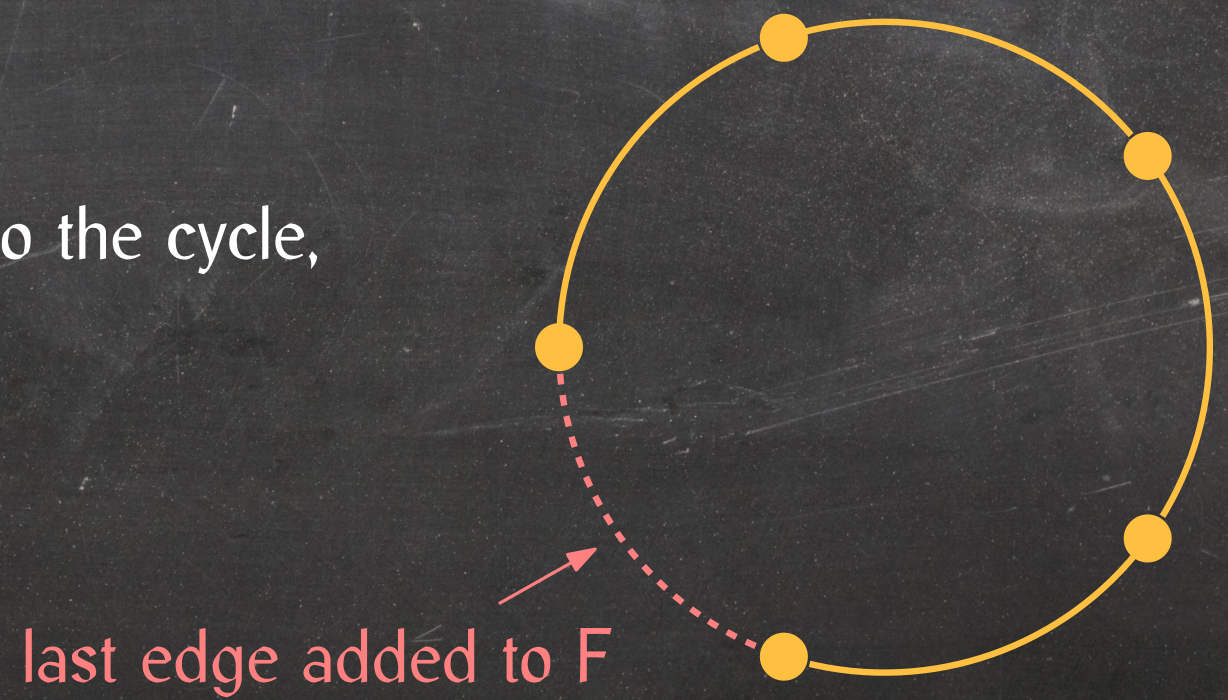
Proof by contradiction:

By the time we add the last edge to the cycle, both its endpoints are explored.

⇒ We would not have added it.



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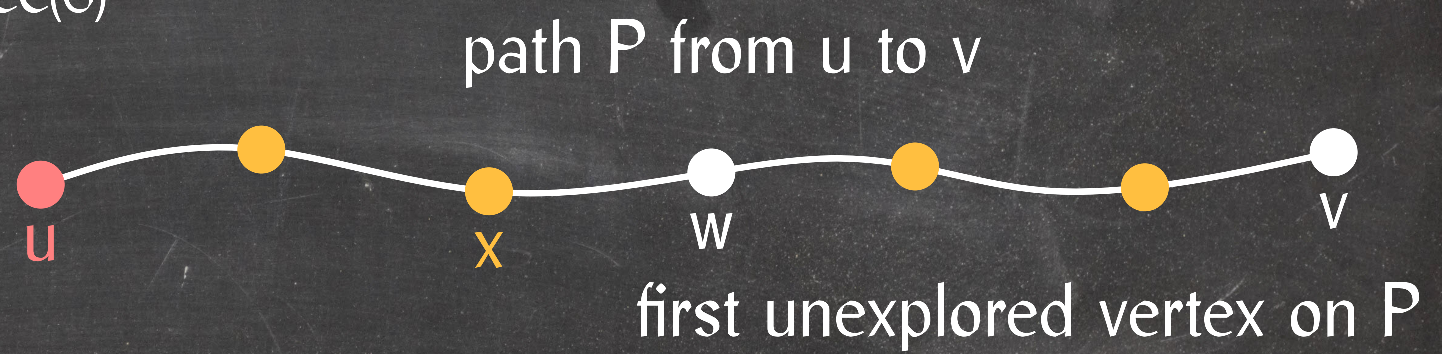

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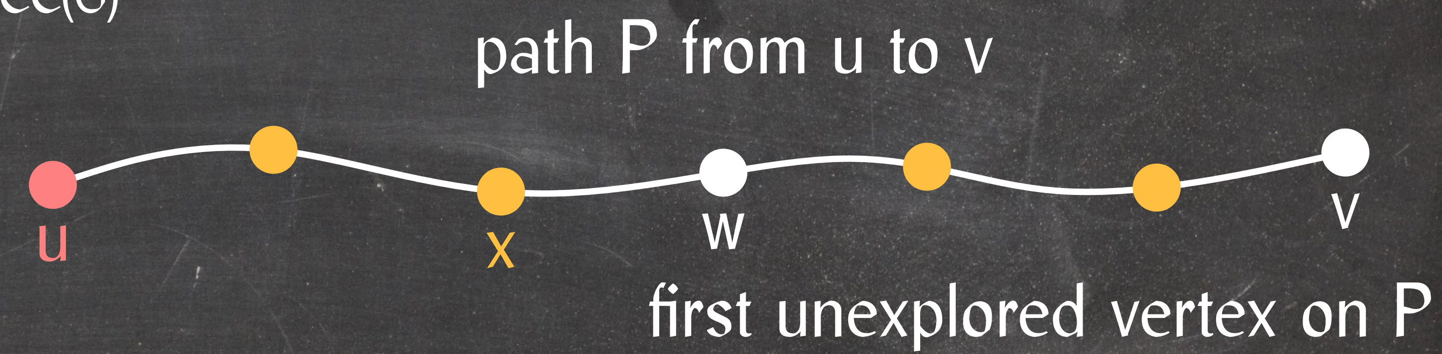
$\Rightarrow$ We'd visit w.



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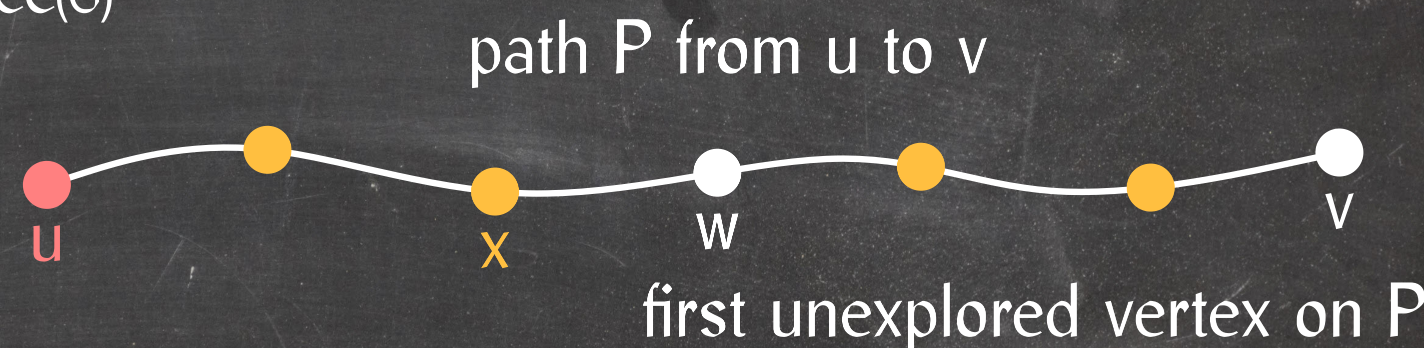

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path P from u to v

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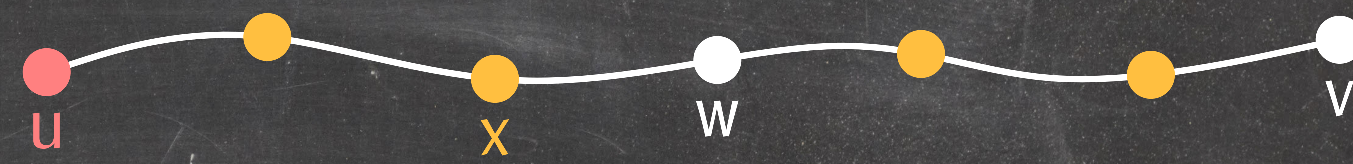

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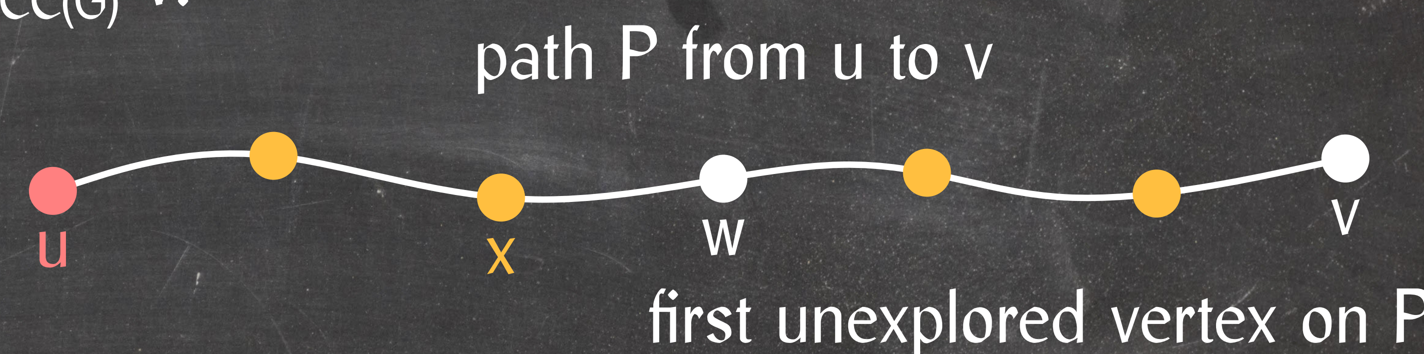

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- v explored because of edge $(w, v) \in Q$.
- w explored before v.

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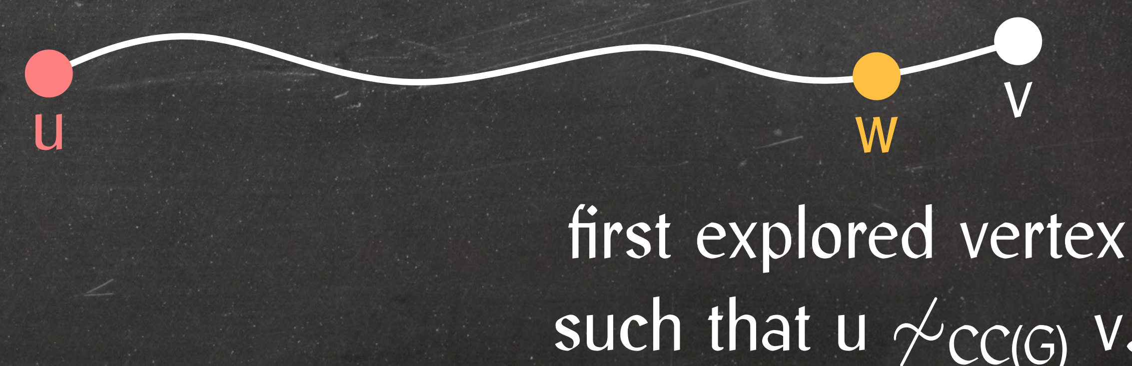

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The Cost of Graph Traversal

TraverseGraph(G)

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Mark every vertex of G as unexplored
F = []
for every vertex u ∈ G
   do if not u.explored
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Lemma: Collecting the vertices of all components takes $O(n)$ time.

Computing Connected Components

Representation using vertex labels:

ComponentLabels(L)

```
i = 0
for every list L′ ∈ L
    do i = i + 1
        for every vertex v ∈ L′
            do v.cc = i
```

Cost: $O(n)$

Computing Connected Components

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We already have the right adjacency lists for the vertices.
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BuildVertexLists(L)

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VL = []
for every list L′ ∈ L
    do VL′ = []
        for every vertex v ∈ L′
            do VL′.append(v)
        VL.append(VL′)
return VL
```

Computing Connected Components

Edge lists:

BuildEdgeLists(G, L)

```
EL = [ ]
for every edge e ∈ G
    do e.collected = False
for every list L′ ∈ L
    do EL′ = [ ]
        for every vertex v ∈ L′
            do for every edge e incident with v
                do if not e.collected
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                         EL′.append(e)
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return EL
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Computing Connected Components

Lemma: The connected components of a graph can be computed in $O(n + m)$ time.

- Building a spanning forest takes $O(n + m + m \cdot (t_a + t_r))$ time.
- Computing the vertex labelling or list of graphs then takes $O(n + m)$ time.
- Using a stack or queue to represent Q, we get $t_a \in O(1)$ and $t_r \in O(1)$.

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BFS forest = spanning forest computed using BFS

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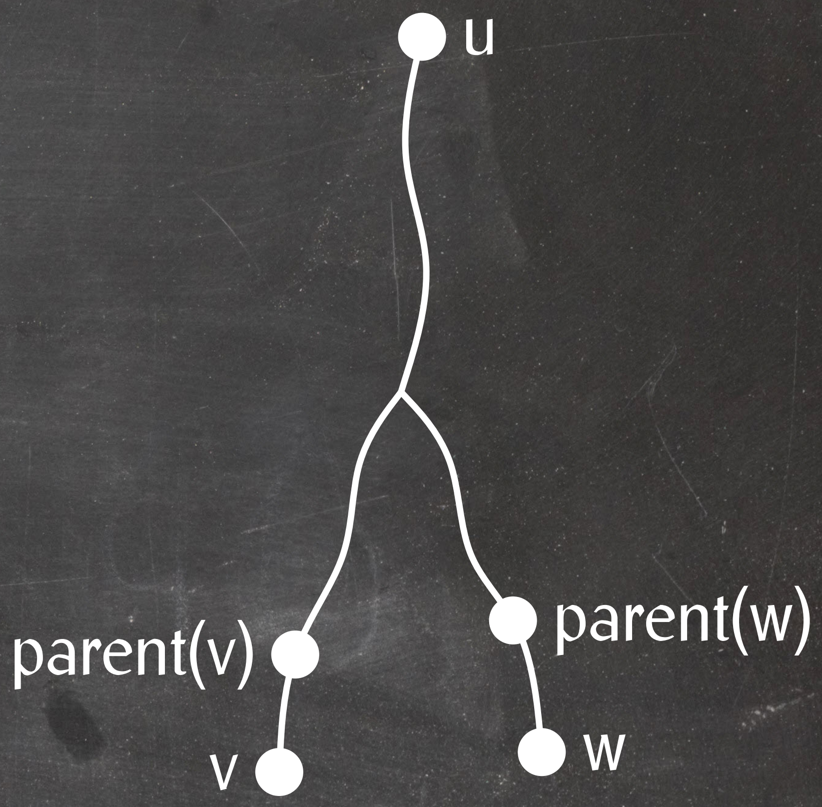

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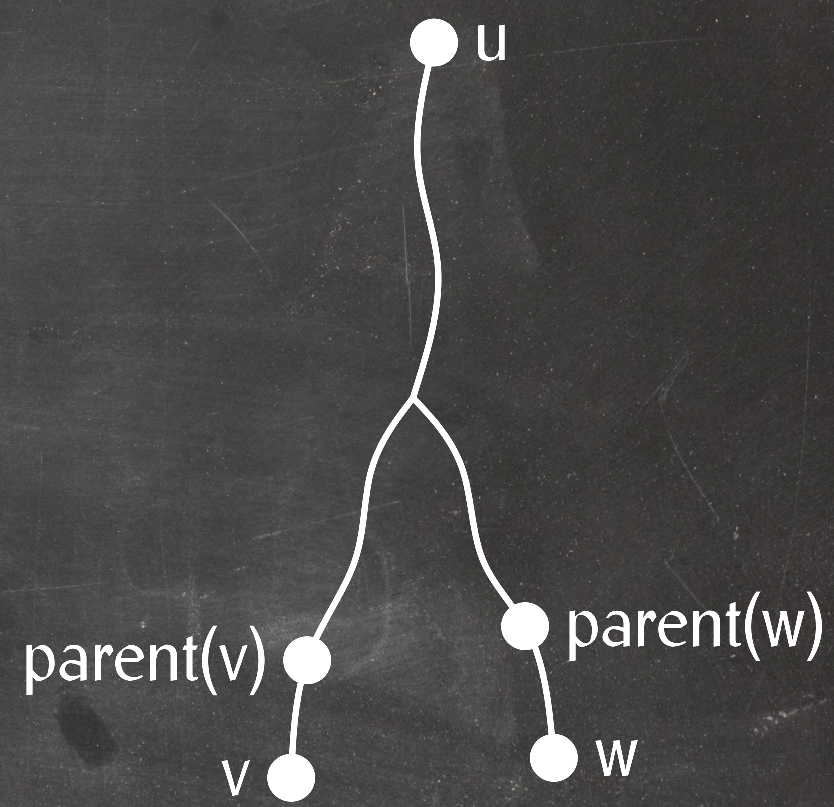

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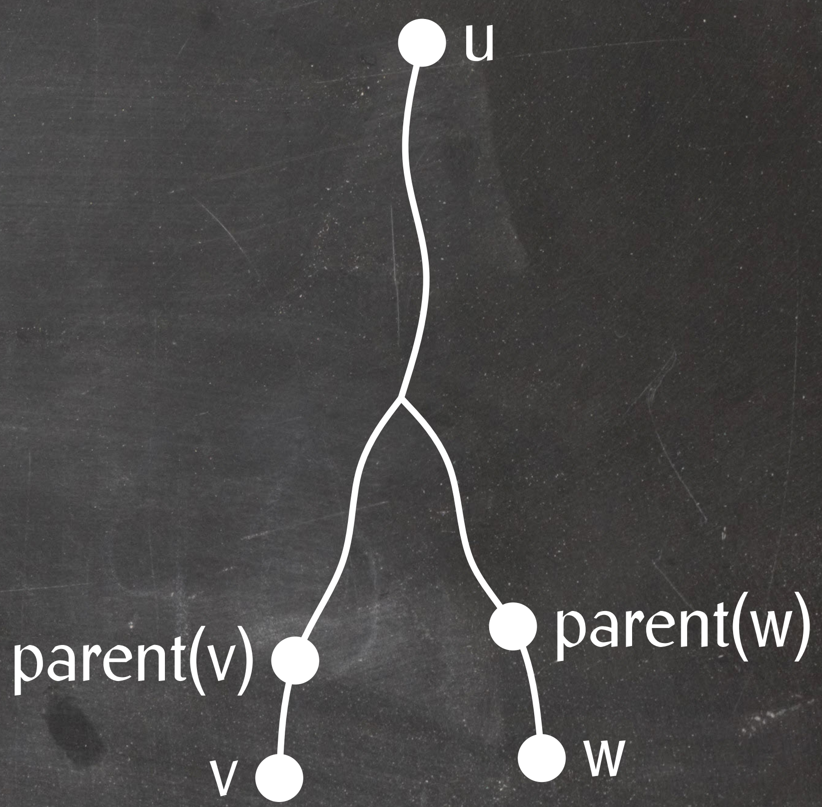

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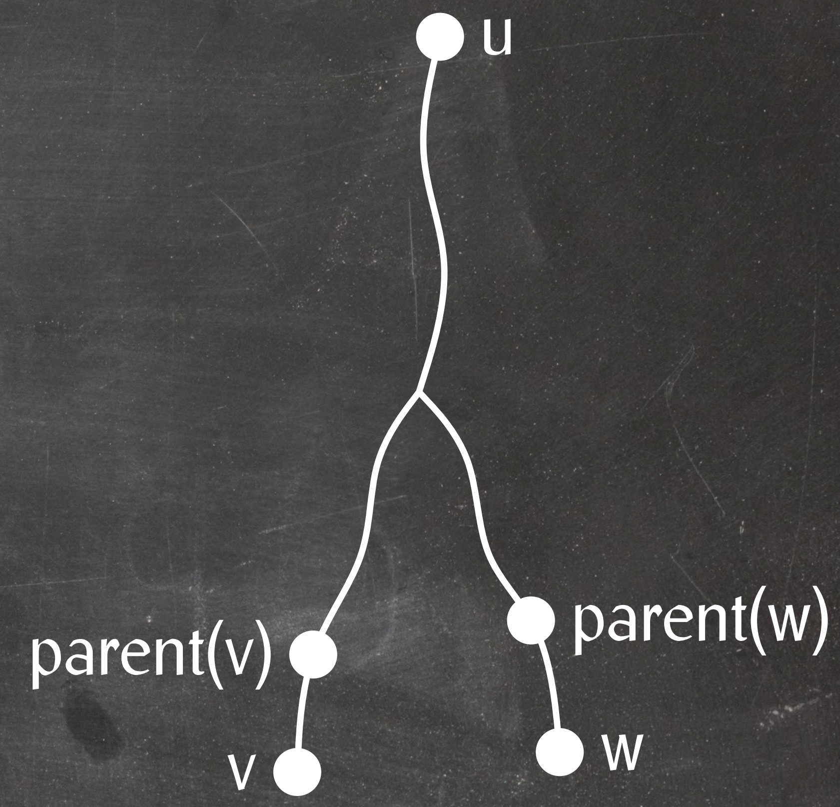

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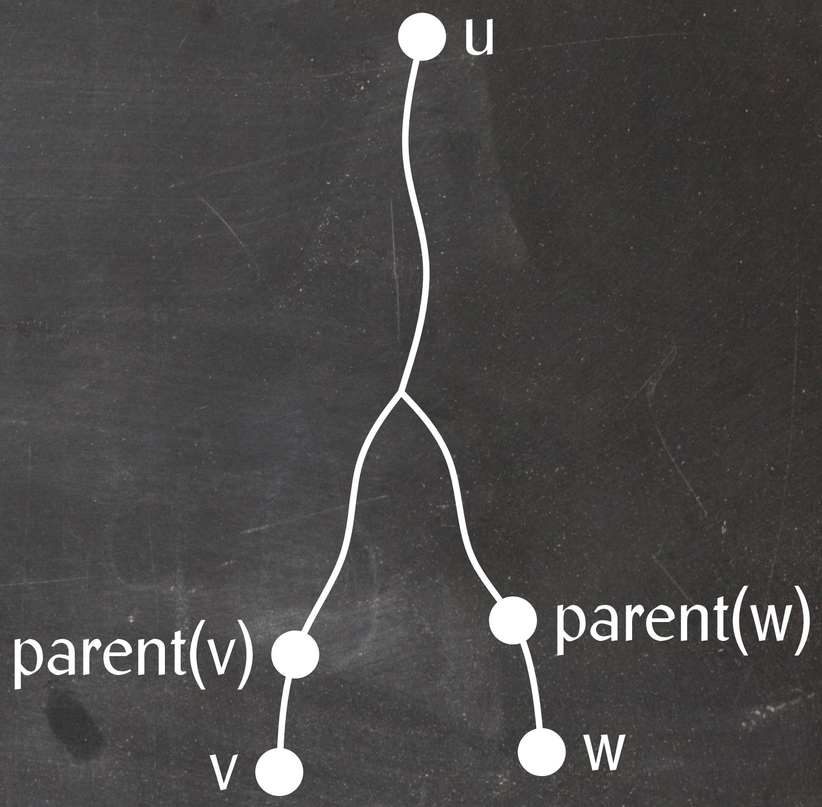

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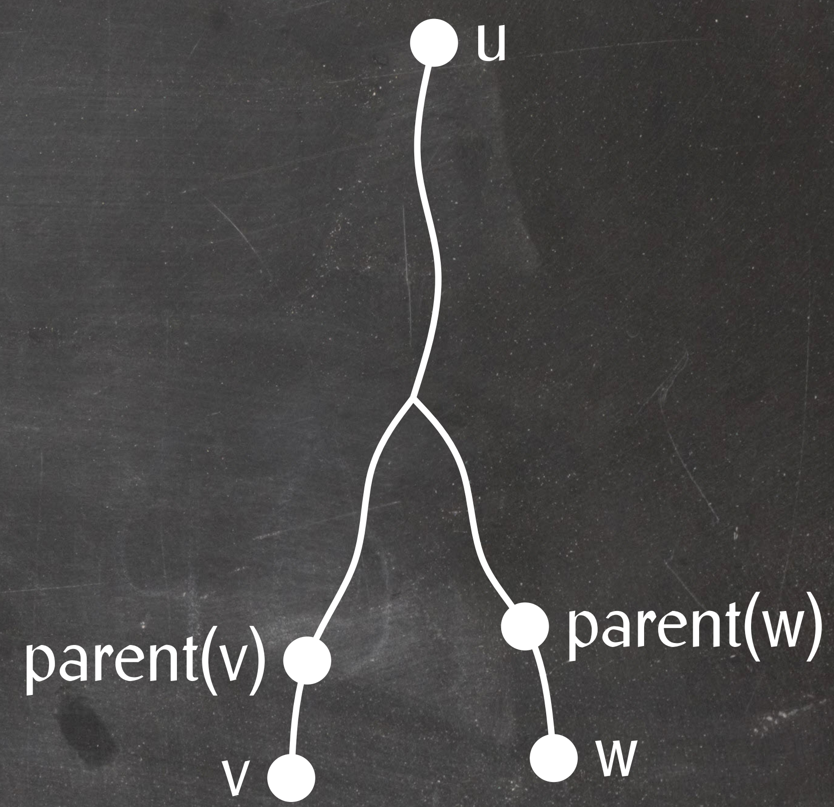

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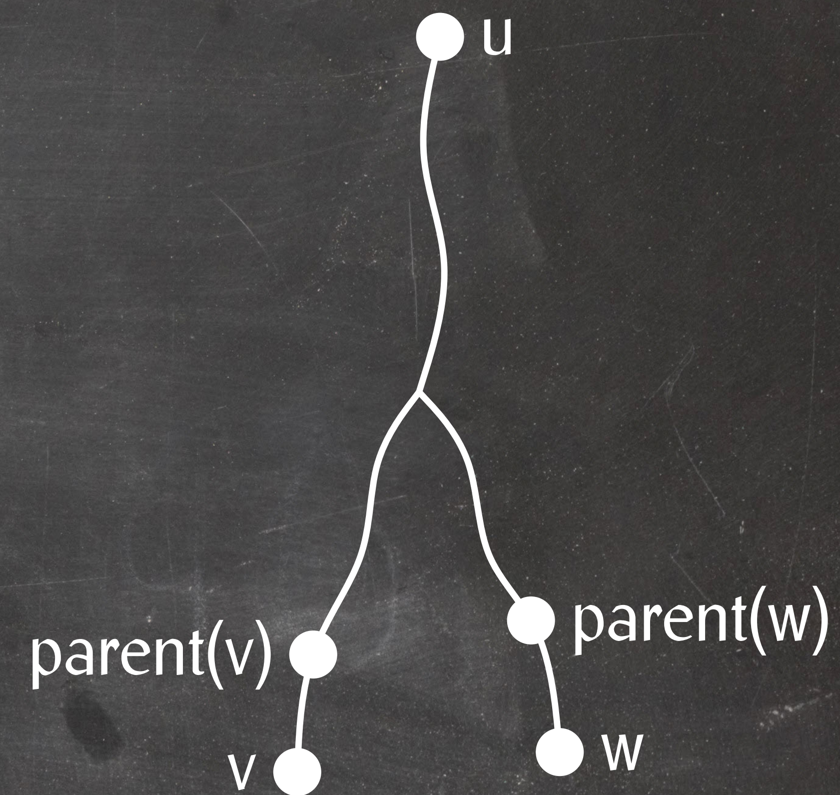

Assume $d_F(v) < d_F(w)$ and w is visited before v.

Choose such a pair (v, w) so that $d_F(w)$ is minimized.

$w \neq u$ because $d_F(w) > d_F(v) \geq 0$ and $d_F(u) = 0$.

$v \neq u$ because u is visited before any other vertex in the same tree.

$\Rightarrow$ parent(v) and parent(w) exist and
$d_F(\text{parent}(v)) = d_F(v) - 1 < d_F(w) - 1 = d_F(\text{parent}(w))$.

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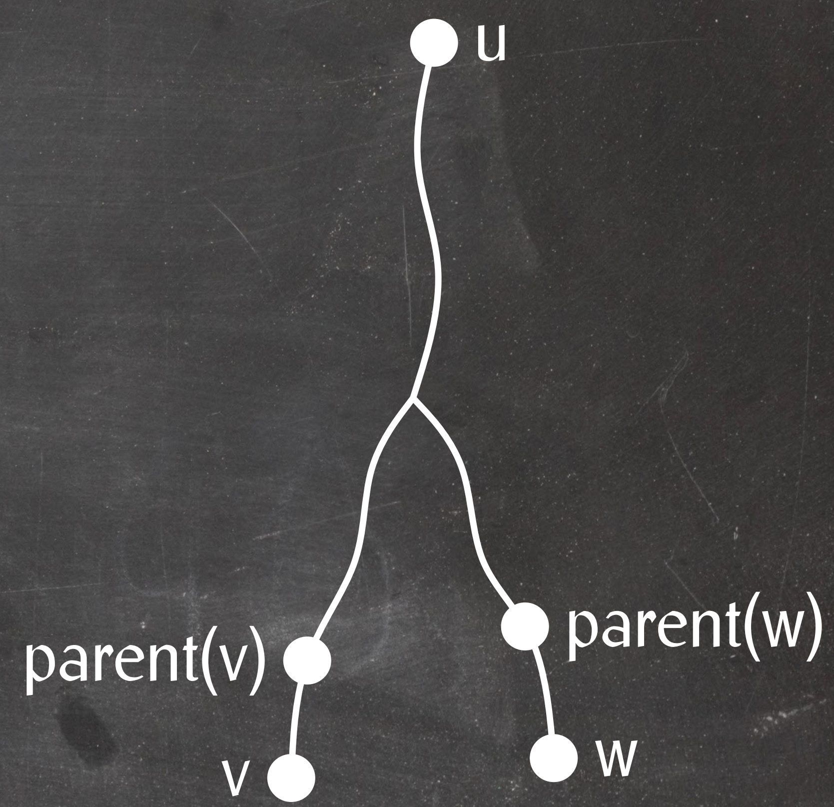
A Property of Undirected BFS Forests

BFS forest = spanning forest computed using BFS



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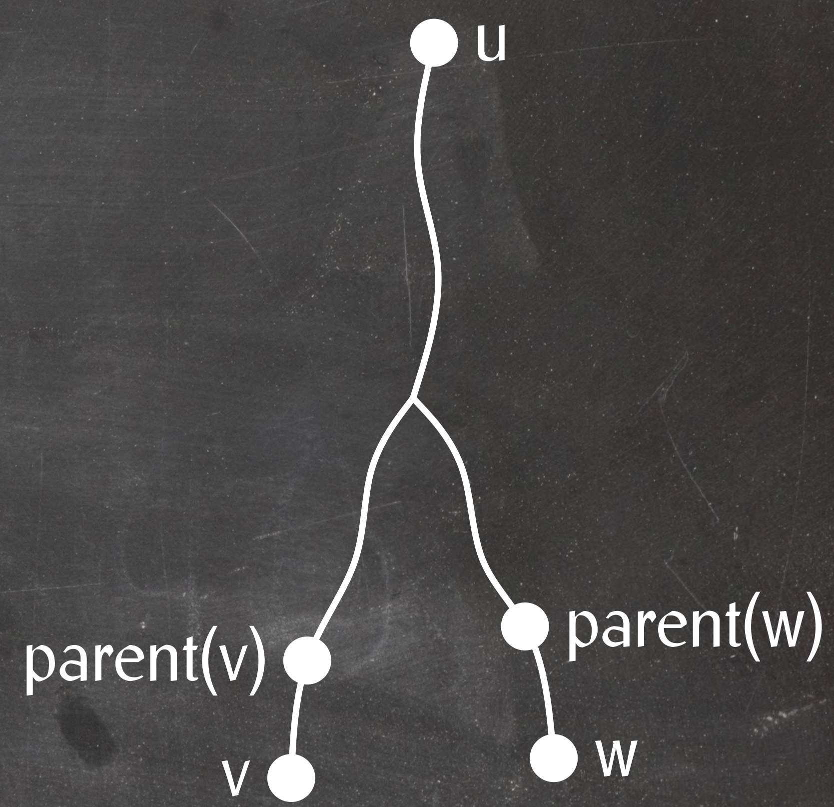

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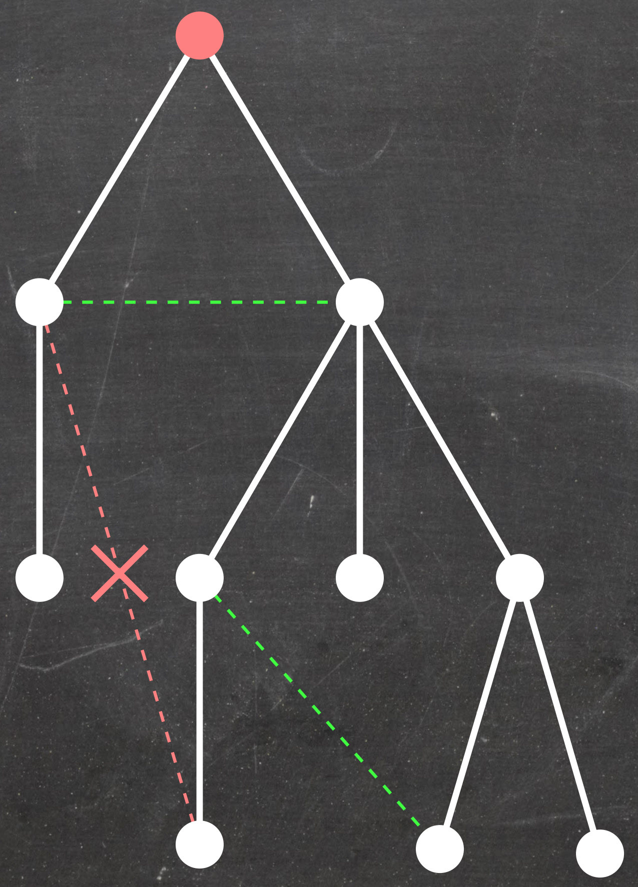

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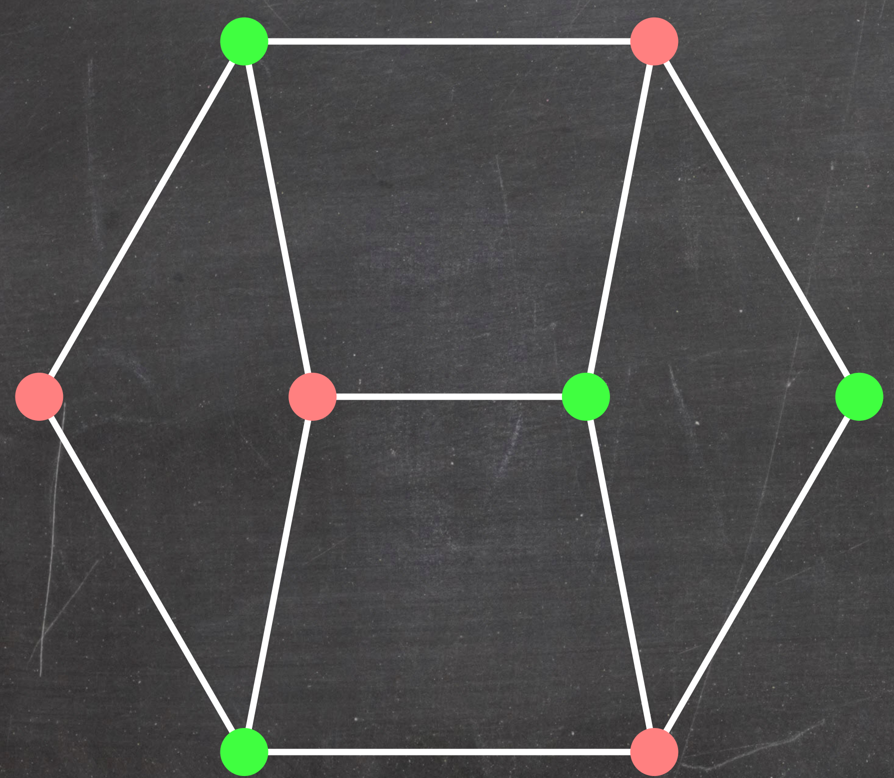

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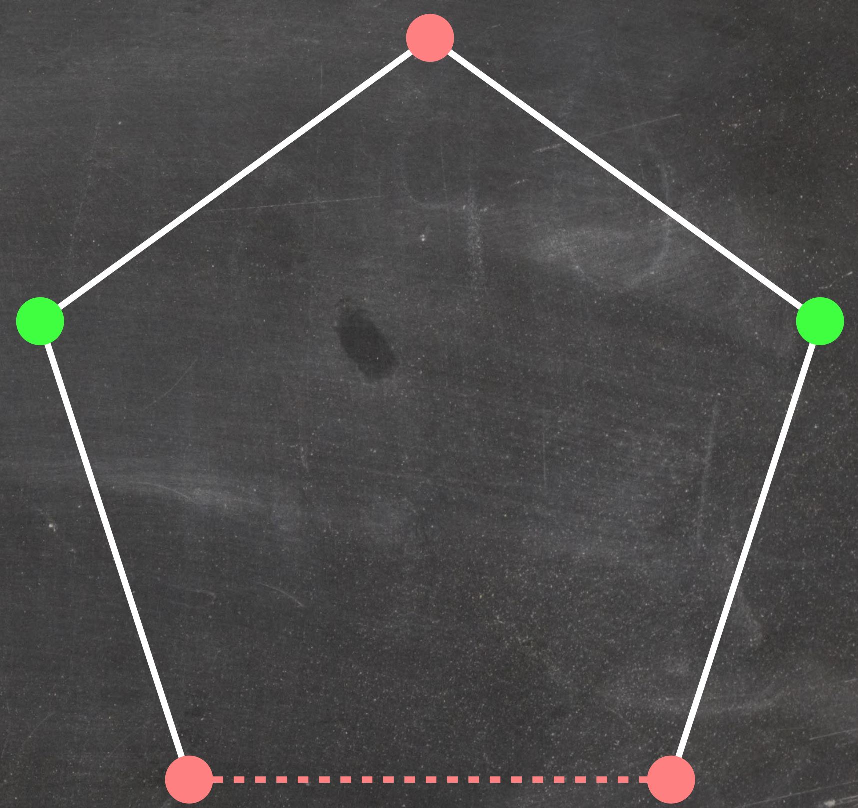

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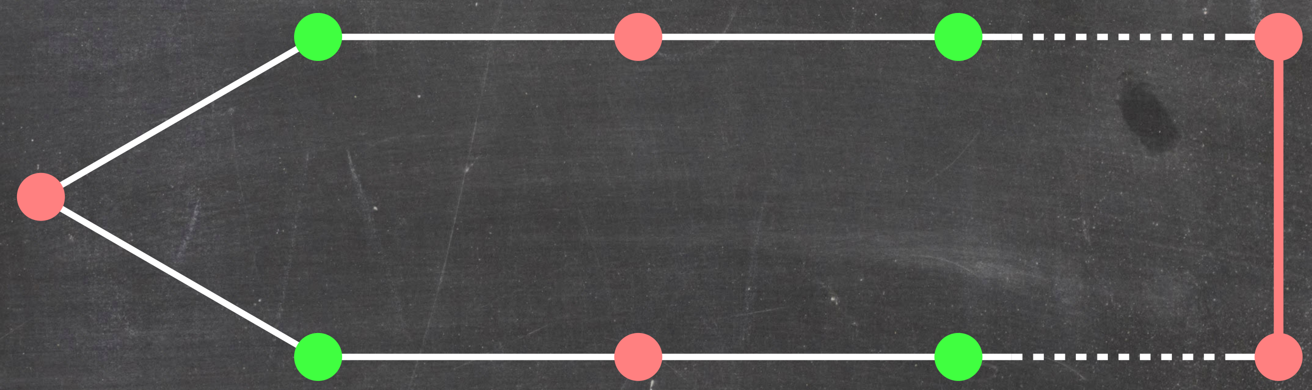

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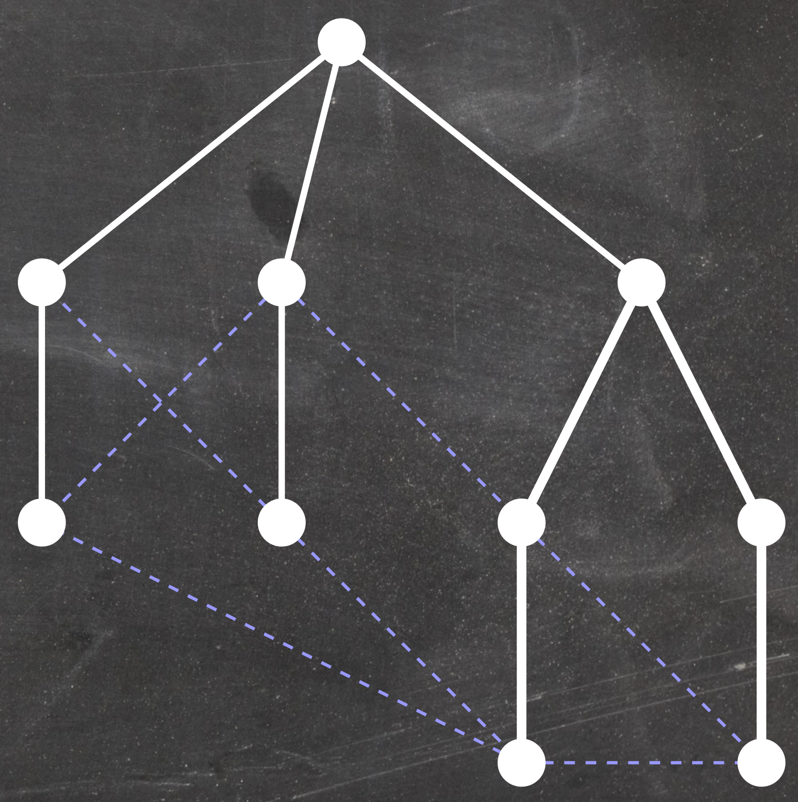

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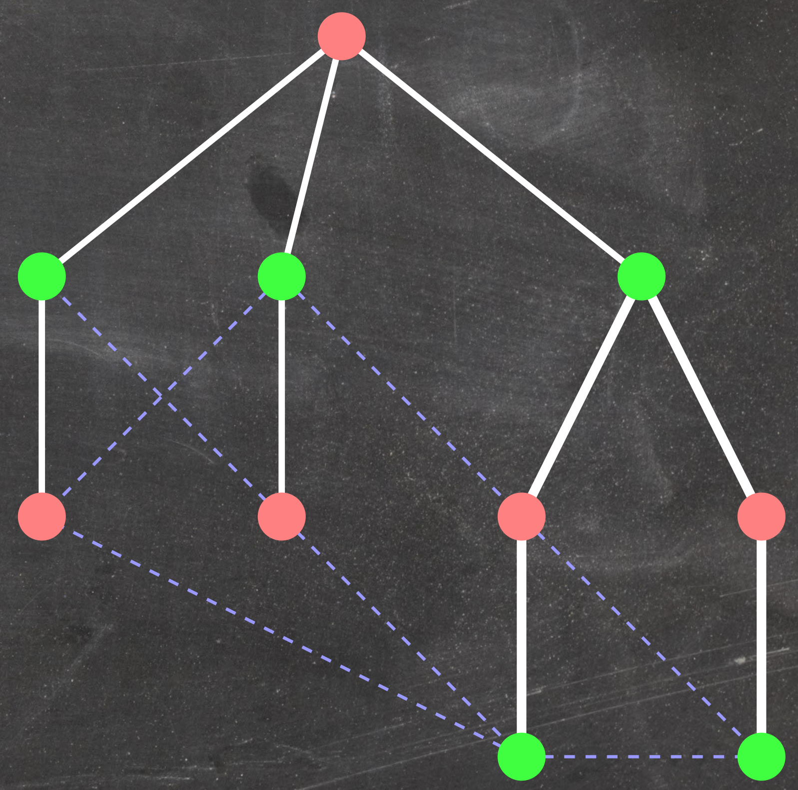

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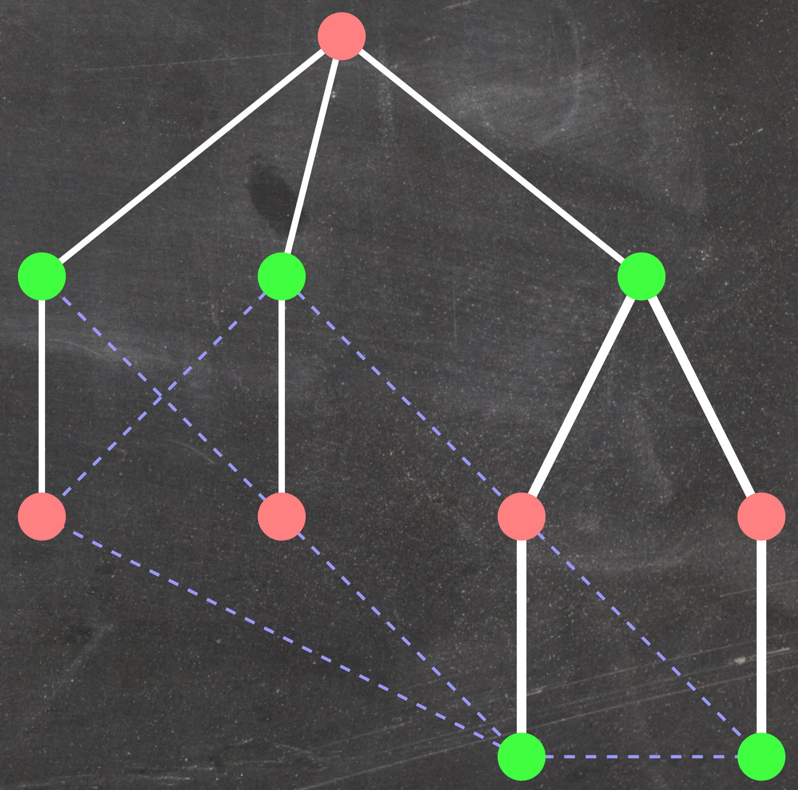

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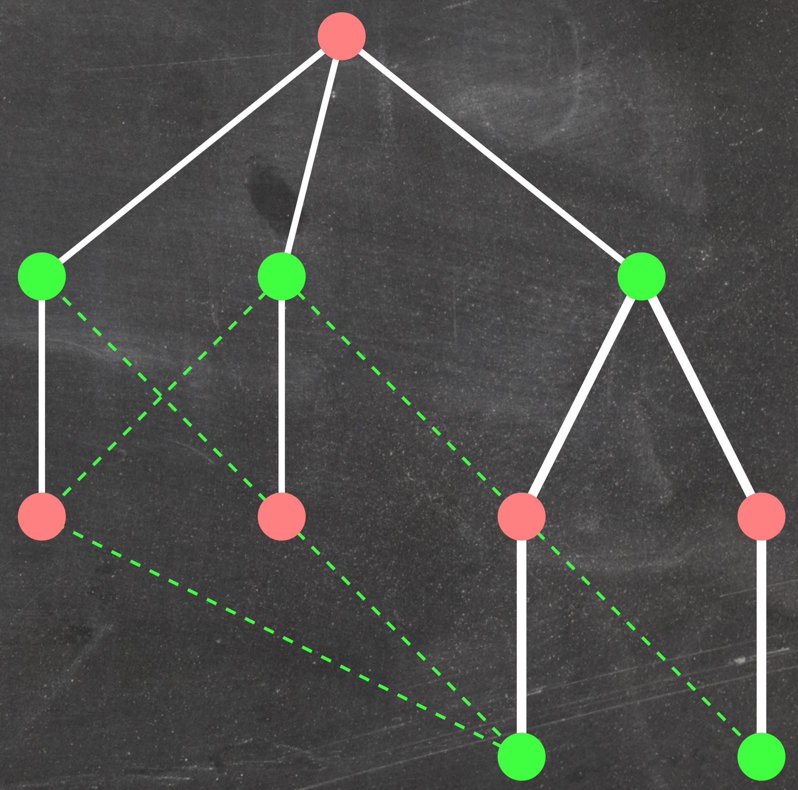

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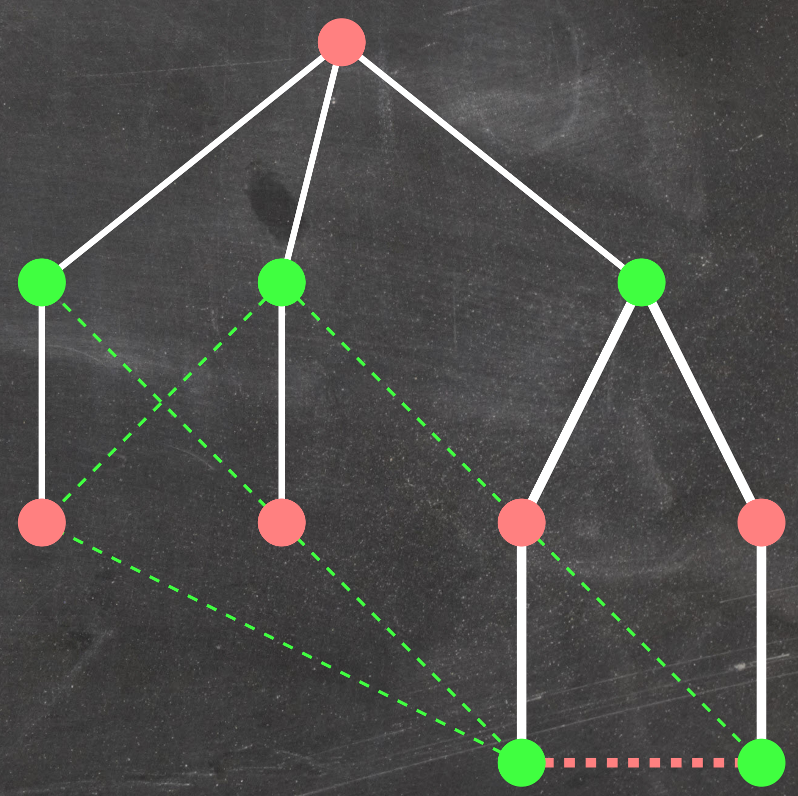

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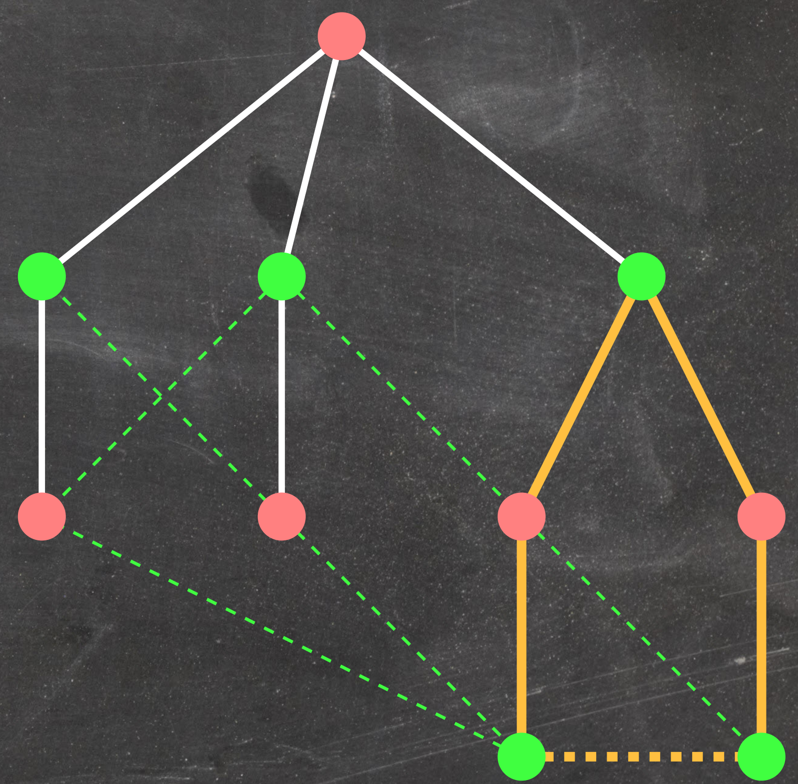
If there is such an edge, there's an odd cycle.



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Lemma: Given a BFS forest F of G, G is bipartite if and only if there is no edge in G with both endpoints on the same level in F.

Bipartiteness Testing

- Compute BFS forest F of G.
- Collect vertices on alternating levels of F into two sets (U, W).
- Test whether any edge has both endpoints in the same set, U or W.
- If so, report the odd cycle induced by such an edge.
- Otherwise, report the bipartition (U, W).

Collecting vertices on alternating levels:

AlternatingLevels(F)

```
U = W = [ ]
for every tree T in F
    do AlternatingLevels′(T, U, W)
return (U, W)
```

AlternatingLevels′(T, U, W)

```
U.append(T.key)
for every child T′ of T
    do AlternatingLevels′(T′, W, U)
```

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Testing for an "odd edge":

OddEdge(G, U, W)

```
A = an array of size n
for every vertex u ∈ U
    do A[u] = "U"
for every vertex w ∈ W
    do A[w] = "W"
for every edge (u, w) ∈ G
    do if A[u] = A[w]
            then return (u, w)
return Nothing
```

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Finding the ancestor edges of all vertices:

AncestorEdges(F)

```
L = an empty list of vertex-vertex list pairs
for every tree T ∈ F
    do AncestorEdges′(T, [ ], L)
return L
```

AncestorEdges′(T, A, L)

```
L = L.append([(T.key, A)])
for every child T′ of T
    do AncestorEdges′(T′, [(T.key, T′.key)] ++ A, L)
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Reporting an odd cycle:

OddCycle(L, (u, w))

```
Find (u, A_u) and (w, A_w) in L
C_u = C_w = []
while A_u.head ≠ A_w.head
    do C_u.append(A_u.head)
       C_w.append(A_w.head)
       A_u = A_u.tail
       A_w = A_w.tail
C_u.reverse().concat([(u, w)]).concat(C_w)
return C_u
```

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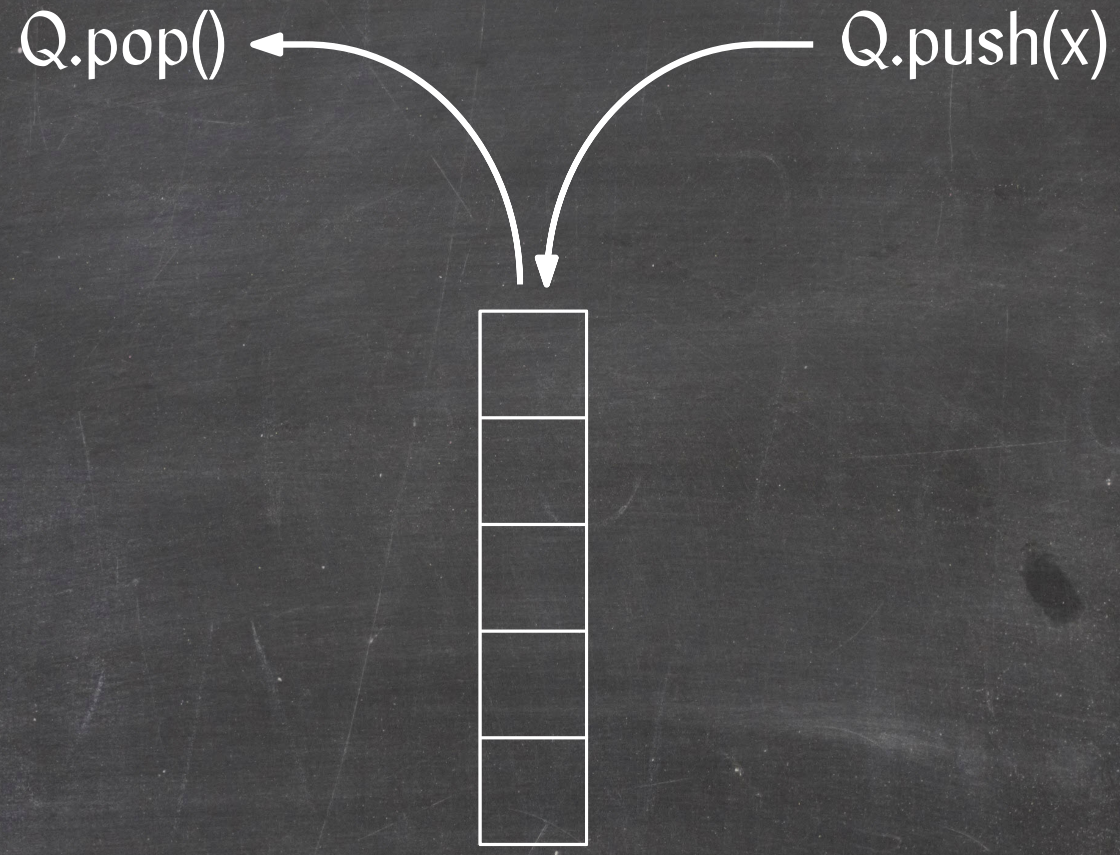
- Otherwise, report the bipartition (U, W).

Lemma: It takes linear time to test whether a graph G is bipartite and either report a valid bipartition or an odd cycle in G.

Depth-First Search

Depth-first search (DFS) = graph traversal using a stack to implement Q.

Stack:



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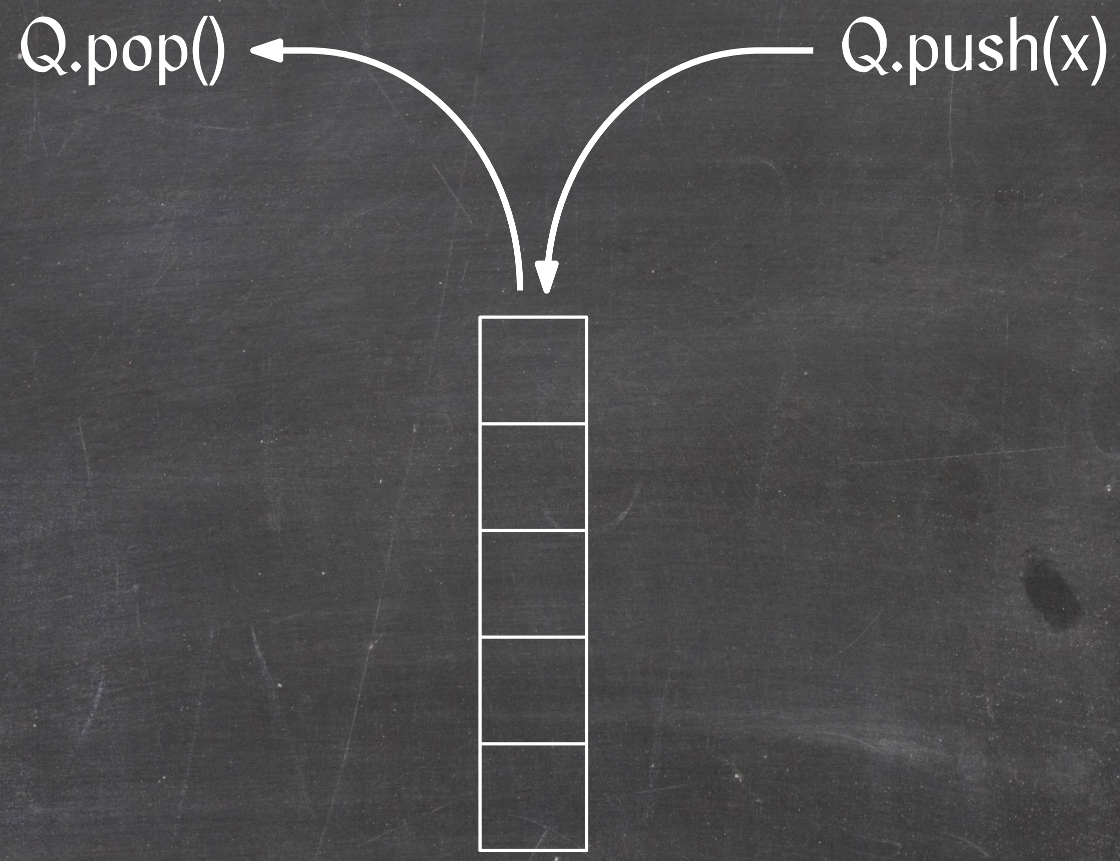

Constant-time implementations:

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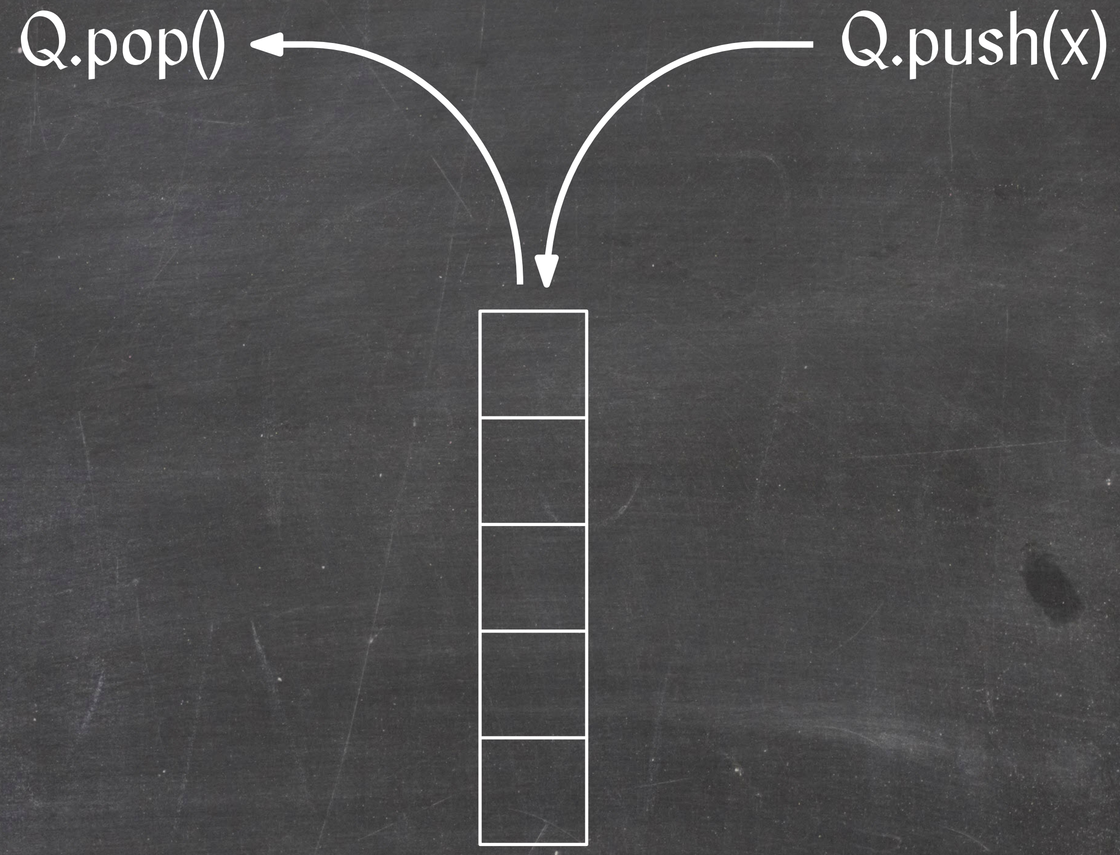

Constant-time implementations:

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Lemma: Depth-first search takes $O(n + m)$ time.

Depth-First Search and Preorder

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It visits every node after its parent:

- v is visited when the edge (parent(v), v) is popped.
- The edge (parent(v), v) must be pushed before this can happen.
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Observation: An edge with one explored and one unexplored endpoint is on the stack.

Depth-First Seach and Preorder

Assume there exist two vertices x and y such that

- y is not a descendant of x,
- y is visited after x, and
- y is visited before some descendant z.

Choose y and z so that

- y is the first visited vertex satisfying the above conditions and
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Case 1: y is a root.

Cannot happen because the edge (parent(z), z) is on the stack when y is visited and the stack is empty when a root is visited.

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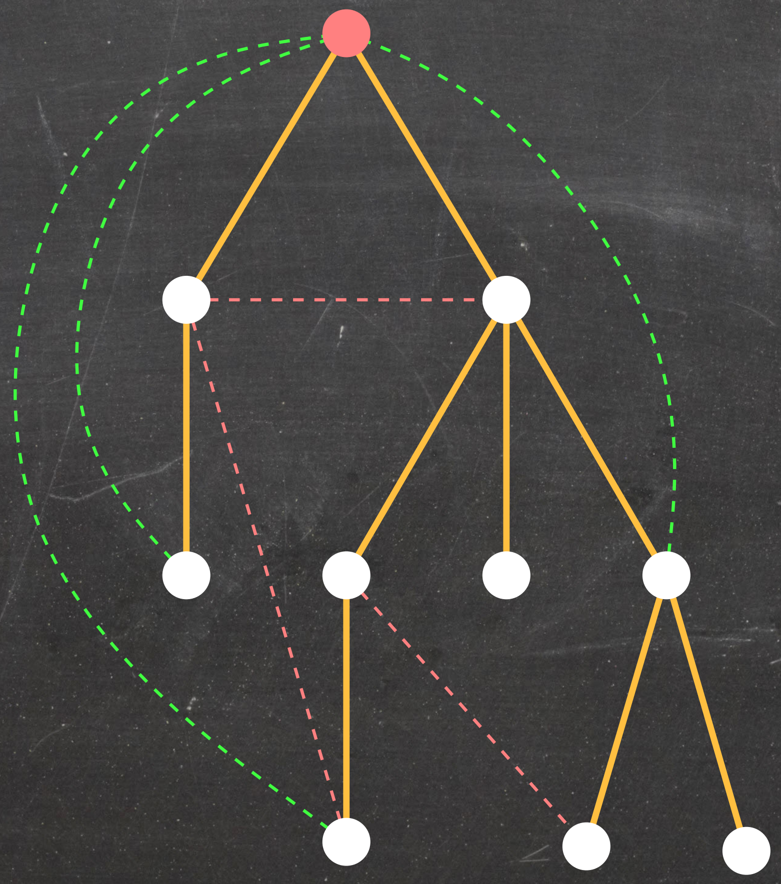
$\Rightarrow$ The edge (parent(z), z) is popped before the edge (parent(y), y).

$\Rightarrow$ z is visited before y, contradiction.

A Property of Undirected DFS Forests

Three types of edges:

- Tree edge (u, w): u is w's parent in F.
- Cross edge (u, w): Neither u nor w is an ancestor of the other.
- Back edge (u, w): u is an ancestor of w but not its parent.



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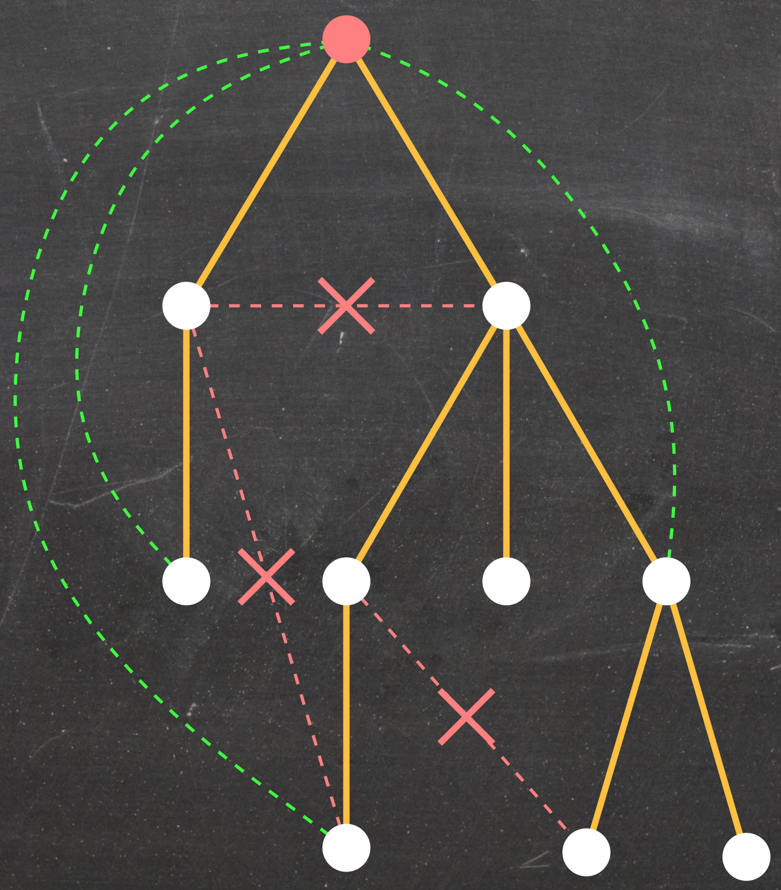
Lemma: All edges of an undirected graph G are tree or back edges with respect to a DFS forest of G.



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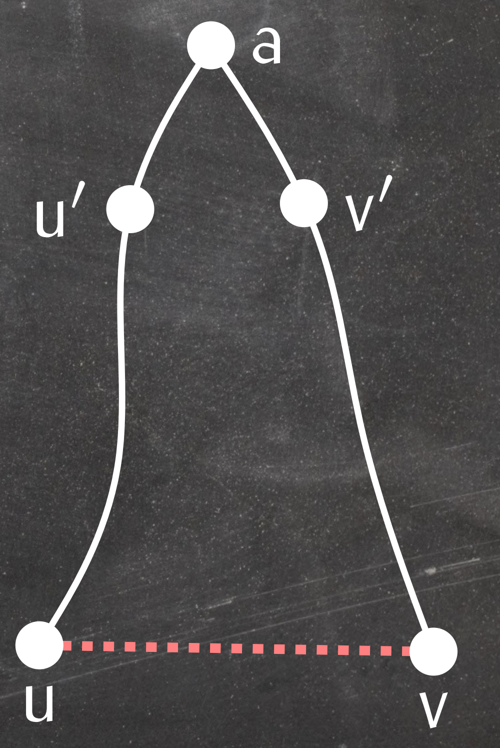
Assume $u < v$ in preorder.



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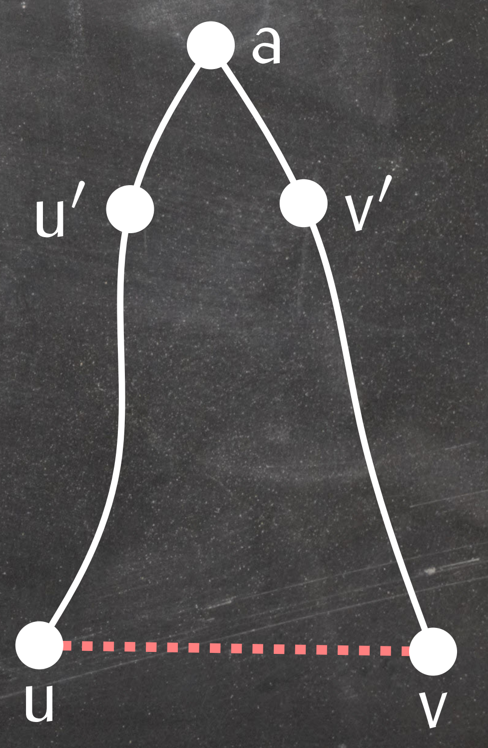

A Property of Undirected DFS Forests

Three types of edges:

- Tree edge (u, w): u is w's parent in F.
- Cross edge (u, w): Neither u nor w is an ancestor of the other.
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Lemma: All edges of an undirected graph G are tree or back edges with respect to a DFS forest of G.

Let a be the LCA of u and v and let u' and v' be the children of a that are ancestors of u and v.

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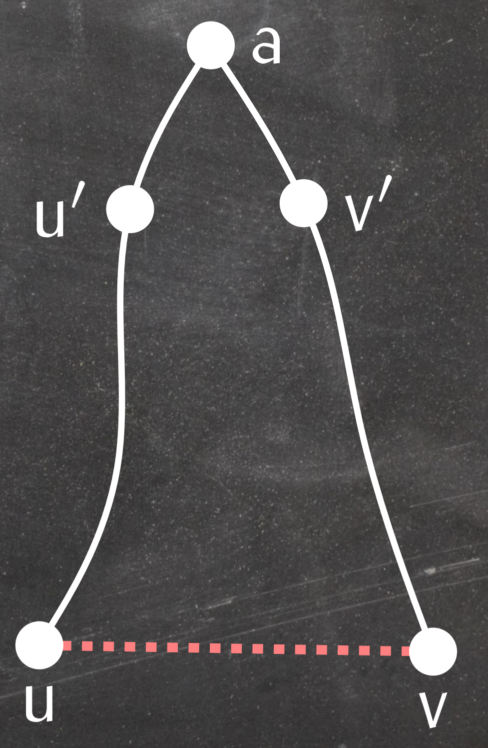
$\Rightarrow$ The edge (a, v') is pushed before u is visited and popped after u is visited.



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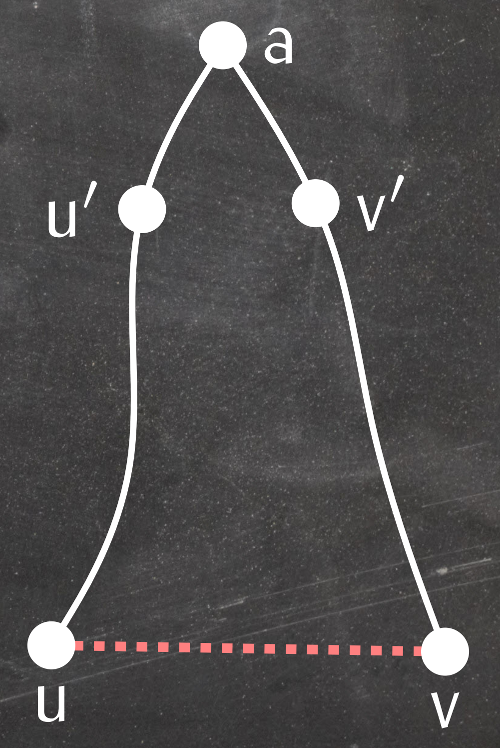

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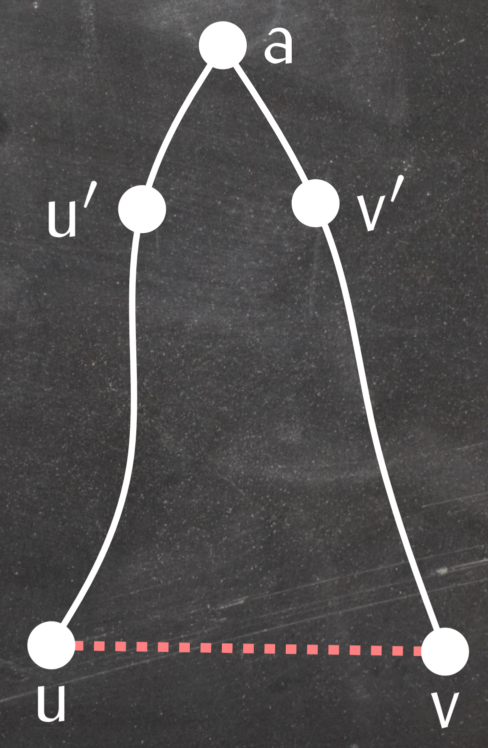

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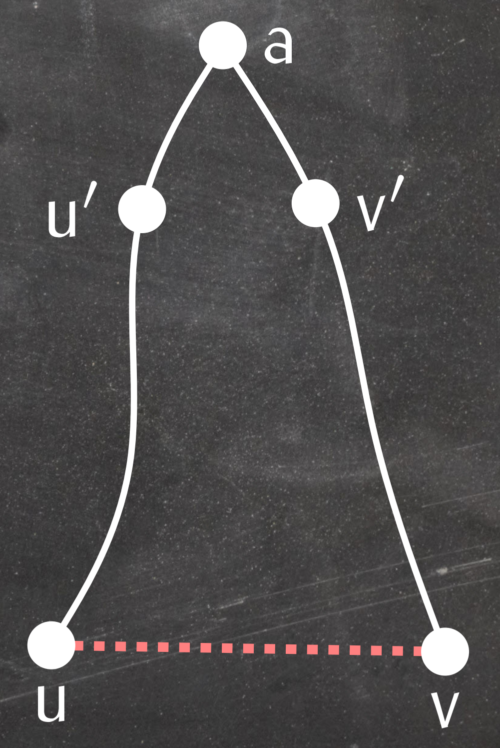
$\Rightarrow$ v is unexplored when the edge (u, v) is popped, a contradiction.



A Property of Directed DFS Forests

Five types of edges:

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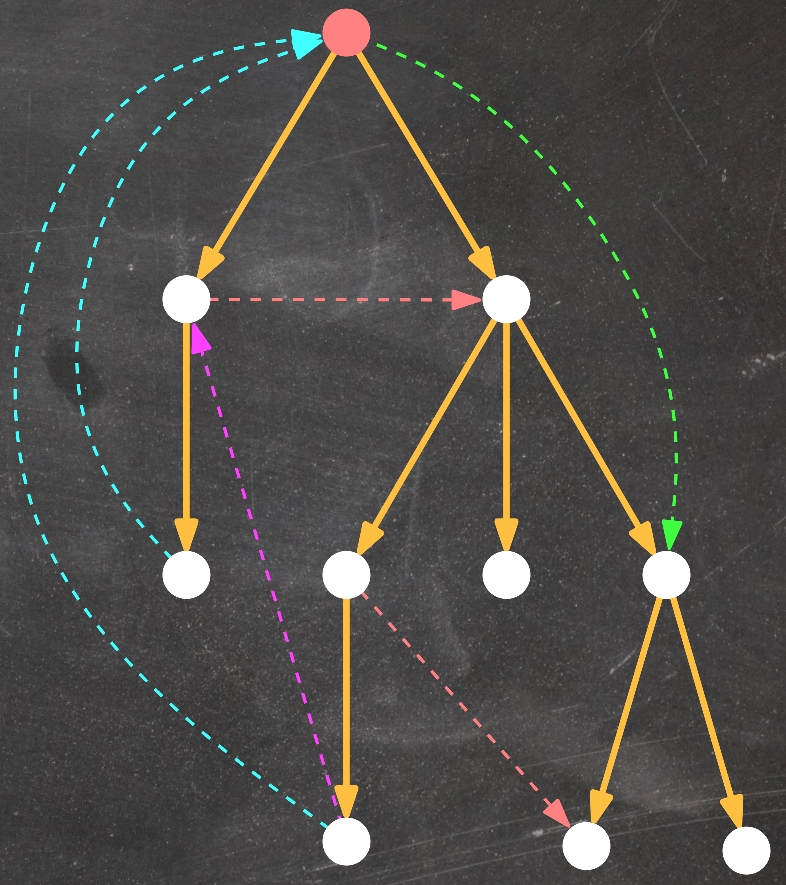

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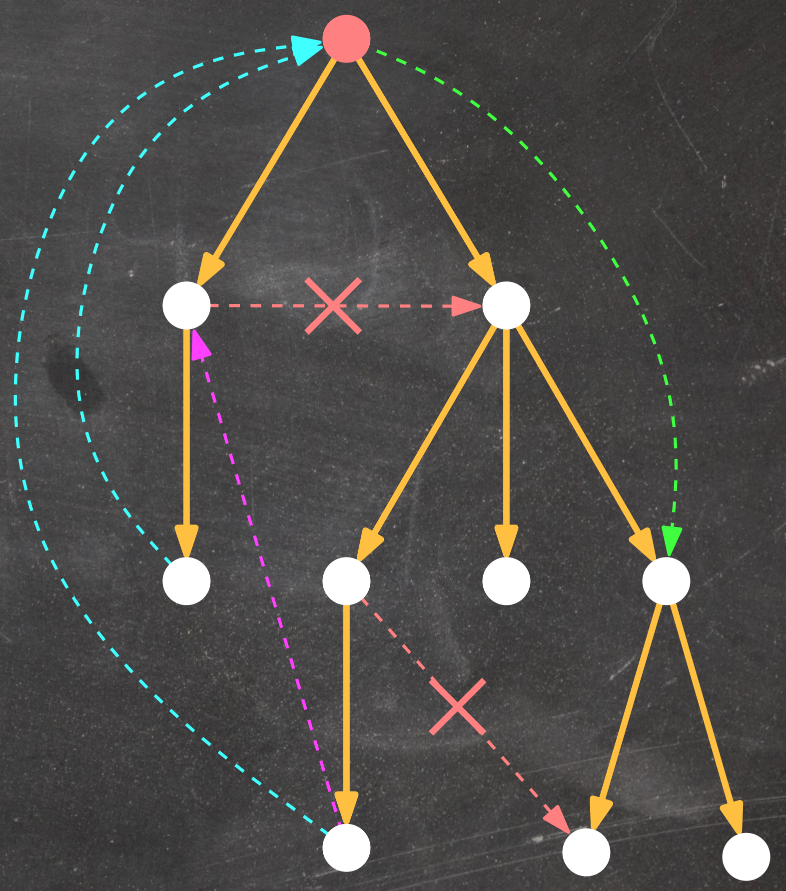
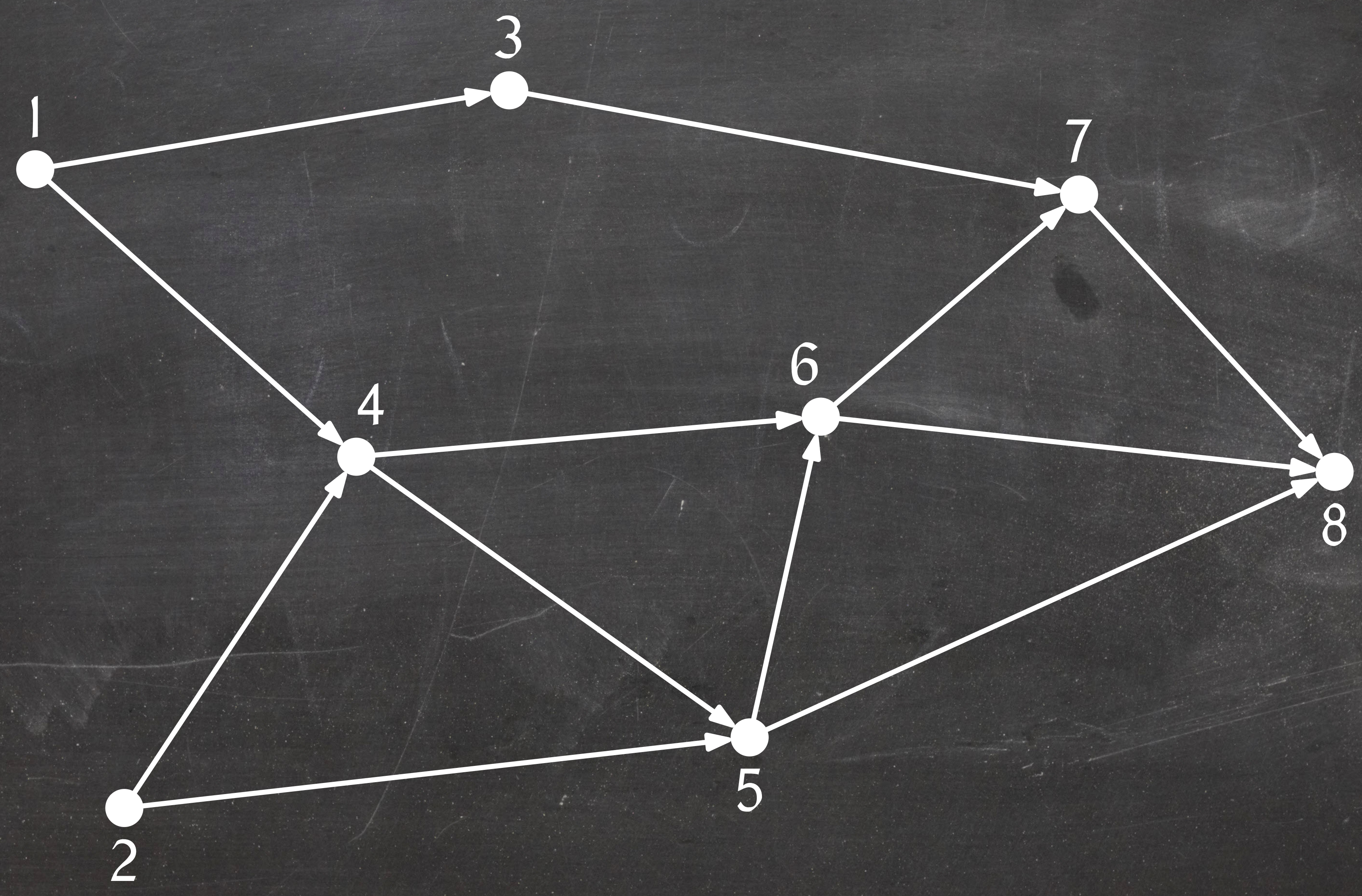


Lemma: A directed graph G does not contain any forward cross edges with respect to a DFS forest of G.

Topological Sorting

A topological ordering of a directed graph is an ordering < of the vertex set of G such that u < v for every edge $(u, v) \in G$.



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Lemma: A graph G has a topological ordering if and only if it contains no directed cycle.

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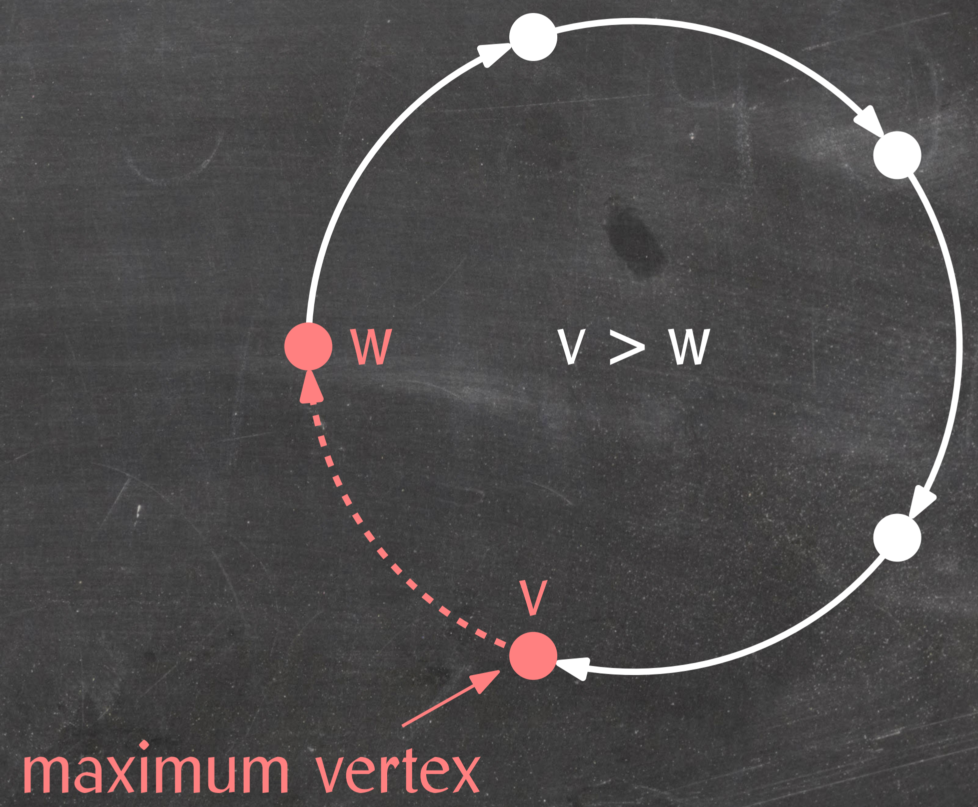
If there's a cycle, there is no topological ordering.



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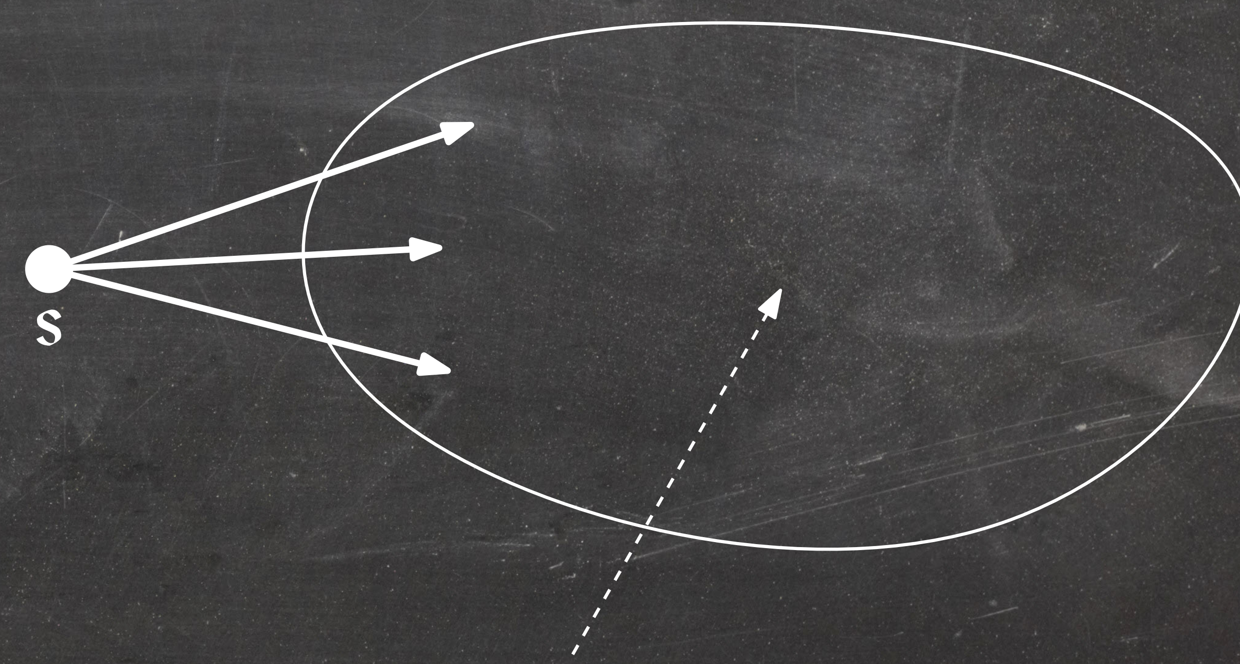
$\Rightarrow$ The following algorithm produces a topological ordering:

- Give s the smallest number.
- Recursively number the rest of the vertices.



Cannot contain a cycle since G contains no cycle.

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Let $R(v)$ be the set of vertices reachable from v.

For an edge (u, v),

- $R(u) \supseteq R(v)$
- $u \in R(u)$
- $u \notin R(v)$ (otherwise there'd be a cycle)

$\Rightarrow R(u) \supset R(v)$.

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Pick a vertex s such that $|R(s)| \geq |R(v)|$ for all $v \in G$.

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Pick a vertex s such that $|R(s)| \geq |R(v)|$ for all $v \in G$.

If s had an in-neighbour u, then $|R(u)| > |R(s)|$, a contradiction.

$\Rightarrow$ s is a source.

Topological Sorting

Lemma: A topological ordering of a directed acyclic graph G can be computed in O(n + m) time.

SimpleTopSort(G)

```
Q = an empty queue
for every vertex v ∈ G
    do label v with its in-degree
        if in-deg(v) = 0
            then Q.enqueue(v)
O = []
while not Q.isEmpty()
    do v = Q.dequeue()
        O.append(v)
        for every out-neighbour w of v
            do in-deg(w) = in-deg(w) − 1
                if in-deg(w) = 0
                    then Q.enqueue(w)
return O
```

Topological Sorting Using DFS

Edges in a DFS forest:

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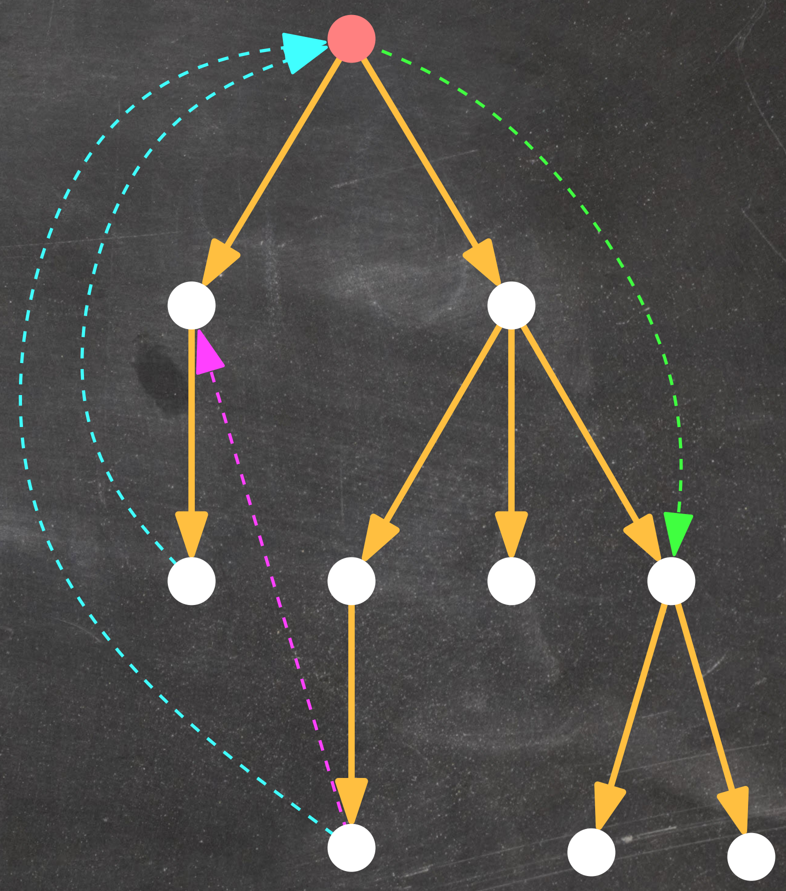

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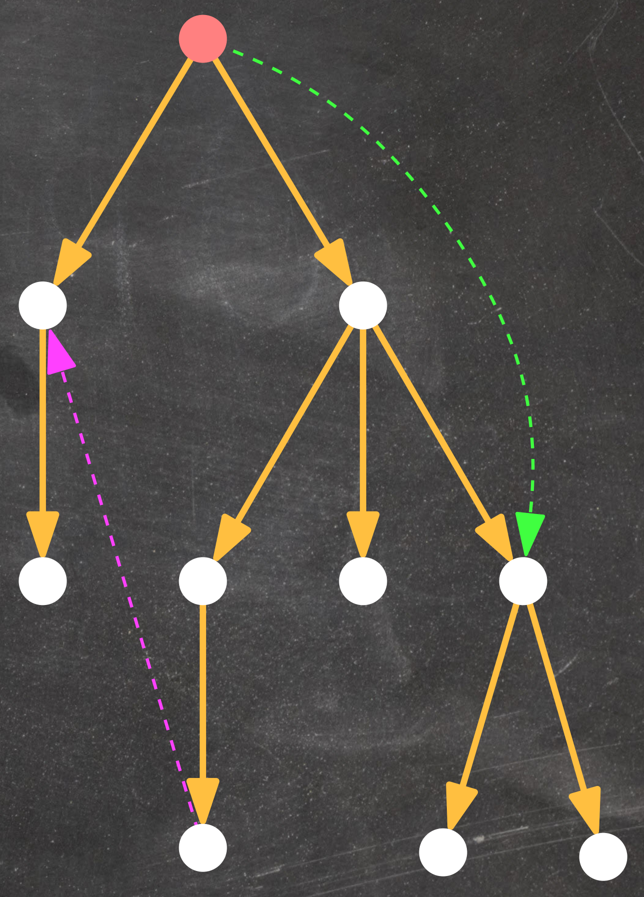

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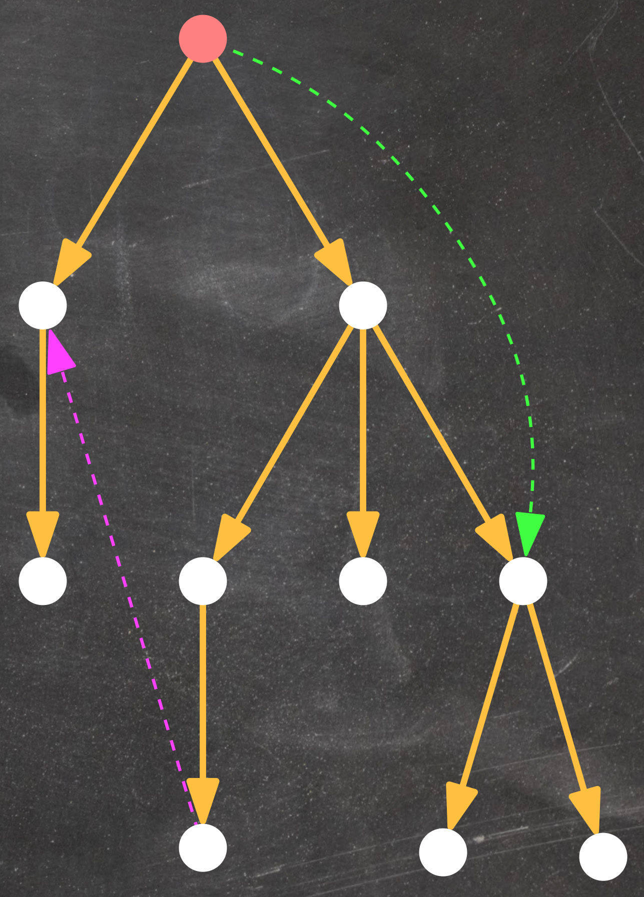
For tree, forward, and backward cross edges (u, v), u > v in postorder.



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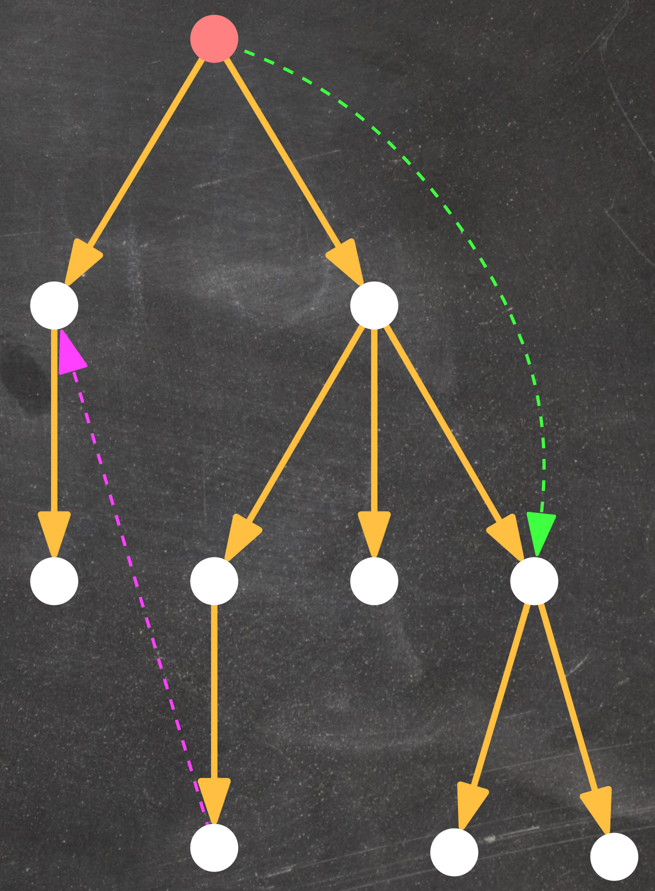
For tree, forward, and backward cross edges (u, v), u > v in postorder.

⇒ Topological sorting algorithm:

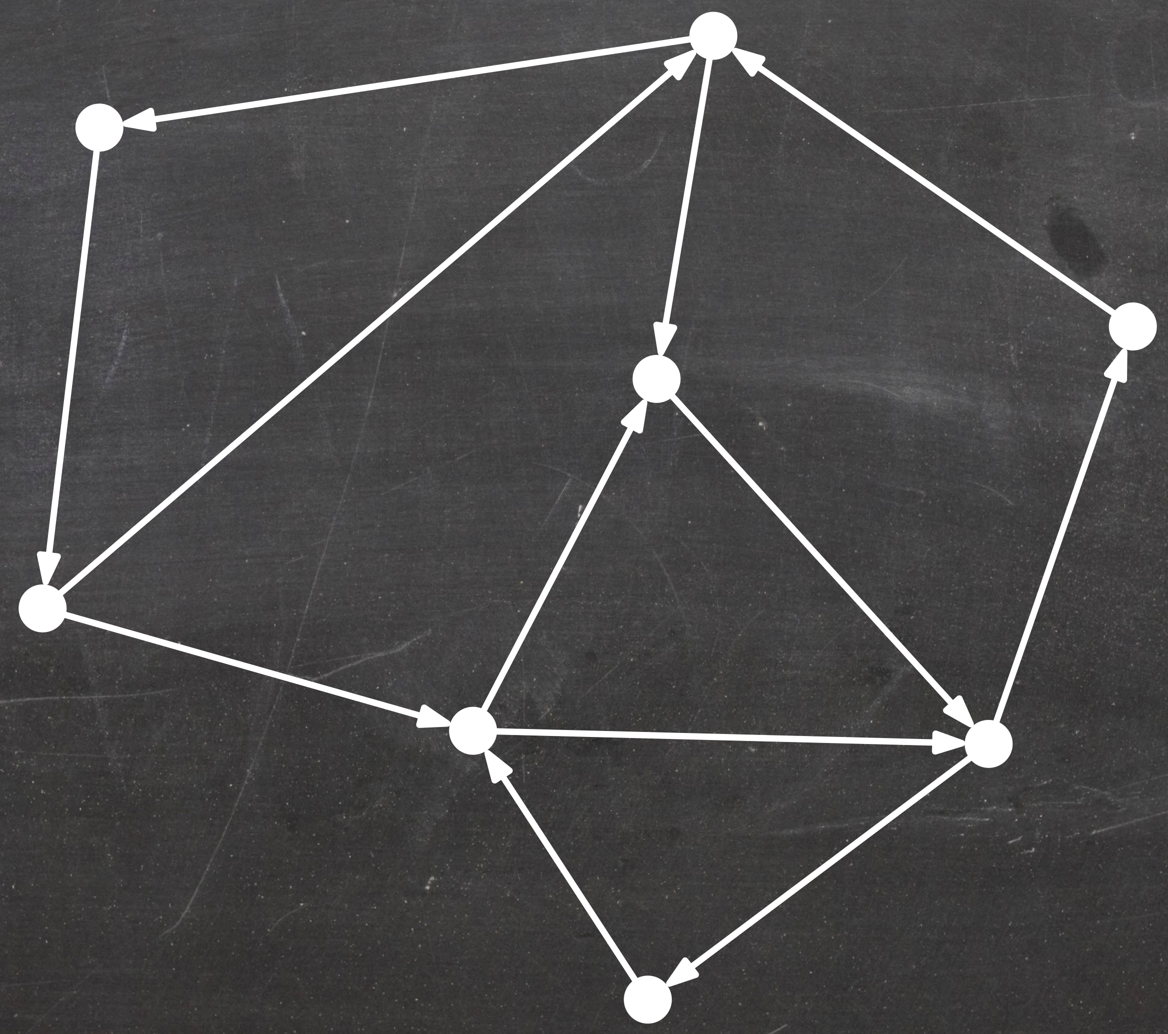
- Compute a DFS forest of G.
- Arrange the vertices in reverse postorder.

This takes $O(n + m)$ time.



Strongly Connected Components

A graph is strongly connected if there exists a path from u to w and from w to u for every pair of vertices $u, w \in G$.



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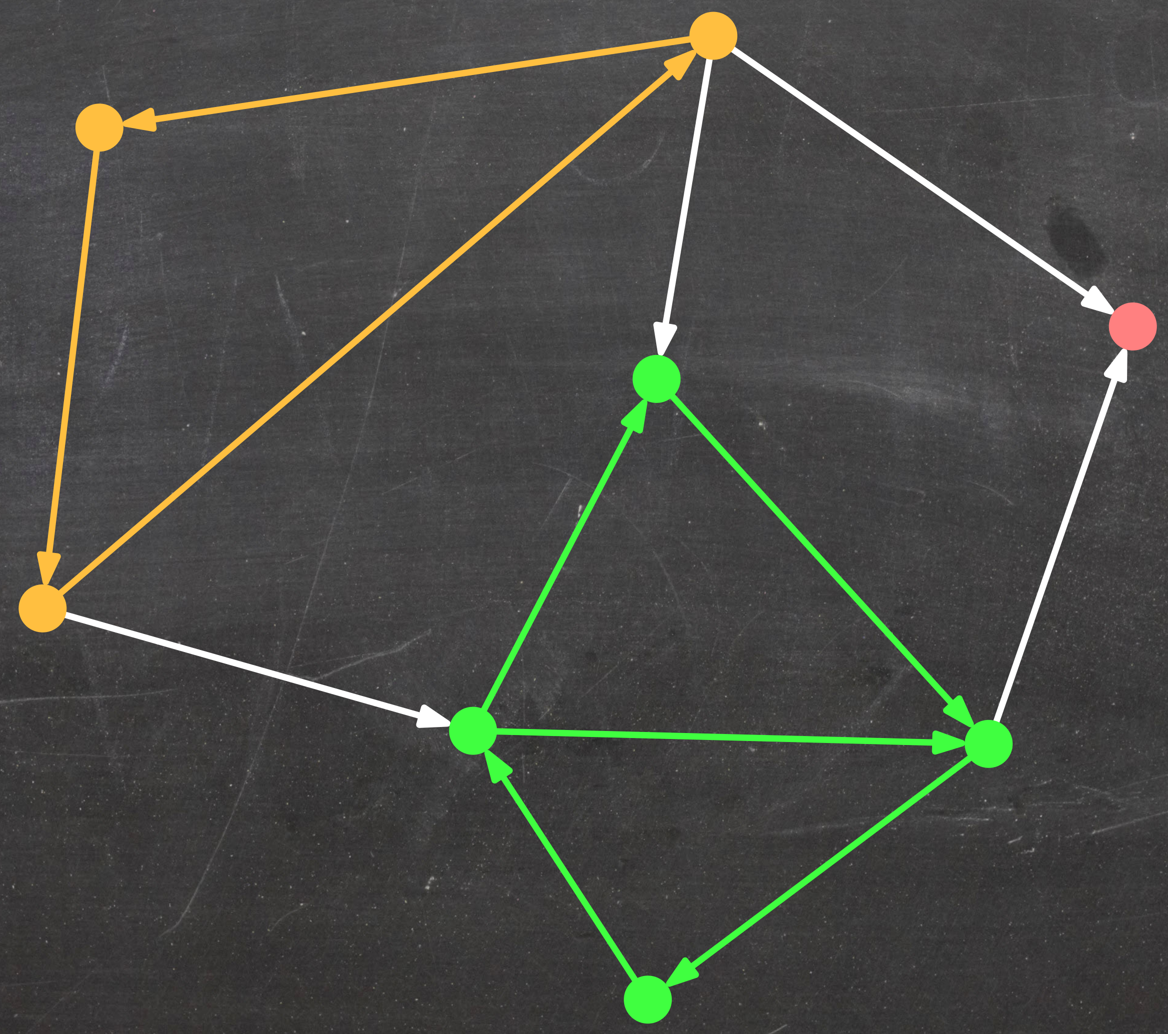

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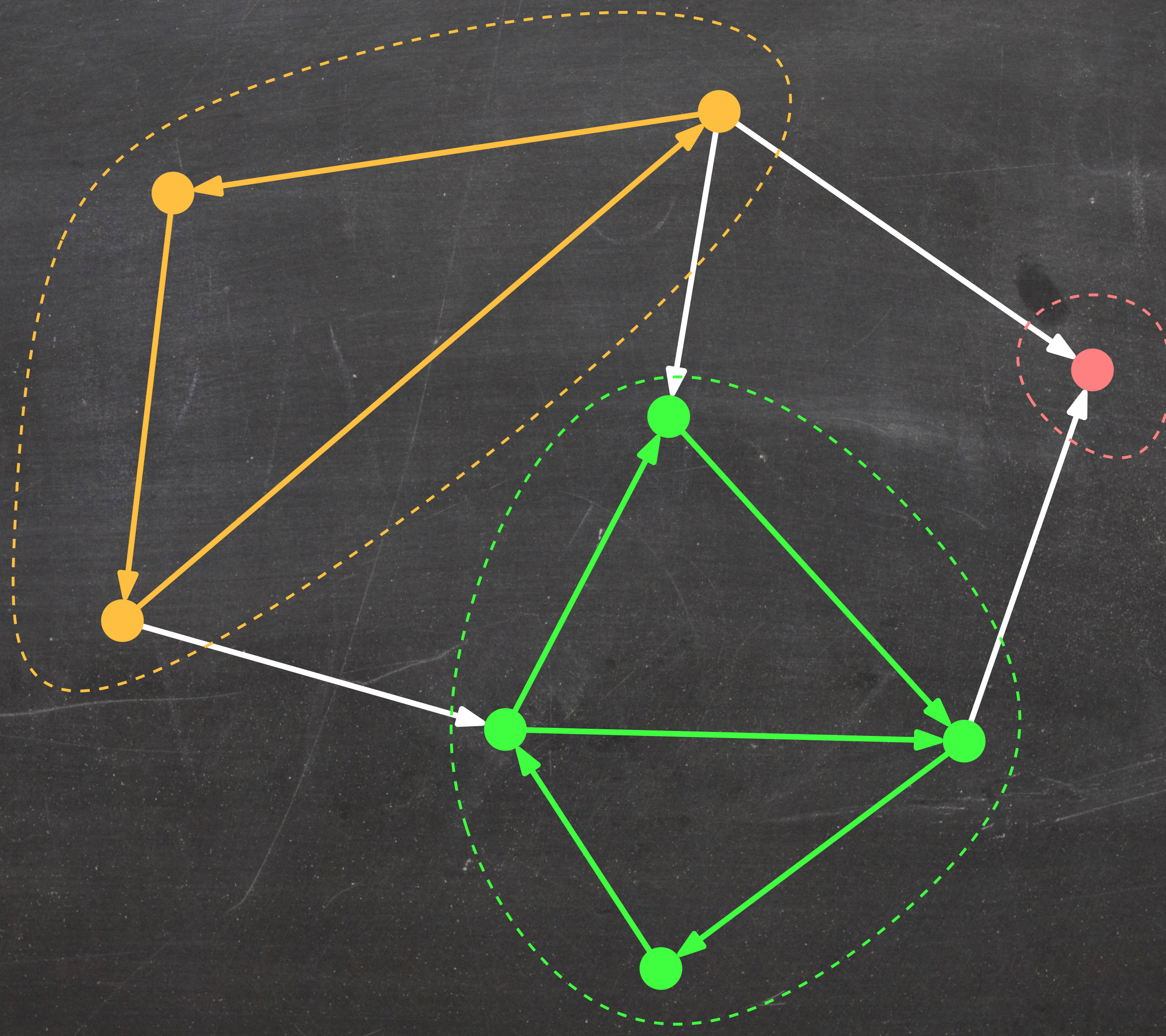
The strongly connected components of G are its maximal strongly connected subgraphs.



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Lemma: For a DFS forest F of G and any two vertices u and w of G, $u \sim_{SCC(G)} w \Rightarrow u \sim_{CC(F)} w$. (The vertices of each strongly connected component of G belong to the same tree of any DFS forest F of G.)

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Let C be the strongly connected component containing u and w and let x be the first vertex in C visited during the construction of F.

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This follows from

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Let $P = \langle x = x_0, x_1, \ldots, x_k = v \rangle$ be such a path from x to v and assume v is not a descendant of x.

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Since x_{i+1} is visited after x and all descendants of x have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.

Since x_{i+1} is no descendant of x, it is not a descendant of x_i.

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Since $x_i < x_{i+1}$ in preorder, this implies that (x_i, x_{i+1}) is a forward cross edge, a contradiction.

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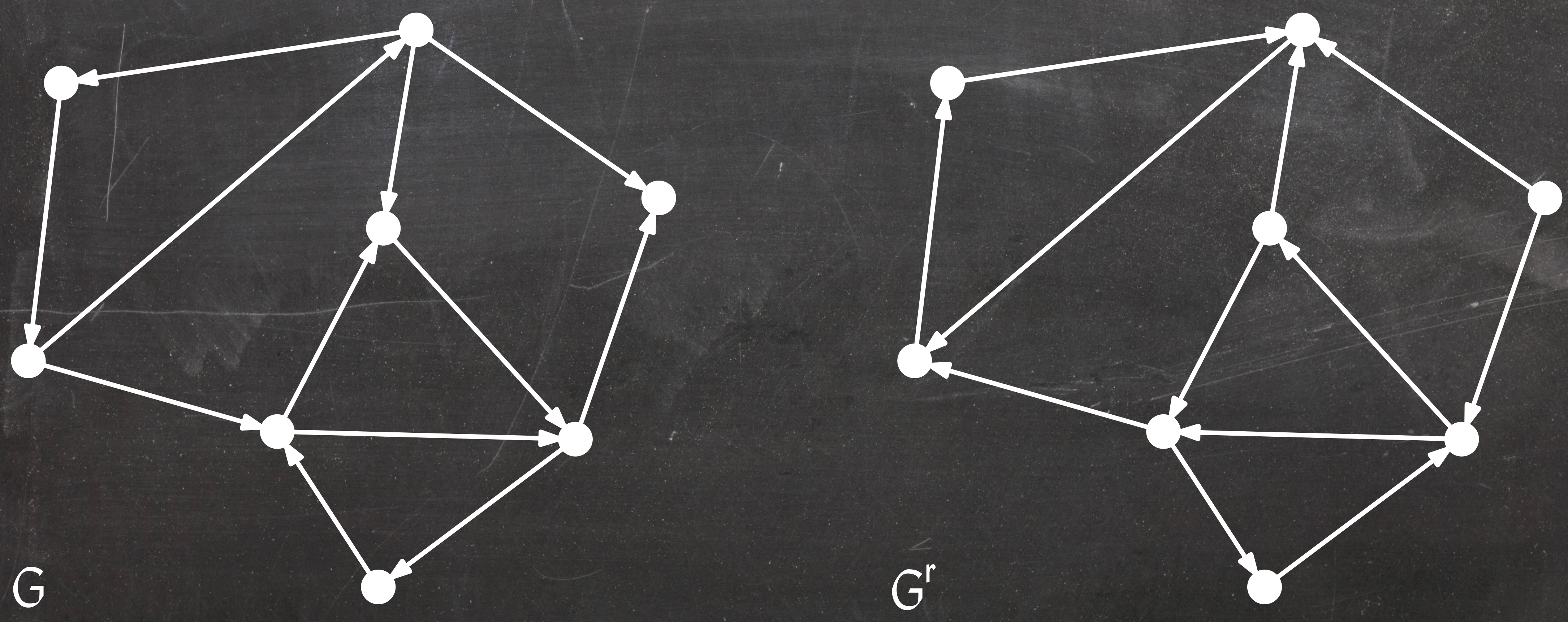
For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.



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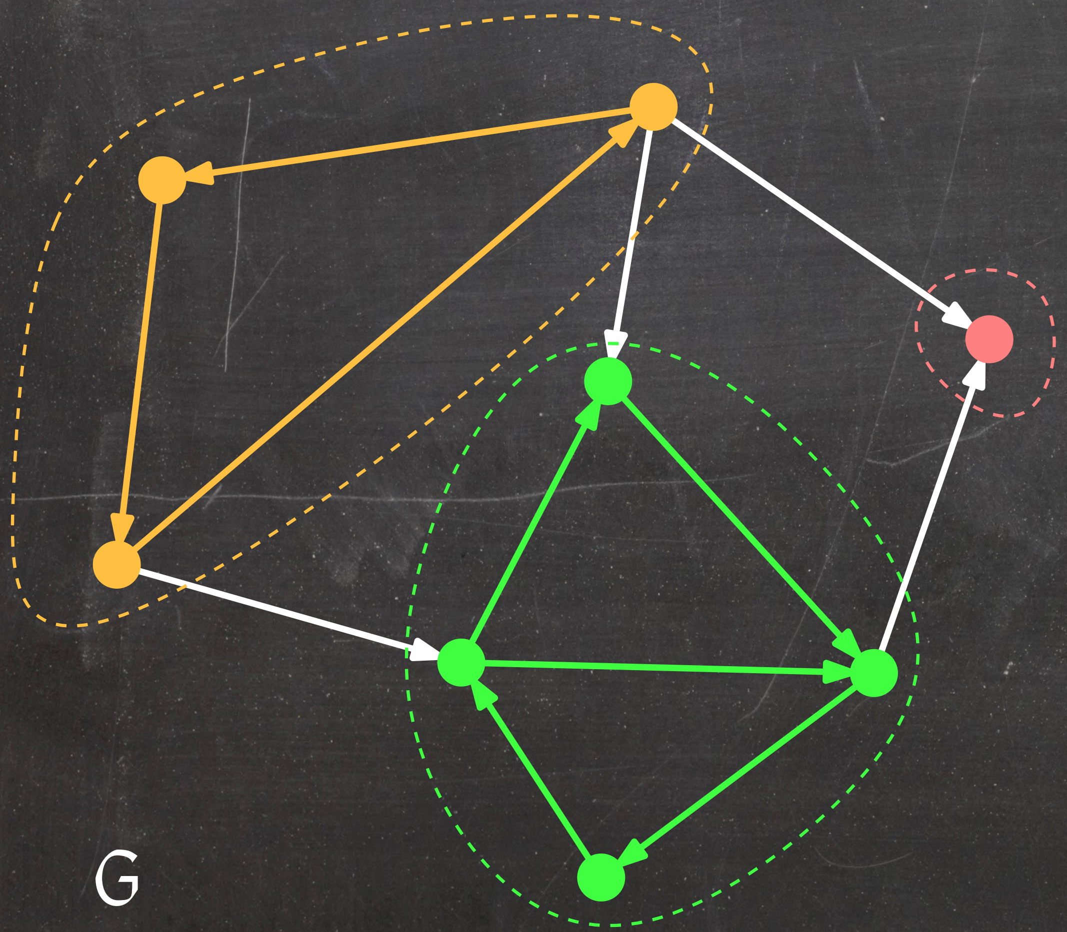

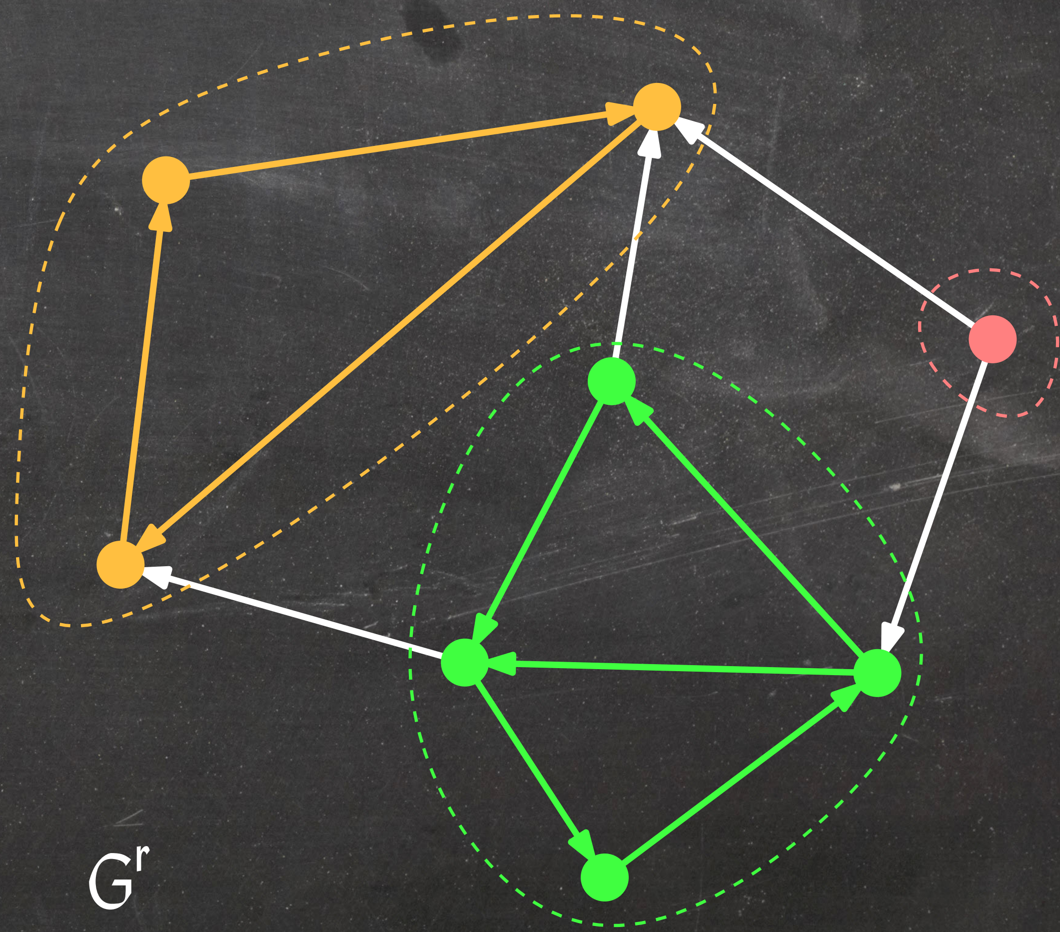

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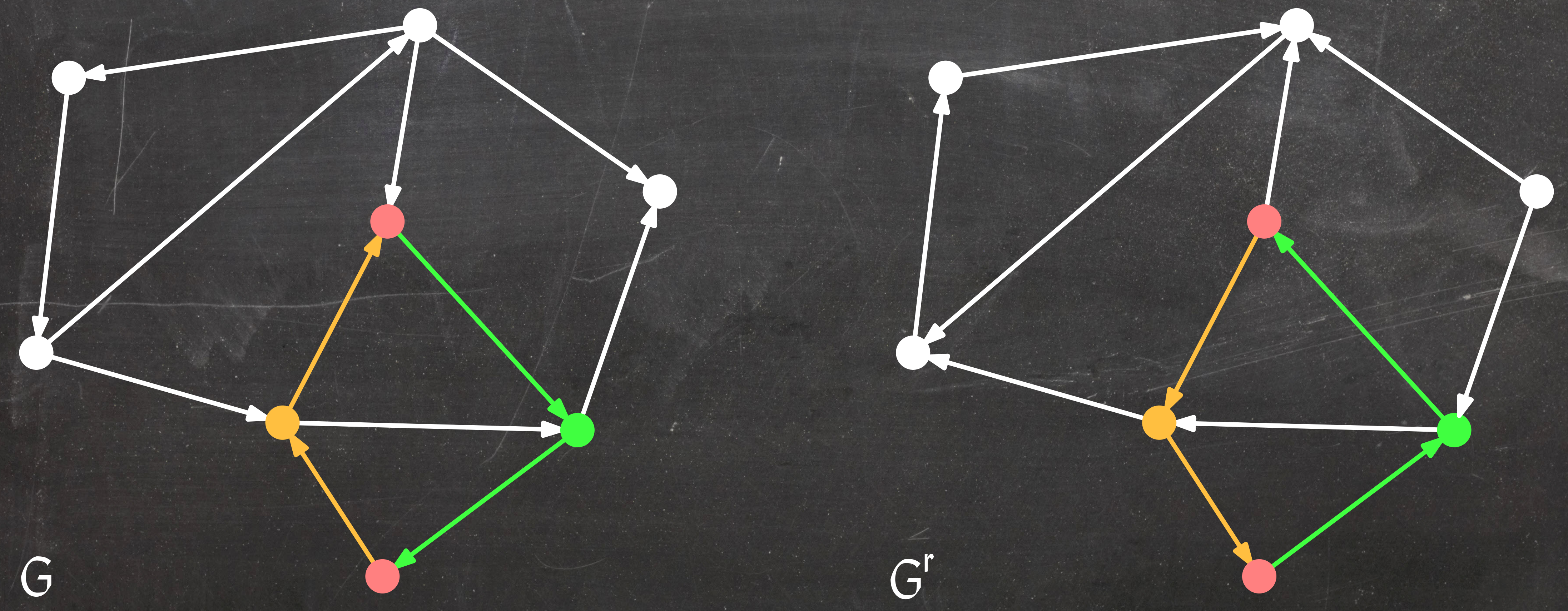
Proof: We have $u \rightsquigarrow_G v$ if and only if $v \rightsquigarrow_{G^r} u$.



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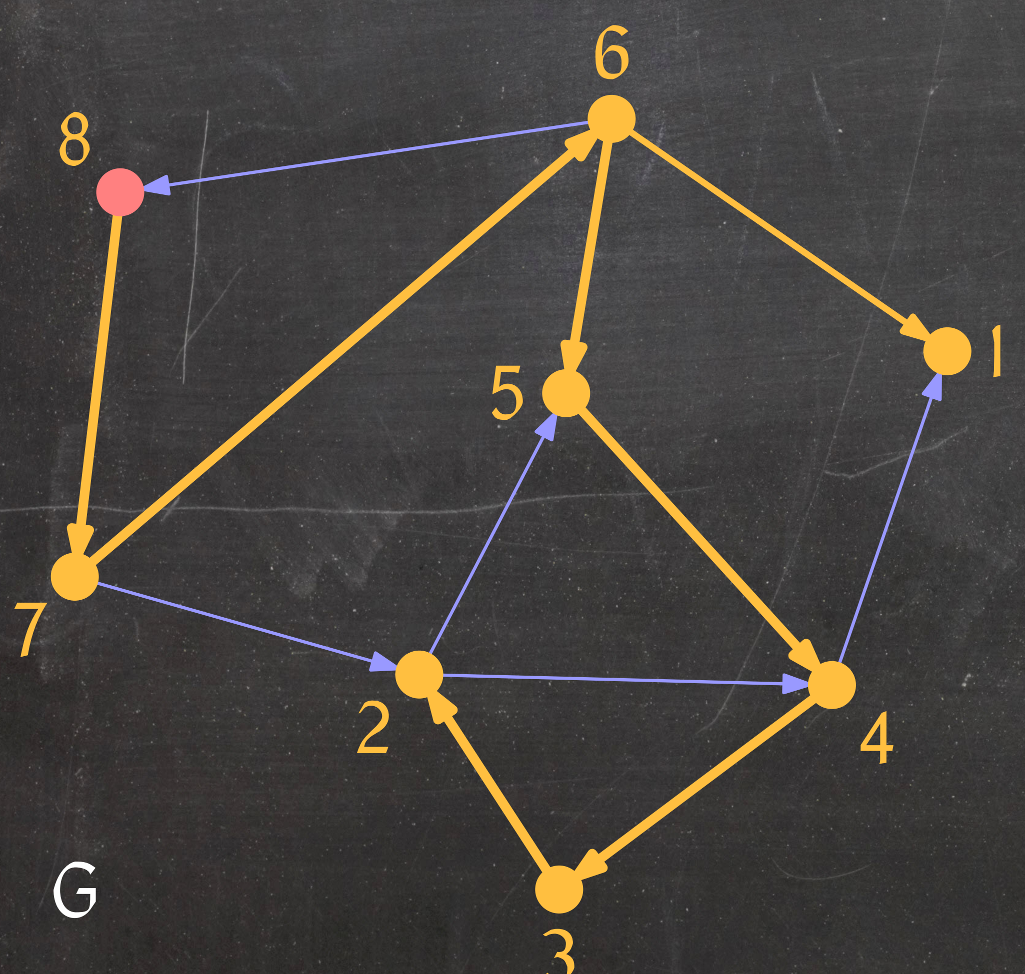

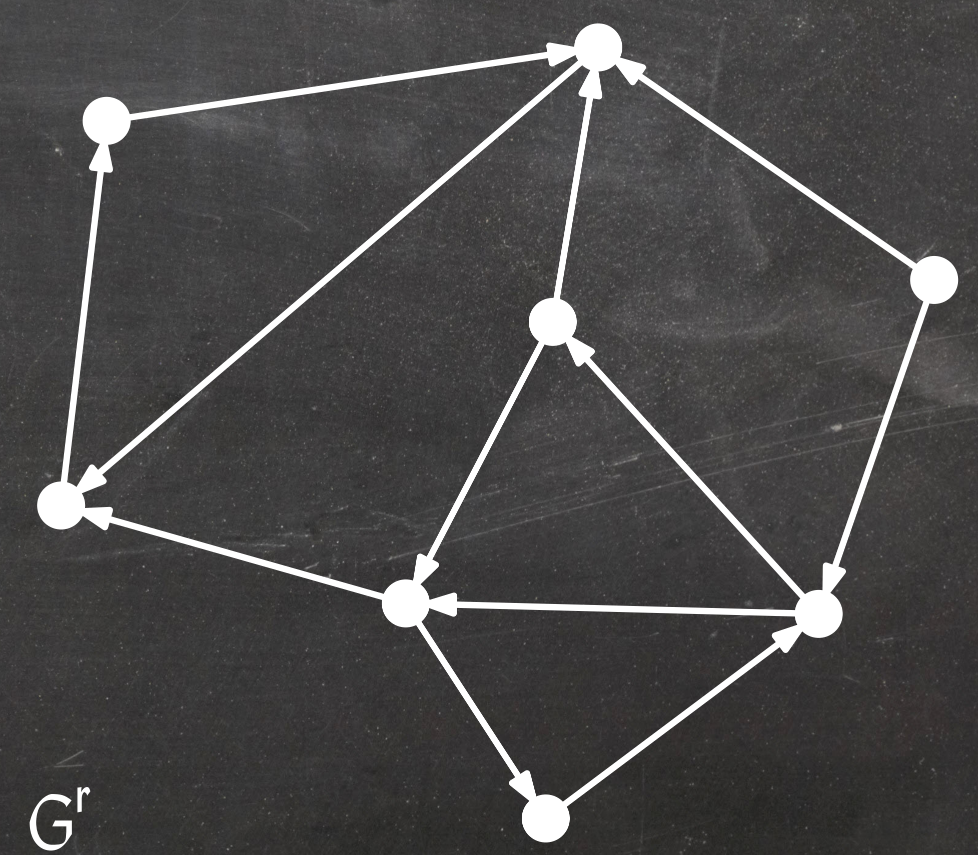

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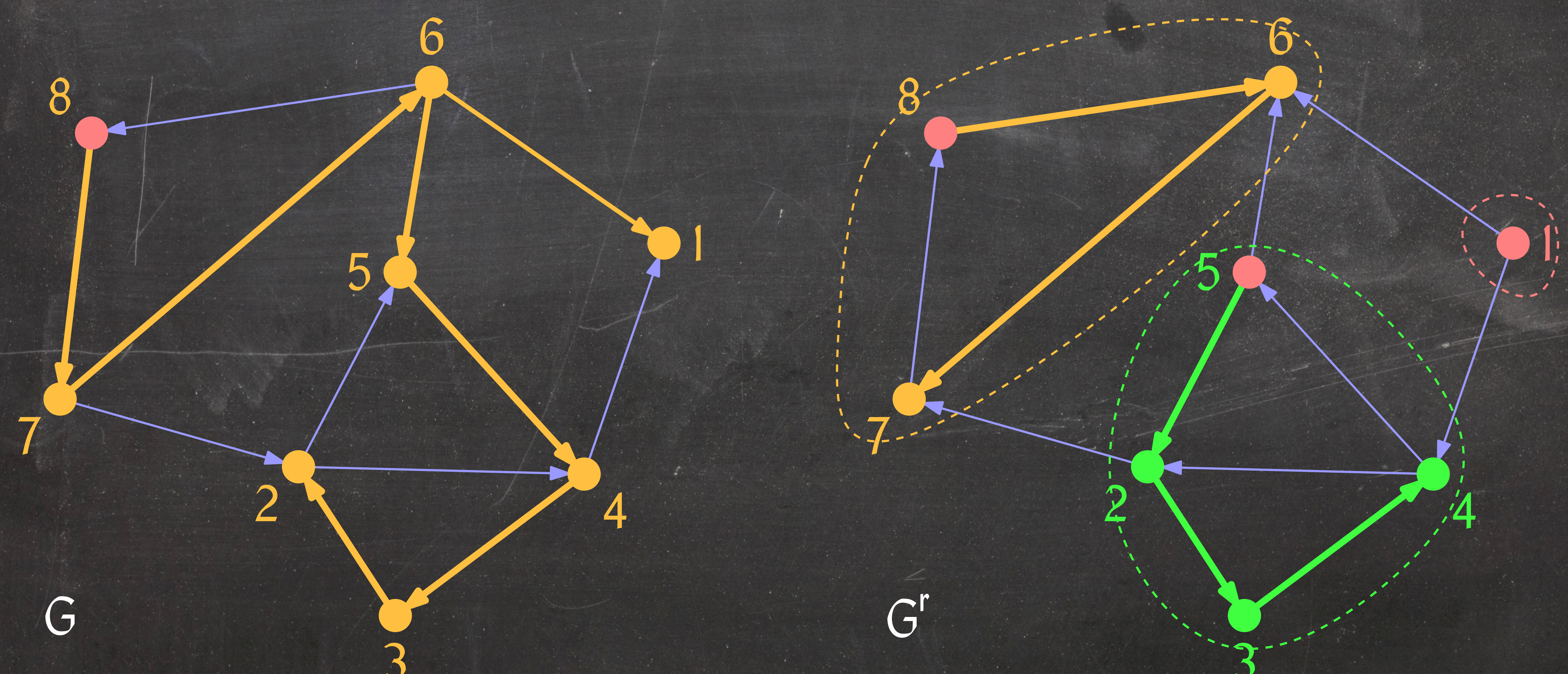

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Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

Lemma: $u \sim_{SCC(G)} v \Leftrightarrow u \sim_{SCC(G^r)} v$.

Proof: We have $u \rightsquigarrow_G v$ if and only if $v \rightsquigarrow_{G^r} u$.

Let F be a DFS forest of G and let $<$ be the postorder of F.

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$\Rightarrow$ Kosaraju's strong connectivity algorithm:

- Compute a DFS forest F of G.
- Compute G^r and arrange the vertices in reverse postorder w.r.t. F.
- Compute a DFS forest F^r of G^r.
- Extract a component labelling of the vertices or the strongly connected components themselves from F^r (almost) as we did for computing connected components.

This takes $O(n + m)$ time.

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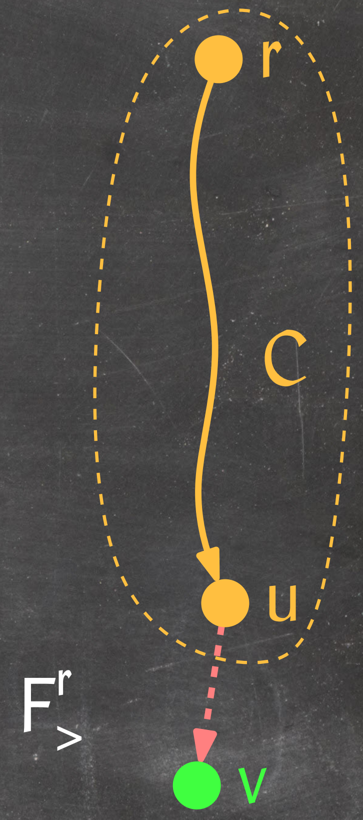

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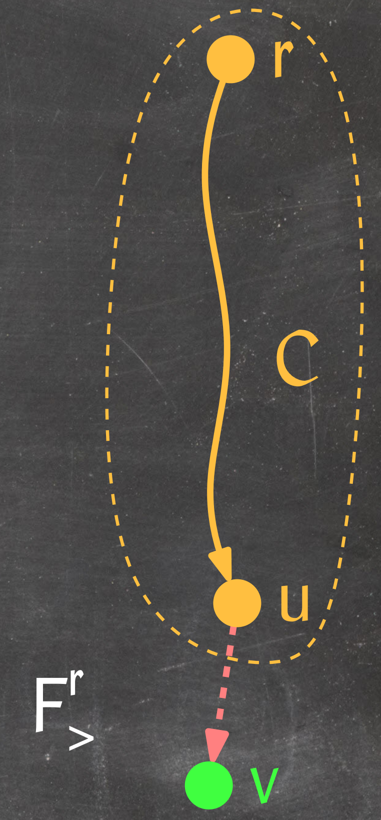

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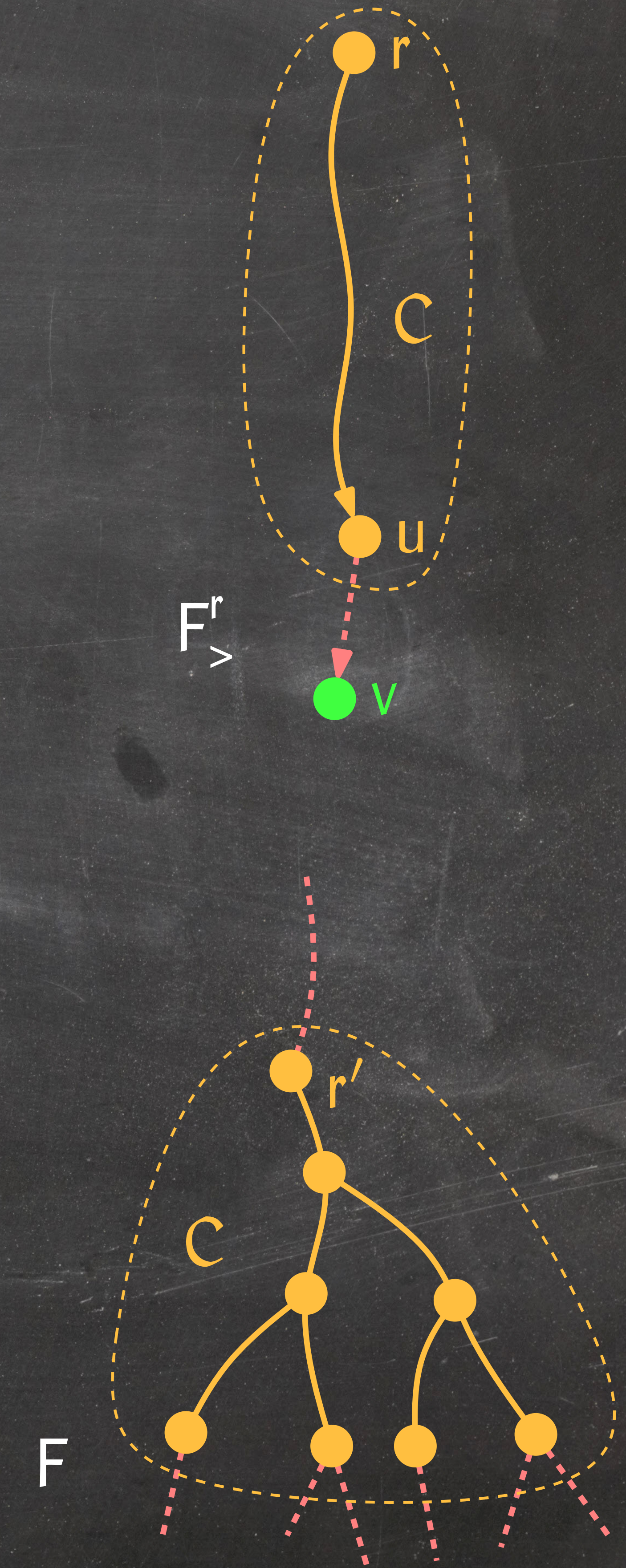

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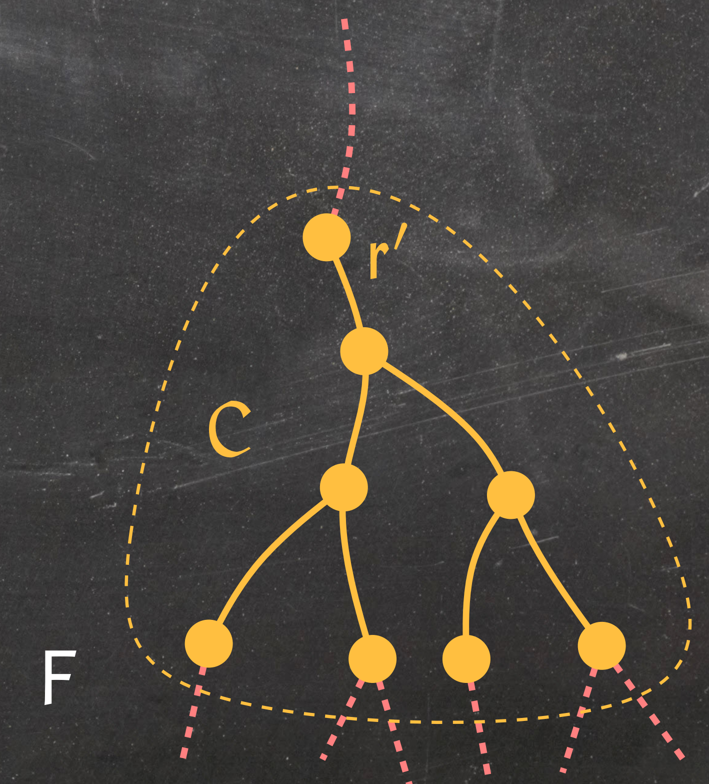
In F, all vertices in C are descendants of some vertex $r' \in C$ and $x \leq r'$ for all $x \in C$.

$\Rightarrow$ $r = r'$ and $u \leq r$.



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Lemma: $u \sim_{SCC(G)} v \Leftrightarrow u \sim_{CC(F'_>)} v$.

If v is a descendant of r in F, then
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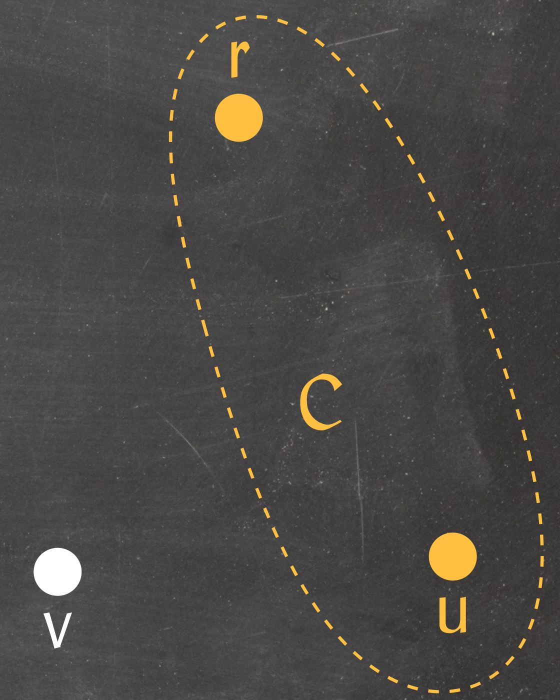

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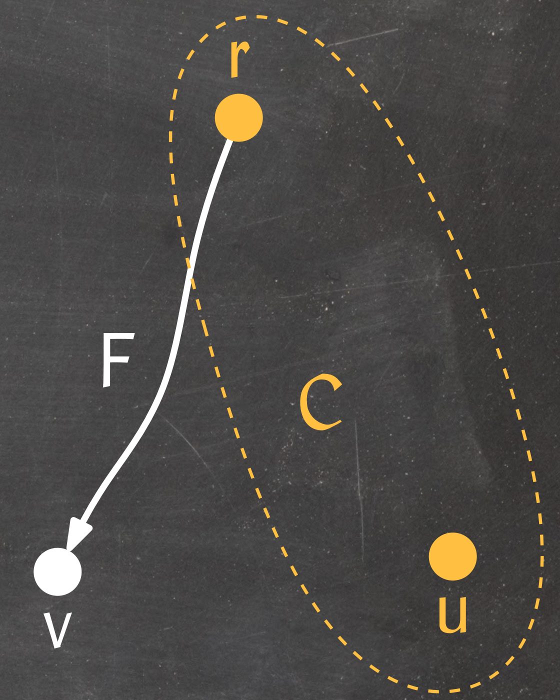

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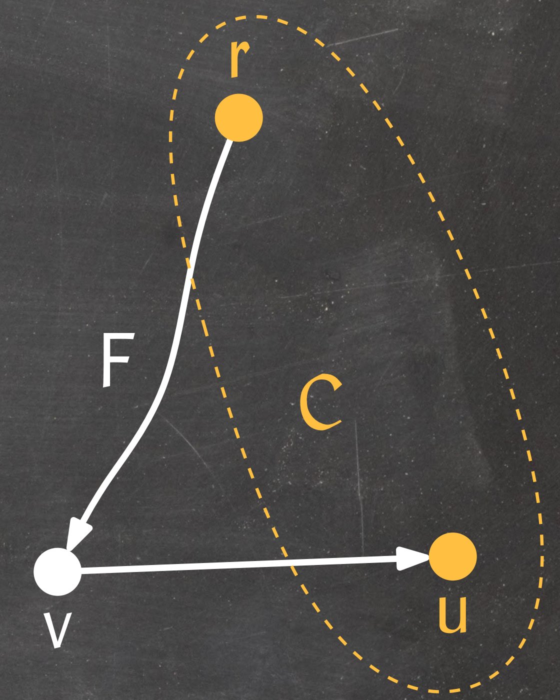

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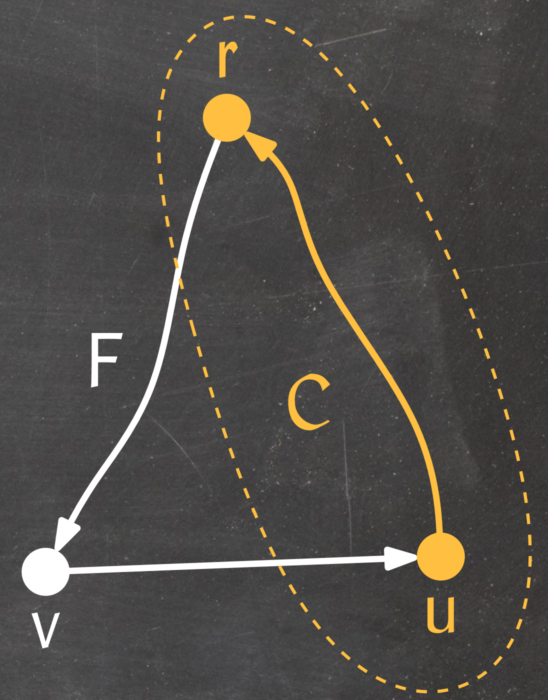

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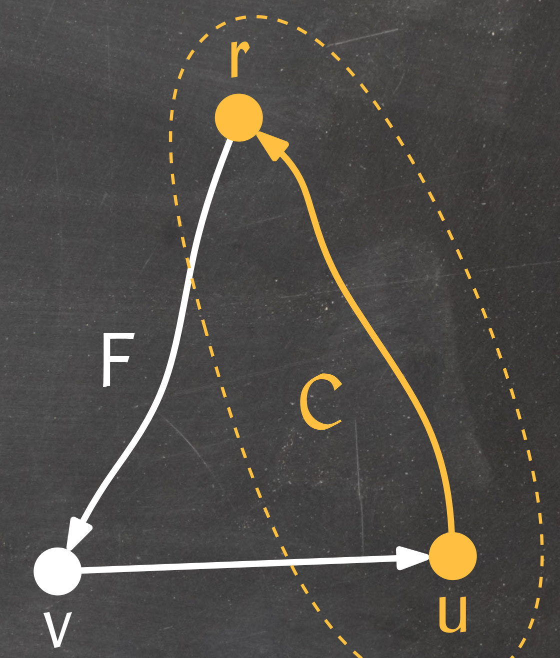

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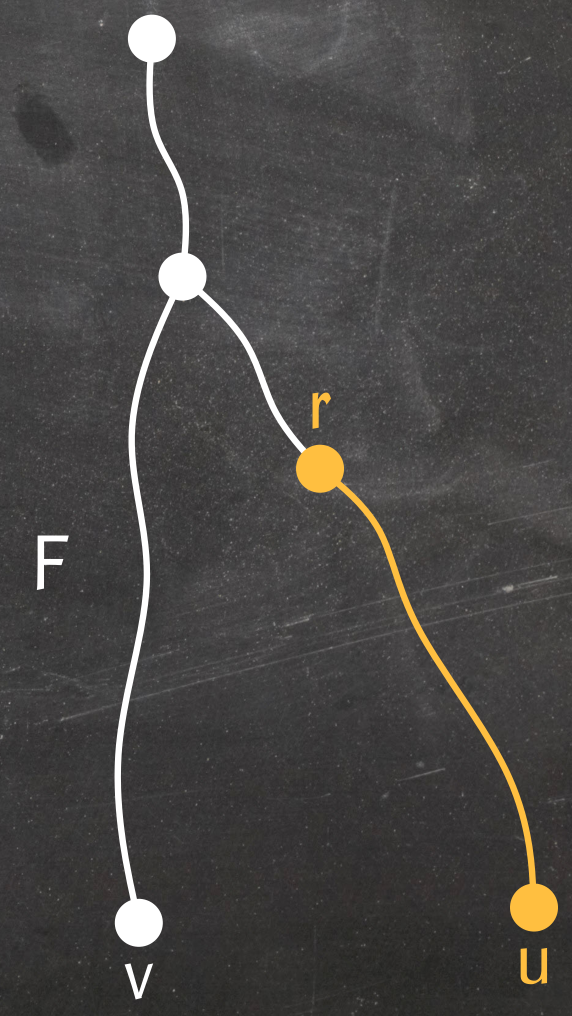

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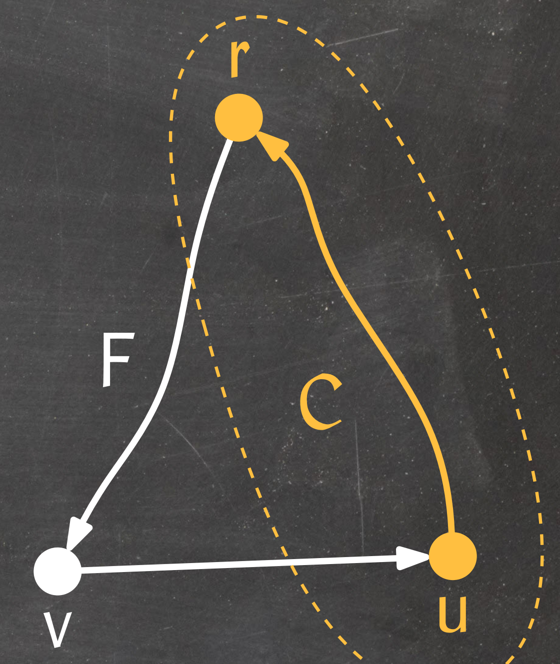
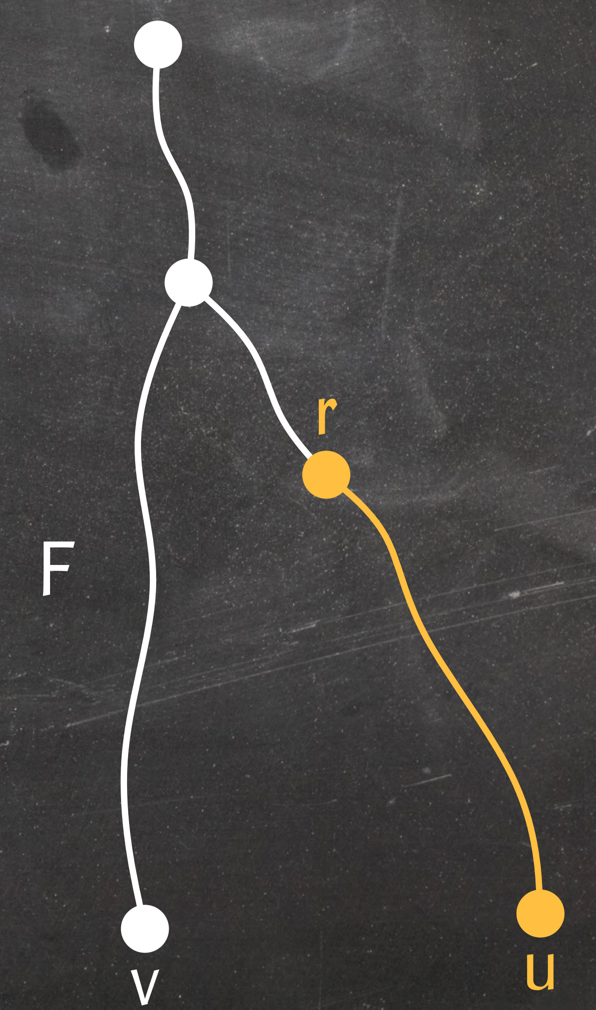
Since $u \leq r$, $v < r$, and the descendants of r are numbered consecutively, we have $v < u$.





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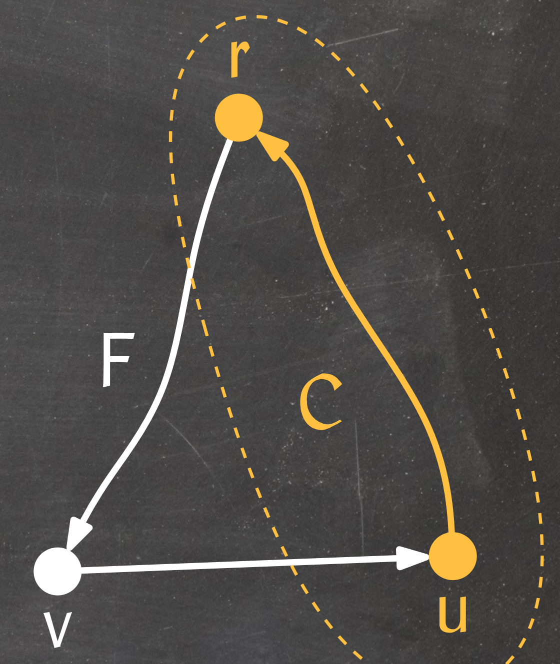

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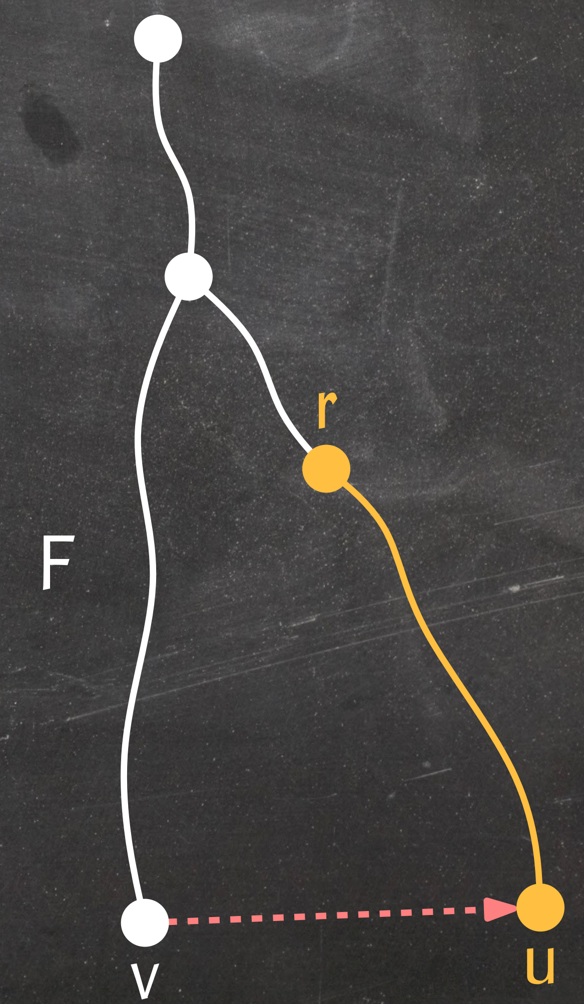
⇒ (v, u) is a forward cross edge w.r.t. F, a contradiction.



Summary

Graphs are fundamental in Computer Science:

Many problems are quite natural to express as graph problems:

- Matching problems
- Scheduling problems
- ...

Data structures are graphs whose nodes store useful information.

Graph exploration lets us learn the structure of a graph:

- Connectivity problems
- Distances between vertices
- Planarity
- ...