Graph Algorithms

Textbook Reading
Chapter 22
Overview

Design principle:
• Learn the structure of the graph by systematic exploration.

Proof technique:
• Proof by contradiction

Problems:
• Connected components
• Bipartiteness testing
• Topological sorting
• Strongly connected components
Graphs, Vertices, and Edges

A graph is an ordered pair $G = (V, E)$.

- $V$ is the set of vertices of $G$.
- $E$ is the set of edges of $G$.
- The elements of $E$ are pairs of vertices $(v, w)$. 
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The degree of a vertex is the number of its incident edges.
Undirected and Directed Graphs

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A directed edge \((v, w)\) is an **out-edge** of \(v\) and an **in-edge** of \(w\).
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A directed edge \((v, w)\) is an out-edge of \(v\) and an in-edge of \(w\).

The **in-degree** and **out-degree** of a vertex are the numbers of its in-edges and out-edges, respectively.
A path from a vertex $s$ to a vertex $t$ is a sequence of vertices $\langle x_0, x_1, \ldots, x_k \rangle$ such that

- $x_0 = s$,
- $x_k = t$, and
- for all $1 \leq i \leq k$, $(x_{i-1}, x_i)$ is an edge of $G$. 
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A **cycle** is a path from a vertex $x$ back to itself.

A path or cycle is **simple** if it contains every vertex of $G$ at most once.
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Adjacency List Representation

- Doubly-linked list of vertices
- Doubly-linked list of edges
- One doubly-linked adjacency list per vertex
- Pointers from adjacency list entries to vertices
- Cross-pointers between edges and adjacency list entries
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Representing Rooted Trees

A rooted tree $T$

- is a tree,
- is a directed graph,
- has one of its vertices, $r$, designated as a root.

There exists a path from $r$ to every vertex in $T$. 
Representing Rooted Trees

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Representation:

Tree $= \text{root}$

Every node stores

- an arbitrary \textit{key}
- a (doubly-linked) list of its \textit{children}. 
Standard Tree Orderings

Preorder:
- Every vertex appears before its children.
- Every vertex appears before its right sibling.
- The vertices in each subtree appear consecutively.

⇒ [a, b, c, d, e, f, g, h, i, j]
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**Lemma:** It takes linear time to arrange the vertices of a forest in preorder or postorder.
Connected Components and Spanning Forests

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A spanning forest of a graph $G$ is a subgraph $F \subseteq G$ with the same number of connected components and which is a forest.
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**Representation:**
- List of graphs or
- Labelling of vertices with component IDs

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A spanning forest of a graph $G$ is a subgraph $F \subseteq G$ with the same number of connected components and which is a forest.

**Representation:** List of rooted trees
Graph Traversal

We use graph traversal to build a spanning forest of $G$. 
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![Graph Diagram]
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Different traversal strategies lead to different spanning forests:

- Breadth-first search
- Depth-first search
- Prim's algorithm for computing minimum spanning trees
- Dijkstra's algorithm for computing shortest paths
Graph Traversal

TraverseGraph(G)

1. Mark every vertex of G as unexplored
2. \( F = [ ] \)
3. For every vertex \( u \in G \)
4. Do if not \( u \).explored
5. Then \( F \).append(TraverseFromVertex(G, u))
6. Return \( F \)
Graph Traversal

**TraverseFromVertex(G, u)**

1. \( u.\text{explored} = \text{True} \)
2. \( u.\text{tree} = \text{Node}(u, []) \)
3. \( Q = \text{an empty edge collection} \)
4. for every out-edge \( (u, v) \) of \( u \) do \( Q.\text{add}((u, v)) \)
5. while not \( Q.\text{isEmpty()} \) do \( (v, w) = Q.\text{remove()} \)
6. if not \( w.\text{explored} \) then \( w.\text{explored} = \text{True} \)
7. \( w.\text{tree} = \text{Node}(w, []) \)
8. \( v.\text{tree}.\text{children}.\text{append}(w.\text{tree}) \)
9. for every out-edge \( (w, x) \) of \( w \) do \( Q.\text{add}((w, x)) \)
10. return \( u.\text{tree} \)
Graph Traversal Computes a Spanning Forest

It computes a subgraph of $G$ because it only adds edges of $G$ to $F$. 
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$$\text{TraverseFromVertex}(G, u)$$

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3. $Q = \text{an empty edge collection}$
4. for every out-edge $(u, v)$ of $u$
   5. do $Q.\text{add}((u, v))$
6. while not $Q.\text{isEmpty}()$
   7. do $(v, w) = Q.\text{remove}()$
   8. if not $w.\text{explored}$
      9. then $w.\text{explored} = \text{True}$
      10. $w.\text{tree} = \text{Node}(w, [])$
      11. $v.\text{tree.}\text{children.}\text{append}(w.\text{tree})$
    12. for every out-edge $(w, x)$ of $w$
        13. do $Q.\text{add}((w, x))$
14. return $u.\text{tree}$
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It computes a subgraph of $G$ because it only adds edges of $G$ to $F$.

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To prove:

- $F$ contains no cycle.
- If $u \sim_{CC(G)} v$ ($u$ and $v$ belong to the same component of $G$), then $u \sim_{CC(F)} v$. 
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Observation: Every edge $(u, v)$ in $Q$ has at least one explored endpoint, namely $u$. 
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TraverseFromVertex(G, u)
1   u.explored = True
2   u.tree = Node(u, [])
3   Q = an empty edge collection
4   for every out-edge $(u, v)$ of u
5     do Q.add((u, v))
6     while not Q.isEmpty()
7       (v, w) = Q.remove()
8       if not w.explored
9         then w.explored = True
10     w.tree = Node(w, [])
11     v.tree.children.append(w.tree)
12     for every out-edge $(w, x)$ of w
13       do Q.add((w, x))
14   return u.tree
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Corollary: Both endpoints of every edge in $F$ are explored.
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Corollary: $F$ contains no cycle.

Proof by contradiction:
By the time we add the last edge to the cycle, both its endpoints are explored.

$\Rightarrow$ We would not have added it.
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**Lemma:** TraverseFromVertex(G, u) visits all vertices v such that $u \sim_{CC(G)} v$ and only those.
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Proof: By induction on the number of invocations of TraverseFromVertex made so far.
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When TraverseFromVertex(G, u) is called, every vertex v such that $u \sim_{CC(G)} v$ is unexplored.
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We visit all vertices v such that $u \sim_{CC(G)} v$:

- Path P from u to v
- First unexplored vertex on P
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We visit all vertices v such that u $\sim_{CC(G)}$ v:

- x adds (x, w) to Q.
- $\Rightarrow$ We'd visit w.
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We do not visit a vertex v such that u \( \not\sim_{CC(G)} v \):

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**Diagram:**

- **Path P from u to v**
- **First unexplored vertex on P**
- **u**
- **x**
- **w**
- **v**
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- first explored vertex such that \( u \not\sim_{CC(G)} v \).
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We visit all vertices v such that $u \sim_{\text{CC}(G)} v$:

- x adds $(x, w)$ to Q.
- $\Rightarrow$ We'd visit w.

We do not visit a vertex v such that $u \not\sim_{\text{CC}(G)} v$:

- v explored because of edge $(w, v) \in Q$.
- w explored before v.
- $\Rightarrow$ w $\sim_{\text{CC}(G)} u$.
- $\Rightarrow$ v $\sim_{\text{CC}(G)} u$.
The Cost of Graph Traversal

**Lemma:** TraverseGraph takes $O(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from $Q$, respectively.
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TraverseGraph itself takes $O(n)$ time.
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1 Mark every vertex of G as unexplored
2 F = []
3 for every vertex u ∈ G
4 do if not u.explored
5 then F.append(TraverseFromVertex(G, u))
6 return F
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$\Rightarrow$ The cost of the for-loops in TraverseFromVertex is $O(m \cdot (1 + t_a))$. 
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1. \(u\).explored = True
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4. for every out-edge \((u, v)\) of \(u\) do \(Q\).add((u, v))
5. while not \(Q\).isEmpty() do
6. \((v, w)\) = \(Q\).remove()
7. if not \(w\).explored then \(w\).explored = True
8. \(w\).tree = Node(w, [])
9. \(v\).tree.children.append(w.tree)
10. for every out-edge \((w, x)\) of \(w\) do \(Q\).add((w, x))
11. return \(u\).tree
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TraverseGraph itself takes $O(n)$ time.

Every edge is added to $Q$ at most once.
$\Rightarrow$ The cost of the for-loops in TraverseFromVertex is $O(m \cdot (1 + t_a))$.

Every edge that is removed must be added first.
$\Rightarrow$ The cost of the while-loop in TraverseFromVertex is $O(m \cdot (1 + t_r))$. 
The Cost of Graph Traversal

Lemma: TraverseGraph takes $O(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from Q, respectively.

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Every edge is added to Q at most once.
⇒ The cost of the for-loops in TraverseFromVertex is $O(m \cdot (1 + t_a))$.

Every edge that is removed must be added first.
⇒ The cost of the while-loop in TraverseFromVertex is $O(m \cdot (1 + t_r))$.

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The Cost of Graph Traversal

**Lemma:** TraverseGraph takes $O(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from $Q$, respectively.

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Computing Connected Components

- Compute a spanning forest $F$.
- Collect vertices of trees in $F$.
- Compute representation of connected components.
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CollectComponentVertices($F$)

1. $L = []$
2. for every tree $T \in F$
3. do $L$.append(CollectDescendantVertices($T$))
4. return $L$
Computing Connected Components

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`CollectComponentVertices(F)`

```python
1 L = []
2 for every tree $T \in F$
3     do L.append(CollectDescendantVertices(T))
4 return L
```

`CollectDescendantVertices(T)`

```python
1 L = [T.key]
2 for every child $T'$ of $T$
3     do L.concat(CollectDescendantVertices(T'))
4 return L
```
Computing Connected Components

- Compute a spanning forest $F$.
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**CollectComponentVertices**(F)

1. $L = []$
2. for every tree $T \in F$
3. do $L.append(\text{CollectDescendantVertices}(T))$
4. return $L$

**CollectDescendantVertices**(T)

1. $L = [T.\text{key}]$
2. for every child $T'$ of $T$
3. do $L.concat(\text{CollectDescendantVertices}(T'))$
4. return $L$

**Lemma:** Collecting the vertices of all components takes $O(n)$ time.
Computing Connected Components

Representation using vertex labels:

\[ \text{ComponentLabels}(L) \]

\[
\begin{align*}
1 & \quad i = 0 \\
2 & \quad \text{for every list } L' \in L \\
3 & \quad \quad \text{do } i = i + 1 \\
4 & \quad \quad \text{for every vertex } v \in L' \\
5 & \quad \quad \quad \text{do } v.cc = i \\
\end{align*}
\]

Cost: \( O(n) \)
Computing Connected Components

Representation as list of graphs:

We already have the right adjacency lists for the vertices. Need to partition the vertex and edge lists into vertex and edge lists for the components.
Computing Connected Components

Representation as list of graphs:

We already have the right adjacency lists for the vertices. Need to partition the vertex and edge lists into vertex and edge lists for the components.

Vertex lists:

BuildVertexLists(L)

```
VL = []
for every list L′ ∈ L
do VL′ = []
for every vertex v ∈ L′
do VL′.append(v)
VL.append(VL′)
return VL
```
Computing Connected Components

Edge lists:

BuildEdgeLists(G, L)

```python
1 EL = []
2 for every edge e ∈ G
do e.collected = False
3 for every list L′ ∈ L
do EL′ = []
4 for every vertex v ∈ L′
do for every edge e incident with v
do if not e.collected
then e.collected = True
5 EL′.append(e)
6 EL.append(EL′)
7 return EL
```
Lemma: The connected components of a graph can be computed in $O(n + m)$ time.

- Building a spanning forest takes $O(n + m + m \cdot (t_a + t_r))$ time.
- Computing the vertex labelling or list of graphs then takes $O(n + m)$ time.
- Using a stack or queue to represent $Q$, we get $t_a \in O(1)$ and $t_r \in O(1)$. 

Breadth-First Search

Breadth-first search (BFS) = graph traversal using a queue to implement Q.

Queue:

Q.dequeue()  Q.enqueue(x)
Breadth-First Search

Breadth-first search (BFS) = graph traversal using a queue to implement Q.

Queue:

Constant-time implementations:
- Doubly-linked list
- Singly-linked list with tail pointer
- “Circular” array (amortized constant cost)
- Pair of singly-linked lists (functional)
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Lemma: Breadth-first search takes $O(n + m)$ time.
A Property of Undirected BFS Forests

**BFS forest** = spanning forest computed using BFS

Let the **depth** $d_F(v)$ of a vertex $v$ in a rooted forest $F$ be the distance from the root of its tree.

**Lemma:** BFS visits the vertices of each component of $F$ in order of increasing depth.
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**Lemma:** BFS visits the vertices of each component of $F$ in order of increasing depth.

Assume $d_F(v) < d_F(w)$ and $w$ is visited before $v$. Choose such a pair $(v, w)$ so that $d_F(w)$ is minimized.

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$v \neq u$ because $u$ is visited before any other vertex in the same tree.

$\Rightarrow$ parent($v$) and parent($w$) exist and

\[ d_F(\text{parent}(v)) = d_F(v) - 1 < d_F(w) - 1 = d_F(\text{parent}(w)). \]
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$d_F($parent($v$)) = $d_F(v) - 1 < d_F(w) - 1 = d_F($parent($w$)).

$\Rightarrow$ parent($v$) is visited before parent($w$).
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\[ d_F(\text{parent}(v)) = d_F(v) - 1 < d_F(w) - 1 = d_F(\text{parent}(w)). \]
$\Rightarrow$ parent$(v)$ is visited before parent$(w)$.
$\Rightarrow$ The edge (parent$(v)$, $v$) is enqueued before the edge (parent$(w)$, $w$).
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$\Rightarrow$ The edge (parent(v), v) is enqueued before the edge (parent(w), w).

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$\Rightarrow$ $v$ is visited before $w$, a contradiction.
Lemma: For every edge \((v, w)\) of \(G\) and any BFS forest \(F\) of \(G\), the depths of \(v\) and \(w\) in \(F\) differ by at most one.
A Property of Undirected BFS Forests

**Lemma:** For every edge \((v, w)\) of \(G\) and any BFS forest \(F\) of \(G\), the depths of \(v\) and \(w\) in \(F\) differ by at most one.

Assume \(d_F(w) > d_F(v) + 1\).
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\[\Rightarrow d_F(parent(w)) > d_F(v).\]
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\(w\) is unexplored when the edge \((\text{parent}(w), w)\) is dequeued.

\[\Rightarrow w\ \text{is unexplored when the edge} \ (v, w)\ \text{is dequeued}.\]

\[\Rightarrow w\ \text{would be added to the list of} \ v\text{'s children, a contradiction.}\]
Bipartite Graphs

A graph is bipartite if its vertices can be partitioned into two sets \((U, W)\) such that every edge has one endpoint in \(U\) and one endpoint in \(W\).
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A graph is bipartite if its vertices can be partitioned into two sets \((U, W)\) such that every edge has one endpoint in \(U\) and one endpoint in \(W\).

**Lemma:** A graph is bipartite if and only if it contains no odd cycle.
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Assume there exists an odd cycle in \(G\).
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Let \(F\) be a BFS forest of \(G\).
Add vertices on odd levels to \(U\), on even levels to \(W\).
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This is the only partition that satisfies the edges of \(F\)!
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This is the only partition that satisfies the edges of \(F\)!

\[\Rightarrow \text{G is bipartite if and only if there is no edge with both endpoints on the same level.}\]

If there is such an edge, there's an odd cycle.
Bipartite Graphs

A graph is bipartite if its vertices can be partitioned into two sets \((U, W)\) such that every edge has one endpoint in \(U\) and one endpoint in \(W\).

**Lemma:** A graph is bipartite if and only if it contains no odd cycle.

**Lemma:** Given a BFS forest \(F\) of \(G\), \(G\) is bipartite if and only if there is no edge in \(G\) with both endpoints on the same level in \(F\).
Bipartiteness Testing

- Compute BFS forest $F$ of $G$.
- Collect vertices on alternating levels of $F$ into two sets $(U, W)$.
- Test whether any edge has both endpoints in the same set, $U$ or $W$.
- If so, report the odd cycle induced by such an edge.
- Otherwise, report the bipartition $(U, W)$.

Collecting vertices on alternating levels:

**AlternatingLevels($F$)**

1. $U = W = []$
2. for every tree $T$ in $F$
3. do AlternatingLevels$'$($T$, $U$, $W$)
4. return $(U, W)$

**AlternatingLevels$'$($T$, $U$, $W$)**

1. $U$.append($T$.key)
2. for every child $T'$ of $T$
3. do AlternatingLevels$'$($T'$, $W$, $U$)
Bipartiteness Testing

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Testing for an “odd edge”:

OddEdge($G$, $U$, $W$)

1. $A$ = an array of size $n$
2. for every vertex $u \in U$
3. do $A[u] = "U"
4. for every vertex $w \in W$
5. do $A[w] = "W"
6. for every edge $(u, w) \in G$
8. then return $(u, w)$
9. return Nothing
Bipartiteness Testing

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Finding the ancestor edges of all vertices:

**AncestorEdges($F$)**

1. $L =$ an empty list of vertex-vertex list pairs
2. for every tree $T \in F$
3. do AncestorEdges$'(T, [], L)$
4. return $L$

**AncestorEdges$'(T, A, L)$**

1. $L =$ $L$.append([[(T.key, A)]]
2. for every child $T'$ of $T$
3. do AncestorEdges$'(T', [(T.key, T'.key)] ++ A, L)$
Bipartiteness Testing

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- Otherwise, report the bipartition $(U, W)$.

Reporting an odd cycle:

\texttt{OddCycle(L, (u, w))}

1. Find $(u, A_u)$ and $(w, A_w)$ in $L$
2. $C_u = C_w = []$
3. \texttt{while} $A_u.head \neq A_w.head$
4. \texttt{do} $C_u.append(A_u.head)$
5. \texttt{do} $C_w.append(A_w.head)$
6. $A_u = A_u.tail$
7. $A_w = A_w.tail$
8. $C_u.reverse().concat([(u, w)]).concat(C_w)$
9. \texttt{return} $C_u$
Bipartiteness Testing

- Compute BFS forest $F$ of $G$.
- Collect vertices on alternating levels of $F$ into two sets $(U, W)$.
- Test whether any edge has both endpoints in the same set, $U$ or $W$.
- If so, report the odd cycle induced by such an edge.
- Otherwise, report the bipartition $(U, W)$.

**Lemma:** It takes linear time to test whether a graph $G$ is bipartite and either report a valid bipartition or an odd cycle in $G$. 
Depth-First Search

Depth-first search (DFS) = graph traversal using a stack to implement Q.

Stack:

- Q.pop()
- Q.push(x)
Depth-First Search

Depth-first search (DFS) = graph traversal using a stack to implement Q.

Stack:

Constant-time implementations:
- Singly-linked list
- Resizeable array (amortized constant cost)
Depth-First Search

Depth-first search (DFS) = graph traversal using a stack to implement Q.

Stack:

Q.pop() ← Q.push(x)

Constant-time implementations:
- Singly-linked list
- Resizeable array (amortized constant cost)

Lemma: Depth-first search takes $O(n + m)$ time.
Depth-First Search and Preorder

**Lemma:** Depth-first search visits the vertices of the spanning forest it creates in preorder.
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It visits the children of every node in left-to-right order. (That’s how we define this order.)
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**Lemma:** Depth-first search visits the vertices of the spanning forest it creates in preorder.

It visits the children of every node in left-to-right order. (That’s how we define this order.)

It visits every node after its parent:

- $v$ is visited when the edge $(\text{parent}(v), v)$ is popped.
- The edge $(\text{parent}(v), v)$ must be pushed before this can happen.
- The edge $(\text{parent}(v), v)$ is pushed when $\text{parent}(v)$ is visited.
**Depth-First Search and Preorder**

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It visits the vertices in each subtree consecutively.
Depth-First Search and Preorder

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- The edge $(\text{parent}(v), v)$ is pushed when parent$(v)$ is visited.

It visits the vertices in each subtree consecutively.

Observation: An edge with one explored and one unexplored endpoint is on the stack.
Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
- $y$ is visited before some descendant $z$.

Choose $y$ and $z$ so that

- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after $\text{parent}(z)$.
Depth-First Search and Preorder

Assume there exist two vertices \( x \) and \( y \) such that
- \( y \) is not a descendant of \( x \),
- \( y \) is visited after \( x \), and
- \( y \) is visited before some descendant \( z \).

Choose \( y \) and \( z \) so that
- \( y \) is the first visited vertex satisfying the above conditions and
- \( y \) is visited after parent(\( z \)).

Case 1: \( y \) is a root.

Cannot happen because the edge (parent(\( z \)), \( z \)) is on the stack when \( y \) is visited and the stack is empty when a root is visited.
Depth-First Search and Preorder

Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
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Choose $y$ and $z$ so that

- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after parent($z$).

**Case 2:** $y$ has a parent parent($y$).
Depth-First Search and Preorder

Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
- $y$ is visited before some descendant $z$.

Choose $y$ and $z$ so that

- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after $\text{parent}(z)$.

Case 2: $y$ has a parent $\text{parent}(y)$.

$\text{parent}(y)$ is visited before $x$ and thus before $\text{parent}(z)$. 
Depth-First Search and Preorder

Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
- $y$ is visited before some descendant $z$.

Choose $y$ and $z$ so that

- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after $\text{parent}(z)$.

Case 2: $y$ has a parent $\text{parent}(y)$.

$\text{parent}(y)$ is visited before $x$ and thus before $\text{parent}(z)$.

$\Rightarrow$ The edge $(\text{parent}(y), y)$ is on the stack when $\text{parent}(z)$ is visited and thus when the edge $(\text{parent}(z), z)$ is pushed.
Depth-First Search and Preorder

Assume there exist two vertices x and y such that
- y is not a descendant of x,
- y is visited after x, and
- y is visited before some descendant z.

Choose y and z so that
- y is the first visited vertex satisfying the above conditions and
- y is visited after parent(z).

Case 2: y has a parent parent(y).

parent(y) is visited before x and thus before parent(z).
⇒ The edge (parent(y), y) is on the stack when parent(z) is visited and thus when the edge (parent(z), z) is pushed.
⇒ The edge (parent(z), z) is popped before the edge (parent(y), y).
Depth-First Search and Preorder

Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
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- $y$ is the first visited vertex satisfying the above conditions and
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$\Rightarrow$ The edge $(\text{parent}(y), y)$ is on the stack when $\text{parent}(z)$ is visited and thus when the edge $(\text{parent}(z), z)$ is pushed.

$\Rightarrow$ The edge $(\text{parent}(z), z)$ is popped before the edge $(\text{parent}(y), y)$.

$\Rightarrow z$ is visited before $y$, contradiction.
A Property of Undirected DFS Forests

Three types of edges:

- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
- **Cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other.
- **Back edge** \((u, w)\): \(u\) is an ancestor of \(w\) but not its parent.
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**Lemma:** All edges of an undirected graph \(G\) are tree or back edges with respect to a DFS forest of \(G\).
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**Lemma:** All edges of an undirected graph \(G\) are tree or back edges with respect to a DFS forest of \(G\).

Let \(a\) be the LCA of \(u\) and \(v\) and let \(u'\) and \(v'\) be the children of \(a\) that are ancestors of \(u\) and \(v\).

Assume \(u < v\) in preorder.
A Property of Undirected DFS Forests

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\[\Rightarrow \text{Vertices } a, u', u, v', v \text{ are visited in this order.}\]
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Assume \(u < v\) in preorder.

\[\Rightarrow\] Vertices \(a, u', u, v', v\) are visited in this order.

\[\Rightarrow\] The edge \((a, v')\) is pushed before \(u\) is visited and popped after \(u\) is visited.

\[\Rightarrow\] The edge \((u, v)\) is pushed after \((a, v')\) is pushed and before \((a, v')\) is popped.
A Property of Undirected DFS Forests

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Assume \(u < v\) in preorder.

\(\Rightarrow\) Vertices \(a, u', u, v', v\) are visited in this order.

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Assume \(u < v\) in preorder.

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\(\Rightarrow\) The edge \((u, v)\) is popped before \((a, v')\) is popped.

\(\Rightarrow\) \(v\) is unexplored when the edge \((u, v)\) is popped, a contradiction.
A Property of Directed DFS Forests

Five types of edges:

- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
- **Forward edge** \((u, w)\): \(u\) is an ancestor of \(w\).
- **Back edge** \((u, w)\): \(w\) is an ancestor of \(u\).
- **Forward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(u < w\) in preorder/postorder.
- **Backward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(w < u\) in preorder/postorder.
A Property of Directed DFS Forests

Five types of edges:

- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
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- **Backward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(w < u\) in preorder/postorder.

**Lemma:** A directed graph \(G\) does not contain any forward cross edges with respect to a DFS forest of \(G\).
A topological ordering of a directed graph is an ordering $<$ of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$. 
Topological Sorting

A topological ordering of a directed graph is an ordering $<$ of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$.

**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.
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**Lemma**: A graph \(G\) has a topological ordering if and only if it contains no directed cycle.

If there's a cycle, there is no topological ordering.
Topological Sorting

A topological ordering of a directed graph is an ordering $<$ of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$.

**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.

We prove that, if there is no cycle, there is always a source (vertex of in-degree 0).
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**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.

We prove that, if there is no cycle, there is always a source (vertex of in-degree 0).

$\Rightarrow$ The following algorithm produces a topological ordering:

- Give $s$ the smallest number.
- Recursively number the rest of the vertices.

Cannot contain a cycle since $G$ contains no cycle.
Topological Sorting

A topological ordering of a directed graph is an ordering $\prec$ of the vertex set of $G$ such that $u \prec v$ for every edge $(u, v) \in G$.

**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.

We prove that, if there is no cycle, there is always a source (vertex of in-degree 0). Let $R(v)$ be the set of vertices reachable from $v$. 

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We prove that, if there is no cycle, there is always a source (vertex of in-degree 0).

Let $R(v)$ be the set of vertices reachable from $v$.

For an edge $(u, v)$,
- $R(u) \supseteq R(v)$
- $u \in R(u)$
- $u \notin R(v)$ (otherwise there'd be a cycle)

$\Rightarrow R(u) \supset R(v)$. 
**Topological Sorting**

A topological ordering of a directed graph is an ordering $< \text{ of the vertex set of } G$ such that $u < v$ for every edge $(u, v) \in G$.

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Pick a vertex $s$ such that $|R(s)| \geq |R(v)|$ for all $v \in G$. 
Topological Sorting

A **topological ordering** of a directed graph is an ordering < of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$.

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For an edge $(u, v)$,
- $R(u) \supseteq R(v)$
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- $u \notin R(v)$ (otherwise there'd be a cycle)
\[\Rightarrow R(u) \supset R(v).\]

Pick a vertex $s$ such that $|R(s)| \geq |R(v)|$ for all $v \in G$.

If $s$ had an in-neighbour $u$, then $|R(u)| > |R(s)|$, a contradiction.
\[\Rightarrow s \text{ is a source.}\]
Topological Sorting

Lemma: A topological ordering of a directed acyclic graph $G$ can be computed in $O(n + m)$ time.

SimpleTopSort($G$)

1. $Q =$ an empty queue
2. for every vertex $v \in G$
3. do label $v$ with its in-degree
4. if in-deg($v$) = 0
5. then $Q$.enqueue($v$)
6. $O =$ []
7. while not $Q$.isEmpty()
8. do $v =$ $Q$.dequeue()
9. $O$.append($v$)
10. for every out-neighbour $w$ of $v$
11. do in-deg($w$) = in-deg($w$) – 1
12. if in-deg($w$) = 0
13. then $Q$.enqueue($w$)
14. return $O$
Topological Sorting Using DFS

Edges in a DFS forest:

- **Tree edge** $(u, w)$: $u$ is $w$'s parent in $F$.
- **Forward edge** $(u, w)$: $u$ is an ancestor of $w$.
- **Back edge** $(u, w)$: $w$ is an ancestor of $u$.
- **Backward cross edge** $(u, w)$: Neither $u$ nor $w$ is an ancestor of the other, $w < u$ in postorder.
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For tree, forward, and backward cross edges \((u, v)\), \(u > v\) in postorder.
Topological Sorting Using DFS

Edges in a DFS forest:
- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
- **Forward edge** \((u, w)\): \(u\) is an ancestor of \(w\).
- **Back edge** \((u, w)\): \(w\) is an ancestor of \(u\).
- **Backward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(w < u\) in postorder.

For tree, forward, and backward cross edges \((u, v)\), \(u > v\) in postorder.

⇒ Topological sorting algorithm:
- Compute a DFS forest of \(G\).
- Arrange the vertices in reverse postorder.

This takes \(O(n + m)\) time.
Strongly Connected Components

A graph is strongly connected if there exists a path from \( u \) to \( w \) and from \( w \) to \( u \) for every pair of vertices \( u, w \in G \).
**Strongly Connected Components**

A graph is **strongly connected** if there exists a path from $u$ to $w$ and from $w$ to $u$ for every pair of vertices $u, w \in G$. 
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A graph is **strongly connected** if there exists a path from $u$ to $w$ and from $w$ to $u$ for every pair of vertices $u, w \in G$.

The **strongly connected components** of $G$ are its maximal strongly connected subgraphs.
**Strongly Connected Components**

A graph is *strongly connected* if there exists a path from $u$ to $w$ and from $w$ to $u$ for every pair of vertices $u, w \in G$.

The *strongly connected components* of $G$ are its maximal strongly connected subgraphs.

**Lemma:** For a DFS forest $F$ of $G$ and any two vertices $u$ and $w$ of $G$, $u \sim_{\text{SCC}(G)} w \Rightarrow u \sim_{\text{CC}(F)} w$. (The vertices of each strongly connected component of $G$ belong to the same tree of any DFS forest $F$ of $G$.)
**Strongly Connected Components**

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Let $C$ be the strongly connected component containing $u$ and $w$ and let $x$ be the first vertex in $C$ visited during the construction of $F$. 
Strongly Connected Components

A graph is strongly connected if there exists a path from \( u \) to \( w \) and from \( w \) to \( u \) for every pair of vertices \( u, w \in G \).

The strongly connected components of \( G \) are its maximal strongly connected subgraphs.

Lemma: For a DFS forest \( F \) of \( G \) and any two vertices \( u \) and \( w \) of \( G \),
\[ u \sim_{\text{SCC}(G)} w \Rightarrow u \sim_{\text{CC}(F)} w. \] (The vertices of each strongly connected component of \( G \) belong to the same tree of any DFS forest \( F \) of \( G \).)

Let \( C \) be the strongly connected component containing \( u \) and \( w \) and let \( x \) be the first vertex in \( C \) visited during the construction of \( F \).

It suffices to prove that \( x \sim_{\text{CC}(F)} v \) for every \( v \in C \).
**Strongly Connected Components**

A graph is strongly connected if there exists a path from \( u \) to \( w \) and from \( w \) to \( u \) for every pair of vertices \( u, w \in G \).

The strongly connected components of \( G \) are its maximal strongly connected subgraphs.

**Lemma:** For a DFS forest \( F \) of \( G \) and any two vertices \( u \) and \( w \) of \( G \),

\[ u \sim_{SCC(G)} w \Rightarrow u \sim_{CC(F)} w. \]  
(The vertices of each strongly connected component of \( G \) belong to the same tree of any DFS forest \( F \) of \( G \)).

Let \( C \) be the strongly connected component containing \( u \) and \( w \) and let \( x \) be the first vertex in \( C \) visited during the construction of \( F \).

It suffices to prove that \( x \sim_{CC(F)} v \) for every \( v \in C \).

This follows from

**Lemma:** If there exists a path from \( x \) to \( v \) consisting of vertices that are unexplored when \( x \) is visited, then \( v \) is a descendant of \( x \) in \( F \).
Strongly Connected Components

Lemma: If there exists a path from $x$ to $v$ consisting of vertices that are unexplored when $x$ is visited, then $v$ is a descendant of $x$ in $F$. 
Strongly Connected Components

Lemma: If there exists a path from $x$ to $v$ consisting of vertices that are unexplored when $x$ is visited, then $v$ is a descendant of $x$ in $F$.

Let $P = \langle x = x_0, x_1, \ldots, x_k = v \rangle$ be such a path from $x$ to $v$ and assume $v$ is not a descendant of $x$. 
**Strongly Connected Components**

Lemma: If there exists a path from $x$ to $v$ consisting of vertices that are unexplored when $x$ is visited, then $v$ is a descendant of $x$ in $F$.

Let $P = \langle x = x_0, x_1, \ldots, x_k = v \rangle$ be such a path from $x$ to $v$ and assume $v$ is not a descendant of $x$.

Since $x$ is a descendant of $x$, there exists a maximal index $0 \leq i < k$ such that $x_0, x_1, \ldots, x_i$ are descendants of $x$ and $x_{i+1}$ is not.
**Strongly Connected Components**

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Since $x_{i+1}$ is visited after $x$ and all descendants of $x$ have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.
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Let $P = \langle x = x_0, x_1, \ldots, x_k = v \rangle$ be such a path from $x$ to $v$ and assume $v$ is not a descendant of $x$.

Since $x$ is a descendant of $x$, there exists a maximal index $0 \leq i < k$ such that $x_0, x_1, \ldots, x_i$ are descendants of $x$ and $x_{i+1}$ is not.

Since $x_{i+1}$ is visited after $x$ and all descendants of $x$ have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.

Since $x_{i+1}$ is no descendant of $x$, it is not a descendant of $x_i$. 
Strongly Connected Components

**Lemma:** If there exists a path from $x$ to $v$ consisting of vertices that are unexplored when $x$ is visited, then $v$ is a descendant of $x$ in $F$.

Let $P = \langle x = x_0, x_1, \ldots, x_k = v \rangle$ be such a path from $x$ to $v$ and assume $v$ is not a descendant of $x$.

Since $x$ is a descendant of $x$, there exists a maximal index $0 \leq i < k$ such that $x_0, x_1, \ldots, x_i$ are descendants of $x$ and $x_{i+1}$ is not.

Since $x_{i+1}$ is visited after $x$ and all descendants of $x$ have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.

Since $x_{i+1}$ is no descendant of $x$, it is not a descendant of $x_i$.

Since $x_i < x_{i+1}$ in preorder, this implies that $(x_i, x_{i+1})$ is a forward cross edge, a contradiction.
**Strongly Connected Components**

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$. 
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$. 

$G$

$G^r$
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \leadsto_G v$ if and only if $v \leadsto_{G^r} u$. 

$G$

$G^r$


**Strongly Connected Components**

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \xleftrightarrow{G} v$ if and only if $v \xleftrightarrow{G^r} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$. 

![Graph G and its reverse G^r with arrows indicating strongly connected components](image_url)
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \ces G v$ if and only if $v \ces_{G^r} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$.

Let $F^r_{\geq}$ be the DFS forest of $G^r$ obtained by calling TraverseFromVertex on unexplored vertices in the opposite order to $<$. 

![Diagram of graphs G and G^r with strongly connected components highlighted in different colors.](image)
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \leadsto_G v$ if and only if $v \leadsto_{Gr} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$.

Let $F^r_>$ be the DFS forest of $G^r$ obtained by calling TraverseFromVertex on unexplored vertices in the opposite order to $<$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F^r_>)} v$. 
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \rightarrow_G v$ if and only if $v \rightarrow_{G^r} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$.

Let $F^r_>$ be the DFS forest of $G^r$ obtained by calling TraverseFromVertex on unexplored vertices in the opposite order to $<$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F^r_>)} v$.

⇒ Kosaraju's strong connectivity algorithm:
- Compute a DFS forest $F$ of $G$.
- Compute $G^r$ and arrange the vertices in reverse postorder w.r.t. $F$.
- Compute a DFS forest $F^r$ of $G^r$.
- Extract a component labelling of the vertices or the strongly connected components themselves from $F^r$ (almost) as we did for computing connected components.

This takes $O(n + m)$ time.
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Lemma: \( u \sim_{SCC(G)} v \iff u \sim_{CC(F_r)} v \).
**Strongly Connected Components**

**Lemma:** \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F')} v. \)

Assume the contrary. Then there exists an edge \((u, v) \in F'_{r}\) such that \(u \not\sim_{\text{SCC}(G)} v.\)
Strongly Connected Components

Lemma: $u \sim_{SCC(G)} v \iff u \sim_{CC(F^r)} v$.

Assume the contrary. Then there exists an edge $(u, v) \in F^r_>$ such that $u \not\sim_{SCC(G)} v$.

$\Rightarrow (v, u) \in G$. 

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\[ \Rightarrow (v, u) \in G. \]

Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{\text{SCC}(G)} y \).
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Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{SCC(G)} y. \)

In particular, \( u \sim_{SCC(G)} r, \) where \( r \) is the root of the tree containing \( u \) and \( v. \)
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**Lemma:** \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F^r)} v. \)

Assume the contrary. Then there exists an edge \( (u, v) \in F^r \rangle \) such that \( u \not\sim_{\text{SCC}(G)} v. \)

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Choose this edge so that each of its ancestor edges \( (x, y) \) satisfies \( x \sim_{\text{SCC}(G)} y. \)

In particular, \( u \sim_{\text{SCC}(G)} r, \) where \( r \) is the root of the tree containing \( u \) and \( v. \)

All vertices in \( C \) are descendants of \( r \) in \( F^r \rangle \) and \( x \leq r \) for all \( x \in C. \)
**Strongly Connected Components**

**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F^r_>)} v. \)

Assume the contrary. Then there exists an edge \((u, v) \in F^r_>\) such that \( u \not\sim_{SCC(G)} v. \)

\( \implies (v, u) \in G. \)

Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{SCC(G)} y. \)

In particular, \( u \sim_{SCC(G)} r, \) where \( r \) is the root of the tree containing \( u \) and \( v. \)

All vertices in \( C \) are descendants of \( r \) in \( F^r_>, \) and \( x \leq r \) for all \( x \in C. \)

Also, \( v < r \) because \( v \) is a descendant of \( r \) in \( F^r_>. \)
**Strongly Connected Components**

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F')} v$.

Assume the contrary. Then there exists an edge $(u, v) \in F'_r$ such that $u \not\sim_{SCC(G)} v$.

$\Rightarrow (v, u) \in G$.

Choose this edge so that each of its ancestor edges $(x, y)$ satisfies $x \sim_{SCC(G)} y$.

In particular, $u \sim_{SCC(G)} r$, where $r$ is the root of the tree containing $u$ and $v$.

All vertices in $C$ are descendants of $r$ in $F'_r$ and $x \leq r$ for all $x \in C$.

Also, $v < r$ because $v$ is a descendant of $r$ in $F'_r$.

In $F$, all vertices in $C$ are descendants of some vertex $r' \in C$ and $x \leq r'$ for all $x \in C$. 
**Strongly Connected Components**

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F^r)} v$.

Assume the contrary. Then there exists an edge $(u, v) \in F^r_\succ$ such that $u \not\sim_{SCC(G)} v$.

$\Rightarrow (v, u) \in G$.

Choose this edge so that each of its ancestor edges $(x, y)$ satisfies $x \sim_{SCC(G)} y$.

In particular, $u \sim_{SCC(G)} r$, where $r$ is the root of the tree containing $u$ and $v$.

All vertices in $C$ are descendants of $r$ in $F^r_\succ$ and $x \leq r$ for all $x \in C$.

Also, $v < r$ because $v$ is a descendant of $r$ in $F^r_\succ$.

In $F$, all vertices in $C$ are descendants of some vertex $r' \in C$ and $x \leq r'$ for all $x \in C$.

$\Rightarrow r = r'$ and $u \leq r$. 
Strongly Connected Components

Lemma: \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r)} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then
\( u \sim_{\text{SCC}(G)} v \), a contradiction.
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**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F')} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim_{SCC(G)} v \), a contradiction.
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**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F_r)} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim_{SCC(G)} v \), a contradiction.
**Strongly Connected Components**

**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F_r^c)} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim_{SCC(G)} v \), a contradiction.
**Strongly Connected Components**

**Lemma:** $u \sim_{SCC(G)} v \Leftrightarrow u \sim_{CC(F^r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{SCC(G)} v$, a contradiction.
**Strongly Connected Components**

**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F^r)} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then 
\( u \sim_{SCC(G)} v \), a contradiction.

If \( v \) is not a descendant of \( r \) in \( F \), then \( v \) is not a descendant of \( u \) because \( u \) is a descendant of \( r \).
**Strongly Connected Components**

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F_r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{SCC(G)} v$, a contradiction.

If $v$ is not a descendant of $r$ in $F$, then $v$ is not a descendant of $u$ because $u$ is a descendant of $r$.

Since $u \leq r$, $v < r$, and the descendants of $r$ are numbered consecutively, we have $v < u$. 
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**Lemma:** $u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{\text{SCC}(G)} v$, a contradiction.

If $v$ is not a descendant of $r$ in $F$, then $v$ is not a descendant of $u$ because $u$ is a descendant of $r$.

Since $u \leq r$, $v < r$, and the descendants of $r$ are numbered consecutively, we have $v < u$.

$\Rightarrow$ $(v, u)$ is a forward cross edge w.r.t. $F$, a contradiction.
Summary

Graphs are fundamental in Computer Science:

Many problems are quite natural to express as graph problems:

- Matching problems
- Scheduling problems
- ...

Data structures are graphs whose nodes store useful information.

Graph exploration lets us learn the structure of a graph:

- Connectivity problems
- Distances between vertices
- Planarity
- ...