Graph Algorithms

Textbook Reading
Chapter 22
Overview

**Design principle:**
- Learn the structure of the graph by systematic exploration.

**Proof technique:**
- Proof by contradiction

**Problems:**
- Connected components
- Bipartiteness testing
- Topological sorting
- Strongly connected components
A graph is an ordered pair $G = (V, E)$.
- $V$ is the set of vertices of $G$.
- $E$ is the set of edges of $G$.
- The elements of $E$ are pairs of vertices $(v, w)$. 
Graphs, Vertices, and Edges

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The endpoints of an edge $e$ are said to be adjacent to each other and incident with $e$.

The degree of a vertex is the number of its incident edges.
A graph is **undirected** if its edges are unordered pairs, that is, \((v, w) = (w, v)\).
Undirected and Directed Graphs

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A directed edge \((v, w)\) is an **out-edge** of \(v\) and an **in-edge** of \(w\).

The **in-degree** and **out-degree** of a vertex are the numbers of its in-edges and out-edges, respectively.
A path from a vertex $s$ to a vertex $t$ is a sequence of vertices $\langle x_0, x_1, \ldots, x_k \rangle$ such that

- $x_0 = s$,
- $x_k = t$, and
- for all $1 \leq i \leq k$, $(x_{i-1}, x_i)$ is an edge of $G$. 
Paths and Cycles

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A cycle is a path from a vertex $x$ back to itself.
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A cycle is a path from a vertex $x$ back to itself.

A path or cycle is **simple** if it contains every vertex of $G$ at most once.
Connected Graphs, Trees, and Forests

A graph is **connected** if there exists a path between every pair of vertices.
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A **forest** is a graph without cycles.
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A tree is a connected forest.
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A **tree** is a connected forest.
Adjacency List Representation

- Doubly-linked list of vertices
- Doubly-linked list of edges
- One doubly-linked adjacency list per vertex
- Pointers from adjacency list entries to vertices
- Cross-pointers between edges and adjacency list entries
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(a, e) → (c, e) → (c, f) → (e, f) → (b, d) →

Diagram:

- Vertices: a, b, c, d, e, f
- Edges: (b, d), (e, f), (c, f), (c, e), (a, e)
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A rooted tree $T$

- is a tree,
- is a directed graph,
- has one of its vertices, $r$, designated as a root.

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**Representation:**

Tree = root

Every node stores
- an arbitrary **key**
- a (doubly-linked) list of its **children**.
**Preorder:**
- Every vertex appears before its children.
- Every vertex appears before its right sibling.
- The vertices in each subtree appear consecutively.

⇒ [a, b, c, d, e, f, g, h, i, j]
Standard Tree Orderings

Preorder:
- Every vertex appears before its children.
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Postorder:
- Every vertex appears after its children.
- Every vertex appears before its right sibling.
- The vertices in each subtree appear consecutively.
⇒ \([c, b, f, e, g, i, j, h, d, a]\)
Standard Tree Orderings

Preorder:
- Every vertex appears before its children.
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Postorder:
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- The vertices in each subtree appear consecutively.
⇒ [c, b, f, e, g, i, j, h, d, a]

Lemma: It takes linear time to arrange the vertices of a forest in preorder or postorder.
The connected components of a graph $G$ are its maximal connected subgraphs.
Connected Components and Spanning Forests

The connected components of a graph $G$ are its maximal connected subgraphs.

A spanning forest of a graph $G$ is a subgraph $F \subseteq G$ with the same number of connected components and which is a forest.
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**Representation:**
- List of graphs or
- Labelling of vertices with component IDs

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**Representation:** List of rooted trees
Graph Traversal

We use graph traversal to build a spanning forest of $G$. 
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![Graph Traversal Diagram]
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Graph Traversal
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Different traversal strategies lead to different spanning forests:
- Breadth-first search
- Depth-first search
- Prim's algorithm for computing minimum spanning trees
- Dijkstra's algorithm for computing shortest paths
**Graph Traversal**

TraverseGraph(G)

1. Mark every vertex of G as unexplored
2. \( F = [] \)
3. for every vertex \( u \in G \)
4. do if not \( u.explored \)
5. then \( F.append(TraverseFromVertex(G, u)) \)
6. return \( F \)

Diagram:

[Graph diagram showing traversal paths]
Graph Traversal

TraverseFromVertex(G, u)

1. u.explored = True
2. u.tree = Node(u, [])
3. Q = an empty edge collection
4. for every out-edge (u, v) of u do Q.add((u, v))
5. while not Q.isEmpty() do (v, w) = Q.remove()
6. if not w.explored then w.explored = True
7. w.tree = Node(w, [])
8. v.tree.children.append(w.tree)
9. for every out-edge (w, x) of v do Q.add((w, x))
10. return u.tree
Graph Traversal Computes a Spanning Forest

It computes a subgraph of $G$ because it only adds edges of $G$ to $F$. 

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M \quad N
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**TraverseFromVertex**($G$, $u$)

1. $u$.explored = True
2. $u$.tree = Node($u$, [])
3. $Q$ = an empty edge collection
4. for every out-edge $(u, v)$ of $u$
   - do $Q$.add($((u, v))$
5. while not $Q$.isEmpty()
6.   (v, w) = $Q$.remove()
7.   if not $w$.explored then
8.     $w$.explored = True
9.     $w$.tree = Node($w$, [])
10.    $v$.tree.children.append($w$.tree)
11.   for every out-edge $(w, x)$ of $v$
12.      do $Q$.add($((w, x))$
13. return $u$.tree
Graph Traversal Computes a Spanning Forest

It computes a subgraph of $G$ because it only adds edges of $G$ to $F$.

$\Rightarrow$ $F$ has at least as many connected components as $G$. 
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$\Rightarrow$ $F$ has at least as many connected components as $G$.

To prove:

- $F$ contains no cycle.
- If $u \sim_{CC(G)} v$ (u and v belong to the same component of $G$), then $u \sim_{CC(F)} v$. 
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Observation: Every edge $(u, v)$ in $Q$ has at least one explored endpoint, namely $u$. 
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6. then $w$.explored = True
9. $w$.tree = Node($w$, [ ])
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Corollary: Both endpoints of every edge in $F$ are explored.
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Corollary: $F$ contains no cycle.

Proof by contradiction:

By the time we add the last edge to the cycle, both its endpoints are explored.

$\Rightarrow$ We would not have added it.
Graph Traversal Computes a Spanning Forest

**Lemma:** TraverseFromVertex(G, u) visits all vertices v such that $u \sim_{CC(G)} v$ and only those.
Graph Traversal Computes a Spanning Forest

**Lemma:** TraverseFromVertex(G, u) visits all vertices v such that u $\sim_{CC(G)}$ v and only those.

**Proof:** By induction on the number of invocations of TraverseFromVertex made so far.
Graph Traversal Computes a Spanning Forest

**Lemma:** TraverseFromVertex$(G, u)$ visits all vertices $v$ such that $u \sim_{CC(G)} v$ and only those.

**Proof:** By induction on the number of invocations of TraverseFromVertex made so far.

When TraverseFromVertex$(G, u)$ is called, every vertex $v$ such that $u \sim_{CC(G)} v$ is unexplored.
Lemma: TraverseFromVertex(G, u) visits all vertices v such that $u \sim_{C(G)} v$ and only those.

Proof: By induction on the number of invocations of TraverseFromVertex made so far.

When TraverseFromVertex(G, u) is called, every vertex v such that $u \sim_{C(G)} v$ is unexplored.

We visit all vertices v such that $u \sim_{C(G)} v$: 
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When TraverseFromVertex(G, u) is called, every vertex v such that \( u \sim_{CC(G)} v \) is unexplored.

We visit all vertices v such that \( u \sim_{CC(G)} v \):

- path P from u to v
- first unexplored vertex on P

\( u \) \hspace{1cm} x \hspace{1cm} w \hspace{1cm} v
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We visit all vertices v such that $u \sim_{CC(G)} v$:

- x adds $(x, w)$ to Q.
- $\Rightarrow$ We'd visit w.
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**Lemma**: TraverseFromVertex\((G, u)\) visits all vertices \(v\) such that \(u \sim_{CC(G)} v\) and only those.

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We visit all vertices \(v\) such that \(u \sim_{CC(G)} v\):

- \(x\) adds \((x, w)\) to \(Q\).
- \(\Rightarrow\) We’d visit \(w\).

We do not visit a vertex \(v\) such that \(u \not\sim_{CC(G)} v\):
Graph Traversal Computes a Spanning Forest

**Lemma:** TraverseFromVertex(G, u) visits all vertices v such that $u \sim_{CC(G)} v$ and only those.

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We visit all vertices v such that $u \sim_{CC(G)} v$:
- x adds (x, w) to Q.
- $\Rightarrow$ We'd visit w.

We do not visit a vertex v such that $u \not\sim_{CC(G)} v$:
- first unexplored vertex on P
- first explored vertex such that $u \not\sim_{CC(G)} v$. 
Graph Traversal Computes a Spanning Forest

**Lemma:** TraverseFromVertex(G, u) visits all vertices v such that u $\sim_{CC(G)}$ v and only those.

**Proof:** By induction on the number of invocations of TraverseFromVertex made so far.

When TraverseFromVertex(G, u) is called, every vertex v such that u $\sim_{CC(G)}$ v is unexplored.

We visit all vertices v such that u $\sim_{CC(G)}$ v:

- x adds (x, w) to Q.
- $\Rightarrow$ We'd visit w.

We do not visit a vertex v such that u $\not\sim_{CC(G)}$ v:

- v explored because of edge (w, v) $\in$ Q.
- w explored before v.
- $\Rightarrow$ w $\sim_{CC(G)}$ u.
- $\Rightarrow$ v $\sim_{CC(G)}$ u.
Lemma: TraverseGraph takes $O(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from $Q$, respectively.
The Cost of Graph Traversal

**Lemma:** TraverseGraph takes $\mathcal{O}(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from Q, respectively.

TraverseGraph itself takes $\mathcal{O}(n)$ time.
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TraverseGraph itself takes $O(n)$ time.

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TraverseGraph(G)
1 Mark every vertex of G as unexplored
2 F = []
3 for every vertex u ∈ G
4 do if not u.explored
5 then F.append(TraverseFromVertex(G, u))
6 return F
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The Cost of Graph Traversal

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TraverseGraph itself takes $O(n)$ time.

Every edge is added to $Q$ at most once.

⇒ The cost of the for-loops in TraverseFromVertex is $O(m \cdot (1 + t_a))$. 
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TraverseFromVertex(G, u)
1 u.explored = True
2 u.tree = Node(u, [])
3 Q = an empty edge collection
4 for every out-edge (u, v) of u
5   do Q.add((u, v))
6   while not Q.isEmpty()
7     do (v, w) = Q.remove()
8     if not w.explored
9       then w.explored = True
10      w.tree = Node(w, [])
11     v.tree.children.append(w.tree)
12   for every out-edge (w, x) of v
13     do Q.add((w, x))
14 return u.tree
```
The Cost of Graph Traversal

**Lemma:** TraverseGraph takes $O(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from $Q$, respectively.

TraverseGraph itself takes $O(n)$ time.

Every edge is added to $Q$ at most once.

⇒ The cost of the for-loops in TraverseFromVertex is $O(m \cdot (1 + t_a))$.

Every edge that is removed must be added first.

⇒ The cost of the while-loop in TraverseFromVertex is $O(m \cdot (1 + t_r))$. 
The Cost of Graph Traversal

Lemma: TraverseGraph takes \( O(n + m + m \cdot (t_a + t_r)) \) time, where \( t_a \) and \( t_r \) are the costs of adding and removing an edge from \( Q \), respectively.

TraverseGraph itself takes \( O(n) \) time.

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Every edge that is removed must be added first.

\[ \Rightarrow \] The cost of the while-loop in TraverseFromVertex is \( O(m \cdot (1 + t_r)) \).

TraverseFromVertex(\( G, u \))

1. \( u\.explored = \text{True} \)
2. \( u\.tree = \text{Node}(u, []) \)
3. \( Q = \text{an empty edge collection} \)
4. for every out-edge \((u, v)\) of \( u \) do \( Q\.add((u, v)) \)
5. while not \( Q\.isEmpty() \) do \( (v, w) = Q\.remove() \)
6. if not \( w\.explored \) then \( w\.explored = \text{True} \)
7. \( w\.tree = \text{Node}(w, []) \)
8. \( v\.tree\.children\.append(w\.tree) \)
9. for every out-edge \((w, x)\) of \( v \) do \( Q\.add((w, x)) \)
10. return \( u\.tree \)
The Cost of Graph Traversal

**Lemma:** TraverseGraph takes $O(n + m + m \cdot (t_a + t_r))$ time, where $t_a$ and $t_r$ are the costs of adding and removing an edge from $Q$, respectively.

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⇒ The cost of the for-loops in TraverseFromVertex is $O(m \cdot (1 + t_a))$.

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Computing Connected Components

- Compute a spanning forest $F$.
- Collect vertices of trees in $F$.
- Compute representation of connected components.
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- Compute representation of connected components.

```
CollectComponentVertices(F)
1 L = []
2 for every tree $T \in F$
3 do L.append(CollectDescendantVertices(T))
4 return L
```
Computing Connected Components

- Compute a spanning forest $F$.
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**CollectComponentVertices($F$)**

1. $L = []$
2. for every tree $T \in F$
3. do $L.append(CollectDescendantVertices(T))$
4. return $L$

**CollectDescendantVertices($T$)**

1. $L = [T.key]$
2. for every child $T'$ of $T$
3. do $L.concat(CollectDescendantVertices(T'))$
4. return $L$
Computing Connected Components

- Compute a spanning forest $F$.
- Collect vertices of trees in $F$.
- Compute representation of connected components.

CollectComponentVertices($F$)

1. $L = []$
2. for every tree $T \in F$
3. do $L$.append(CollectorsDescendantVertices($T$))
4. return $L$

CollectDescendantVertices($T$)

1. $L = [T.key]$
2. for every child $T'$ of $T$
3. do $L$.concat(CollectorsDescendantVertices($T'$))
4. return $L$

Lemma: Collecting the vertices of all components takes $O(n)$ time.
Computing Connected Components

Representation using vertex labels:

ComponentLabels(L)

1. \( i = 0 \)
2. for every list \( L' \in L \)
3. \( \quad i = i + 1 \)
4. for every vertex \( v \in L' \)
5. \( \quad v.cc = i \)

Cost: \( O(n) \)
Computing Connected Components

Representation as list of graphs:

We already have the right adjacency lists for the vertices. Need to partition the vertex and edge lists into vertex and edge lists for the components.
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Vertex lists:

BuildVertexLists(L)

```python
VL = []
for every list L′ ∈ L do VL′ = []
for every vertex v ∈ L′ do VL′.append(v)
VL.append(VL′)
return VL
```
Computing Connected Components

Edge lists:

BuildEdgeLists(G, L)

1. \( EL = [ ] \)
2. for every edge \( e \in G \) do \( e\).collected = False
3. for every list \( L' \in L \) do \( EL' = [ ] \)
4. for every vertex \( v \in L' \) do for every edge \( e \) incident with \( v \) do if not \( e\).collected then \( e\).collected = True
5. \( EL'.append(e) \)
6. \( EL.append(EL') \)
7. return \( EL \)
Lemma: The connected components of a graph can be computed in $O(n + m)$ time.

- Building a spanning forest takes $O(n + m + m \cdot (t_a + t_r))$ time.
- Computing the vertex labelling or list of graphs then takes $O(n + m)$ time.
- Using a stack or queue to represent $Q$, we get $t_a \in O(1)$ and $t_r \in O(1)$. 
Breadth-First Search

Breadth-first search (BFS) = graph traversal using a queue to implement Q.

Queue:

Q.enqueue(x)

Q.dequeue()
Breadth-First Search

Breadth-first search (BFS) = graph traversal using a queue to implement Q.

Queue:

Constant-time implementations:
- Doubly-linked list
- Singly-linked list with tail pointer
- “Circular” array (amortized constant cost)
- Pair of singly-linked lists (functional)
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A Property of Undirected BFS Forests

BFS forest = spanning forest computed using BFS

Let the depth $d_F(v)$ of a vertex $v$ in a rooted forest $F$ be the distance from the root of its tree.

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![Diagram of BFS forest and vertex relationships]
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![Diagram](image)
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$d_F($parent$(v)) = d_F(v) - 1 < d_F(w) - 1 = d_F($parent$(w))$. 

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\[ \Rightarrow v \text{ is visited before } w, \text{ a contradiction.} \]
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Lemma: For every edge \((v, w)\) of \(G\) and any BFS forest \(F\) of \(G\), the depths of \(v\) and \(w\) in \(F\) differ by at most one.
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\[\Rightarrow w\text{ is unexplored when the edge } (v, w)\text{ is dequeued.}\]

\[\Rightarrow w\text{ would be added to the list of } v\text{'s children, a contradiction.}\]
A graph is **bipartite** if its vertices can be partitioned into two sets \((U, W)\) such that every edge has one endpoint in \(U\) and one endpoint in \(W\).
Bipartite Graphs

A graph is bipartite if its vertices can be partitioned into two sets \((U, W)\) such that every edge has one endpoint in \(U\) and one endpoint in \(W\).

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\[ \Rightarrow \quad G \text{ is bipartite if and only if there is no edge with both endpoints on the same level.} \]

If there is such an edge, there's an odd cycle.
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**Lemma:** Given a BFS forest \(F\) of \(G\), \(G\) is bipartite if and only if there is no edge in \(G\) with both endpoints on the same level in \(F\).
Bipartiteness Testing

- Compute BFS forest $F$ of $G$.
- Collect vertices on alternating levels of $F$ into two sets $(U, W)$.
- Test whether any edge has both endpoints in the same set, $U$ or $W$.
- If so, report the odd cycle induced by such an edge.
- Otherwise, report the bipartition $(U, W)$.

Collecting vertices on alternating levels:

$\text{AlternatingLevels}(F)$

1. $U = W = []$
2. for every tree $T$ in $F$
3. do $\text{AlternatingLevels'}(T, U, W)$
4. return $(U, W)$

$\text{AlternatingLevels'}(T, U, W)$

1. $U$.append($T$.key)
2. for every child $T'$ of $T$
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**Testing for an “odd edge”:**

**OddEdge**(G, U, W)

```plaintext
1 A = an array of size n
2 for every vertex $u \in U$
3 do A[u] = “U”
4 for every vertex $w \in W$
5 do A[w] = “W”
6 for every edge $(u, w) \in G$
7 do if A[u] = A[w]
8 then return $(u, w)$
9 return Nothing
```
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Finding the ancestor edges of all vertices:

AncestorEdges($F$)

1. $L$ = an empty list of vertex-vertex list pairs
2. for every tree $T \in F$
   3. do AncestorEdges$′(T, [], L)$
4. return $L$

AncestorEdges$′(T, A, L)$

1. $L = L.append([(T.key, A)])$
2. for every child $T′$ of $T$
   3. do AncestorEdges$′(T′, [(T.key, T′.key)] + A, L)$
Bipartiteness Testing

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- Otherwise, report the bipartition \((U, W)\).

**Reporting an odd cycle:**

\[
\text{OddCycle}(L, (u, w))
\]

1. Find \((u, A_u)\) and \((w, A_w)\) in \( L \)
2. \( C_u = C_w = [\] \)
3. while \( A_u.\text{head} \neq A_w.\text{head} \) do
   4. \( C_u.\text{append}(A_u.\text{head}) \)
   5. \( C_w.\text{append}(A_w.\text{head}) \)
6. \( A_u = A_u.\text{tail} \)
7. \( A_w = A_w.\text{tail} \)
8. \( C_u.\text{reverse()}.\text{concat([[(u, w)]].concat(C_w))} \)
9. return \( C_u \)
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- Otherwise, report the bipartition $(U, W)$.

**Lemma:** It takes linear time to test whether a graph $G$ is bipartite and either report a valid bipartition or an odd cycle in $G$. 
Depth-First Search

Depth-first search (DFS) = graph traversal using a stack to implement Q.

Stack:

Q.pop() ← arrow

Q.push(x) ← arrow
Depth-First Search

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Stack:

Constant-time implementations:
- Singly-linked list
- Resizeable array (amortized constant cost)
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Depth-First Search and Preorder

Lemma: Depth-first search visits the vertices of the spanning forest it creates in preorder.
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It visits every node after its parent:

- \( v \) is visited when the edge \((\text{parent}(v), v)\) is popped.
- The edge \((\text{parent}(v), v)\) must be pushed before this can happen.
- The edge \((\text{parent}(v), v)\) is pushed when \(\text{parent}(v)\) is visited.
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It visits the vertices in each subtree consecutively.
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It visits the vertices in each subtree consecutively.

**Observation:** An edge with one explored and one unexplored endpoint is on the stack.
Depth-First Search and Preorder

Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
- $y$ is visited before some descendant $z$.

Choose $y$ and $z$ so that

- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after $\text{parent}(z)$.
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Choose \( y \) and \( z \) so that

- \( y \) is the first visited vertex satisfying the above conditions and
- \( y \) is visited after \( \text{parent}(z) \).

Case 1: \( y \) is a root.

Cannot happen because the edge \((\text{parent}(z), z)\) is on the stack when \( y \) is visited and the stack is empty when a root is visited.
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Case 2: $y$ has a parent $\text{parent}(y)$. 
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Choose $y$ and $z$ so that
- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after $\text{parent}(z)$.

**Case 2:** $y$ has a parent $\text{parent}(y)$.

$\text{parent}(y)$ is visited before $x$ and thus before $\text{parent}(z)$. 
Depth-First Search and Preorder

Assume there exist two vertices $x$ and $y$ such that

- $y$ is not a descendant of $x$,
- $y$ is visited after $x$, and
- $y$ is visited before some descendant $z$.

Choose $y$ and $z$ so that

- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after parent($z$).

Case 2: $y$ has a parent parent($y$).

parent($y$) is visited before $x$ and thus before parent($z$).

⇒ The edge (parent($y$), $y$) is on the stack when parent($z$) is visited and thus when the edge (parent($z$), $z$) is pushed.
Assume there exist two vertices $x$ and $y$ such that
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Choose $y$ and $z$ so that
- $y$ is the first visited vertex satisfying the above conditions and
- $y$ is visited after $\text{parent}(z)$.

**Case 2:** $y$ has a parent $\text{parent}(y)$.

$\text{parent}(y)$ is visited before $x$ and thus before $\text{parent}(z)$.

$\Rightarrow$ The edge $(\text{parent}(y), y)$ is on the stack when $\text{parent}(z)$ is visited and thus when the edge $(\text{parent}(z), z)$ is pushed.

$\Rightarrow$ The edge $(\text{parent}(z), z)$ is popped before the edge $(\text{parent}(y), y)$.
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$\Rightarrow$ $z$ is visited before $y$, contradiction.
A Property of Undirected DFS Forests

Three types of edges:

- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
- **Cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other.
- **Back edge** \((u, w)\): \(u\) is an ancestor of \(w\) but not its parent.
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Lemma: All edges of an undirected graph \(G\) are tree or back edges with respect to a DFS forest of \(G\).
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Lemma: All edges of an undirected graph \(G\) are tree or back edges with respect to a DFS forest of \(G\).

Let \(a\) be the LCA of \(u\) and \(v\) and let \(u'\) and \(v'\) be the children of \(a\) that are ancestors of \(u\) and \(v\).

Assume \(u < v\) in preorder.
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\[\Rightarrow\] Vertices \(a, u', u, v', v\) are visited in this order.
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Let \(a\) be the LCA of \(u\) and \(v\) and let \(u'\) and \(v'\) be the children of \(a\) that are ancestors of \(u\) and \(v\).

Assume \(u < v\) in preorder.

\[
\Rightarrow \text{Vertices } a, u', u, v', v \text{ are visited in this order.}
\]

\[
\Rightarrow \text{The edge } (a, v') \text{ is pushed before } u \text{ is visited and popped after } u \text{ is visited.}
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\Rightarrow \text{The edge } (u, v) \text{ is pushed after } (a, v') \text{ is pushed and before } (a, v') \text{ is popped.}
\]

\[
\Rightarrow \text{The edge } (u, v) \text{ is popped before } (a, v') \text{ is popped.}
\]

\[
\Rightarrow v \text{ is unexplored when the edge } (u, v) \text{ is popped, a contradiction.}
\]
A Property of Directed DFS Forests

Five types of edges:
- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
- **Forward edge** \((u, w)\): \(u\) is an ancestor of \(w\).
- **Back edge** \((u, w)\): \(w\) is an ancestor of \(u\).
- **Forward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(u < w\) in preorder/postorder.
- **Backward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(w < u\) in preorder/postorder.
A Property of Directed DFS Forests

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- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
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- **Backward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(w < u\) in preorder/postorder.

Lemma: A directed graph \(G\) does not contain any forward cross edges with respect to a DFS forest of \(G\).
A topological ordering of a directed graph is an ordering < of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$. 
Topological Sorting

A topological ordering of a directed graph is an ordering $<$ of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$.

Lemma: A graph $G$ has a topological ordering if and only if it contains no directed cycle.
Topological Sorting

A topological ordering of a directed graph is an ordering < of the vertex set of G such that u < v for every edge (u, v) ∈ G.

Lemma: A graph G has a topological ordering if and only if it contains no directed cycle.

If there's a cycle, there is no topological ordering.
Topological Sorting

A topological ordering of a directed graph is an ordering $\prec$ of the vertex set of $G$ such that $u \prec v$ for every edge $(u, v) \in G$.

**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.

We prove that, if there is no cycle, there is always a source (vertex of in-degree 0).
A **topological ordering** of a directed graph is an ordering $<$ of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$.

**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.

We prove that, if there is no cycle, there is always a source (vertex of in-degree 0).

⇒ The following algorithm produces a topological ordering:

- Give $s$ the smallest number.
- Recursively number the rest of the vertices.

Cannot contain a cycle since $G$ contains no cycle.
Topological Sorting

A topological ordering of a directed graph is an ordering $<$ of the vertex set of $G$ such that $u < v$ for every edge $(u, v) \in G$.

**Lemma:** A graph $G$ has a topological ordering if and only if it contains no directed cycle.

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Let $R(v)$ be the set of vertices reachable from $v$. 

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Let $R(v)$ be the set of vertices reachable from $v$.

For an edge $(u, v)$,
- $R(u) \supseteq R(v)$
- $u \in R(u)$
- $u \notin R(v)$ (otherwise there'd be a cycle)

$\Rightarrow R(u) \supset R(v)$. 
Topological Sorting

A topological ordering of a directed graph is an ordering \( < \) of the vertex set of \( G \) such that \( u < v \) for every edge \((u, v) \in G\).

**Lemma:** A graph \( G \) has a topological ordering if and only if it contains no directed cycle.

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Pick a vertex \( s \) such that \( |R(s)| \geq |R(v)| \) for all \( v \in G \).
Topological Sorting

A topological ordering of a directed graph is an ordering \(<\) of the vertex set of \(G\) such that \(u < v\) for every edge \((u, v) \in G\).

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We prove that, if there is no cycle, there is always a source (vertex of in-degree 0).

Let \(R(v)\) be the set of vertices reachable from \(v\).

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\(\Rightarrow R(u) \supset R(v)\).

Pick a vertex \(s\) such that \(|R(s)| \geq |R(v)|\) for all \(v \in G\).

If \(s\) had an in-neighbour \(u\), then \(|R(u)| > |R(s)|\), a contradiction.

\(\Rightarrow s\) is a source.
Topological Sorting

Lemma: A topological ordering of a directed acyclic graph $G$ can be computed in $O(n + m)$ time.

SimpleTopSort($G$)

1. $Q = \text{an empty queue}$
2. for every vertex $v \in G$
   3. do label $v$ with its in-degree
      4. if in-deg($v$) = 0
         5. then $Q$.enqueue($v$)
6. $O = []$
7. while not $Q$.isEmpty()
   8. do $v = Q$.dequeue()
      9. $O$.append($v$)
     10. for every out-neighbour $w$ of $v$
        11. do in-deg($w$) = in-deg($w$) − 1
            if in-deg($w$) = 0
               then $Q$.enqueue($w$)
14. return $O$
Topological Sorting Using DFS

Edges in a DFS forest:
- **Tree edge** \((u, w)\): \(u\) is \(w\)'s parent in \(F\).
- **Forward edge** \((u, w)\): \(u\) is an ancestor of \(w\).
- **Back edge** \((u, w)\): \(w\) is an ancestor of \(u\).
- **Backward cross edge** \((u, w)\): Neither \(u\) nor \(w\) is an ancestor of the other, \(w < u\) in postorder.
Topological Sorting Using DFS

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For tree, forward, and backward cross edges $(u, v)$, $u > v$ in postorder.
Topological Sorting Using DFS

Edges in a DFS forest:

- **Tree edge** \((u, w)\): u is w’s parent in F.
- **Forward edge** \((u, w)\): u is an ancestor of w.
- **Back edge** \((u, w)\): w is an ancestor of u.
- **Backward cross edge** \((u, w)\): Neither u nor w is an ancestor of the other, w < u in postorder.

For tree, forward, and backward cross edges \((u, v)\), \(u > v\) in postorder.

⇒ Topological sorting algorithm:

- Compute a DFS forest of G.
- Arrange the vertices in reverse postorder.

This takes \(O(n + m)\) time.
Strongly Connected Components

A graph is strongly connected if there exists a path from $u$ to $w$ and from $w$ to $u$ for every pair of vertices $u, w \in G$. 
**Strongly Connected Components**

A graph is *strongly connected* if there exists a path from \( u \) to \( w \) and from \( w \) to \( u \) for every pair of vertices \( u, w \in G \).
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A graph is strongly connected if there exists a path from $u$ to $w$ and from $w$ to $u$ for every pair of vertices $u, w \in G$.

The strongly connected components of $G$ are its maximal strongly connected subgraphs.
**Strongly Connected Components**

A graph is **strongly connected** if there exists a path from \(u\) to \(w\) and from \(w\) to \(u\) for every pair of vertices \(u, w \in G\).

The **strongly connected components** of \(G\) are its maximal strongly connected subgraphs.

**Lemma:** For a DFS forest \(F\) of \(G\) and any two vertices \(u\) and \(w\) of \(G\),
\[ u \sim_{\text{SCC}(G)} w \Rightarrow u \sim_{\text{CC}(F)} w. \]
(The vertices of each strongly connected component of \(G\) belong to the same tree of any DFS forest \(F\) of \(G\).)
Strongly Connected Components

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**Lemma:** For a DFS forest $F$ of $G$ and any two vertices $u$ and $w$ of $G$, $u \sim_{SCC(G)} w \Rightarrow u \sim_{CC(F)} w$. (The vertices of each strongly connected component of $G$ belong to the same tree of any DFS forest $F$ of $G$.)

Let $C$ be the strongly connected component containing $u$ and $w$ and let $x$ be the first vertex in $C$ visited during the construction of $F$. 
**Strongly Connected Components**

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Let $C$ be the strongly connected component containing $u$ and $w$ and let $x$ be the first vertex in $C$ visited during the construction of $F$.

It suffices to prove that $x \sim_{CC(F)} v$ for every $v \in C$. 
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Let $C$ be the strongly connected component containing $u$ and $w$ and let $x$ be the first vertex in $C$ visited during the construction of $F$.

It suffices to prove that $x \sim_{CC(F)} v$ for every $v \in C$.

This follows from

**Lemma:** If there exists a path from $x$ to $v$ consisting of vertices that are unexplored when $x$ is visited, then $v$ is a descendant of $x$ in $F$. 
Strongly Connected Components

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**Lemma:** If there exists a path from \( x \) to \( v \) consisting of vertices that are unexplored when \( x \) is visited, then \( v \) is a descendant of \( x \) in \( F \).

Let \( P = \langle x = x_0, x_1, \ldots, x_k = v \rangle \) be such a path from \( x \) to \( v \) and assume \( v \) is not a descendant of \( x \).
Strongly Connected Components

Lemma: If there exists a path from \( x \) to \( v \) consisting of vertices that are unexplored when \( x \) is visited, then \( v \) is a descendant of \( x \) in \( F \).

Let \( P = \langle x = x_0, x_1, \ldots, x_k = v \rangle \) be such a path from \( x \) to \( v \) and assume \( v \) is not a descendant of \( x \).

Since \( x \) is a descendant of \( x \), there exists a maximal index \( 0 \leq i < k \) such that \( x_0, x_1, \ldots, x_i \) are descendants of \( x \) and \( x_{i+1} \) is not.
**Strongly Connected Components**

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Since $x$ is a descendant of $x$, there exists a maximal index $0 \leq i < k$ such that $x_0, x_1, \ldots, x_i$ are descendants of $x$ and $x_{i+1}$ is not.

Since $x_{i+1}$ is visited after $x$ and all descendants of $x$ have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.
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Since $x$ is a descendant of $x$, there exists a maximal index $0 \leq i < k$ such that $x_0, x_1, \ldots, x_i$ are descendants of $x$ and $x_{i+1}$ is not.

Since $x_{i+1}$ is visited after $x$ and all descendants of $x$ have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.

Since $x_{i+1}$ is no descendant of $x$, it is not a descendant of $x_i$. 
**Strongly Connected Components**

**Lemma:** If there exists a path from $x$ to $v$ consisting of vertices that are unexplored when $x$ is visited, then $v$ is a descendant of $x$ in $F$.

Let $P = \langle x = x_0, x_1, \ldots, x_k = v \rangle$ be such a path from $x$ to $v$ and assume $v$ is not a descendant of $x$.

Since $x$ is a descendant of $x$, there exists a maximal index $0 \leq i < k$ such that $x_0, x_1, \ldots, x_i$ are descendants of $x$ and $x_{i+1}$ is not.

Since $x_{i+1}$ is visited after $x$ and all descendants of $x$ have consecutive preorder numbers, we have $x_i < x_{i+1}$ in preorder.

Since $x_{i+1}$ is no descendant of $x$, it is not a descendant of $x_i$.

Since $x_i < x_{i+1}$ in preorder, this implies that $(x_i, x_{i+1})$ is a forward cross edge, a contradiction.
**Strongly Connected Components**

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$. 
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$. 

![Diagram showing Strongly Connected Components](image)
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \leadsto_G v$ if and only if $v \leadsto_{G^r} u$. 
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

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Proof: We have $u \rightsquigarrow_G v$ if and only if $v \rightsquigarrow_{G^r} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$. 
### Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

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Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$.

Let $F^r_<$ be the DFS forest of $G^r$ obtained by calling `TraverseFromVertex` on unexplored vertices in the opposite order to $<$.
Strongly Connected Components

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \sim_G v$ if and only if $v \sim_{G^r} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$.

Let $F^r_{>}$ be the DFS forest of $G^r$ obtained by calling `TraverseFromVertex` on unexplored vertices in the opposite order to $<$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F^r_{>})} v$. 
**Strongly Connected Components**

For a graph $G = (V, E)$, let $G^r = (V, E^r)$, where $E^r = \{(v, u) \mid (u, v) \in E\}$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{SCC(G^r)} v$.

**Proof:** We have $u \sim_G v$ if and only if $v \sim_{G^r} u$.

Let $F$ be a DFS forest of $G$ and let $<$ be the postorder of $F$.

Let $F_r>$ be the DFS forest of $G^r$ obtained by calling TraverseFromVertex on unexplored vertices in the opposite order to $<$.

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F_r>)} v$.

⇒ Kosaraju's strong connectivity algorithm:

- Compute a DFS forest $F$ of $G$.
- Compute $G^r$ and arrange the vertices in reverse postorder w.r.t. $F$.
- Compute a DFS forest $F^r$ of $G^r$.
- Extract a component labelling of the vertices or the strongly connected components themselves from $F^r$ (almost) as we did for computing connected components.

This takes $O(n + m)$ time.
Strongly Connected Components

Lemma: $u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r^*)} v$. 
**Strongly Connected Components**

**Lemma:** \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F')} v. \)

Assume the contrary. Then there exists an edge \((u, v) \in F'_>\) such that \( u \not\sim_{\text{SCC}(G)} v. \)
**Strongly Connected Components**

**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F_\succ^r)} v. \)

Assume the contrary. Then there exists an edge \((u, v) \in F_\succ^r\) such that \( u \not\sim_{SCC(G)} v. \)

\[\Rightarrow (v, u) \in G.\]
**Strongly Connected Components**

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F')} v$.

Assume the contrary. Then there exists an edge $(u, v) \in F'_r$ such that $u \not\sim_{SCC(G)} v$.

$\implies (v, u) \in G$.

Choose this edge so that each of its ancestor edges $(x, y)$ satisfies $x \sim_{SCC(G)} y$.

\[ \forall (x, y) \in F'_r, x \sim_{SCC(G)} y. \]
**Strongly Connected Components**

**Lemma:** $u \sim_{SCC(G)} v \iff u \sim_{CC(F^r)} v$.

Assume the contrary. Then there exists an edge $(u, v) \in F^r_>$ such that $u \not\sim_{SCC(G)} v$.

$\Rightarrow (v, u) \in G$.

Choose this edge so that each of its ancestor edges $(x, y)$ satisfies $x \sim_{SCC(G)} y$.

In particular, $u \sim_{SCC(G)} r$, where $r$ is the root of the tree containing $u$ and $v$. 
**Strongly Connected Components**

**Lemma:** \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r^\succ)} v. \)

Assume the contrary. Then there exists an edge \((u, v) \in F_r^\succ\) such that \( u \not\sim_{\text{SCC}(G)} v. \)

\( \Rightarrow (v, u) \in G. \)

Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{\text{SCC}(G)} y. \)

In particular, \( u \sim_{\text{SCC}(G)} r, \) where \( r \) is the root of the tree containing \( u \) and \( v. \)

All vertices in \( C \) are descendants of \( r \) in \( F_r^\succ \) and \( x \leq r \) for all \( x \in C. \)
**Strongly Connected Components**

**Lemma:** \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r^\succ)} v \).

Assume the contrary. Then there exists an edge \((u, v) \in F_r^\succ\) such that \( u \not\sim_{\text{SCC}(G)} v \).

\[\Rightarrow (v, u) \in G.\]

Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{\text{SCC}(G)} y \).

In particular, \( u \sim_{\text{SCC}(G)} r \), where \( r \) is the root of the tree containing \( u \) and \( v \).

All vertices in \( C \) are descendants of \( r \) in \( F_r^\succ \) and \( x \leq r \) for all \( x \in C \).

Also, \( v < r \) because \( v \) is a descendant of \( r \) in \( F_r^\succ \).
**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F_r^>)} v. \)

Assume the contrary. Then there exists an edge \((u, v) \in F_r^>\) such that \( u \not\sim_{SCC(G)} v. \)

\[ \Rightarrow (v, u) \in G. \]

Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{SCC(G)} y. \)

In particular, \( u \sim_{SCC(G)} r, \) where \( r \) is the root of the tree containing \( u \) and \( v. \)

All vertices in \( C \) are descendants of \( r \) in \( F_r^> \) and \( x \leq r \) for all \( x \in C. \)

Also, \( v < r \) because \( v \) is a descendant of \( r \) in \( F_r^>. \)

In \( F, \) all vertices in \( C \) are descendants of some vertex \( r' \in C \) and \( x \leq r' \) for all \( x \in C. \)
Strongly Connected Components

**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F_r^>)} v \).

Assume the contrary. Then there exists an edge \((u, v) \in F_r^>\) such that \( u \not\sim_{SCC(G)} v \).

\[ \Rightarrow (v, u) \in G. \]

Choose this edge so that each of its ancestor edges \((x, y)\) satisfies \( x \sim_{SCC(G)} y \).

In particular, \( u \sim_{SCC(G)} r \), where \( r \) is the root of the tree containing \( u \) and \( v \).

All vertices in \( C \) are descendants of \( r \) in \( F_r^> \) and \( x \leq r \) for all \( x \in C \).

Also, \( v < r \) because \( v \) is a descendant of \( r \) in \( F_r^> \).

In \( F \), all vertices in \( C \) are descendants of some vertex \( r' \in C \) and \( x \leq r' \) for all \( x \in C \).

\[ \Rightarrow r = r' \text{ and } u \leq r. \]
Strongly Connected Components

Lemma: \( u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r)} v \).

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim_{\text{SCC}(G)} v \), a contradiction.
**Strongly Connected Components**

**Lemma:** $u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{\text{SCC}(G)} v$, a contradiction.
Strongly Connected Components

Lemma: $u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{\text{SCC}(G)} v$, a contradiction.
Strongly Connected Components

**Lemma:** \( u \sim^{\text{SCC}(G)} v \iff u \sim^{\text{CC}(F_r)} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim^{\text{SCC}(G)} v \), a contradiction.
**Strongly Connected Components**

**Lemma:** $u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F^r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{\text{SCC}(G)} v$, a contradiction.
**Strongly Connected Components**

**Lemma:** $u \sim_{\text{SCC}(G)} v \iff u \sim_{\text{CC}(F_r)} v$.

If $v$ is a descendant of $r$ in $F$, then $u \sim_{\text{SCC}(G)} v$, a contradiction.

If $v$ is not a descendant of $r$ in $F$, then $v$ is not a descendant of $u$ because $u$ is a descendant of $r$. 
**Strongly Connected Components**

**Lemma:** \( u \sim_{SCC(G)} v \iff u \sim_{CC(F)} v. \)

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim_{SCC(G)} v \), a contradiction.

If \( v \) is not a descendant of \( r \) in \( F \), then \( v \) is not a descendant of \( u \) because \( u \) is a descendant of \( r \).

Since \( u \leq r, v < r \), and the descendants of \( r \) are numbered consecutively, we have \( v < u \).
Strongly Connected Components

Lemma: \( u \sim_{SCC(G)} v \iff u \sim_{CC(F')} v \).

If \( v \) is a descendant of \( r \) in \( F \), then \( u \sim_{SCC(G)} v \), a contradiction.

If \( v \) is not a descendant of \( r \) in \( F \), then \( v \) is not a descendant of \( u \) because \( u \) is a descendant of \( r \).

Since \( u \leq r, v < r \), and the descendants of \( r \) are numbered consecutively, we have \( v < u \).

\( \Rightarrow (v, u) \) is a forward cross edge w.r.t. \( F \), a contradiction.
Summary

Graphs are fundamental in Computer Science:

Many problems are quite natural to express as graph problems:

- Matching problems
- Scheduling problems
- ...

Data structures are graphs whose nodes store useful information.

Graph exploration lets us learn the structure of a graph:

- Connectivity problems
- Distances between vertices
- Planarity
- ...