Foundations

Textbook Reading

Chapters 2 & 3

Overview

Review of things you should know

- Proof by contradiction
- Arrays, linked lists, stacks, and queues

Analysis of algorithms

- Worst-case and average-case running time
- Asymptotic notation

Stable Matching: The Gale-Shapley Algorithm

StableMatching(M, W)

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while there exists an unmarried man m
    do m proposes to the most preferable woman w he has not proposed to yet
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w would have married m then.

Contradiction.

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Contradiction.

More Questions

Does the final matching depend on the order in which the men propose?

Is the process fair?

Can the algorithm be implemented efficiently?

Can we implement a faster algorithm?

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Is the process fair?
No! The men fare much better than the women.

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No!

Is the process fair?
No! The men fare much better than the women.

Can the algorithm be implemented efficiently?

Can we implement a faster algorithm?
Yes, using randomization.

Computational Tractability

Informally, we consider a problem computationally tractable if it can be solved using reasonable resources.

Resources:

- Running time
- Memory usage
- Disk usage
- Number of messages sent across the network
- Energy
- ...

Model of Computation: The RAM Model

We would like to be able to predict the running time of algorithms before implementing them.

We would like our analysis to be applicable to a wide range of machines.

$\Rightarrow$ We need to base our analysis on a model of computation that captures the characteristics of a wide range of machines.

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The Random Access Machine (RAM) model:

Elementary operations take constant time:

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$\Rightarrow$ By counting elementary operations, we can compare the actual running times of two algorithms up to constant factors.

Efficient Algorithm = Polynomial Running Time

Most algorithms are fast for small inputs. We care about their behaviour for non-trivial (i.e., large) inputs.

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Justification: Overwhelmingly, polynomial-time algorithms are fast in practice and exponential-time algorithms are not.

Running Time May Depend on Specific Input

InsertionSort(A, n)

```
for i = 2 to n
    do x = A[i]
       j = i − 1
       while j > 0 and A[j] > x
           do A[j + 1] = A[j]
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How do we unify this into one function $T(n)$?

Worst-Case and Average-Case Running Time

The worst-case running time of an algorithm A is a function T(n) defined as the maximum running time of A over all possible inputs of size n.

The average-case running time of an algorithm A is a function T(n) defined as the average running time of A over all possible inputs of size n.

Asymptotic Running Time

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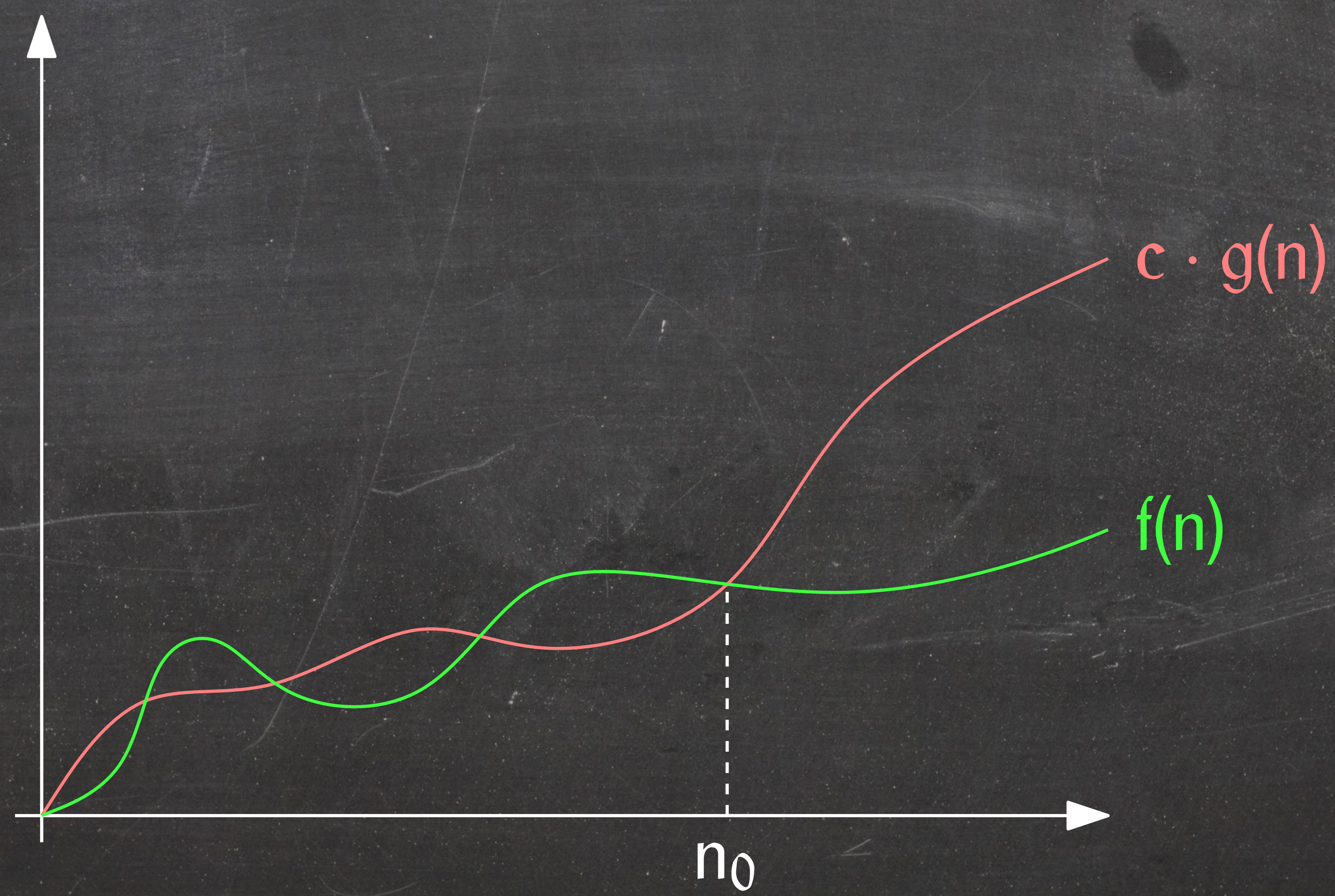
Formally: We want $T_A(n) < T_B(n)$ for all $n \geq n_0$, where n_0 is the smallest input size we consider to be "large".

O-Notation

$f(n) \in O(g(n))$ means that $f(n)$ is at most a constant factor larger than $g(n)$ for large enough n.

Formally:

$$f(n) \in O(g(n))$$
$$\Updownarrow$$
$$\exists c > 0, n_0 \geq 0 \ \forall n \geq n_0 : f(n) \leq c \cdot g(n)$$



Ω-Notation

$f(n) \in \Omega(g(n))$ means that $f(n)$ is at most a constant factor smaller than $g(n)$ for large enough n.

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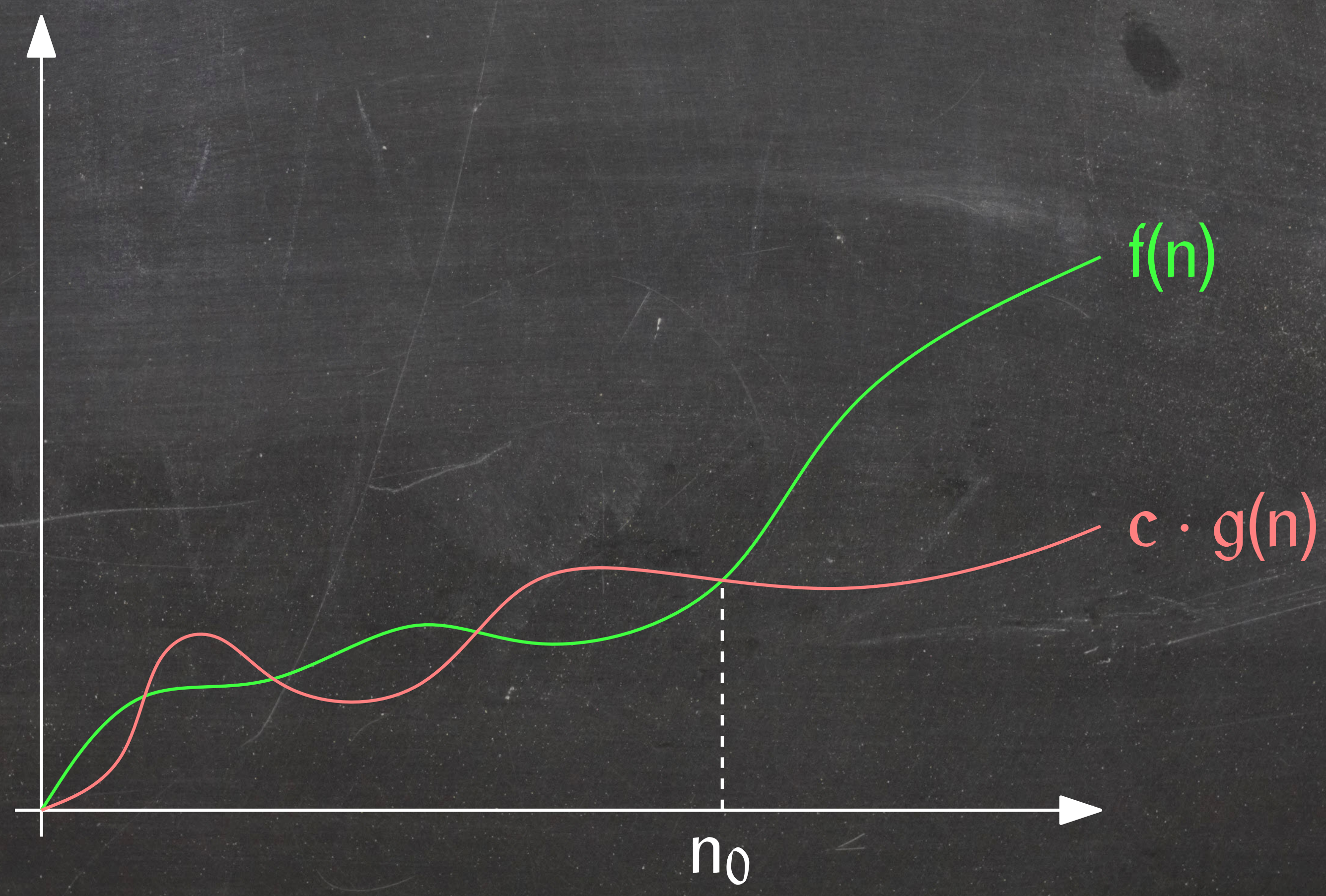
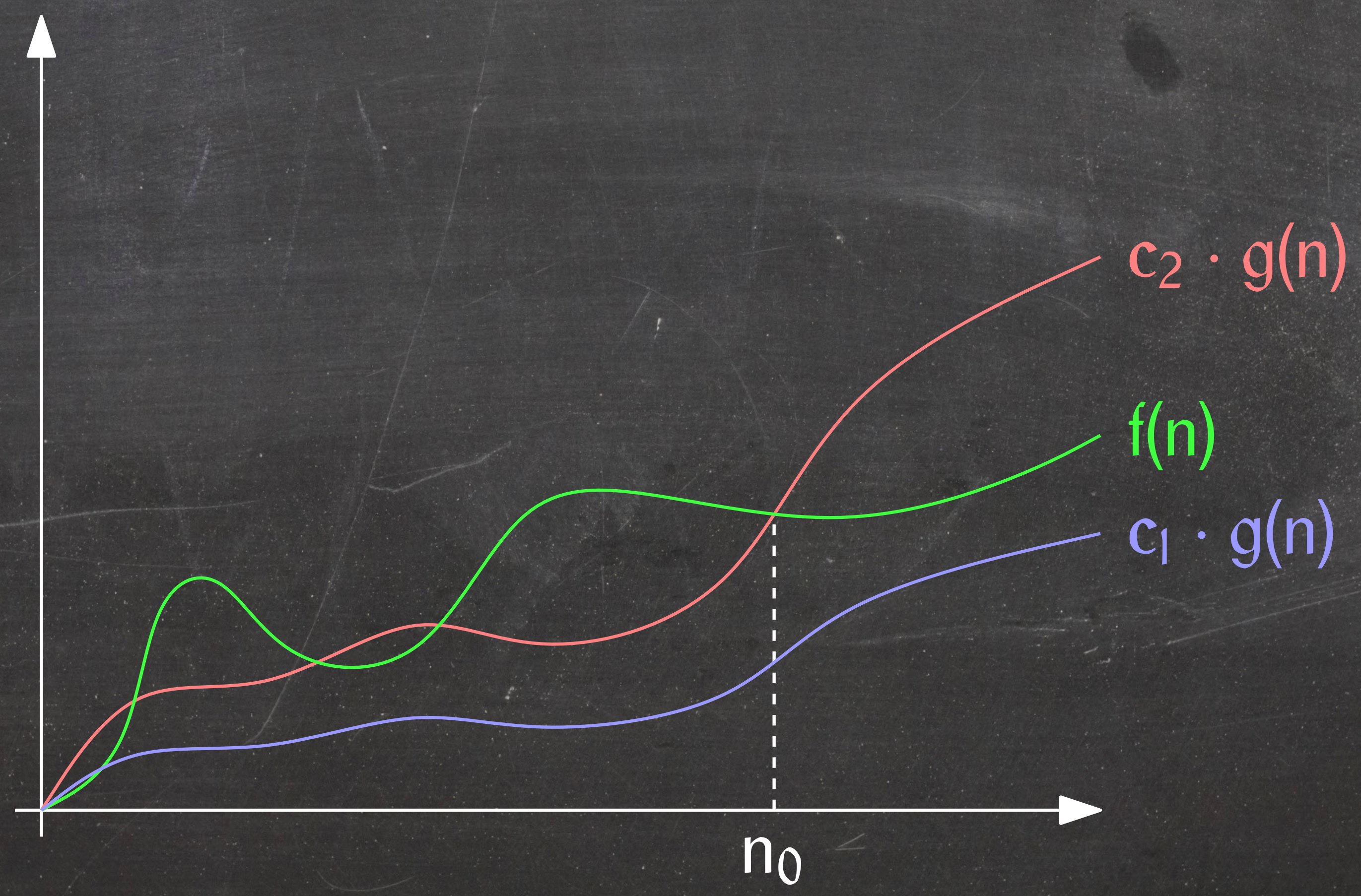


Θ-Notation

$f(n) \in \Theta(g(n))$ means that the difference between $f(n)$ and $g(n)$ is at most a constant factor for large enough n.

Formally:

$$f(n) \in \Theta(g(n))$$
$$\Updownarrow$$
$$\exists c_1 > 0, c_2 > 0, n_0 \geq 0 \; \forall n \geq n_0 : c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$



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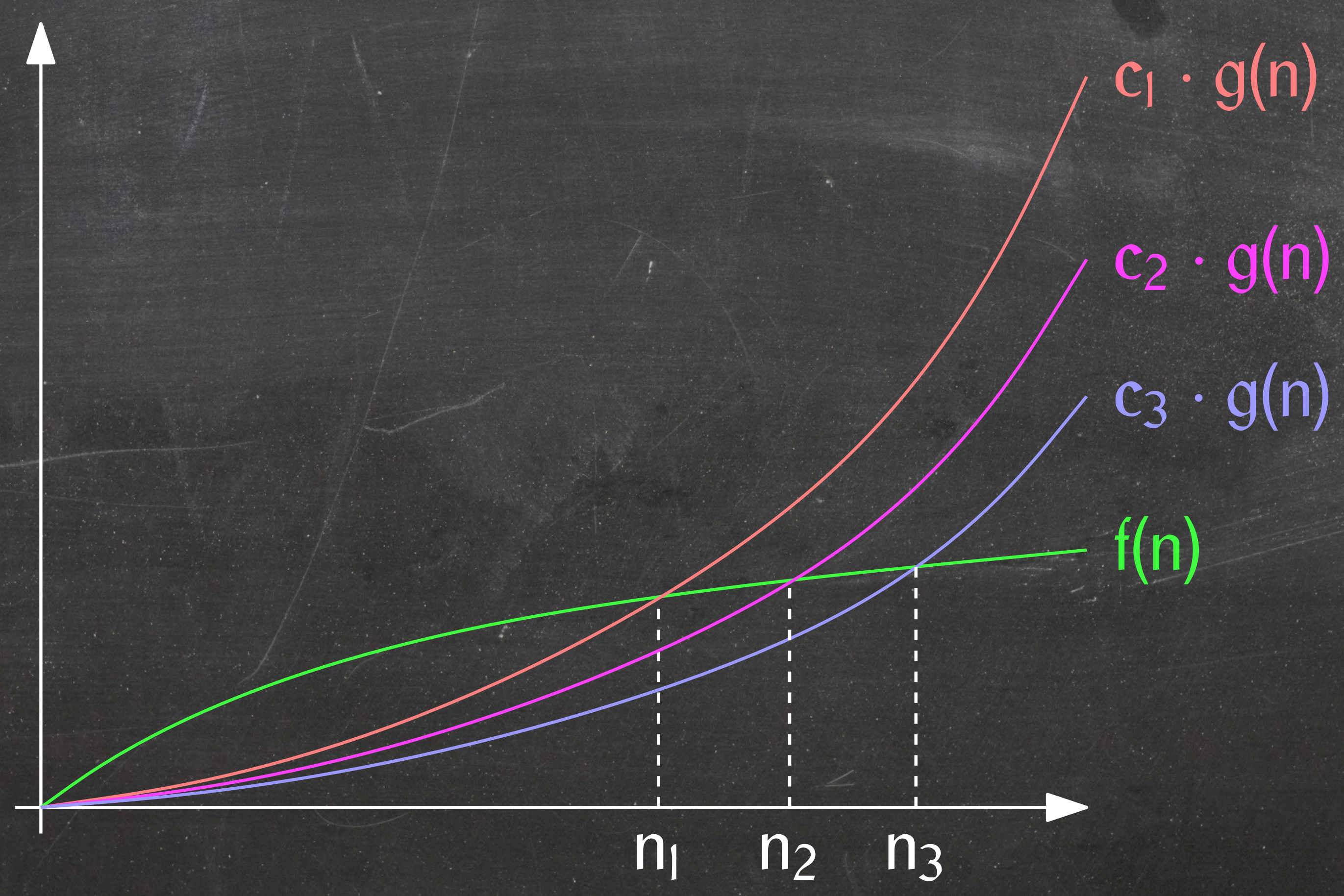

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$f(n) \in \omega(g(n))$ means that the ratio between $f(n)$ and $g(n)$ grows without bounds as n grows. An algorithm with running time $g(n)$ is much faster than one with running time $f(n)$ for large enough inputs, even if run on a slower computer!

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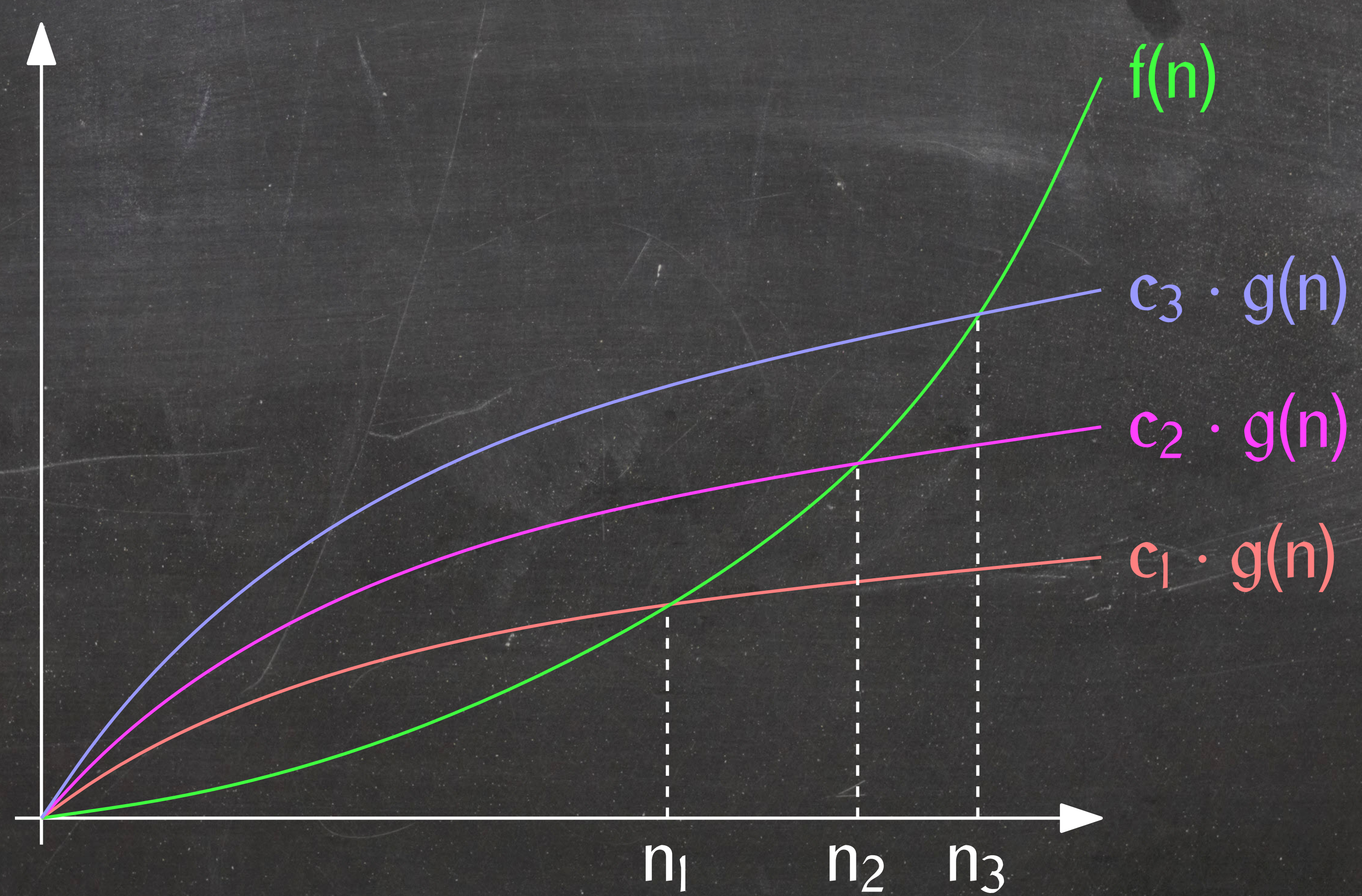

A Few Simple Facts

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$$f_1(n) \in O(g_1(n)) \text{ and } f_2(n) \in O(g_2(n)) \Longrightarrow f_1(n) + f_2(n) \in O(g_1(n) + g_2(n))$$

$$f(n) \in O(g(n)) \Longrightarrow f(n) + g(n) \in O(g(n))$$

Asymptotic Analysis and Limits

The following relationships hold for positive increasing functions f(n) and g(n). Since the running times of algorithms are positive and increasing, we can use these rules when analyzing algorithms.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \iff f(n) \in o(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0 \implies f(n) \in \Theta(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \implies a^{f(n)} \in o(a^{g(n)}) \text{ for any } a > 1$$

$$f(n) \in o(g(n)) \implies a^{f(n)} \in o(a^{g(n)}) \text{ for any } a > 1$$

$$f(n) \in \Theta(g(n)) \not\Longrightarrow a^{f(n)} \in \Theta(a^{g(n)})$$

Asymptotic Analysis and Algorithm Performance

What does it mean if $T_A(n) \in O(T_B(n))$?

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Algorithm A is at most a constant factor slower than algorithm B.
The constant factor may be large!!!

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In a first filter step to select possible candidate algorithms and during algorithm design, this is helpful.

Subsequent choices have to be based on our experience, analyses that do take constants into account, or experimental evaluation.

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What do we gain?

A simple, succinct expression of the performance of an algorithm.

Implementation of the Gale-Shapley Algorithm

StableMatching(M, W)

```
while there exists an unmarried man m
    do m proposes to the most preferable woman w he has not proposed to yet
        if w is unmarried or likes m better than her current partner m′
            then if w is married
                    then w divorces m′
                 w marries m
```

Questions we can and should ask about the algorithm:

- Is there always a stable matching?
- Does the algorithm always terminate?
- Does the algorithm always produce a stable matching?
- How efficient is the algorithm? Can we bound its running time?

Implementation of the Gale-Shapley Algorithm

StableMatching(M : Array[Man], W : Array[Woman])

```
Q = an empty queue
for every man m ∈ M
    do Q.enqueue(m)
while not Q.isEmpty()
    do m = Q.dequeue()
       w = W[m.nextOnList()]
       if not w.isMarried()
          then w.marry(m)
          else m′ = w.partner()
               if w.prefers(m, m′)
                  then w.marry(m)
                       Q.enqueue(m′)
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Queue:

- $O(1)$ time per operation

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```

Queue:

- $O(1)$ time per operation

Man:

- Preference list = array + current index/list
- nextOnList = access + increase index or pointer jump on list

$\Rightarrow$ $O(1)$ time

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```

Woman:

- Stores pointer to her partner

$\Rightarrow$ isMarried/marry/partner take O(1) time

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```

Woman:

- Stores pointer to her partner

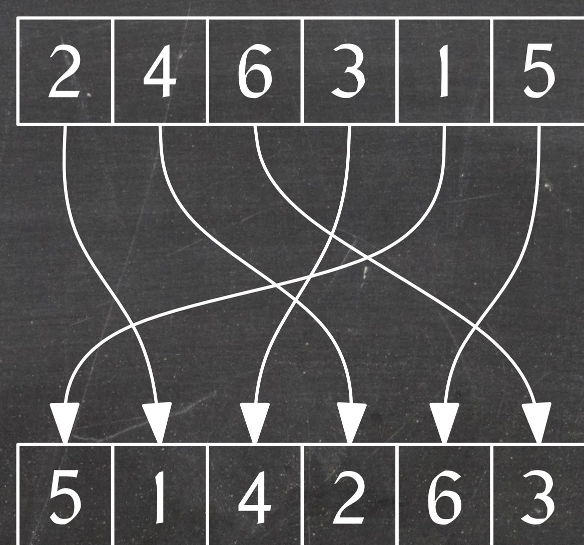
$\Rightarrow$ isMarried/marry/partner take $O(1)$ time

- prefers takes $O(1)$ time if we have an inverted preference list:
  - Map every man to his rank in the preference list.

Inverting a Preference List

InvertPreflist(w : Woman)

```
L = new array of size |w.preflist|
for i = 1 to |w.preflist|
    do L[w.preflist[i]] = i
w.preflist = L
```



This takes linear time.

Implementation of the Gale-Shapley Algorithm

StableMatching(M : Array[Man], W : Array[Woman])

```
Q = an empty queue
for every man m ∈ M
    do Q.enqueue(m)
for every woman w ∈ W
    do InvertPreflist(w)
while not Q.isEmpty()
    do m = Q.dequeue()
       w = W[m.nextOnList()]
       if not w.isMarried()
           then w.marry(m)
           else m′ = w.partner()
                if w.prefers(m, m′)
                    then w.marry(m)
                         Q.enqueue(m′)
                    else Q.enqueue(m)
```

The Gale-Shapley algorithm can be implemented to run in $O(n^2)$ time.

Summary

Review of things you should know

- Proof by contradiction
- Arrays, linked lists, stacks, and queues

Analysis of algorithms

- Worst-case and average-case running time
- Asymptotic notation