Dynamic Programming

Textbook Reading
Chapters 15, 24 & 25
Overview

Design principle

- Recursively break the problem into smaller subproblems.
- Avoid repeatedly solving the same subproblems by caching their solutions.

Important tool

- Recurrence relations

Problems

- Weighted interval scheduling
- Sequence alignment
- Optimal binary search trees
- Shortest paths
Given:
A set of activities competing for time intervals on a certain resource (E.g., classes to be scheduled competing for a classroom)

Goal:
Schedule non-conflicting activities so that the total time the resource is in use is maximized.
W. I. S.: A Naïve Solution

• Try all possible subsets.
• Check each subset for conflicts.
• Out of the non-conflicting ones, remember the one with maximal total length.
W. I. S.: A Naïve Solution

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Cost:
W. I. S.: A Naïve Solution

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- Check each subset for conflicts.
- Out of the non-conflicting ones, remember the one with maximal total length.

Cost: $O(2^n \cdot n^2)$
W. 1. S.: Towards a Better Solution

General idea:

• Try to make **one choice at a time**, just as in a greedy algorithm.
• In each step, what are the options we can choose from?
• What can we say about the subproblem we obtain after choosing each option?
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What options do we have?
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What options do we have?
An interval is in the optimal solution or it isn't.
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Towards a recurrence for the cost of an optimal solution:
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Towards a recurrence for the cost of an optimal solution:

If the maximal-length subset of \{l_1, l_2, \ldots, l_n\} does not include \( l_n \), then it must be a maximal-length subset of \{l_1, l_2, \ldots, l_{n-1}\}. 
W. I. S.: Towards a Better Solution

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- Try to make one choice at a time, just as in a greedy algorithm.
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If the maximal-length subset of \{l_1, l_2, \ldots, l_n\} does not include \(l_n\), then it must be a maximal-length subset of \{l_1, l_2, \ldots, l_{n-1}\}.

If the maximal-length subset of \{l_1, l_2, \ldots, l_n\} includes \(l_n\), then it must be \(O \cup \{l_n\}\), where \(O\) is a maximal-length subset of all intervals in \{l_1, l_2, \ldots, l_n\} that do not overlap \(l_n\).
W. I. S.: Cleaning Up the Model

Number the intervals by increasing ending times:
W. I. S.: Cleaning Up the Model

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```
<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>p_j</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>
```

For $1 \leq j \leq n$, let $p_j = \max\{0\} \cup \{k | 1 \leq k < j \text{ and } l_k \text{ does not overlap } l_j\}$. 
W. I. S.: Cleaning Up the Model

Number the intervals by increasing ending times:

For $1 \leq j \leq n$, let $p_j = \max\{\{0\} \cup \{k \mid 1 \leq k < j \text{ and } I_k \text{ does not overlap } I_j\}\}$.

<table>
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<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>3</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

If the maximal-length subset of $\{I_1, I_2, \ldots, I_n\}$ includes $I_n$, then it is $O_{p_n} \cup \{I_n\}$, where $O_{p_n}$ is a maximal-length subset of the intervals $\{I_1, I_2, \ldots, I_{p_n}\}$.
Let $|I_j|$ be the length of interval $I_j$. 

W. I. S.: A Recurrence for the Optimal Solution
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Let $|I_j|$ be the length of interval $I_j$.

Let $\ell(j)$ be maximal total length of any subset of non-overlapping intervals in $\{I_1, I_2, \ldots, I_j\}$. 
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Let $\ell(j)$ be maximal total length of any subset of non-overlapping intervals in \{l_1, l_2, \ldots, l_j\}.

What we're interested in is $\ell(n)$!
Let $|I_j|$ be the length of interval $I_j$.

Let $\ell(j)$ be maximal total length of any subset of non-overlapping intervals in $
\{I_1, I_2, \ldots, I_j\}$.

What we're interested in is $\ell(n)!$

\[
\ell(j) = \begin{cases} 
0 & j = 0 \\
\max(\ell(j - 1), |I_j| + \ell(p_j)) & j > 0
\end{cases}
\]
W. I. S.: A Recursive Algorithm

FindScheduleLength(l, p, j)

1. if \( j = 0 \) then return 0
2. else return \( \max(\text{FindScheduleLength}(l, p, p[j]) + |l[j]|, \text{FindScheduleLength}(l, p, j - 1)) \)
W. I. S.: A Recursive Algorithm

FindScheduleLength(l, p, j)

1  if j = 0
2   then return 0
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Running time:
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Running time: \(O(2^n)\)
W. I. S.: A Recursive Algorithm

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2 then return 0
3 else return max(FindScheduleLength(I, p, p[j]) + |I[j]|, FindScheduleLength(I, p, j – 1))

Running time: $O(2^n)$
W. I. S.: A Recursive Algorithm

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The recursive algorithm computes many values repeatedly.
**W. I. S.: A Recursive Algorithm**

**FindScheduleLength(l, p, j)**

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2. then return 0  
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**Running time:** \( O(2^n) \)

The recursive algorithm computes many values repeatedly.

There are only \( n \) values to compute!
Memoization: Store already computed values in a table to avoid recomputing them.
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Here, initialize a table $\ell$ where $\ell[j]$ is the length of the optimal schedule for $\{I_1, I_2, \ldots, I_j\}$. Initially, $\ell[j] = -\infty$ for all $j$. 
**Memoization**: Store already computed values in a table to avoid recomputing them.

Here, initialize a table $\ell$ where $\ell[j]$ is the length of the optimal schedule for $\{I_1, I_2, \ldots, I_j\}$. Initially, $\ell[j] = -\infty$ for all $j$.

```python
def FindScheduleLength(l, ell, p, j):
    if j == 0:
        return 0
    else if ell[j] < 0:
        ell[j] = max(FindScheduleLength(l, p, p[j]) + |I[j]|, FindScheduleLength(l, p, j - 1))
    return ell[j]
```
Memoization: Store already computed values in a table to avoid recomputing them.

Here, initialize a table \( \ell \) where \( \ell[j] \) is the length of the optimal schedule for \( \{I_1, I_2, \ldots, I_j\} \). Initially, \( \ell[j] = -\infty \) for all \( j \).

**FindScheduleLength(I, \ell, p, j)**

1. if \( j = 0 \)
2. then return 0
3. else if \( \ell[j] < 0 \)
4. then \( \ell[j] = \max(\text{FindScheduleLength}(I, p, p[j]) + |I[j]|, \text{FindScheduleLength}(I, p, j - 1)) \)
5. return \( \ell[j] \)

Running time: \( O(n) \)
**W. I. S.: Iterative Table Fill-In**

**FindScheduleLength**(l, p)

1. \( l[0] = 0 \)
2. for \( j = 1 \) to \( n \) do
3. \( l[j] = \max(l[j - 1], l[p[j]] + |I[j]|) \)
4. return \( l[n] \)
W. I. S.: Iterative Table Fill-In

FindScheduleLength(I, p)

1 \[ \ell[0] = 0 \]
2 \[ \text{for } j = 1 \text{ to } n \]
3 \[ \text{do } \ell[j] = \max(\ell[j - 1], \ell[p[j]] + ||I[i]|) \]
4 \[ \text{return } \ell[n] \]

Running time: \( O(n) \)
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Running time: \( O(n) \)

Advantage over memoization:
- No need for recursion.
- Algorithm is often simpler.
W. I. S.: Iterative Table Fill-In

FindScheduleLength(I, p)

1 \( \ell[0] = 0 \)
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4 \( \text{return } \ell[n] \)

Running time: \( O(n) \)

Advantage over memoization:
- No need for recursion.
- Algorithm is often simpler.

Disadvantage over memoization:
- Need to worry about the order in which the table entries are computed:
  - All entries needed to compute the current entry need to be computed first.
- Memoization computes table entries as needed.
FindSchedule(l, p)

1 \( \ell[0] = 0 \)
2 \( S[0] = [] \)
3 for \( j = 1 \) to \( n \)
4 do if \( \ell[j - 1] > \ell[p[j]] + |I[j]| \)
5 then \( \ell[j] = \ell[j - 1] \)
6 \( S[j] = S[j - 1] \)
7 else \( \ell[j] = \ell[p[j]] + |I[j]| \)
8 \( S[j] = [I[j]] ++ S[p[j]] \)
9 return \( S[n] \)
W. I. S.: Computing the Set of Intervals

FindSchedule(I, p)

1 \( \ell[0] = 0 \)
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8 \( S[j] = [I[j]] ++ S[p[j]] \)
9 return \( S[n] \)

Running time: \( \mathcal{O}(n) \)
FindSchedule(I, p)

1 \ell[0] = 0
2 S[0] = []
3 for j = 1 to n
4     do if \ell[j – 1] > \ell[p[j]] + |I[j]|
5           then \ell[j] = \ell[j – 1]
6             S[j] = S[j – 1]
7     else \ell[j] = \ell[p[j]] + |I[j]|
8       S[j] = [I[j]] ++ S[p[j]]
9 return S[n]

Running time: \(O(n)\)

This computes the sequence of intervals ordered from last to first.

This list is of course easy to reverse in linear time.
W. I. S.: The Missing Details

What's missing?

- Sort the intervals by their ending times.
- Compute the predecessor array $p$. 
What’s missing?

- Sort the intervals by their ending times.
- Compute the predecessor array $p$.

Solution:

- Sorting is easily done in $O(n \lg n)$ time.
- To compute $p[j]$, perform binary search with $I[j]$’s starting time on the sorted array of ending times.
What's missing?

- Sort the intervals by their ending times.
- Compute the predecessor array \( p \).

Solution:

- Sorting is easily done in \( O(n \log n) \) time.
- To compute \( p[j] \), perform binary search with \( I[j] \)’s starting time on the sorted array of ending times.

Theorem: The weighted interval scheduling problem can be solved in \( O(n \log n) \) time.
The Dynamic Programming Technique

The technique:

- Develop a recurrence expressing the optimal solution for a given problem instance in terms of optimal solutions for smaller problem instances:
- Evaluate this recurrence
  - Recursively using memoization or
  - Using iterative table fill-in.
The Dynamic Programming Technique

**The technique:**

- Develop a **recurrence** expressing the **optimal solution** for a given problem instance in terms of optimal solutions for smaller problem instances:
- Evaluate this recurrence
  - Recursively using **memoization** or
  - Using iterative **table fill-in**.

For this to work, the problem must exhibit the **optimal substructure property**: The optimal solution to a problem instance must be composed of optimal solutions to smaller problem instances.
The Dynamic Programming Technique

The technique:

- Develop a recurrence expressing the optimal solution for a given problem instance in terms of optimal solutions for smaller problem instances:
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For this to work, the problem must exhibit the optimal substructure property: The optimal solution to a problem instance must be composed of optimal solutions to smaller problem instances.

A speed-up over the naïve recursive algorithm is achieved if the problem exhibits overlapping subproblems: The same subproblem occurs over and over again in the recursive evaluation of the recurrence.
Developing a Dynamic Programming Algorithm

Step 1: Think top-down:
- Consider an optimal solution (without worrying about how to compute it).
- Identify how the optimal solution of any problem instance decomposes into optimal solutions to smaller problem instances.
- Write down a recurrence based on this analysis.

Step 2: Formulate the algorithm, which computes the solution bottom-up:
- Since an optimal solution depends on optimal solutions to smaller problem instances, we need to compute those first.
Sequence Alignment

Given the search term “Dalhusy Computer Science”, Google suggests the correction “Dalhousie Computer Science”.
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Can Google read your mind?
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Can Google read your mind? No!

They use a clever algorithm to match your mistyped query against the phrases they have in their database.

“Dalhousie” is the closest match to “Dalhusy” they find.
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They use a clever algorithm to match your mistyped query against the phrases they have in their database.

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What’s a good similarity criterion?
Sequence Alignment

**Problem:** Given two strings $X = x_1x_2 \cdots x_m$ and $Y = y_1y_2 \cdots y_n$, extend them to two strings $X' = x'_1x'_2 \cdots x'_t$ and $Y' = y'_1y'_2 \cdots y'_t$ of the same length by inserting gaps so that the following dissimilarity measure $D(X', Y')$ is minimized:

$$D(X', Y') = \sum_{i=1}^{t} d(x'_i, y'_i)$$

$$d(x, y) = \begin{cases} 
\delta & x = \_ \text{ or } y = \_ \text{ (gap penalty)} \\
\mu_{x,y} & \text{otherwise (mismatch penalty)}
\end{cases}$$
Sequence Alignment

**Problem:** Given two strings $X = x_1x_2 \cdots x_m$ and $Y = y_1y_2 \cdots y_n$, extend them to two strings $X' = x_1'x_2' \cdots x_t'$ and $Y' = y_1'y_2' \cdots y_t'$ of the same length by inserting gaps so that the following dissimilarity measure $D(X', Y')$ is minimized:

$$D(X', Y') = \sum_{i=1}^{t} d(x'_i, y'_i)$$

$$d(x, y) = \begin{cases} 
\delta & x = _- \text{ or } y = _- (\text{gap penalty}) \\
\mu_{x,y} & \text{otherwise (mismatch penalty)} 
\end{cases}$$

**Example:**

$$\text{Dalh} \underline{\text{usu}} \underline{\text{y}} \text{Dalhousie}$$

$$D(X', Y') = 2\delta + \mu_{iy}$$
Sequence Alignment

Problem: Given two strings \( X = x_1x_2 \cdots x_m \) and \( Y = y_1y_2 \cdots y_n \), extend them to two strings \( X' = x'_1x'_2 \cdots x'_t \) and \( Y' = y'_1y'_2 \cdots y'_t \) of the same length by inserting gaps so that the following dissimilarity measure \( D(X', Y') \) is minimized:

\[
D(X', Y') = \sum_{i=1}^{t} d(x'_i, y'_i)
\]

\[
d(x, y) = \begin{cases} 
\delta & \text{if } x = _- \text{ or } y = _- \text{ (gap penalty)} \\
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\end{cases}
\]

Example:

\[
\begin{align*}
\text{Dalh} & \quad \text{usy} \\
\text{Dalhousie}
\end{align*}
\]

\[
D(X', Y') = 2\delta + \mu_{iy}
\]

Another (more important?) application:
DNA sequence alignment to measure the similarity between different DNA samples.
Sequence Alignment: Problem Analysis

Assume $(x'_1 x'_2 \cdots x'_t, y'_1 y'_2 \cdots y'_t)$ is an optimal alignment for $(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_n)$.

What choices do we have for the final pair $(x'_t, y'_t)$?
Sequence Alignment: Problem Analysis

Assume \((x'_1x'_2\cdots x'_t, y'_1y'_2\cdots, y'_t)\) is an optimal alignment for \((x_1x_2\cdots x_m, y_1y_2\cdots y_n)\).

What choices do we have for the final pair \((x'_t, y'_t)\)?

- \(x'_t = x_m\) and \(y'_t = y_n\)
Sequence Alignment: Problem Analysis

Assume \((x_1', x_2', \ldots, x_t', y_1', y_2', \ldots, y_t')\) is an optimal alignment for \((x_1 x_2 \ldots x_m, y_1 y_2 \ldots y_n)\).

What choices do we have for the final pair \((x_t', y_t')\)?

- \(x_t' = x_m\) and \(y_t' = y_n\)
- \(x_t' = x_m\) and \(y_t' = \_\)
Sequence Alignment: Problem Analysis

Assume \((x_1'x_2' \cdots x_t', y_1'y_2' \cdots, y_t')\) is an optimal alignment for \((x_1x_2 \cdots x_m, y_1y_2 \cdots y_n)\).

What choices do we have for the final pair \((x_t', y_t')\)?

- \(x_t' = x_m\) and \(y_t' = y_n\)
- \(x_t' = x_m\) and \(y_t' = \_\)
- \(x_t' = \_\) and \(y_t' = y_n\)
Sequence Alignment: Problem Analysis

Assume \((x'_1 x'_2 \cdots x'_t, y'_1 y'_2 \cdots y'_t)\) is an optimal alignment for \((x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_n)\).

What choices do we have for the final pair \((x'_t, y'_t)\)?

- \(x'_t = x_m\) and \(y'_t = y_n\)

\((x'_1 x'_2 \cdots x'_{t-1}, y'_1 y'_2 \cdots y'_{t-1})\) must be an optimal alignment for \((x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_{n-1})\).
Sequence Alignment: Problem Analysis

Assume \((x_1'x_2' \cdots x_t', y_1'y_2' \cdots, y_t')\) is an optimal alignment for \((x_1x_2 \cdots x_m, y_1y_2 \cdots y_n)\).

What choices do we have for the final pair \((x_t', y_t')\)?

- \(x_t' = x_m\) and \(y_t' = y_n\)

\((x_1'x_2' \cdots x_{t-1}', y_1'y_2' \cdots y_{t-1}')\) must be an optimal alignment for \((x_1x_2 \cdots x_{m-1}, y_1y_2 \cdots y_{n-1})\).

Assume there's a better alignment \((x_1''x_2'' \cdots x_s'', y_1''y_2'' \cdots y_s'')\) with dissimilarity

\[
\sum_{i=1}^{s} d(x_i'', y_i'') < \sum_{i=1}^{t-1} d(x_i', y_i').
\]
Sequence Alignment: Problem Analysis

Assume \((x_1'x_2' \cdots x_t', y_1'y_2' \cdots , y_t')\) is an optimal alignment for \((x_1x_2 \cdots x_m, y_1y_2 \cdots y_n)\). What choices do we have for the final pair \((x_t', y_t')\)?

- \(x_t' = x_m\) and \(y_t' = y_n\)

\((x_1'x_2' \cdots x_{t-1}', y_1'y_2' \cdots y_{t-1}')\) must be an optimal alignment for \((x_1x_2 \cdots x_{m-1}, y_1y_2 \cdots y_{n-1})\).

Assume there's a better alignment \((x_1''x_2'' \cdots x_s'', y_1'y_2' \cdots y_s'')\) with dissimilarity

\[
\sum_{i=1}^{s} d(x_i'', y_i'') < \sum_{i=1}^{t-1} d(x_i', y_i').
\]

Then \((x_1''x_2'' \cdots x_s'x_t', y_1'y_2' \cdots y_s'y_t')\) is an alignment for \((x_1x_2 \cdots x_m, y_1y_2 \cdots y_n)\) with dissimilarity

\[
\sum_{i=1}^{s} d(x_i'', y_i'') + d(x_t', y_t') < \sum_{i=1}^{t-1} d(x_i', y_i') + d(x_t', y_t') = \sum_{i=1}^{t} d(x_i', y_i'),
\]

a contradiction.
Sequence Alignment: Problem Analysis

Assume \((x'_1 x'_2 \cdots x'_t, y'_1 y'_2 \cdots, y'_t)\) is an optimal alignment for \((x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_n)\).

What choices do we have for the final pair \((x'_t, y'_t)\)?

- \(x'_t = x_m\) and \(y'_t = y_n\)
  
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**Sequence Alignment: Problem Analysis**

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  \((x'_1 x'_2 \cdots x'_{t-1}, y'_1 y'_2 \cdots y'_{t-1})\) must be an optimal alignment for \((x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_{n})\).

- \(x'_t = \_\) and \(y'_t = y_n\)
  
  \((x'_1 x'_2 \cdots x'_{t-1}, y'_1 y'_2 \cdots y'_{t-1})\) must be an optimal alignment for \((x_1 x_2 \cdots x_{m}, y_1 y_2 \cdots y_{n-1})\).
Sequence Alignment: The Recurrence

Let $D(i, j)$ be the dissimilarity of the strings $x_1x_2\cdots x_i$ and $y_1y_2\cdots y_j$. 
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We are interested in $D(m, n)$. 
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Let $D(i, j)$ be the dissimilarity of the strings $x_1x_2\cdots x_i$ and $y_1y_2\cdots y_j$.

We are interested in $D(m, n)$.

**Recurrence:**

$$D(i, j) = \begin{cases} 
\delta \cdot j & i = 0 \\
\delta \cdot i & j = 0 \\
\min(D(i - 1, j - 1) + \mu_{x_i, y_j}, D(i, j - 1) + \delta, D(i - 1, j) + \delta) & \text{otherwise}
\end{cases}$$
Sequence Alignment: The Algorithm

SequenceAlignment(X, Y, μ, δ)

1. \( D[0, 0] = 0 \)
2. \( A[0, 0] = [] \)
3. for \( i = 1 \) to \( m \)
   4. do \( D[i, 0] = D[i - 1, 0] + \delta \)
   5. \( A[i, 0] = [(X[i], \_)] ++ A[i - 1, 0] \)
6. for \( j = 1 \) to \( n \)
   7. do \( D[0, j] = D[0, j - 1] + \delta \)
   8. \( A[0, j] = [(Y[j])] ++ A[0, j - 1] \)
9. for \( i = 1 \) to \( m \)
   10. do for \( j = 1 \) to \( n \)
      11. do \( D[i, j] = D[i - 1, j - 1] + \mu[X[i], Y[j]] \)
      12. \( A[i, j] = [(X[i], Y[j])] ++ A[i - 1, j - 1] \)
      13. if \( D[i, j] > D[i - 1, j] + \delta \)
          then \( D[i, j] = D[i - 1, j] + \delta \)
          \( A[i, j] = [(X[i], \_)] ++ A[i - 1, j] \)
      14. if \( D[i, j] > D[i, j - 1] + \delta \)
          then \( D[i, j] = D[i, j - 1] + \delta \)
19. return \( A[m, n] \)
Sequence Alignment: The Algorithm

SequenceAlignment(X, Y, µ, δ)

1. \( D[0, 0] = 0 \)
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   do \( D[i, 0] = D[i - 1, 0] + \delta \)
   \( A[i, 0] = [[X[i], \_]] \) ++ \( A[i - 1, 0] \)
4. for \( j = 1 \) to \( n \)  
   do \( D[0, j] = D[0, j - 1] + \delta \)
   \( A[0, j] = [[\_, Y[j]]] \) ++ \( A[0, j - 1] \)
5. for \( i = 1 \) to \( m \)  
   do for \( j = 1 \) to \( n \)  
      do \( D[i, j] = D[i - 1, j - 1] + \mu[X[i], Y[j]] \)
      \( A[i, j] = [[X[i], Y[j]]] \) ++ \( A[i - 1, j - 1] \)
      if \( D[i, j] > D[i - 1, j] + \delta \)  
         then \( D[i, j] = D[i - 1, j] + \delta \)
         \( A[i, j] = [[X[i], \_]] \) ++ \( A[i - 1, j] \)
      if \( D[i, j] > D[i, j - 1] + \delta \)  
         then \( D[i, j] = D[i, j - 1] + \delta \)
6. return \( A[m, n] \)

Running time: \( O(mn) \)
**Sequence Alignment: The Algorithm**

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   10. do for \(j = 1\) to \(n\)
       11. do \(D[i, j] = D[i - 1, j - 1] + \mu[X[i], Y[j]]\)
       12. \(A[i, j] = [(X[i], Y[j])] ++ A[i - 1, j - 1]\)
       13. if \(D[i, j] > D[i - 1, j] + \delta\)
           then \(D[i, j] = D[i - 1, j] + \delta\)
           \(A[i, j] = [(X[i], \_) ++ A[i - 1, j]\)
       16. if \(D[i, j] > D[i, j - 1] + \delta\)
           then \(D[i, j] = D[i, j - 1] + \delta\)
           \(A[i, j] = [\(_\), Y[j]]] ++ A[i, j - 1]\)
5. return \(A[m, n]\)

**Running time:** \(O(mn)\)

Again, the sequence alignment is reported back-to-front and can be reversed in \(O(m + n)\) time.
Optimal Binary Search Trees

Balanced binary search trees (red-black trees, AVL trees, ...) guarantee $O(\lg n)$ time to find an element.
Optimal Binary Search Trees

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Can we do better?
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Can we do better? Not in the worst case.
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Let $x_1 < x_2 < \cdots < x_n$ be the elements to be stored in the tree.
Optimal Binary Search Trees

Balanced binary search trees (red-black trees, AVL trees, ...) guarantee $O(\lg n)$ time to find an element.

Can we do better? Not in the worst case.

Let $x_1 < x_2 < \cdots < x_n$ be the elements to be stored in the tree.

Let $P = \{p_1, p_2, \ldots, p_n\}$ be the probabilities of searching for these elements.
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For a binary search tree $T$, let $d_T(x_i)$ denote the depth of element $x_i$ in $T$. 
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Balanced binary search trees (red-black trees, AVL trees, ...) guarantee $O(\log n)$ time to find an element.

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For a binary search tree $T$, let $d_T(x_i)$ denote the \textbf{depth} of element $x_i$ in $T$.

The cost of searching for element $x_i$ is in $O(d_T(x_i))$. 
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The expected cost of a random query is in $O(C_P(T))$, where

$$C_P(T) = \sum_{i=1}^{n} p_i d_T(x_i).$$
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The expected cost of a random query is in $O(C_P(T))$, where

$$C_P(T) = \sum_{i=1}^{n} p_i d_T(x_i).$$

An optimal binary search tree is a binary search tree $T$ that minimizes $C_P(T)$. 
Balancing Is Not Necessarily Optimal

Assume $n = 2^k - 1$ and $p_i = 2^{-i}$ for all $1 \leq i \leq n - 1$ and $p_n = 2^{-n+1}$. 
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Balanced tree:
Balancing Is Not Necessarily Optimal

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$x_1$ is at depth $\lg n$.

$\Rightarrow$ Expected cost $\geq \frac{\lg n}{2}$.
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Long path:
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Depth of $x_i$ is $i$. 
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**Balanced tree:**
- $x_1$ is at depth $\lg n$.

$\Rightarrow$ Expected cost $\geq \frac{\lg n}{2}$.

**Long path:**
- Depth of $x_i$ is $i$.

$\Rightarrow$ Expected cost

$$= \sum_{i=1}^{n} \frac{i}{2^i} + \frac{n}{2^n} < \sum_{i=1}^{\infty} \frac{i}{2^i} + \frac{n}{2^n}$$

$$= \frac{1/2}{(1 - 1/2)^2} + \frac{n}{2^n} = 2 + \frac{n}{2^n} < 3$$
Optimal Binary Search Trees: Problem Analysis

The structure of a binary search tree:
Assume we want to store elements $x_\ell, x_{\ell+1}, \ldots, x_r$. 
Optimal Binary Search Trees: Problem Analysis

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Assume we want to store elements $x_\ell, x_{\ell+1}, \ldots, x_r$. 

![Diagram of a binary search tree with elements $x_\ell, x_{\ell+1}, \ldots, x_m$, $x_{m+1}, x_{m+2}, \ldots, x_r$, and the optimal binary search tree $T_\ell$ and $T_r$.]
Optimal Binary Search Trees: Problem Analysis

The structure of a binary search tree:
Assume we want to store elements $x_\ell, x_{\ell+1}, \ldots, x_r$.

Let $p_{i,j} = \sum_{h=i}^{j} p_h$.

$C_P(T) = p_{\ell,r} + C_P(T_\ell) + C_P(T_r)$
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\[
C_P(T) = p_{\ell,r} + C_P(T_\ell) + C_P(T_r)
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\(\Rightarrow\) \(T_\ell\) and \(T_r\) are optimal search trees for \(x_\ell, x_{\ell+1}, \ldots, x_{m-1}\) and \(x_{m+1}, x_{m+2}, \ldots, x_r\), respectively.
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\(\Rightarrow\) \(T_\ell\) and \(T_r\) are optimal search trees for \(x_\ell, x_{\ell+1}, \ldots, x_{m-1}\) and \(x_{m+1}, x_{m+2}, \ldots, x_r\), respectively.

We need to figure out which element to store at the root!
Let $C(\ell, r)$ be the cost of an optimal binary search tree for $x_\ell, x_{\ell+1}, \ldots, x_r$.

We are interested in $C(1, n)$. 

Optimal Binary Search Trees: The Recurrence
Optimal Binary Search Trees: The Recurrence

Let $C(\ell, r)$ be the cost of an optimal binary search tree for $x_\ell, x_{\ell+1}, \ldots, x_r$.

We are interested in $C(1, n)$.

$$C(\ell, r) = \begin{cases} 0 & r < \ell \\ p_{\ell, r} + \min_{\ell \leq m \leq r} (C_{\ell, m-1} + C_{m+1, r}) & \text{otherwise} \end{cases}$$
Optimal Binary Search Trees: The Algorithm

OptimalBinarySearchTree(X, P)

1. for i = 1 to n
2. do P'[i, i] = P[i]
3. for j = i + 1 to n
4. do P'[i, j] = P'[i, j - 1] + P[j]
5. for i = 1 to n + 1
6. do C[i, i - 1] = 0
7. T[i, i - 1] = ∅
8. for ℓ = 0 to n - 1
9. do for i = 1 to n - ℓ
10. do C[i, i + ℓ] = ∞
11. for j = i to i + ℓ
12. do if C[i, i + ℓ] > C[i, j - 1] + C[j + 1, i + ℓ]
13. then C[i, i + ℓ] = C[i, j - 1] + C[j + 1, i + ℓ]
14. T[i, i + ℓ] = new node storing X[j]
15. T[i, i + ℓ].left = T[i, j - 1]
16. T[i, i + ℓ].right = T[j + 1, i + ℓ]
17. C[i, i + ℓ] = C[i, i + ℓ] + P'[i, i + ℓ]
18. return T[1, n]
Optimal Binary Search Trees: The Algorithm

OptimalBinarySearchTree(X, P)

1. for i = 1 to n
   2. do \( P'[i, i] = P[i] \)
   3. for j = i + 1 to n
      4. do \( P'[i, j] = P'[i, j - 1] + P[j] \)
   5. for i = 1 to n + 1
   6. do \( C[i, i - 1] = 0 \)
   7. \( T[i, i - 1] = \emptyset \)
   8. for \( \ell = 0 \) to \( n - 1 \)
      9. do for i = 1 to \( n - \ell \)
         10. do \( C[i, i + \ell] = \infty \)
         11. for j = i to \( i + \ell \)
             12. do if \( C[i, i + \ell] > C[i, j - 1] + C[j + 1, i + \ell] \)
                 then \( C[i, i + \ell] = C[i, j - 1] + C[j + 1, i + \ell] \)
                 \( T[i, i + \ell] = \) new node storing \( X[j] \)
                 \( T[i, i + \ell].left = T[i, j - 1] \)
                 \( T[i, i + \ell].right = T[j + 1, i + \ell] \)
         13. \( C[i, i + \ell] = C[i, i + \ell] + P'[i, i + \ell] \)
   14. return \( T[1, n] \)

Lemma: An optimal binary search tree for \( n \) elements can be computed in \( O(n^3) \) time.
Dijkstra's algorithm may fail in the presence of negative-weight edges:

Dijkstra

Correct
Single-Source Shortest Paths

Dijkstra's algorithm may fail in the presence of negative-weight edges:

We need an algorithm that can deal with negative-length edges.
Lemma: If $P = \langle u_0, v_1, \ldots, u_k \rangle$ is a shortest path from $u_0 = s$ to $u_k = v$, then $P' = (u_0, u_1, \ldots, u_{k-1})$ is a shortest path from $u_0$ to $u_{k-1}$. 
Lemma: If $P = \langle u_0, v_1, \ldots, u_k \rangle$ is a shortest path from $u_0 = s$ to $u_k = v$, then $P' = \langle u_0, u_1, \ldots, u_{k-1} \rangle$ is a shortest path from $u_0$ to $u_{k-1}$. 

Image: Diagram showing a shortest path from $u_0$ to $u_{k-1}$. 

Shortest path from $u_0$ to $u_{k-1}$
Lemma: If $P = \langle u_0, v_1, \ldots, u_k \rangle$ is a shortest path from $u_0 = s$ to $u_k = v$, then $P' = (u_0, u_1, \ldots, u_{k-1})$ is a shortest path from $u_0$ to $u_{k-1}$.

Observation: $P'$ has one less edge than $P$. 
Single-Source Shortest Paths: The Recurrence

Let $d_i(s, v)$ be the length of the shortest path $P_i(s, v)$ from $s$ to $v$ that has at most $i$ edges.
Single-Source Shortest Paths: The Recurrence

Let $d_i(s,v)$ be the length of the shortest path $P_i(s,v)$ from $s$ to $v$ that has at most $i$ edges.

$d_i(s,v) = \infty$ if there is no path with at most $i$ edges from $s$ to $v$. 
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$d(s, v) = d_{n-1}(s, v)$
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$d(s, v) = d_{n-1}(s, v)$

**Recurrence:**

If $i = 0$, then there exists a path from $s$ to $v$ with at most $i$ edges only if $v = s$:

$$d_0(s, v) = \begin{cases} 
0 & v = s \\
\infty & \text{otherwise}
\end{cases}$$
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Let $d_i(s, v)$ be the length of the shortest path $P_i(s, v)$ from $s$ to $v$ that has at most $i$ edges.

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- $P_i(s, v)$ has at most $i - 1$ edges or
Single-Source Shortest Paths: The Recurrence

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d_i(s,v) = \infty \text{ if there is no path with at most } i \text{ edges from } s \text{ to } v.

d(s,v) = d_{n-1}(s,v)

Recurrence:

If $i = 0$, then there exists a path from $s$ to $v$ with at most $i$ edges only if $v = s$:

\[ d_0(s,v) = \begin{cases} 0 & v = s \\ \infty & \text{otherwise} \end{cases} \]

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Let \( d_i(s, v) \) be the length of the shortest path \( P_i(s, v) \) from \( s \) to \( v \) that has at most \( i \) edges.

\[ d_i(s, v) = \infty \] if there is no path with at most \( i \) edges from \( s \) to \( v \).

\[ d(s, v) = d_{n-1}(s, v) \]

Recurrence:
If \( i = 0 \), then there exists a path from \( s \) to \( v \) with at most \( i \) edges only if \( v = s \):

\[ d_0(s, v) = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{otherwise} \end{cases} \]

If \( i > 0 \), then

- \( P_i(s, v) \) has at most \( i - 1 \) edges or
- \( P_i(s, v) \) has \( i \) edges.

\[ P_i(s, v) = P_{i-1}(s, v) \rightarrow P_i(s, v) = P_{i-1}(s, v) \]

- \( P_i(s, v) \) has \( i \) edges.
Single-Source Shortest Paths: The Recurrence

Let \( d_i(s, v) \) be the length of the shortest path \( P_i(s, v) \) from \( s \) to \( v \) that has at most \( i \) edges.

\[ d_i(s, v) = \infty \] if there is no path with at most \( i \) edges from \( s \) to \( v \).

\[ d(s, v) = d_{n-1}(s, v) \]

**Recurrence:**

If \( i = 0 \), then there exists a path from \( s \) to \( v \) with at most \( i \) edges only if \( v = s \):

\[ d_0(s, v) = \begin{cases} 
0 & \text{if } v = s \\
\infty & \text{otherwise}
\end{cases} \]

If \( i > 0 \), then

- \( P_i(s, v) \) has at most \( i - 1 \) edges or

\[ \Rightarrow P_i(s, v) = P_{i-1}(s, v) \]

- \( P_i(s, v) \) has \( i \) edges.

\[ \Rightarrow P_i(s, v) = P_{i-1}(s, u) \circ \langle (u, v) \rangle \text{ for some in-neighbour } u \text{ of } v. \]
Single-Source Shortest Paths: The Recurrence

Let $d_i(s, v)$ be the length of the shortest path $P_i(s, v)$ from $s$ to $v$ that has at most $i$ edges.

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$d(s, v) = d_{n-1}(s, v)$

**Recurrence:**

If $i = 0$, then there exists a path from $s$ to $v$ with at most $i$ edges only if $v = s$:

$$d_0(s, v) = \begin{cases} 0 & v = s \\ \infty & \text{otherwise} \end{cases}$$

If $i > 0$, then

$$d_i(s, v) = \min(d_{i-1}(s, v), \min\{d_{i-1}(s, u) + w(u, v) \mid (u, v) \in E\})$$
BellmanFord(G, s)

1. for every vertex v ∈ G
2. do d[v] = ∞
3. P[v] = ∅
4. d[s] = 0
5. P[s] = [s]
6. for i = 1 to n – 1
7. do for every vertex v ∈ G
8. do for every in-edge e of v
9. do if d[e.tail] + e.weight < d[v]
10. then d[v] = d[e.tail] + e.weight
11. P[v] = [v] ++ P[e.tail]
12. return (d, P)
Single-Source Shortest Paths: The Bellman-Ford Algorithm

BellmanFord(G, s)

1. for every vertex v ∈ G
2. do d[v] = ∞
3. P[v] = ∅
4. d[s] = 0
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6. for i = 1 to n – 1
7. do for every vertex v ∈ G
8. do for every in-edge e of v
9. do if d[e.tail] + e.weight < d[v]
10. then d[v] = d[e.tail] + e.weight
11. P[v] = [v] ++ P[e.tail]
12. return (d, P)

Lemma: The single-source shortest paths problem can be solved in O(nm) time on any weighted graph, provided there are no negative cycles.
All-Pairs Shortest Paths

Goal: Compute the distance $d(u, v)$ (and the corresponding shortest path), for every pair of vertices $u, v \in G$. 
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Improved algorithms:
- Floyd-Warshall: \( O(n^3) \)
- Johnson: \( O(n^2 \log n + nm) \) (really cool!)
  - Run Bellman-Ford from an arbitrary vertex \( s \) in \( O(nm) \) time.
  - Change edge weights so they are all non-negative but shortest paths don't change!
  - Run Dijkstra \( n \) times.
All-Pairs Shortest Paths: The Recurrence

Number the vertices 1, 2, . . . , n.

Let $d_i(u, v)$ be the length of the shortest path $P_i(u, v)$ that visits only vertices in $\{1, 2, \ldots, i\} \cup \{u, v\}$.
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\[
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\]

If \( i = 0 \), \( P_0(u, v) \) cannot visit any vertices other than \( u \) and \( v \):

\[
d_0(u, v) = \begin{cases} 
  w(u, v) & (u, v) \in E \\
  \infty & \text{otherwise}
\end{cases}
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If $i \notin P_i(u, v)$, then $P_i(u, v) = P_{i-1}(u, v)$. 

$1, 2, \ldots , i - 1$
All-Pairs Shortest Paths: The Recurrence

Number the vertices 1, 2, . . ., n.

Let $d_i(u, v)$ be the length of the shortest path $P_i(u, v)$ that visits only vertices in \{1, 2, . . ., i\} ∪ \{u, v\}.

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If $i = 0$, $P_0(u, v)$ cannot visit any vertices other than u and v:

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$$d_i(u, v) = \min(d_{i-1}(u, v), d_{i-1}(u, i) + d_{i-1}(i, v))$$
All-Pairs Shortest Paths: The Floyd-Warshall Algorithm

FloydWarshall(G)

1. for every pair of vertices $u, v \in G$
2. do $d[u, v] = \infty$
3. do $p[u, v] = \text{Nothing}$
4. for every vertex $v \in G$
5. do $d[v, v] = 0$
6. do $p[v, v] = v$
7. for every edge $e \in G$
8. do $d[e\.tail, e\.head] = e\.weight$
9. do $p[e\.tail, e\.head] = e\.tail$
10. for $i = 1 \text{ to } n$
11. do for every pair of vertices $u, v \in G$ such that $i \notin \{u, v\}$
12. do if $d[u, v] > d[u, i] + d[i, v]$
13. then $d[u, v] = d[u, i] + d[i, v]$
15. return $(d, p)$
FloydWarshall(G)
  1 for every pair of vertices \(u, v \in G\)
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  3 \(p[u, v] = \text{Nothing}\)
  4 for every vertex \(v \in G\)
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  6 \(p[v, v] = v\)
  7 for every edge \(e \in G\)
  8 do \(d[e.\text{tail}, e.\text{head}] = e.\text{weight}\)
  9 \(p[e.\text{tail}, e.\text{head}] = e.\text{tail}\)
  10 for \(i = 1\) to \(n\)
  11 do for every pair of vertices \(u, v \in G\) such that \(i \notin \{u, v\}\)
  12 do if \(d[u, v] > d[u, i] + d[i, v]\)
  13 then \(d[u, v] = d[u, i] + d[i, v]\)
  14 \(p[u, v] = p[i, v]\)
  15 return \((d, p)\)

ReportPath\((p, u, v)\)
  1 if \(p[u, v] = \text{Nothing}\)
  2 then return \(\text{Nothing}\)
  3 \(P = [v]\)
  4 while \(v \neq u\)
  5 do \(v = p[u, v]\)
  6 \(P.\text{prepend}(v)\)
  7 return \(P\)
**All-Pairs Shortest Paths: The Floyd-Warshall Algorithm**

**FloydWarshall**(G)

1. for every pair of vertices \( u, v \in G \)
2. \( d[u, v] = \infty \)
3. \( p[u, v] = \text{Nothing} \)
4. for every vertex \( v \in G \)
5. \( d[v, v] = 0 \)
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10. for \( i = 1 \) to \( n \)
11. do for every pair of vertices \( u, v \in G \) such that \( i \notin \{u, v\} \)
12. \( \text{do if } d[u, v] > d[u, i] + d[i, v] \)
13. \( \text{then } d[u, v] = d[u, i] + d[i, v] \)
15. return \( (d, p) \)

**ReportPath**(\( p, u, v \))

1. if \( p[u, v] = \text{Nothing} \) then return \( \text{Nothing} \)
2. \( P = [v] \)
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4. do \( v = p[u, v] \)
5. \( P.\text{prepend}(v) \)
6. return \( P \)

**Lemma:** The all-pairs shortest paths problem can be solved in \( \mathcal{O}(n^3) \) time, provided there are no negative cycles.
Summary

Both greedy algorithms and dynamic programming are applicable when the problem has optimal substructure:

The optimal solution for a given input instance contains within it optimal solutions to smaller input instances.
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Both greedy algorithms and dynamic programming are applicable when the problem has **optimal substructure**:

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**Greedy algorithms** are applicable when an optimal solution can be obtained by making a **locally optimal choice** and then solving the resulting subproblem.

**Dynamic programming** exhaustively explores all possible choices and chooses the one that gives the best solution.
Summary

Both greedy algorithms and dynamic programming are applicable when the problem has optimal substructure:

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Greedy algorithms are applicable when an optimal solution can be obtained by making a locally optimal choice and then solving the resulting subproblem.

Dynamic programming exhaustively explores all possible choices and chooses the one that gives the best solution.

Dynamic programming yields a faster solution than the naïve recursive algorithm when there are lots of overlapping subproblems.
The design of a dynamic programming algorithm proceeds in two phases:

1. Analyze the structure of an optimal solution to develop a recurrence for the cost of an optimal solution.

2. Develop an algorithm that uses the recurrence to compute an optimal solution
   - Recursively using memoization or
   - Iteratively by populating a table with the costs of the solutions to all possible subproblems.

Both types of algorithms compute optimal solutions bottom-up.