Divide and Conquer

Textbook Reading
Chapters 4, 7 & 33.4
Overview

Design principle
- Divide and conquer

Proof technique
- Induction, induction, induction

Analysis technique
- Recurrence relations

Problems
- Sorting (Merge Sort and Quick Sort)
- Selection
- Matrix multiplication
- Finding the two closest points
Merge Sort

\textbf{MergeSort}(A, \ell, r)

1. if \( r \leq \ell \) then return
2. \( m = \lfloor (\ell + r)/2 \rfloor \)
3. \textbf{MergeSort}(A, \ell, m)
4. \textbf{MergeSort}(A, m + 1, r)
5. \textbf{Merge}(A, \ell, m, r)
Merging Two Sorted Lists

Merge(A, \( \ell \), m, r)

1. \( n_1 = m - \ell + 1 \)
2. \( n_2 = r - m \)
3. for i = 1 to \( n_1 \)
   do \( L[i] = A[\ell + i - 1] \)
4. for i = 1 to \( n_2 \)
   do \( R[i] = A[m + i] \)
5. \( L[n_1 + 1] = R[n_2 + 1] = +\infty \)
6. \( i = j = 1 \)
7. for k = \( \ell \) to \( r \)
   do if \( L[i] \leq R[j] \)
      then \( A[k] = L[i] \)
      i = i + 1
   else \( A[k] = R[j] \)
5. j = j + 1
Divide and Conquer

Three steps:

- **Divide** the input into smaller parts.
- **Recursively** solve the same problem on these smaller parts.
- **Combine** the solutions computed by the recursive calls to obtain the final solution.
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This can be viewed as a reduction technique, reducing the original problem to simpler problems to be solved in the divide and/or combine steps.
Divide and Conquer

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- **Divide** the input into smaller parts.
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- **Combine** the solutions computed by the recursive calls to obtain the final solution.

This can be viewed as a reduction technique, reducing the original problem to simpler problems to be solved in the divide and/or combine steps.

**Example:** Once we unfold the recursion of Merge Sort, we're left with nothing but Merge steps. Thus, we reduce sorting to the simpler problem of merging sorted lists.
Loop Invariants

... are a technique for proving the correctness of an iterative algorithm.

The invariant states conditions that should hold before and after each iteration.

**Correctness proof using a loop invariant:**

**Initialization:** Prove the invariant holds before the first iteration.

**Maintenance:** Prove each iteration maintains the invariant.

**Termination:** Prove that the correctness of the invariant after the last iteration implies correctness of the algorithm.
Correctness of Merge

\textbf{Merge}(A, \ell, m, r)

1. \(n_1 = m - \ell + 1\)
2. \(n_2 = r - m\)
3. for \(i = 1\) to \(n_1\)
   - \(\text{do } L[i] = A[l + i - 1]\)
4. for \(i = 1\) to \(n_2\)
   - \(\text{do } R[i] = A[m + i]\)
5. \(L[n_1 + 1] = R[n_2 + 1] = +\infty\)
6. \(i = j = 1\)
7. for \(k = \ell\) to \(r\)
   - \(\text{do if } L[i] \leq R[j]\)
     - \(\text{then } A[k] = L[i]\)
     - \(i = i + 1\)
   - \(\text{else } A[k] = R[j]\)
     - \(j = j + 1\)

\textbf{Loop invariant:}

- \(A[\ell \ldots k - 1] \cup L[i \ldots n_1] \cup R[j \ldots n_2]\) is the set of elements originally in \(A[\ell \ldots r]\).
- \(A[\ell \ldots k - 1], L[i \ldots n_1],\) and \(R[j \ldots n_2]\) are sorted.
- \(x \leq y\) for all \(x \in A[\ell \ldots k - 1]\) and \(y \in L[i \ldots n_1] \cup R[j \ldots n_2]\).
Correctness of Merge

Merge(A, ℓ, m, r)

1. \( n_1 = m - ℓ + 1 \)
2. \( n_2 = r - m \)
3. for \( i = 1 \) to \( n_1 \)
   - do \( L[i] = A[ℓ + i - 1] \)
4. for \( i = 1 \) to \( n_2 \)
   - do \( R[i] = A[m + i] \)
5. \( L[n_1 + 1] = R[n_2 + 1] = +∞ \)
6. \( i = j = 1 \)
7. for \( k = ℓ \) to \( r \)
   - do if \( L[i] \leq R[j] \)
     - then \( A[k] = L[i] \)
     - \( i = i + 1 \)
   - else \( A[k] = R[j] \)
   - \( j = j + 1 \)

Initialization:
- \( A[ℓ \ldots m] \) is copied to \( L[1 \ldots n_1] \).
- \( A[m + 1 \ldots r] \) is copied to \( R[1 \ldots n_2] \).
- \( i = 1, j = 1, k = 1 \).
Correctness of Merge

Merge(A, ℓ, m, r)

```
1 \quad n_1 = m - \ell + 1
2 \quad n_2 = r - m
3 \quad \text{for } i = 1 \text{ to } n_1
4 \quad \quad \text{do } L[i] = A[\ell + i - 1]
5 \quad \text{for } i = 1 \text{ to } n_2
6 \quad \quad \text{do } R[i] = A[m + i]
7 \quad L[n_1 + 1] = R[n_2 + 1] = +\infty
8 \quad i = j = 1
9 \quad \text{for } k = \ell \text{ to } r
10 \quad \quad \text{do if } L[i] \leq R[j]
11 \quad \quad \quad \text{then } A[k] = L[i]
12 \quad \quad \quad i = i + 1
13 \quad \quad \text{else } A[k] = R[j]
14 \quad \quad j = j + 1
```

Termination:

- k = r + 1

⇒ A[ℓ .. r] contains all items it contained initially, in sorted order.
Correctness of Merge

Merge($A, \ell, m, r$)

1. $n_1 = m - \ell + 1$
2. $n_2 = r - m$
3. for $i = 1$ to $n_1$
   4. do $L[i] = A[l + i - 1]$
4. for $i = 1$ to $n_2$
5. do $R[i] = A[m + i]$
6. $L[n_1 + 1] = R[n_2 + 1] = +\infty$
7. $i = j = 1$
8. for $k = \ell$ to $r$
9. do if $L[i] \leq R[j]$
10. then $A[k] = L[i]$
11. $i = i + 1$
12. else $A[k] = R[j]$
13. $j = j + 1$

Maintenance:

- $A[k'] \leq L[i]$ for all $k' < k$
- $L[i] \leq L[i']$ for all $i' > i$
- $L[i] \leq R[j] \leq R[j']$ for all $j' > j$
Correctness of Merge

Merge(A, ℓ, m, r)

1. \[ n_1 = m - ℓ + 1 \]
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5. for \( i = 1 \) to \( n_2 \)
   
   6. do \( R[i] = A[m + i] \)
7. \( L[n_1 + 1] = R[n_2 + 1] = \infty \)
8. \( i = j = 1 \)
9. for \( k = ℓ \) to \( r \)
   
   10. do if \( L[i] \leq R[j] \)
     
     then \( A[k] = L[i] \)
     
     11. \( i = i + 1 \)
    
    else \( A[k] = R[j] \)
    
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Maintenance:

- \( A[k'] \leq L[i] \) for all \( k' < k \)
- \( L[i] \leq L[i'] \) for all \( i' > i \)
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- \( A[k'] \leq L[i] \) for all \( k' < k \)
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- \( L[i] \leq R[j] \leq R[j'] \) for all \( j' > j \)
Correctness of Merge Sort

**Lemma:** Merge Sort correctly sorts any input array.

```
MergeSort(A, ℓ, r)
1  if r ≤ ℓ  
2    then return 
3  m = ⌊(ℓ + r)/2⌋ 
4  MergeSort(A, ℓ, m) 
5  MergeSort(A, m + 1, r) 
6  Merge(A, ℓ, m, r)
```
Correctness of Merge Sort

Lemma: Merge Sort correctly sorts any input array.

Proof by induction on n.
Correctness of Merge Sort

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Proof by induction on \( n \).

Base case: \( (n = 1) \)
- An empty or one-element array is already sorted.
- Merge sort does nothing.

MergeSort(A, \( \ell \), \( r \))

1. if \( r \leq \ell \)
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3. \( m = \lfloor (\ell + r)/2 \rfloor \)
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6. Merge(A, \( \ell \), \( m \), \( r \))
Correctness of Merge Sort

**Lemma:** Merge Sort correctly sorts any input array.

Proof by induction on $n$.

**Base case:** ($n = 1$)
- An empty or one-element array is already sorted.
- Merge sort does nothing.

**Inductive step:** ($n > 1$)
- The left and right halves have size less than $n$ each.
- By the inductive hypothesis, the recursive calls sort them correctly.
- Merge correctly merges the two sorted sequences.

```plaintext
MergeSort(A, ℓ, r)
1  if r ≤ ℓ
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3  m = ⌊(ℓ + r)/2⌋
4  MergeSort(A, ℓ, m)
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6  Merge(A, ℓ, m, r)
```
Correctness of Divide and Conquer Algorithms

Divide and conquer algorithms are the algorithmic incarnation of induction:

**Base case:** Solve trivial instances directly, without recursing.

**Inductive step:** Reduce the solution of a given instance to the solution of smaller instances, by recursing.
Correctness of Divide and Conquer Algorithms

Divide and conquer algorithms are the algorithmic incarnation of induction:

**Base case:** Solve trivial instances directly, without recursing.

**Inductive step:** Reduce the solution of a given instance to the solution of smaller instances, by recursing.

⇒ Induction is the natural proof method for divide and conquer algorithms.
A recurrence relation defines the value $f(n)$ of a function $f(\cdot)$ for argument $n$ in terms of the values of $f(\cdot)$ for arguments smaller than $n$. 
Recurrence Relations

A recurrence relation defines the value $f(n)$ of a function $f(\cdot)$ for argument $n$ in terms of the values of $f(\cdot)$ for arguments smaller than $n$.

Examples:

Fibonacci numbers:

$$F_n = \begin{cases} 
1 & \text{if } n = 0 \text{ or } n = 1 \\
F_{n-1} + F_{n-2} & \text{otherwise}
\end{cases}$$
Recurrence Relations

A recurrence relation defines the value \( f(n) \) of a function \( f(\cdot) \) for argument \( n \) in terms of the values of \( f(\cdot) \) for arguments smaller than \( n \).

Examples:

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\]

**Binomial coefficients**

\[
B(n, k) = \begin{cases} 
1 & \text{if } k = 1 \text{ or } k = n \\
B(n - 1, k - 1) + B(n - 1, k) & \text{otherwise}
\end{cases}
\]
A Recurrence for Merge Sort

MergeSort(A, ℓ, r)

1 if \( r \leq ℓ \)
2 then return
3 \( \text{m} = \lfloor (\ell + r)/2 \rfloor \)
4 MergeSort(A, ℓ, m)
5 MergeSort(A, m + 1, r)
6 Merge(A, ℓ, m, r)

Recurrence:

\[ T(n) = \]
Analysis:
If \( n = 0 \) or \( n = 1 \), we spend constant time to figure out that there is nothing to do and then exit.

Recurrence:
\[
T(n) = \begin{cases} 
\Theta(1) & n \leq 1 
\end{cases}
\]
A Recurrence for Merge Sort

Analysis:

If \( n = 0 \) or \( n = 1 \), we spend constant time to figure out that there is nothing to do and then exit.

If \( n > 1 \), we

Recurrence:

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T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
& \text{if } n > 1
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MergeSort(A, \( \ell \), \( r \))

1 if \( r \leq \ell \)
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A Recurrence for Merge Sort

Analysis:

If \( n = 0 \) or \( n = 1 \), we spend constant time to figure out that there is nothing to do and then exit.

If \( n > 1 \), we

- Spend constant time to determine the middle index \( m \),

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If \( n = 0 \) or \( n = 1 \), we spend constant time to figure out that there is nothing to do and then exit.

If \( n > 1 \), we
- Spend constant time to determine the middle index \( m \),
- Make one recursive call on the left half, which has size \( \lceil n/2 \rceil \),

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T(n) = \begin{cases} 
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If \( n = 0 \) or \( n = 1 \), we spend constant time to figure out that there is nothing to do and then exit.

If \( n > 1 \), we

- Spend constant time to determine the middle index \( m \),
- Make one recursive call on the left half, which has size \( \lceil n/2 \rceil \),
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T(n) = \begin{cases} 
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If $n > 1$, we

- Spend constant time to determine the middle index $m$,
- Make one recursive call on the left half, which has size $\lceil n/2 \rceil$,
- Make one recursive call on the right half, which has size $\lfloor n/2 \rfloor$, and
- Spend linear time to merge the two resulting sorted sequences.

Recurrence:

\[
T(n) = \begin{cases} 
\Theta(1) & n \leq 1 \\
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$$T(n) = \begin{cases} \Theta(1) & n \leq 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & n > 1 \end{cases}$$
A Recurrence for Binary Search

BinarySearch(A, ℓ, r, x)

1  if r < ℓ  
2     then return “no”  
3  m = ⌊(ℓ + r)/2⌋  
4  if x = A[m]  
5     then return “yes”  
6  if x < A[m]  
7     then return BinarySearch(A, ℓ, m – 1, x)  
8  else return BinarySearch(A, m + 1, r, x)  

Recurrence:

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If \( n > 0 \), we
- Spend constant time to find the middle element and compare it to \( x \).

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- Spend constant time to find the middle element and compare it to \( x \).
- Make one recursive call on one of the two halves, which has size at most \( \lfloor n/2 \rfloor \),

Recurrence:

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Simplified Recurrence Notation

The recurrences we use to analyze algorithms all have a base case of the form

\[ T(n) \leq c \forall n \leq n_0. \]

The exact choices of \( c \) and \( n_0 \) affect the value of \( T(n) \) for any \( n \) by only a constant factor.
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Floors and ceilings usually affect the value of $T(n)$ by only a constant factor.
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So we are lazy and write

- **Merge Sort**: \( T(n) = 2T(n/2) + \Theta(n) \)
- **Binary search**: \( T(n) = T(n/2) + \Theta(1) \)
“Solving” Recurrences

Given two algorithms A and B with running times

\[ T_A(n) = 2T(n/2) + \Theta(n) \]
\[ T_B(n) = 3T(n/2) + \Theta(lg n), \]

which one is faster?
“Solving” Recurrences

Given two algorithms $A$ and $B$ with running times

$$T_A(n) = 2T(n/2) + \Theta(n)$$
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A recurrence for $T(n)$ precisely defines $T(n)$, but it is hard for us to look at the function and say which one grows faster.
“Solving” Recurrences

Given two algorithms A and B with running times

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A recurrence for \( T(n) \) precisely defines \( T(n) \), but it is hard for us to look at the function and say which one grows faster.

\[ \Rightarrow \text{ We want a closed-form expression for } T(n), \text{ that is, one of the form } T(n) \in \Theta(n), \]
\[ T(n) \in \Theta(n^2), \ldots, \text{ one that does not depend on } T(n') \text{ for any } n' < n. \]
Methods for Solving Recurrences

Substitution:

- Guess the solution.  
  (Intuition, experience, black magic, recursion trees, trial and error)
- Use induction to prove that the guess is correct.
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**Recursion trees:**
- Draw a tree that visualizes how the recurrence unfolds.
- Sum up the costs of the nodes in the tree to
  - Obtain an exact answer if the analysis is done rigorously enough or
  - Obtain a guess that can then be verified rigorously using substitution.
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**Master Theorem:**
- Cook book recipe for solving common recurrences.
- Immediately tells us the solution after we verify some simple conditions to determine which case of the theorem applies.
Lemma: The running time of Merge Sort is in $O(n \lg n)$. 
Substitution: Merge Sort

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Recurrence:

$$T(n) = 2T(n/2) + O(n)$$
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Recurrence:

\[
T(n) = 2T(n/2) + O(n), \quad \text{that is,} \\
T(n) \leq 2T(n/2) + an, \quad \text{for some } a > 0 \text{ and all } n \geq n_0.
\]
Substitution: Merge Sort

Lemma: The running time of Merge Sort is in $O(n \lg n)$.

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Guess:

$$T(n) \leq cn \lg n, \text{ for some } c > 0 \text{ and all } n \geq n_1.$$
Substitution: Merge Sort

Lemma: The running time of Merge Sort is in $\mathcal{O}(n \lg n)$.

Recurrence:

\[ T(n) = 2T(n/2) + \mathcal{O}(n), \text{ that is,} \]
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Guess:

\[ T(n) \leq cn \lg n, \text{ for some } c > 0 \text{ and all } n \geq n_1. \]

Base case:

For $2 \leq n < 4$, $T(n) \leq c' \leq c'n \leq c'n \lg n$, for some $c' > 0$. 
**Substitution: Merge Sort**

**Lemma:** The running time of Merge Sort is in $O(n \log n)$.

**Recurrence:**

$$T(n) = 2T(n/2) + O(n),$$

that is,

$$T(n) \leq 2T(n/2) + an,$$

for some $a > 0$ and all $n \geq n_0$.

**Guess:**

$$T(n) \leq cn \log n,$$

for some $c > 0$ and all $n \geq n_1$.

**Base case:**

For $2 \leq n < 4$, $T(n) \leq c' \leq c' n \leq c' n \log n$, for some $c' > 0$.

$\Rightarrow$ $T(n) \leq cn \log n$ as long as $c \geq c'$. 
Substitution: Merge Sort

Inductive step: \((n \geq 4)\)
Substitution: Merge Sort

Inductive step: \((n \geq 4)\)

\[ T(n) \leq 2T(n/2) + an \]
Substitution: Merge Sort

Inductive step: \( n \geq 4 \)

\[
T(n) \leq 2T(n/2) + an \\
\leq 2 \cdot \left( \frac{cn}{2} \log \frac{n}{2} \right) + an
\]
Substitution: Merge Sort

**Inductive step:** \((n \geq 4)\)

\[
T(n) \leq 2T(n/2) + an
\]

\[
\leq 2 \cdot \left( \frac{cn}{2} \cdot \lg \frac{n}{2} \right) + an
\]

\[
= cn(\lg n - 1) + an
\]
Substitution: Merge Sort

Inductive step: \((n \geq 4)\)

\[
T(n) \leq 2T(n/2) + an
\leq 2 \cdot \left( \frac{cn}{2} \lg \frac{n}{2} \right) + an
= cn(\lg n - 1) + an
= cn \lg n + (a - c)n
\]
Substitution: Merge Sort

Inductive step: \((n \geq 4)\)

\[
T(n) \leq 2T(n/2) + an \\
\leq 2 \cdot \left( \frac{cn}{2} \log \frac{n}{2} \right) + an \\
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= cn \log n + (a - c)n \\
\leq cn \log n, \text{ for all } c \geq a.
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Substitution: Merge Sort

Inductive step: \((n \geq 4)\)

\[
T(n) \leq 2T(n/2) + an
\]

\[
\leq 2 \cdot \left( \frac{cn}{2} \log \frac{n}{2} \right) + an
\]

\[
= cn(\log n - 1) + an
\]

\[
= cn \log n + (a - c)n
\]

\[
\leq cn \log n, \text{ for all } c \geq a.
\]

Notes:
- We only proved the upper bound. The lower bound, \(T(n) \in \Omega(n \log n)\) can be proven analogously.
Substitution: Merge Sort

Inductive step: \( (n \geq 4) \)

\[
T(n) \leq 2T(n/2) + an
\leq 2 \cdot \left( \frac{cn}{2} \lg \frac{n}{2} \right) + an
= cn(\lg n - 1) + an
= cn \lg n + (a - c)n
\leq cn \lg n, \text{ for all } c \geq a.
\]

Notes:

- We only proved the upper bound. The lower bound, \( T(n) \in \Omega(n \lg n) \) can be proven analogously.

- Since the base case is valid only for \( n \geq 2 \) and we use the inductive hypothesis for \( n/2 \) in the inductive step, the inductive step is valid only for \( n \geq 4 \). Hence, a base case for \( 2 \leq n < 4 \).
**Substitution: Binary Search**

**Lemma:** The running time of binary search is in $O(\lg n)$. 
**Lemma:** The running time of binary search is in $O(\lg n)$.

**Recurrence:**

\[
T(n) = T(n/2) + O(1), \text{ that is,} \\
T(n) \leq T(n/2) + a, \text{ for some } a > 0 \text{ and all } n \geq n_0.
\]
Substitution: Binary Search

Lemma: The running time of binary search is in $O(\lg n)$.

Recurrence:

$$T(n) = T(n/2) + O(1), \text{ that is,}$$

$$T(n) \leq T(n/2) + a, \text{ for some } a > 0 \text{ and all } n \geq n_0.$$

Guess:

$$T(n) \leq c \lg n, \text{ for some } c > 0 \text{ and all } n \geq n_1.$$
Lemma: The running time of binary search is in $O(\lg n)$.

Recurrence:

$$T(n) = T(n/2) + O(1), \text{ that is,}$$
$$T(n) \leq T(n/2) + a, \text{ for some } a > 0 \text{ and all } n \geq n_0.$$  

Guess:

$$T(n) \leq c \lg n, \text{ for some } c > 0 \text{ and all } n \geq n_1.$$  

Base case:

For $2 \leq n < 4$, $T(n) \leq c' \leq c' \lg n$, for some $c' > 0$. 

**Substitution: Binary Search**

**Lemma:** The running time of binary search is in $O(\lg n)$.

**Recurrence:**

$$T(n) = T(n/2) + O(1),$$

that is,

$$T(n) \leq T(n/2) + a,$$

for some $a > 0$ and all $n \geq n_0$.

**Guess:**

$$T(n) \leq c \lg n,$$

for some $c > 0$ and all $n \geq n_1$.

**Base case:**

For $2 \leq n < 4$, $T(n) \leq c' \leq c' \lg n$, for some $c' > 0$.

$\Rightarrow$ $T(n) \leq c \lg n$ as long as $c \geq c'$. 
Substitution: Binary Search

Inductive step: \((n \geq 4)\)
Substitution: Binary Search

Inductive step: \((n \geq 4)\)

\[ T(n) \leq T(n/2) + a \]
Inductive step: \((n \geq 4)\)

\[
T(n) \leq T(n/2) + a \\
\leq c \lg \frac{n}{2} + a
\]
**Substitution: Binary Search**

**Inductive step:** \( n \geq 4 \)

\[
T(n) \leq T(n/2) + a \\
\leq c \lg \frac{n}{2} + a \\
= c(\lg n - 1) + a
\]
Substitution: Binary Search

Inductive step: \((n \geq 4)\)

\[
T(n) \leq T(n/2) + a \\
\leq c \lg \frac{n}{2} + a \\
= c(lg n - 1) + a \\
= c \lg n + (a - c)
\]
Substitution: Binary Search

Inductive step: \( n \geq 4 \)

\[
T(n) \leq T(n/2) + a \\
\leq c \log \frac{n}{2} + a \\
= c(\log n - 1) + a \\
= c \log n + (a - c) \\
\leq c \log n, \text{ for all } c \geq a.
\]
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

$$T(n) = 2T(n/2) + O(n) \quad \text{to} \quad T(n) \leq 2T(n/2) + an$$

and the claim

$$T(n) \in O(n \lg n) \quad \text{to} \quad T(n) \leq cn \lg n?$$
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

\[ T(n) = 2T(n/2) + O(n) \quad \text{to} \quad T(n) \leq 2T(n/2) + an \]

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\[ T(n) \in O(n \log n) \quad \text{to} \quad T(n) \leq cn \log n? \]

If we're not careful, we may “prove” nonsensical results:

Recurrence: \( T(n) = T(n - 1) + n \)

Claim: \( T(n) \in O(n) \)

Note that \( T(n) \in \Theta(n^2)! \)
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

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**Base case:** \( (n = 1) \)
\[ T(n) = T(1) = 1 \in O(n) \]
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

\[ T(n) = 2T(n/2) + O(n) \] to \[ T(n) \leq 2T(n/2) + an \]

and the claim

\[ T(n) \in O(n \log n) \] to \[ T(n) \leq cn \log n? \]

If we're not careful, we may "prove" nonsensical results:

**Recurrence:** \[ T(n) = T(n - 1) + n \]

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\[ T(n) = T(1) = 1 \in O(n) \]

**Inductive step:** \( n > 1 \)

\[ T(n) = T(n - 1) + n = O(n - 1) + n = O(n) + n = O(n) \]
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

\[ T(n) = 2T(n/2) + O(n) \]

... to

\[ T(n) \leq 2T(n/2) + an \]

and the claim

\[ T(n) \in O(n \lg n) \]

... to

\[ T(n) \leq cn \lg n? \]

If we're not careful, we may “prove” nonsensical results:

**Recurrence:** \[ T(n) = T(n - 1) + n \]

**Claim:** \[ T(n) \in O(n) \quad T(n) \leq cn \]

**Base case:** \( n = 1 \)
\[ T(1) = T(1) = 1 \in O(n) \]

**Inductive step:** \( n > 1 \)
\[ T(n) = T(n - 1) + n = O(n - 1) + n = \]
\[ O(n) + n = O(n) \]
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

\[ T(n) = 2T(n/2) + O(n) \]

to

\[ T(n) \leq 2T(n/2) + an \]

and the claim

\[ T(n) \in O(n \log n) \]

to

\[ T(n) \leq cn \log n? \]

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**Inductive step:** \( n > 1 \)

\( T(n) = T(n - 1) + n = O(n - 1) + n = O(n) + n = O(n) \)
Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

\[ T(n) = 2T(n/2) + O(n) \quad \text{to} \quad T(n) \leq 2T(n/2) + an \]

and the claim

\[ T(n) \in O(n \lg n) \quad \text{to} \quad T(n) \leq cn \lg n? \]

If we're not careful, we may “prove” nonsensical results:

**Recurrence:** \( T(n) = T(n - 1) + n \)

**Claim:** \( T(n) \in O(n) \quad T(n) \leq cn \)

**Base case:** \( (n = 1) \)

\[ T(n) = T(1) = 1 \in O(n) \quad T(n) = T(1) = c \leq cn \]

**Inductive step:** \( (n > 1) \)

\[ T(n) = T(n - 1) + n = O(n - 1) + n = O(n) + n = O(n) \quad T(n) = T(n - 1) + n \leq c(n - 1) + n = cn + (n - c) > cn! \]
A Recursion Tree for Merge Sort

Recurrence: \( T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = 2T(n/2) + an \)
A Recursion Tree for Merge Sort

**Recurrence:** $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = 2T(n/2) + an$

**Strategy:** Expand the recurrence all the way down to the base case
A Recursion Tree for Merge Sort

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\( T(n) \)
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A Recursion Tree for Merge Sort

Recurrence: \( T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = 2T(n/2) + an \)

Strategy: Expand the recurrence all the way down to the base case
A Recursion Tree for Merge Sort

**Recurrence:** \( T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = 2T(n/2) + an \)

**Strategy:** Expand the recurrence all the way down to the base case

**Solution:** \( T(n) \in \Theta(n \lg n) \)
A Recursion Tree for Binary Search

Recurrence: \( T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a \)
A Recursion Tree for Binary Search

Recurrence: \( T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a \)

\[ T(n) \]
A Recursion Tree for Binary Search

Recurrence: $T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a$

```
     a
    /\  
   /   
  T(n/2)
```
A Recursion Tree for Binary Search

Recurrence: \( T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a \)

```
a
  \
  a
  \
  a
  \
  T\left(\frac{n}{4}\right)
```
A Recursion Tree for Binary Search

Recurrence: \( T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a \)

\[ \begin{align*}
T(n) &= T(n/2) + a \\
    &= T(n/4) + a \\
    &= \ldots \\
    &= T\left(\frac{n}{8}\right) + a \\
\end{align*} \]
A Recursion Tree for Binary Search

**Recurrence:** \( T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a \)
A Recursion Tree for Binary Search

Recurrence: $T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a$

```
\text{lg} n
```

```
\text{a}
```

```
\text{a}
```

```
\text{a}
```

```
\text{a}
```

```
\text{b}
```

A Recursion Tree for Binary Search

**Recurrence:** \( T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = T(n/2) + a \)

**Solution:** \( T(n) \in \Theta(\lg n) \)
A Less Obvious Recursion Tree

Recurrence: \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an \)
A Less Obvious Recursion Tree

Recurrence: \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an \)
A Less Obvious Recursion Tree

Recurrence: \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \implies T(n) = T(2n/3) + T(n/3) + an \)

\[
\begin{align*}
\log_{\frac{3}{2}} n & \quad a\left(\frac{2n}{3}\right) \\
\quad & \quad an \\
\quad & \quad a\left(\frac{n}{3}\right) \\
\quad & \quad a\left(\frac{4n}{9}\right) \\
\quad & \quad an \\
\quad & \quad a\left(\frac{n}{9}\right) \\
\quad & \quad a\left(\frac{8n}{27}\right) \\
\quad & \quad an \\
\quad & \quad a\left(\frac{n}{27}\right) \\
& \quad b \\
& \quad bn \\
& \quad b
\end{align*}
\]
A Less Obvious Recursion Tree

Recurrence: \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an \)
A Less Obvious Recursion Tree

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**Recurrence:** \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an \)
Recurrence: \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an \)
**A Less Obvious Recursion Tree**

**Recurrence:** \( T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an \)

**Solution:** \( T(n) \in \Theta(n \log n) \)
Sometimes Only Substitution Will Do

**Recurrence:** \( T(n) = T(2n/3) + T(n/2) + \Theta(n) \)

**Lower bound:** \( T(n) \in \Omega(n^{1 + \log_2(7/6)}) \approx \Omega(n^{1.22}) \)

**Upper bound:** \( T(n) \in O(n^{1 + \log_{3/2}(7/6)}) \approx O(n^{1.38}) \)
Master Theorem

Master Theorem: Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).$$

(i) If $f(n) \in \mathcal{O}(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a \log n})$.

(iii) If $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and $a \cdot f(n/b) \leq cf(n)$, for some $\epsilon > 0$ and $c < 1$ and for all $n \geq n_0$, then $T(n) \in \Theta(f(n))$. 
Master Theorem

**Master Theorem:** Let \( a \geq 1 \) and \( b > 1 \), let \( f(n) \) be a positive function and let \( T(n) \) be given by the following recurrence:

\[
T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).
\]

(i) If \( f(n) \in O(n^{\log_b a - \epsilon}) \), for some \( \epsilon > 0 \), then \( T(n) \in \Theta(n^{\log_b a}) \).

(ii) If \( f(n) \in \Theta(n^{\log_b a}) \), then \( T(n) \in \Theta(n^{\log_b a \lg n}) \).

(iii) If \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq cf(n) \), for some \( \epsilon > 0 \) and \( c < 1 \) and for all \( n \geq n_0 \), then \( T(n) \in \Theta(f(n)) \).

**Example 1:** Merge Sort again

\[
T(n) = 2T(n/2) + \Theta(n)
\]
Master Theorem

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).$$

(i) If $f(n) \in O(n^\log_b a - \epsilon)$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^\log_b a)$.

(ii) If $f(n) \in \Theta(n^\log_b a)$, then $T(n) \in \Theta(n^\log_b a \log n)$.

(iii) If $f(n) \in \Omega(n^\log_b a + \epsilon)$ and $a \cdot f(n/b) \leq cf(n)$, for some $\epsilon > 0$ and $c < 1$ and for all $n \geq n_0$, then $T(n) \in \Theta(f(n))$.

**Example 1:** Merge Sort again

$$T(n) = 2T(n/2) + \Theta(n)$$

$a = 2 \quad b = 2 \quad f(n) \in \Theta(n)$
Master Theorem

**Master Theorem:** Let \( a \geq 1 \) and \( b > 1 \), let \( f(n) \) be a positive function and let \( T(n) \) be given by the following recurrence:

\[
T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n).
\]

(i) If \( f(n) \in O(n^{\log_b a - \epsilon}) \), for some \( \epsilon > 0 \), then \( T(n) \in \Theta(n^{\log_b a}) \).

(ii) If \( f(n) \in \Theta(n^{\log_b a}) \), then \( T(n) \in \Theta(n^{\log_b a \log n}) \).

(iii) If \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq cf(n) \), for some \( \epsilon > 0 \) and \( c < 1 \) and for all \( n \geq n_0 \), then \( T(n) \in \Theta(f(n)) \).

**Example 1:** Merge Sort again

\[
T(n) = 2T(n/2) + \Theta(n)
\]

\[
a = 2 \quad b = 2 \quad f(n) \in \Theta(n) = \Theta(n^{\log_2 2})
\]
Master Theorem

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n).$$

(i) If $f(n) \in O(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a \lg n})$.

(iii) If $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and $a \cdot f(n/b) \leq cf(n)$, for some $\epsilon > 0$ and $c < 1$ and for all $n \geq n_0$, then $T(n) \in \Theta(f(n))$.

**Example 1:** Merge Sort again

$$T(n) = 2T(n/2) + \Theta(n)$$

$a = 2$  $b = 2$  $f(n) \in \Theta(n) = \Theta(n^{\log_2 2})$

$T(n) \in \Theta(n \lg n)$
**Master Theorem**

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).$$

(i) If $f(n) \in \mathcal{O}(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a \lg n})$.

(iii) If $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and $a \cdot f(n/b) \leq cf(n)$, for some $\epsilon > 0$ and $c < 1$ and for all $n \geq n_0$, then $T(n) \in \Theta(f(n))$.

**Example 2:** Matrix multiplication

$$T(n) = 7T(n/2) + \Theta(n^2)$$
Master Theorem

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).$$

(i) If $f(n) \in \mathcal{O}(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a \lg n})$.

(iii) If $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and $a \cdot f(n/b) \leq cf(n)$, for some $\epsilon > 0$ and $c < 1$ and for all $n \geq n_0$, then $T(n) \in \Theta(f(n))$.

**Example 2:** Matrix multiplication

$$T(n) = 7T(n/2) + \Theta(n^2)$$

$a = 7 \quad b = 2 \quad f(n) \in \Theta(n^2)$
Master Theorem

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(i) If $f(n) \in O(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

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**Example 2:** Matrix multiplication

$$T(n) = 7T(n/2) + \Theta(n^2)$$

$a = 7$ \quad $b = 2$ \quad $f(n) \in \Theta(n^2) \subseteq O(n^{\log_2 7 - \epsilon})$ for all $0 < \epsilon \leq \log_2 7 - 2$
Master Theorem

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n).$$

(i) If $f(n) \in O(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a \lg n})$.

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$T(n) \in \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81})$
Master Theorem

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

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(i) If $f(n) \in \mathcal{O}(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a \lg n})$.

(iii) If $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and $a \cdot f(n/b) \leq cf(n)$, for some $\epsilon > 0$ and $c < 1$ and for all $n \geq n_0$, then $T(n) \in \Theta(f(n))$.

**Example 3:**

$$T(n) = T(n/2) + n$$
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**Master Theorem:** Let \( a \geq 1 \) and \( b > 1 \), let \( f(n) \) be a positive function and let \( T(n) \) be given by the following recurrence:

\[
T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n).
\]

(i) If \( f(n) \in O(n^{\log_b a - \epsilon}) \), for some \( \epsilon > 0 \), then \( T(n) \in \Theta(n^{\log_b a}) \).

(ii) If \( f(n) \in \Theta(n^{\log_b a}) \), then \( T(n) \in \Theta(n^{\log_b a \log n}) \).

(iii) If \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq cf(n) \), for some \( \epsilon > 0 \) and \( c < 1 \) and for all \( n \geq n_0 \), then \( T(n) \in \Theta(f(n)) \).

**Example 3:**

\[
T(n) = T(n/2) + n
\]

\[
a = 1 \quad b = 2
\]

\[
f(n) = n
\]
Master Theorem

Master Theorem: Let \( a \geq 1 \) and \( b > 1 \), let \( f(n) \) be a positive function and let \( T(n) \) be given by the following recurrence:

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T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).
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(i) If \( f(n) \in O(n^{\log_b a - \epsilon}) \), for some \( \epsilon > 0 \), then \( T(n) \in \Theta(n^{\log_b a}) \).
(ii) If \( f(n) \in \Theta(n^{\log_b a}) \), then \( T(n) \in \Theta(n^{\log_b a \log n}) \).
(iii) If \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq cf(n) \), for some \( \epsilon > 0 \) and \( c < 1 \) and for all \( n \geq n_0 \), then \( T(n) \in \Theta(f(n)) \).

Example 3:

\[
T(n) = T(n/2) + n
\]

\[
a = 1 \quad b = 2
\]

\[
f(n) = n \in \Omega(n^{\log_2 1 + \epsilon}) \text{ for all } 0 < \epsilon \leq 1
\]
Master Theorem

**Master Theorem:** Let \( a \geq 1 \) and \( b > 1 \), let \( f(n) \) be a positive function and let \( T(n) \) be given by the following recurrence:

\[
T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).
\]

(i) If \( f(n) \in \mathcal{O}(n^{\log_b a - \epsilon}) \), for some \( \epsilon > 0 \), then \( T(n) \in \Theta(n^{\log_b a}) \).

(ii) If \( f(n) \in \Theta(n^{\log_b a}) \), then \( T(n) \in \Theta(n^{\log_b a \log n}) \).

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**Example 3:**

\[
T(n) = T(n/2) + n
\]

\[
a = 1 \quad b = 2
\]

\[
f(n) = n \in \Omega(n^{\log_2 1+\epsilon}) \text{ for all } 0 < \epsilon \leq 1 \quad f(n/2) = n/2 \leq f(n)/2
\]
**Master Theorem**

**Master Theorem:** Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

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**Example 3:**

$$T(n) = T(n/2) + n$$

$a = 1$  
$b = 2$

$f(n) = n \in \Omega(n^{\log_2 1 + \epsilon})$ for all $0 < \epsilon \leq 1$  
$f(n/2) = n/2 \leq f(n)/2$

$T(n) \in \Theta(n)$
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]
Master Theorem: Proof

\[ T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n) \]

**Case 1:** \( f(n) \in O(n^{\log_b a - \epsilon}) \)
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 1:** \( f(n) \in O(n^{\log_b a - \epsilon}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in O \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right) = O(n^{\log_b a - \epsilon} \cdot b^{i\epsilon}) \]

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in O \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right) = O(n^{\log_b a - \epsilon} \cdot b^{i\epsilon}) \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 1:** \( f(n) \in \Theta(n^{\log_b a - \epsilon}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in \Theta \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right) \]

\[ = \Theta(n^{\log_b a - \epsilon} \cdot b^{i\epsilon}) \]

\[ f(n) \]

\[ \Theta(\ldots) \cdot b^\epsilon \]

\[ \Theta(\ldots) \cdot b^{2\epsilon} \]

\[ \Theta(\ldots) \cdot b^{i\epsilon} \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 1:** \( f(n) \in \mathcal{O}(n^{\log_b a - \epsilon}) \)

\[
a^i \cdot f \left( \frac{n}{b^i} \right) \in \mathcal{O} \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right)
= \mathcal{O}(n^{\log_b a - \epsilon} \cdot b^{i\epsilon}) \cdot f(n)
\]

\[ T(n) = \Theta(n^{\log_b a}) + \mathcal{O}(n^{\log_b a - \epsilon}) \cdot \sum_{i=1}^{\log_b n - 1} b^{i\epsilon} \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 1:** \( f(n) \in O(n^{\log_b a - \epsilon}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in O \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right) = O(n^{\log_b a - \epsilon} \cdot b^{i\epsilon}) \cdot f(n) \]

\[ T(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \sum_{i=1}^{\log_b n-1} b^{i\epsilon} \]

\[ = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \frac{(b^\epsilon)^{\log_b n - 1}}{b^\epsilon - 1} \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 1:** \( f(n) \in O(n^{\log_b a - \epsilon}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in O \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right) = O(n^{\log_b a - \epsilon} \cdot b^i \epsilon) \]

\[ T(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \sum_{i=1}^{\log_b n - 1} b^i \epsilon \]

\[ = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \frac{(b^\epsilon)^{\log_b n} - 1}{b^\epsilon - 1} \]

\[ = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot n^\epsilon \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[
T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n)
\]

**Case 1:** \( f(n) \in O(n^{\log_b a - \epsilon}) \)

\[
a^i \cdot f \left( \frac{n}{b^i} \right) \in O \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a - \epsilon} \right) = O(n^{\log_b a - \epsilon} \cdot b^i \epsilon) \cdot f(n)
\]

\[
T(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \sum_{i=1}^{\log_b n - 1} b^i \epsilon
\]

\[
= \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \frac{(b^\epsilon)^{\log_b n} - 1}{b^\epsilon - 1}
\]

\[
= \Theta(n^{\log_b a}) \cdot n^\epsilon
\]

\[
= \Theta(n^{\log_b a})
\]

\[
a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a})
\]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

Case 2: \( f(n) \in \Theta(n^{\log_b a}) \)

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 2:** \( f(n) \in \Theta(n^{\log_b a}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in \Theta \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a} \right) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

Case 2: \( f(n) \in \Theta(n^{\log_b a}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in \Theta \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a} \right) = \Theta(n^{\log_b a}) \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n) \]

**Case 2:** \( f(n) \in \Theta(n^{\log_b a}) \)

\[ a^i \cdot f\left(\frac{n}{b^i}\right) \in \Theta\left(a^i \cdot \left(\frac{n}{b^i}\right)^{\log_b a}\right) = \Theta(n^{\log_b a}) \]

\[ T(n) = \Theta(n^{\log_b a}) \cdot \log_b n \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 2:** \( f(n) \in \Theta(n^{\log_b a}) \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \in \Theta \left( a^i \cdot \left( \frac{n}{b^i} \right)^{\log_b a} \right) = \Theta(n^{\log_b a}) \]

\[ T(n) = \Theta(n^{\log_b a}) \cdot \log_b n = \Theta(n^{\log_b a} \log n) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)
**Master Theorem: Proof**

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left(\frac{n}{b}\right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

Claim: \( a^i \cdot f \left(\frac{n}{b^i}\right) \leq c^i \cdot f(n) \)

For \( i = 0 \), \( a^0 \cdot f \left(\frac{n}{b^0}\right) = f(n) = c^0 \cdot f(n) \).
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

For \( i > 0, \)

\[ a^i \cdot f \left( \frac{n}{b^i} \right) = a^{i-1} \cdot \left( a \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \]

\[ a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

Claim: \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

For \( i > 0, \)
\[
\begin{align*}
a^i \cdot f \left( \frac{n}{b^i} \right) &= a^{i-1} \cdot \left( a \cdot f \left( \frac{n/b^{i-1}}{b} \right) \right) \\
&\leq a^{i-1} \cdot \left( c \cdot f \left( \frac{n}{b^{i-1}} \right) \right)
\end{align*}
\]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

For \( i > 0 \),

\[
\begin{align*}
    a^i \cdot f \left( \frac{n}{b^i} \right) &= a^{i-1} \cdot \left( a \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
    &\leq a^{i-1} \cdot \left( c \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
    &= c \cdot \left( a^{i-1} \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
    &= c \cdot \left( a^{i-1} \cdot f \left( \frac{n}{b^{i-1}} \right) \right)
\end{align*}
\]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

For \( i > 0 \),
\[
\begin{align*}
a^i \cdot f \left( \frac{n}{b^i} \right) &= a^{i-1} \cdot \left( a \cdot f \left( \frac{n/b^{i-1}}{b} \right) \right) \\
&\leq a^{i-1} \cdot \left( c \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
&= c \cdot \left( a^{i-1} \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
&\leq c \cdot (c^{i-1} \cdot f(n))
\end{align*}
\]

\( a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \)
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

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For \( i > 0, \)

\[
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a^i \cdot f \left( \frac{n}{b^i} \right) &= a^{i-1} \cdot \left( a \cdot f \left( \frac{n/b^{i-1}}{b} \right) \right) \\
&\leq a^{i-1} \cdot \left( c \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
&= c \cdot \left( a^{i-1} \cdot f \left( \frac{n}{b^{i-1}} \right) \right) \\
&\leq c \cdot (c^{i-1} \cdot f(n)) \\
&= c^i \cdot f(n)
\end{align*}
\]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)
Master Theorem: Proof

\[ T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \varepsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f\left(\frac{n}{b^i}\right) \leq c^i \cdot f(n) \)

\[ T(n) \in \Omega(n^{\log_b a + f(n)}) = \Omega(f(n)) \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

Claim: \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

\[ T(n) \in \Omega(n^{\log_b a} + f(n)) = \Omega(f(n)) \]

\[ T(n) \in \Theta \left( n^{\log_b a} + \sum_{i=0}^{\infty} c^i \cdot f(n) \right) \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

Claim: \( a^i \cdot f\left(\frac{n}{b^i}\right) \leq c^i \cdot f(n) \)

\[ T(n) \in \Omega(n^{\log_b a} + f(n)) = \Omega(f(n)) \]

\[ T(n) \in O\left(n^{\log_b a} + \sum_{i=0}^{\infty} c^i \cdot f(n)\right) \]

\[ = O\left(n^{\log_b a} + f(n) \cdot \sum_{i=0}^{\infty} c^i\right) \]

\[ a^{\log_b n} \cdot \Theta(1) = \Theta(n^{\log_b a}) \]
Master Theorem: Proof

\[ T(n) = a \cdot T \left( \frac{n}{b} \right) + f(n) \]

**Case 3:** \( f(n) \in \Omega(n^{\log_b a + \epsilon}) \) and \( a \cdot f(n/b) \leq c \cdot f(n) \) for some \( c < 1 \)

**Claim:** \( a^i \cdot f \left( \frac{n}{b^i} \right) \leq c^i \cdot f(n) \)

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\[ = O(f(n)) \]
Quick Sort

QuickSort(A, ℓ, r)

1 if r ≤ ℓ
2 then return
3 m = Partition(A, ℓ, r)
4 QuickSort(A, ℓ, m – 1)
5 QuickSort(A, m + 1, r)
Quick Sort

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Worst case:
Quick Sort

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1. if \( r \leq ℓ \) then return
2. \( m = \text{Partition}(A, ℓ, r) \)
3. QuickSort(A, ℓ, m – 1)
4. QuickSort(A, m + 1, r)

Worst case:

\[ T(n) = \Theta(n) + T(n - 1) \]
Quick Sort

**QuickSort**(A, ℓ, r)

1. if r ≤ ℓ then return
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**Worst case:**

\[ T(n) = \Theta(n) + T(n - 1) = \Theta(n^2) \]
Quick Sort

QuickSort(A, ℓ, r)

1. if r ≤ ℓ then return
2. m = Partition(A, ℓ, r)
3. QuickSort(A, ℓ, m – 1)
4. QuickSort(A, m + 1, r)

Worst case:

T(n) = Θ(n) + T(n – 1) = Θ(n²)

Best case:
Quick Sort

QuickSort(A, ℓ, r)

1. if r ≤ ℓ then return
2. m = Partition(A, ℓ, r)
3. QuickSort(A, ℓ, m – 1)
4. QuickSort(A, m + 1, r)

Worst case:

$$T(n) = \Theta(n) + T(n - 1) = \Theta(n^2)$$

Best case:

$$T(n) = \Theta(n) + 2T(\lfloor n/2 \rfloor)$$
Quick Sort

QuickSort(A, ℓ, r)

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5 QuickSort(A, m + 1, r)

Worst case:

T(n) = \Theta(n) + T(n – 1) = \Theta(n^2)

Best case:

T(n) = \Theta(n) + 2T(\lfloor n/2 \rfloor) = \Theta(n \lg n)
**Quick Sort**

QuickSort(A, ℓ, r)

1. if r ≤ ℓ then return
2. m = Partition(A, ℓ, r)
3. QuickSort(A, ℓ, m – 1)
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**Worst case:**

\[ T(n) = \Theta(n) + T(n - 1) = \Theta(n^2) \]

**Best case:**

\[ T(n) = \Theta(n) + 2T(\lfloor n/2 \rfloor) = \Theta(n \lg n) \]

**Average case:**

\[ T(n) = \Theta(n \lg n) \]
Two Partitioning Algorithms

HoarePartition(A, l, r)

1  x = A[r]
2  i = l - 1
3  j = r + 1
4  while True
5     do repeat i = i + 1
6        until A[i] ≥ x
7     repeat j = j - 1
8        until A[j] ≤ x
9    if i < j
10       then swap A[i] and A[j]
11     else return j

Loop invariants:

<table>
<thead>
<tr>
<th>≤ x</th>
<th>?</th>
<th>≥ x</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td></td>
<td>j</td>
</tr>
</tbody>
</table>
Two Partitioning Algorithms

HoarePartition(A, l, r)

1. \( x = A[r] \)
2. \( i = l - 1 \)
3. \( j = r + 1 \)
4. while True
   5. do repeat \( i = i + 1 \)
      6. until \( A[i] \geq x \)
   7. repeat \( j = j - 1 \)
      8. until \( A[j] \leq x \)
   9. if \( i < j \)
      10. then swap \( A[i] \) and \( A[j] \)
      11. else return \( j \)

Loop invariants:

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<tr>
<td>i</td>
<td>?</td>
<td>j</td>
</tr>
</tbody>
</table>

LomutoPartition(A, l, r)

1. \( i = l - 1 \)
2. for \( j = l \) to \( r - 1 \)
3. do if \( A[j] \leq A[r] \)
4. then \( i = i + 1 \)
5. swap \( A[i] \) and \( A[j] \)
6. swap \( A[i + 1] \) and \( A[r] \)
7. return \( i + 1 \)
Two Partitioning Algorithms

HoarePartition(A, l, r)

1. \( x = A[r] \)
2. \( i = l - 1 \)
3. \( j = r + 1 \)
4. while True do
   5. repeat \( i = i + 1 \)
   6. until \( A[i] \geq x \)
   7. repeat \( j = j - 1 \)
   8. until \( A[j] \leq x \)
   9. if \( i < j \) then
      10. \( \text{swap } A[i] \text{ and } A[j] \)
   11. else return \( j \)

LomutoPartition(A, l, r)

1. \( i = l - 1 \)
2. for \( j = l \) to \( r - 1 \) do
   3. if \( A[j] \leq A[r] \) then \( i = i + 1 \)
   4. swap \( A[i] \) and \( A[j] \)
   5. swap \( A[i + 1] \) and \( A[r] \)
   6. return \( i + 1 \)

HoarePartition is more efficient in practice.

LomutoPartition has some properties that make average-case analysis easier.

HoarePartition is more convenient for worst-case Quick Sort.
Two Partitioning Algorithms

HoarePartition($A, l, r, x$)

1. $i = l - 1$
2. $j = r + 1$
3. while True
4.   do repeat $i = i + 1$
5.     until $A[i] \geq x$
6.   repeat $j = j - 1$
7.     until $A[j] \leq x$
8.   if $i < j$
9.     then swap $A[i]$ and $A[j]$
10. else return $j$
11. return $i + 1$

LomutoPartition($A, l, r$)

1. $i = l - 1$
2. for $j = l$ to $r - 1$
3.   do if $A[j] \leq A[r]$
4.      then $i = i + 1$
5.      swap $A[i]$ and $A[j]$
6. swap $A[i + 1]$ and $A[r]$
7. return $i + 1$

HoarePartition is more efficient in practice.

LomutoPartition has some properties that make average-case analysis easier.

HoarePartition is more convenient for worst-case Quick Sort.
Selection

**Goal:** Given an *unsorted* array $A$ and an integer $1 \leq k \leq n$, find the $k$th smallest element in $A$. 

```
| 8 | 17 | 5 | 43 | 3 | 12 | 64 | 21 |
```

4th smallest element
Selection

**Goal:** Given an unsorted array $A$ and an integer $1 \leq k \leq n$, find the $k$th smallest element in $A$.

First idea: Sort the array and then return the element in the $k$th position.

$\Rightarrow O(n \lg n)$ time
Selection

Goal: Given an unsorted array A and an integer $1 \leq k \leq n$, find the kth smallest element in A.

First idea: Sort the array and then return the element in the kth position.

$\Rightarrow O(n \lg n)$ time

We can find the minimum ($k = 1$) or the maximum ($k = n$) in $O(n)$ time!
**Selection**

**Goal:** Given an *unsorted* array $A$ and an integer $1 \leq k \leq n$, find the $k$th smallest element in $A$.

First idea: Sort the array and then return the element in the $k$th position.

$\Rightarrow O(n \log n)$ time

We can find the minimum ($k = 1$) or the maximum ($k = n$) in $O(n)$ time!

It would be nice if we were able to find the $k$th smallest element, for any $k$, in $O(n)$ time.
The Sorting Idea Isn't Half Bad, But . . .

To find the kth smallest element, we don't need to sort the input completely.

We only need to verify that there are exactly \( k - 1 \) elements smaller than the element we return.
The Sorting Idea Isn’t Half Bad, But …

To find the kth smallest element, we don’t need to sort the input completely.

We only need to verify that there are exactly $k - 1$ elements smaller than the element we return.

\[ \leq p \quad p \quad \geq p \]

Partition

L

R
The Sorting Idea Isn't Half Bad, But . . .

To find the $k$th smallest element, we don't need to sort the input completely.

We only need to verify that there are exactly $k - 1$ elements smaller than the element we return.

If $|L| = k - 1$, then $p$ is the $k$th smallest element.
The Sorting Idea Isn’t Half Bad, But . . .

To find the kth smallest element, we don’t need to sort the input completely.

We only need to verify that there are exactly $k - 1$ elements smaller than the element we return.

If $|L| = k - 1$, then $p$ is the kth smallest element.

If $|L| \geq k$, then the kth smallest element in $L$ is the kth smallest element in $A$. 
The Sorting Idea Isn't Half Bad, But . . .

To find the kth smallest element, we don't need to sort the input completely.

We only need to verify that there are exactly \( k - 1 \) elements smaller than the element we return.

If \( |L| = k - 1 \), then \( p \) is the kth smallest element.

If \( |L| \geq k \), then the kth smallest element in \( L \) is the kth smallest element in \( A \).

If \( |L| < k - 1 \), then the \( (k - |L| + 1) \)st element in \( R \) is the kth smallest element in \( A \).
Quick Select

QuickSelect(A, ℓ, r, k)

1. if r ≤ ℓ
2. then return A[ℓ]
3. m = Partition(A, ℓ, r)
4. if m – ℓ = k – 1
5. then return A[m]
6. else if m – ℓ ≥ k
7. then return QuickSelect(A, ℓ, m – 1, k)
8. else return QuickSelect(A, m + 1, r, k – (m + 1 – ℓ))
Quick Select

QuickSelect(A, \ell, r, k)

1. if \( r \leq \ell \)
2. then return \( A[\ell] \)
3. \( m = \text{Partition}(A, \ell, r) \)
4. if \( m - \ell = k - 1 \)
5. then return \( A[m] \)
6. else if \( m - \ell \geq k \)
7. then return \( \text{QuickSelect}(A, \ell, m - 1, k) \)
8. else return \( \text{QuickSelect}(A, m + 1, r, k - (m + 1 - \ell)) \)

Worst case:
Quick Select

QuickSelect(A, ℓ, r, k)

1. if \( r \leq \ell \)
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Worst case:

\[
T(n) = \Theta(n) + T(n - 1) = \Theta(n^2)
\]
Quick Select

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1. if r ≤ ℓ then return A[ℓ]
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Worst case:

\[ T(n) = \Theta(n) + T(n - 1) = \Theta(n^2) \]

Best case:
Quick Select

QuickSelect(A, \ell, r, k)

1. if r \leq \ell
2. then return A[\ell]
3. m = Partition(A, \ell, r)
4. if m – \ell = k – 1
5. then return A[m]
6. else if m – \ell \geq k
7. then return QuickSelect(A, \ell, m – 1, k)
8. else return QuickSelect(A, m + 1, r, k – (m + 1 – \ell))

Worst case:
\[ T(n) = \Theta(n) + T(n – 1) = \Theta(n^2) \]

Best case:
\[ T(n) = \Theta(n) \]
Quick Select

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Worst case:

T(n) = \Theta(n) + T(n – 1) = \Theta(n^2)

Best case:

T(n) = \Theta(n) + T(n/2) = \Theta(n)
Quick Select

QuickSelect(A, ℓ, r, k)

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3 m = Partition(A, ℓ, r)
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8 else return QuickSelect(A, m + 1, r, k – (m + 1 – ℓ))

Worst case:

\[ T(n) = \Theta(n) + T(n - 1) = \Theta(n^2) \]

Best case:

\[ T(n) = \Theta(n) + T(n/2) = \Theta(n) \]

Average case:

\[ T(n) = \Theta(n) \]
Worst-Case Selection

QuickSelect(A, ℓ, r, k)

1. if r ≤ ℓ
2. then return A[ℓ]
3. p = FindPivot(A, ℓ, r)
4. m = HoarePartition(A, ℓ, r, p)
5. if m – ℓ + 1 ≥ k
6. then return QuickSelect(A, ℓ, m, k)
7. else return QuickSelect(A, m + 1, r, k – (m + 1 – ℓ))
Worst-Case Selection

QuickSelect(A, ℓ, r, k)

1  if r ≤ ℓ
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3  p = FindPivot(A, ℓ, r)
4  m = HoarePartition(A, ℓ, r, p)
5  if m − ℓ + 1 ≥ k
6    then return QuickSelect(A, ℓ, m, k)
7  else return QuickSelect(A, m + 1, r, k − (m + 1 − ℓ))

If we could guarantee that p is the median of A[ℓ .. r], then we'd recurse on at most n/2 elements.
Worst-Case Selection

QuickSelect(A, ℓ, r, k)

1. if r ≤ ℓ then return A[ℓ]
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If we could guarantee that p is the median of A[ℓ..r], then we'd recurse on at most n/2 elements.

⇒ T(n) = Θ(n) + T(n/2) = Θ(n).
Worst-Case Selection

QuickSelect(\(A, \ell, r, k\))

1. if \(r \leq \ell\)
2. then return \(A[\ell]\)
3. \(p = \text{FindPivot}(A, \ell, r)\)
4. \(m = \text{HoarePartition}(A, \ell, r, p)\)
5. if \(m - \ell + 1 \geq k\)
6. then return QuickSelect(\(A, \ell, m, k\))
7. else return QuickSelect(\(A, m + 1, r, k - (m + 1 - \ell)\))

If we could guarantee that \(p\) is the median of \(A[\ell \ldots r]\), then we'd recurse on at most \(n/2\) elements.

\[ T(n) = \Theta(n) + T(n/2) = \Theta(n). \]

Alas, finding the median is selection!
If there are at least $\epsilon n$ elements smaller than $p$ and at least $\epsilon n$ elements greater than $p$, then

$$T(n) \leq \Theta(n) + T((1 - \epsilon)n) = \Theta(n).$$
Finding An Approximate Median

\textbf{FindPivot}(A, \ell, r)

1. \[ n' = \lfloor (r - \ell)/5 \rfloor + 1 \]
2. \textbf{for} i = 0 \textbf{to} n' - 1
3. \textbf{do} InsertionSort(A, \ell + 5 \cdot i, \min(\ell + 5 \cdot i + 4, r))
4. \textbf{if} \ell + 5i + 4 \leq r
5. \textbf{then} B[i + 1] = A[\ell + 5 \cdot i + 2]
6. \textbf{else} B[i + 1] = A[\ell + 5 \cdot i]
7. \textbf{return} QuickSelect(B, 1, n', \lceil n'/2 \rceil)

Approximate median
Finding An Approximate Median
Finding An Approximate Median

There are at least \( \lceil n'/2 \rceil - 1 \) medians smaller than the median of medians.
Finding An Approximate Median

There are at least \( \lceil n'/2 \rceil - 1 \) medians smaller than the median of medians.

For at least \( \lceil n'/2 \rceil - 1 \) of the medians, there are two elements in each of their groups that are smaller than the median of medians.
Finding An Approximate Median

There are at least \( \lceil \frac{n'}{2} \rceil - 1 \) medians smaller than the median of medians.

For at least \( \lceil \frac{n'}{2} \rceil - 1 \) of the medians, there are two elements in each of their groups that are smaller than the median of medians.
Finding An Approximate Median

There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.

For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.

$n' = \lceil n/5 \rceil$
Finding An Approximate Median

There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.

For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.

$n' = \lceil n/5 \rceil$

**Total number of elements smaller than the median of medians:**

$$3 \left( \left\lceil \frac{n'}{2} \right\rceil - 1 \right) = 3 \left\lfloor \frac{n/5}{2} \right\rfloor - 3 \geq \frac{3n}{10} - 3$$
Finding An Approximate Median

There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.

For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.

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Total number of elements smaller than the median of medians:

$$3 \left( \left\lceil \frac{n'}{2} \right\rceil - 1 \right) = 3 \left\lceil \frac{n}{5} \right\rceil - 3 \geq \frac{3n}{10} - 3$$

The same analysis holds for counting the number of elements greater than the median of medians.
Finding An Approximate Median

There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.

For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.

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**Total number of elements smaller than the median of medians:**

$$3 \left( \left\lfloor \frac{n'}{2} \right\rfloor - 1 \right) = 3 \left\lfloor \frac{\lceil n/5 \rceil}{2} \right\rfloor - 3 \geq \frac{3n}{10} - 3$$

The same analysis holds for counting the number of elements greater than the median of medians.
Worst-Case Selection: Analysis

FindPivot (excluding the recursive call to QuickSelect) and Partition take $O(n)$ time.
Worst-Case Selection: Analysis

FindPivot (excluding the recursive call to QuickSelect) and Partition take $O(n)$ time.
FindPivot recurses on $\lceil n/5 \rceil < n/5 + 1$ elements.
Worst-Case Selection: Analysis

FindPivot (excluding the recursive call to QuickSelect) and Partition take $O(n)$ time.

FindPivot recurses on $\lceil \frac{n}{5} \rceil < \frac{n}{5} + 1$ elements.

QuickSelect itself recurses on at most $\frac{7n}{10} + 3$ elements.
Worst-Case Selection: Analysis

FindPivot (excluding the recursive call to QuickSelect) and Partition take $O(n)$ time. FindPivot recurses on $\left\lceil \frac{n}{5} \right\rceil < \frac{n}{5} + 1$ elements.
QuickSelect itself recurses on at most $\frac{7n}{10} + 3$ elements.

$$T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + O(n)$$
Worst-Case Selection: Analysis

**Claim:** $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$. 
Worst-Case Selection: Analysis

Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

Base case: $(n < 80)$
Worst-Case Selection: Analysis

Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

Base case: $(n < 80)$

We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$. 
Worst-Case Selection: Analysis

Claim: \( T(n) \in O(n) \), that is, \( T(n) \leq cn \), for some \( c > 0 \) and all \( n \geq 1 \).

Base case: \( (n < 80) \)

We already observed that the running time is in \( O(n^2) \) in the worst case. Since \( n \in O(1) \), \( n^2 \in O(1) \).

\( \Rightarrow T(n) \leq c' \leq cn \) for \( c \) sufficiently large.
**Worst-Case Selection: Analysis**

**Claim:** $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

**Base case:** ($n < 80$)

We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$.

$\Rightarrow T(n) \leq c' \leq cn$ for $c$ sufficiently large.

**Inductive Step:** ($n \geq 80$)
Worst-Case Selection: Analysis

Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

Base case: $(n < 80)$

We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$.

$\Rightarrow T(n) \leq c' \leq cn$ for $c$ sufficiently large.

Inductive Step: $(n \geq 80)$

$$T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an$$
Worst-Case Selection: Analysis

Claim: \( T(n) \in \mathcal{O}(n) \), that is, \( T(n) \leq cn \), for some \( c > 0 \) and all \( n \geq 1 \).

Base case: \((n < 80)\)

We already observed that the running time is in \( \mathcal{O}(n^2) \) in the worst case. Since \( n \in \mathcal{O}(1) \), \( n^2 \in \mathcal{O}(1) \).

\[ \Rightarrow T(n) \leq c' \leq cn \] for \( c \) sufficiently large.

Inductive Step: \((n \geq 80)\)

\[
T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an
\]

\[
\leq c \left(\frac{n}{5} + 1\right) + c \left(\frac{7n}{10} + 3\right) + an
\]
Worst-Case Selection: Analysis

Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

Base case: $(n < 80)$

We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$.

$\Rightarrow T(n) \leq c' \leq cn$ for $c$ sufficiently large.

Inductive Step: $(n \geq 80)$

$$T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an$$

$$\leq c \left(\frac{n}{5} + 1\right) + c \left(\frac{7n}{10} + 3\right) + an$$

$$\leq c \left(\frac{4n}{20} + \frac{14n}{20} + \frac{n}{20}\right) + an$$
Worst-Case Selection: Analysis

Claim: \( T(n) \in O(n) \), that is, \( T(n) \leq cn \), for some \( c > 0 \) and all \( n \geq 1 \).

Base case: \((n < 80)\)

We already observed that the running time is in \( O(n^2) \) in the worst case. Since \( n \in O(1) \), \( n^2 \in O(1) \).

\[ \Rightarrow T(n) \leq c' \leq cn \text{ for } c \text{ sufficiently large.} \]

Inductive Step: \((n \geq 80)\)

\[
T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an
\leq c\left(\frac{n}{5} + 1\right) + c\left(\frac{7n}{10} + 3\right) + an
\leq c\left(\frac{4n}{20} + \frac{14n}{20} + \frac{n}{20}\right) + an
= \left(\frac{19c}{20} + a\right)n
\]
Worst-Case Selection: Analysis

Claim: \( T(n) \in \mathcal{O}(n) \), that is, \( T(n) \leq cn \), for some \( c > 0 \) and all \( n \geq 1 \).

Base case: \((n < 80)\)

We already observed that the running time is in \( \mathcal{O}(n^2) \) in the worst case. Since \( n \in \mathcal{O}(1) \), \( n^2 \in \mathcal{O}(1) \).

\[ \Rightarrow T(n) \leq c' \leq cn \text{ for } c \text{ sufficiently large.} \]

Inductive Step: \((n \geq 80)\)

\[
T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an
\]

\[
\leq c \left(\frac{n}{5} + 1\right) + c \left(\frac{7n}{10} + 3\right) + an
\]

\[
\leq c \left(\frac{4n}{20} + \frac{14n}{20} + \frac{n}{20}\right) + an
\]

\[
= \left(\frac{19c}{20} + a\right)n
\]

\[
\leq cn \quad \forall c \geq 20a
\]
Worst-Case Quick Sort

QuickSort(A, ℓ, r)

1. if r ≤ ℓ
2. then return
3. p = FindPivot(A, ℓ, r)
4. m = HoarePartition(A, ℓ, r, p)
5. return QuickSort(A, ℓ, m)
6. return QuickSort(A, m + 1, r)
Worst-Case Quick Sort

QuickSort(A, ℓ, r)

1 if r ≤ ℓ
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3 p = FindPivot(A, ℓ, r)
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6 return QuickSort(A, m + 1, r)

T(n) = Θ(n) + T(n₁) + T(n₂), where n₁ + n₂ = n and n₁, n₂ ≤ \(\frac{7n}{10} + 3\)
Worst-Case Quick Sort

QuickSort(A, ℓ, r)

1 if r ≤ ℓ
2 then return
3 p = FindPivot(A, ℓ, r)
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6 return QuickSort(A, m + 1, r)

T(n) = Θ(n) + T(n_1) + T(n_2), where n_1 + n_2 = n and n_1, n_2 ≤ \frac{7n}{10} + 3

Claim: T(n) ∈ O(n \lg n), that is, T(n) ≤ cn \lg n, for some c > 0 and all n ≥ 2.
Worst-Case Quick Sort

QuickSort(A, ℓ, r)
1 if r ≤ ℓ
2 then return
3 p = FindPivot(A, ℓ, r)
4 m = HoarePartition(A, ℓ, r, p)
5 return QuickSort(A, ℓ, m)
6 return QuickSort(A, m + 1, r)

T(n) = \Theta(n) + T(n_1) + T(n_2), where n_1 + n_2 = n and n_1, n_2 \leq \frac{7n}{10} + 3

Claim: T(n) \in O(n \lg n), that is, T(n) \leq cn \lg n, for some c > 0 and all n \geq 2.

Base case: (n < 30)

We already observed that the running time is in O(n^2) in the worst case. Since n \in O(1), n^2 \in O(1).
⇒ T(n) \leq c' \leq cn for c sufficiently large.
Worst-Case Quick Sort

Inductive Step: \( n \geq 30 \)
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[ T(n) \leq T(n_1) + T(n_2) + an \]
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[
T(n) \leq T(n_1) + T(n_2) + an
\leq cn_1 \log n_1 + cn_2 \log n_2 + an
\]
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[
T(n) \leq T(n_1) + T(n_2) + an \\
\leq cn_1 \log n_1 + cn_2 \log n_2 + an \\
\leq cn_1 \log \left(\frac{7n}{10} + 3\right) + cn_2 \log \left(\frac{7n}{10} + 3\right) + an
\]
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[
T(n) \leq T(n_1) + T(n_2) + an \\
\leq cn_1 \log n_1 + cn_2 \log n_2 + an \\
\leq cn_1 \log \left(\frac{7n}{10} + 3\right) + cn_2 \log \left(\frac{7n}{10} + 3\right) + an \\
= cn \log \left(\frac{7n}{10} + 3\right) + an
\]
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[
T(n) \leq T(n_1) + T(n_2) + an \\
\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
\leq cn_1 \lg \left(\frac{7n}{10} + 3\right) + cn_2 \lg \left(\frac{7n}{10} + 3\right) + an \\
= cn \lg \left(\frac{7n}{10} + 3\right) + an \\
\leq cn \lg \left(\frac{7n}{10} + \frac{n}{10}\right) + an
\]
Worst-Case Quick Sort

Inductive Step: \( n \geq 30 \)

\[
T(n) \leq T(n_1) + T(n_2) + an
\]

\[
\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an
\]

\[
\leq cn_1 \lg \left( \frac{7n}{10} + 3 \right) + cn_2 \lg \left( \frac{7n}{10} + 3 \right) + an
\]

\[
= cn \lg \left( \frac{7n}{10} + 3 \right) + an
\]

\[
\leq cn \lg \left( \frac{7n}{10} + \frac{n}{10} \right) + an
\]

\[
= cn \lg \frac{4n}{5} + an
\]
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[ T(n) \leq T(n_1) + T(n_2) + an \]
\[ \leq cn_1 \log n_1 + cn_2 \log n_2 + an \]
\[ \leq cn_1 \log \left( \frac{7n}{10} + 3 \right) + cn_2 \log \left( \frac{7n}{10} + 3 \right) + an \]
\[ = cn \log \left( \frac{7n}{10} + 3 \right) + an \]
\[ \leq cn \log \left( \frac{7n}{10} + \frac{n}{10} \right) + an \]
\[ = cn \log \frac{4n}{5} + an \]
\[ = cn \left( \log n - \log \frac{5}{4} \right) + an \]
Worst-Case Quick Sort

Inductive Step: \( n \geq 30 \)

\[
T(n) \leq T(n_1) + T(n_2) + an \\
\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
\leq cn_1 \lg \left( \frac{7n}{10} + 3 \right) + cn_2 \lg \left( \frac{7n}{10} + 3 \right) + an \\
= cn \lg \left( \frac{7n}{10} + 3 \right) + an \\
\leq cn \lg \left( \frac{7n}{10} + \frac{n}{10} \right) + an \\
= cn \lg \frac{4n}{5} + an \\
= cn \left( \lg n - \lg \frac{5}{4} \right) + an \\
= cn \lg n + \left( a - c \lg \frac{5}{4} \right)n
\]
Worst-Case Quick Sort

Inductive Step: \((n \geq 30)\)

\[
T(n) \leq T(n_1) + T(n_2) + an \\
\leq cn_1 \log n_1 + cn_2 \log n_2 + an \\
\leq cn_1 \log \left(\frac{7n}{10} + 3\right) + cn_2 \log \left(\frac{7n}{10} + 3\right) + an \\
= cn \log \left(\frac{7n}{10} + 3\right) + an \\
\leq cn \log \left(\frac{7n}{10} + \frac{n}{10}\right) + an \\
= cn \log \frac{4n}{5} + an \\
= cn \left(\log n - \log \frac{5}{4}\right) + an \\
= cn \log n + \left(a - c \log \frac{5}{4}\right)n \\
\leq cn \log n \quad \forall c \geq \frac{a}{\log(5/4)} \approx 3.1a
\]
Matrix Multiplication

We want to compute $C = A \times B$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices and hence $C = (c_{ij})$ is too.
Matrix Multiplication

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Definition:

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \forall \ 1 \leq i, k \leq n.$$
Matrix Multiplication

We want to compute $C = A \times B$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices and hence $C = (c_{ij})$ is too.

**Definition:**

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} \quad \forall 1 \leq i, k \leq n.$$ 

The naïve algorithm implementing the definition:

**MatrixProduct(A, B)**

1. $C = \text{an } n \times n \text{ array}$
2. for $i = 1$ to $n$
3.  do for $k = 1$ to $n$
4.  do $C[i, k] = 0$
5.  do for $j = 1$ to $n$
6.  do $C[i, k] = C[i, k] + A[i, j] \cdot B[j, k]$
7. return $C$
Matrix Multiplication

We want to compute $C = A \times B$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices and hence $C = (c_{ij})$ is too.

Definition:

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \forall 1 \leq i, k \leq n.$$ 

The naïve algorithm implementing the definition:

MatrixProduct($A$, $B$)

1. $C$ = an $n \times n$ array
2. for $i = 1$ to $n$
3. do for $k = 1$ to $n$
4. do $C[i, k] = 0$
5. do $j = 1$ to $n$
6. do $C[i, k] = C[i, k] + A[i, j] \cdot B[j, k]$
7. return $C$

Cost: $\Theta(n^3)$
Matrix Multiplication: Divide and Conquer

For simplicity, assume $n = 2^t$ for some integer $t$. 
Matrix Multiplication: Divide and Conquer

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**Base case:** $t = 0$

$$c_{11} = a_{11} \cdot b_{11}$$
Matrix Multiplication: Divide and Conquer

For simplicity, assume $n = 2^t$ for some integer $t$.

**Base case:** $t = 0$

$$c_{11} = a_{11} \cdot b_{11}$$

**Inductive step:** $t > 0$

$$\begin{array}{cc}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{array} = \begin{array}{cc}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array} \times \begin{array}{cc}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{array}$$

For $1 \leq i, k \leq 2$,

$$C_{ik} = \sum_{j=1}^{2} (A_{ij} \times B_{jk}).$$
Matrix Multiplication: Divide and Conquer

MatrixProductDNC(A, B, C, i_l, i_u, j_l, j_u, k_l, k_u)

1. if \( i_l = i_u \)
2. then \( C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] \times B[j_l, k_l] \)
3. else \( i_m = (i_l + i_u - 1)/2 \)
4. \( j_m = (j_l + j_u - 1)/2 \)
5. \( k_m = (k_l + k_u - 1)/2 \)
6. MatrixProductDNC(A, B, C, i_l, i_m, j_l, j_m, k_l, k_m)
7. MatrixProductDNC(A, B, C, i_l, i_m, j_m + 1, j_u, k_l, k_m)
8. MatrixProductDNC(A, B, C, i_l, i_m, j_l, j_m, k_m + 1, k_u)
9. MatrixProductDNC(A, B, C, i_l, i_m, j_m + 1, j_u, k_m + 1, k_u)
10. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m, k_l, k_m)
11. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l, k_m)
12. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m, k_m + 1, k_u)
13. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_m + 1, k_u)
Matrix Multiplication: Divide and Conquer

MatrixProductDNC(A, B, C, i_l, i_u, j_l, j_u, k_l, k_u)

1. if $i_l = i_u$
2. then $C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] \times B[j_l, k_l]$
3. else $i_m = (i_l + i_u - 1)/2$
4. $j_m = (j_l + j_u - 1)/2$
5. $k_m = (k_l + k_u - 1)/2$
6. MatrixProductDNC(A, B, C, i_l, i_m, j_l, j_m, k_l, k_m)
7. MatrixProductDNC(A, B, C, i_l, i_m, j_m + 1, j_u, k_l, k_m)
8. MatrixProductDNC(A, B, C, i_l, i_m, j_l, j_m, k_m + 1, k_u)
9. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m, k_l, k_m)
10. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l, k_m)
11. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m + 1, j_u, k_m + 1, k_u)
12. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_u, j_m + 1, k_m + 1, k_u)
13. MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m + 1, j_u, k_m + 1, k_u)

Cost: $T(n) = 8T(n/2) + \Theta(1)$
Matrix Multiplication: Divide and Conquer

MatrixProductDNC(A, B, C, i_l, i_u, j_l, j_u, k_l, k_u)

1 if i_l = i_u
2 then C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] × B[j_l, k_l]
3 else
4 i_m = (i_l + i_u - 1)/2
5 j_m = (j_l + j_u - 1)/2
6 k_m = (k_l + k_u - 1)/2
7 MatrixProductDNC(A, B, C, i_l, i_m, j_l, j_m, k_l, k_m)
8 MatrixProductDNC(A, B, C, i_l, i_m, j_m + 1, j_u, k_l, k_m)
9 MatrixProductDNC(A, B, C, i_l, i_m, j_l, j_m, k_m + 1, k_u)
10 MatrixProductDNC(A, B, C, i_l, i_m, j_m + 1, j_u, k_m + 1, k_u)
11 MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m, k_l, k_m)
12 MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l, k_m)
13 MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l, j_m + 1, j_u, k_m + 1, k_u)

Cost: T(n) = 8T(n/2) + Θ(1) ∈ Θ(n³)
Matrix Multiplication: Strassen's Algorithm

Goal:

\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\lg 7}) \approx \Theta(n^{2.81}) \]
Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$$
Matrix Multiplication: Strassen’s Algorithm

Goal:

\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\log_2{7}}\right) \approx \Theta(n^{2.81}) \]

Idea:

\[ C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = A_{11} A_{12} A_{21} A_{22} \times \]

\[
\begin{array}{ccc}
1 & 1 \\
\end{array}
\]

\times

\[
\begin{array}{c}
B_{11} \\
B_{21} \\
B_{12} \\
B_{22}
\end{array}
\]
Matrix Multiplication: Strassen's Algorithm

Goal:

\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81}) \]

Idea:

\[ C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = A_{11} A_{12} A_{21} A_{22} \times \]

\[ M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12} \]
Matrix Multiplication: Strassen's Algorithm

Goal:

\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\lg 7}) \approx \Theta(n^{2.81}) \]

Idea:

\[ C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = A_{11} A_{12} A_{21} A_{22} \times \]

\[ M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12} \]

\[ = A_{11} A_{12} A_{21} A_{22} \times \]

\[ \times B_{11} B_{21} B_{12} B_{22} \]
Matrix Multiplication: Strassen's Algorithm

Goal:

\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81}) \]

Idea:

\[ C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \end{bmatrix} \]

\[ M = A_{11} \times B_{11} + A_{12} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12} \]

\[ = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \end{bmatrix} \]

\[ = (A_{11} + A_{21}) \times (B_{11} + B_{12}) \]
Matrix Multiplication: Strassen’s Algorithm

Goal:
\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81}) \]

Idea:

\[
C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = A_{11}A_{12}A_{21}A_{22} \times A \times B
\]

\[
M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}
\]

\[
= (A_{11} + A_{21}) \times (B_{11} + B_{12})
\]

2 inconveniently placed ones = 2 multiplications
Matrix Multiplication: Strassen’s Algorithm

**Goal:**
\[ T(n) = 7T(n/2) + \Theta(n^2) \in \Theta(n^{\lg 7}) \approx \Theta(n^{2.81}) \]

**Idea:**
- 4 conveniently placed ones = 1 multiplication
- 2 inconveniently placed ones = 2 multiplications

\[ C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = A_{11} | A_{12} | A_{21} | A_{22} \times 1 | 1 \times \]

\[ M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12} \]

\[ = A_{11} | A_{12} | A_{21} | A_{22} \times 1 | 1 \times (B_{11} + B_{21}) (B_{12} + B_{22}) \]

2 inconveniently placed ones = 2 multiplications
4 conveniently placed ones = 1 multiplication
Matrix Multiplication: Strassen's Alg.

\[ M_1 = (A_{11} + A_{22}) \times (B_{11} + B_{22}) \]
\[ M_2 = A_{11} \times (B_{12} + B_{22}) \]
\[ M_3 = A_{22} \times (B_{11} + B_{21}) \]
\[ M_4 = (A_{11} - A_{12}) \times B_{22} \]
\[ M_5 = (A_{22} - A_{21}) \times B_{11} \]
\[ M_6 = (A_{11} + A_{21}) \times (B_{11} - B_{12}) \]
\[ M_7 = (A_{12} + A_{22}) \times (B_{21} - B_{22}) \]
Matrix Multiplication: Strassen's Alg.
Matrix Multiplication: Strassen's Alg.

\[ C_{11} = M_1 + M_7 - M_4 - M_3 \]
Matrix Multiplication: Strassen’s Alg.

\[ C_{11} = M_1 + M_7 - M_4 - M_3 \]

\[ C_{12} = M_2 - M_4 \]
Matrix Multiplication: Strassen's Alg.

\[C_{11} = M_1 + M_7 - M_4 - M_3\]

\[C_{12} = M_2 - M_4\]

\[C_{21} = M_3 - M_5\]
Matrix Multiplication: Strassen's Alg.

\[ C_{11} = M_1 + M_7 - M_4 - M_3 \]
\[ C_{12} = M_2 - M_4 \]
\[ C_{21} = M_3 - M_5 \]
\[ C_{22} = M_1 - M_6 - M_2 - M_5 \]
Matrix Multiplication: Strassen's Algorithm

\textbf{Strassen}(A, B)

1. \textbf{let} $n \times n$ be the dimension of $A$ and $B$
2. \textbf{if} $n = 1$
   3. \textbf{then} \textbf{return} $A[1, 1] \cdot B[1, 1]$
4. \textbf{else} partition $A$ into submatrices $A_{11}, A_{12}, A_{21}, A_{22}$
5. partition $B$ into submatrices $B_{11}, B_{12}, B_{21}, B_{22}$
6. $M_1 = \text{Strassen}(A_{11} + A_{22}, B_{11} + B_{22})$
7. $M_2 = \text{Strassen}(A_{11}, B_{12} + B_{22})$
8. $M_3 = \text{Strassen}(A_{22}, B_{11} + B_{21})$
9. $M_4 = \text{Strassen}(A_{11} - A_{12}, B_{22})$
10. $M_5 = \text{Strassen}(A_{22} - A_{21}, B_{11})$
11. $M_6 = \text{Strassen}(A_{11} + A_{21}, B_{11} - B_{12})$
12. $M_7 = \text{Strassen}(A_{12} + A_{22}, B_{21} - B_{22})$
13. $C_{11} = M_1 + M_7 - M_4 - M_3$
14. $C_{12} = M_2 - M_4$
15. $C_{21} = M_3 - M_5$
16. $C_{22} = M_1 - M_6 - M_2 - M_5$
17. assemble $C$ from $C_{11}, C_{12}, C_{21}, C_{22}$
18. \textbf{return} $C$
Matrix Multiplication: Strassen's Algorithm

Strassen(A, B)

1  let n × n be the dimension of A and B
2  if n = 1
3     then return A[1, 1] · B[1, 1]
4  else partition A into submatrices A₁₁, A₁₂, A₂₁, A₂₂
5      partition B into submatrices B₁₁, B₁₂, B₂₁, B₂₂
6      M₁ = Strassen(A₁₁ + A₂₂, B₁₁ + B₂₂)
7      M₂ = Strassen(A₁₁, B₁₂ + B₂₂)
8      M₃ = Strassen(A₂₂, B₁₁ + B₂₁)
9      M₄ = Strassen(A₁₁ − A₁₂, B₂₂)
10     M₅ = Strassen(A₂₂ − A₂₁, B₁₁)
11     M₆ = Strassen(A₁₁ + A₂₁, B₁₁ − B₁₂)
12     M₇ = Strassen(A₁₂ + A₂₂, B₂₁ − B₂₂)
13     C₁₁ = M₁ + M₇ − M₄ − M₃
14     C₁₂ = M₂ − M₄
15     C₂₁ = M₃ − M₅
16     C₂₂ = M₁ − M₆ − M₂ − M₅
17     assemble C from C₁₁, C₁₂, C₂₁, C₂₂
18     return C

Cost: T(n) = 7T(n/2) + Θ(n²) ∈ Θ(n⁴ log 7)
Closest Pair

Given a point set $P$ in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $\|p - q\|_2$ (the Euclidean distance from $p$ to $q$).
Closest Pair

Given a point set $P$ in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $\|p - q\|_2$ (the Euclidean distance from $p$ to $q$).

Can be computed in $O(n^2)$ time. How?
Closest Pair

Given a point set $P$ in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $\|p - q\|_2$ (the Euclidean distance from $p$ to $q$).

Can be computed in $O(n^2)$ time. How?

Can we do better?
Closest Pair: Divide and Conquer

If we divide the point set into the leftmost ⌈n/2⌉ points (L) and the rightmost ⌊n/2⌋ points (R), then the closest pair has

- both points in L,
- both points in R or
- one point in L and one point in R.
Closest Pair: Divide and Conquer

ClosestPair(P)

1 if |P| ≤ 1
2 \hspace{1em} then return Nothing
3 Split P into two sets L and R containing the leftmost \(\lceil n/2 \rceil\) and the rightmost \(\lfloor n/2 \rfloor\) points, respectively.
4 \((p_\ell, q_\ell) = \text{ClosestPair}(L)\)
5 \((p_r, q_r) = \text{ClosestPair}(R)\)
6 \((p_m, q_m) = \text{ClosestPairLR}(L, R)\)
7 return the pair \((p_i, q_i), i \in \{\ell, r, m\}\), that minimizes \(\|p_i - q_i\|_2\)
Closest Pair: Divide and Conquer

ClosestPair(P)

1. if |P| ≤ 1
2. then return Nothing
3. Split P into two sets L and R containing the leftmost ⌈n/2⌉ and the rightmost ⌊n/2⌋ points, respectively.
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5. \((p_r, q_r) = \text{ClosestPair}(R)\)
6. \((p_m, q_m) = \text{ClosestPairLR}(L, R)\)
7. return the pair \((p_i, q_i), i \in \{\ell, r, m\}\), that minimizes \(\|p_i - q_i\|_2\)

Naïve implementation of ClosestPairLR: \(\Theta(n^2)\) time
Closest Pair: Divide and Conquer

ClosestPair(P)

1. if |P| ≤ 1 then return Nothing
2. Split P into two sets L and R containing the leftmost ⌈n/2⌉ and the rightmost ⌊n/2⌋ points, respectively.
3. (p_ℓ, q_ℓ) = ClosestPair(L)
4. (p_r, q_r) = ClosestPair(R)
5. (p_m, q_m) = ClosestPairLR(L, R)
6. return the pair (p_i, q_i), i ∈ {ℓ, r, m}, that minimizes ||p_i - q_i||_2

Naïve implementation of ClosestPairLR: \(\Theta(n^2)\) time

Running time: \(T(n) = 2T(n/2) + \Theta(n^2)\)
Closest Pair: Divide and Conquer

ClosestPair(P)

1. if |P| \leq 1 then return Nothing
2. Split P into two sets L and R containing the leftmost \( \lceil n/2 \rceil \) and the rightmost \( \lfloor n/2 \rfloor \) points, respectively.
3. \((p_\ell, q_\ell) = \text{ClosestPair}(L)\)
4. \((p_r, q_r) = \text{ClosestPair}(R)\)
5. \((p_m, q_m) = \text{ClosestPairLR}(L, R)\)
6. return the pair \((p_i, q_i), i \in \{\ell, r, m\},\) that minimizes \(\|p_i - q_i\|_2\)

Naïve implementation of ClosestPairLR: \(\Theta(n^2)\) time

Running time: \(T(n) = 2T(n/2) + \Theta(n^2) \in \Theta(n^2)\)
Closest Pair: Divide and Conquer

ClosestPair(P)

1. if |P| ≤ 1
2. then return Nothing
3. Split P into two sets L and R containing the leftmost ⌈n/2⌉ and the rightmost ⌊n/2⌋ points, respectively.
4. (p_ℓ, q_ℓ) = ClosestPair(L)
5. (p_r, q_r) = ClosestPair(R)
6. (p_m, q_m) = ClosestPairLR(L, R)
7. return the pair (p_i, q_i), i ∈ {ℓ, r, m}, that minimizes ∥p_i − q_i∥_2

Better implementation of ClosestPairLR: Θ(n) time

Running time: T(n) = 2T(n/2) + Θ(n) ∈ Θ(n lg n)
Closest Pair: Preprocessing

ClosestPair(P)

1. Make two copies X and Y of P
2. Sort the points in X by their x-coordinates
3. Sort the points in Y by their y-coordinates
4. return ClosestPairRec(X, Y)
Closest Pair: Preprocessing

ClosestPair(P)

1. Make two copies X and Y of P
2. Sort the points in X by their x-coordinates
3. Sort the points in Y by their y-coordinates
4. return ClosestPairRec(X, Y)

We prove that ClosestPairRec takes $O(n \log n)$ time.

⇒ The whole algorithm takes $O(n \log n)$ time.
Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

1  if |Y| ≤ 1
2     then return (Nothing, ∞)
3  p = the middle element in X
4  X_ℓ = the part of X up to and including p
5  X_r = the part of X after x
6  Y_ℓ = \{q ∈ Y | q.x ≤ p.x\}
7  Y_r = \{q ∈ Y | q.x > p.x\}
8  (pair_ℓ, δ_ℓ) = ClosestPairRec(X_ℓ, Y_ℓ)
9  (pair_r, δ_r) = ClosestPairRec(X_r, Y_r)
10 (pair_m, δ_m) = ClosestPairLR(Y_ℓ, Y_r, p.x, min(δ_ℓ, δ_r))
11 return the pair (pair_i, δ_i), i ∈ \{ℓ, r, m\}, that minimizes δ_i
Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

1 if |Y| ≤ 1
2 then return (Nothing, ∞)
3 p = the middle element in X
4 X_ℓ = the part of X up to and including p
5 X_r = the part of X after p
6 Y_ℓ = \{q ∈ Y | q.x ≤ p.x\}
7 Y_r = \{q ∈ Y | q.x > p.x\}
8 (pair_ℓ, δ_ℓ) = ClosestPairRec(X_ℓ, Y_ℓ)
9 (pair_r, δ_r) = ClosestPairRec(X_r, Y_r)
10 (pair_m, δ_m) = ClosestPairLR(Y_ℓ, Y_r, p.x, min(δ_ℓ, δ_r))
11 return the pair (pair_i, δ_i), i ∈ \{ℓ, r, m\}, that minimizes δ_i

We already have a pair with distance δ = min(δ_ℓ, δ_r).
⇒ only need to look for pairs with distances < δ.
Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

1. if $|Y| \leq 1$
   - then return (Nothing, $\infty$)
2. $p =$ the middle element in $X$
3. $X_\ell =$ the part of $X$ up to and including $p$
4. $X_r =$ the part of $X$ after $x$
5. $Y_\ell =$ $\langle q \in Y \mid q.x \leq p.x \rangle$
6. $Y_r =$ $\langle q \in Y \mid q.x > p.x \rangle$
7. $(\text{pair}_\ell, \delta_\ell) =$ ClosestPairRec($X_\ell$, $Y_\ell$)
8. $(\text{pair}_r, \delta_r) =$ ClosestPairRec($X_r$, $Y_r$)
9. $(\text{pair}_m, \delta_m) =$ ClosestPairLR($Y_\ell$, $Y_r$, $p.x$, $\min(\delta_\ell, \delta_r)$)
10. return the pair $(\text{pair}_i, \delta_i)$, $i \in \{\ell, r, m\}$, that minimizes $\delta_i$

We prove that ClosestPairLR($Y_\ell$, $Y_r$, $x$, $\delta$) takes $O(n)$ time.

$\Rightarrow \quad T(n) = 2T(n/2) + O(n) \in O(n \lg n)$
Closest Pair: One Left, One Right

ClosestPairLR(Y_ℓ, Y_r, x, δ)

1. \[ Z_ℓ = \langle p \in Y_ℓ \mid x - p.x \leq \delta \rangle \]
2. \[ Z_r = \langle p \in Y_r \mid p.x - x \leq \delta \rangle \]
3. \[ \text{pair} = \text{Nothing} \]
4. \[ \delta' = \infty \]
5. \[ j = 1 \]
6. \[ \text{for } i = 1 \text{ to } |Z_ℓ| \]
7. \[ \text{do while } j < |Z_r| \text{ and } Z_r[j].y < Z_ℓ[i].y - \delta \]
8. \[ \text{do } j = j + 1 \]
9. \[ k = j \]
10. \[ \text{while } k \leq |Z_r| \text{ and } Z_r[k].y \leq Z_ℓ[i].y + \delta \]
11. \[ \text{do if } ||Z_ℓ[i] - Z_r[k]|| < \delta' \]
12. \[ \text{then } \delta' = ||Z_ℓ[i] - Z_r[k]|| \]
13. \[ \text{pair} = (Z_ℓ[i], Z_r[k]) \]
14. \[ k = k + 1 \]
15. \[ \text{return } (\text{pair}, \delta') \]
Closest Pair: One Left, One Right

ClosestPairLR(\(Y_\ell, Y_r, x, \delta\))

1. \(Z_\ell = \langle p \in Y_\ell \mid x - p.x \leq \delta \rangle\)
2. \(Z_r = \langle p \in Y_r \mid p.x - x \leq \delta \rangle\)
3. pair = Nothing
4. \(\delta' = \infty\)
5. \(j = 1\)
6. for \(i = 1\) to \(|Z_\ell|\)
   7. do while \(j < |Z_r|\) and \(Z_r[j].y < Z_\ell[i].y - \delta\)
      8. do \(j = j + 1\)
      9. \(k = j\)
   10. while \(k \leq |Z_r|\) and \(Z_r[k].y \leq Z_\ell[i].y + \delta\)
      11. do if \(\|Z_\ell[i] - Z_r[k]\| < \delta'\)
         12. then \(\delta' = \|Z_\ell[i] - Z_r[k]\|
         13. pair = \((Z_\ell[i], Z_r[k])\)
      14. \(k = k + 1\)
6. return \((\text{pair}, \delta')\)
Closest Pair: One Left, One Right

ClosestPairLR($Y_\ell$, $Y_r$, $x$, $\delta$)

1. $Z_\ell = \langle p \in Y_\ell \mid x - p.x \leq \delta \rangle$
2. $Z_r = \langle p \in Y_r \mid p.x - x \leq \delta \rangle$
3. pair = Nothing
4. $\delta' = \infty$
5. $j = 1$
6. for i = 1 to $|Z_\ell|$
   7. do while $j < |Z_r|$ and $Z_r[j].y < Z_\ell[i].y - \delta$
      8. do $j = j + 1$
   9. $k = j$
   10. while $k \leq |Z_r|$ and $Z_r[k].y \leq Z_\ell[i].y + \delta$
         11. do if $\|Z_\ell[i] - Z_r[k]\| < \delta'$
              then $\delta' = \|Z_\ell[i] - Z_r[k]\|$
              pair = ($Z_\ell[i]$, $Z_r[k]$)
         12. $k = k + 1$
13. return (pair, $\delta'$)
Closest Pair: One Left, One Right

ClosestPairLR($Y_ℓ$, $Y_r$, $x$, $δ$)

1. $Z_ℓ = \langle p ∈ Y_ℓ \mid x - p.x ≤ δ \rangle$
2. $Z_r = \langle p ∈ Y_r \mid p.x - x ≤ δ \rangle$
3. pair = Nothing
4. $δ' = \infty$
5. $j = 1$
6. for $i = 1$ to $|Z_ℓ|$
7. do while $j < |Z_r|$ and $Z_r[j].y < Z_ℓ[i].y - δ$
8. do $j = j + 1$
9. $k = j$
10. while $k ≤ |Z_r|$ and $Z_r[k].y ≤ Z_ℓ[i].y + δ$
11. do if $||Z_ℓ[i] - Z_r[k]|| < δ'$
12. then $δ' = ||Z_ℓ[i] - Z_r[k]||$
13. pair = ($Z_ℓ[i]$, $Z_r[k]$)
14. $k = k + 1$
15. return (pair, $δ'$)
Closest Pair: One Left, One Right

**Lemma:** For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$. 
Closest Pair: One Left, One Right

**Lemma:** For every point \( p \in Z_l \), there exist at most 8 points \( q \in Z_r \) such that 
\[ |q.y - p.y| \leq \delta. \]

**Corollary:** The running time of ClosestPairLR is in \( O(n) \).
**Closest Pair: One Left, One Right**

**Lemma:** For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

**Corollary:** The running time of ClosestPairLR is in $O(n)$. 

[Diagram of points and distances]
Closest Pair: One Left, One Right

Lemma: For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

Corollary: The running time of ClosestPairLR is in $O(n)$. 
**Closest Pair: One Left, One Right**

**Lemma:** For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

**Corollary:** The running time of ClosestPairLR is in $O(n)$.

Two points in the same square have distance at most $\delta/\sqrt{2} < \delta$ from each other.
Closest Pair: One Left, One Right

**Lemma:** For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

**Corollary:** The running time of ClosestPairLR is in $O(n)$.

Two points in the same square have distance at most $\delta/\sqrt{2} < \delta$ from each other.

$\Rightarrow$ At most one point in each of the 8 squares.
Summary

The Divide and Conquer paradigm:

- **Divide** the input into smaller instances of the same problem.
- **Solve** these instances recursively.
- **Combine** the obtained solutions to obtain a solution to the original input.

Divide-and-conquer algorithms always recurse on smaller inputs.

⇒ Natural expression of running time using recurrence relations.
⇒ Natural strategy to prove correctness is induction.

Solving recurrence relations:

- **Substitution**
- **Recursion trees**
- **Master Theorem**