Divide and Conquer

Textbook Reading

Chapters 4, 7 & 33.4

Overview

Design principle

- Divide and conquer

Proof technique

- Induction, induction, induction

Analysis technique

- Recurrence relations

Problems

- Sorting (Merge Sort and Quick Sort)
- Selection
- Matrix multiplication
- Finding the two closest points

Merge Sort

```
MergeSort(A, ℓ, r)
1  if r ≤ ℓ
2       then return
3  m = ⌊(ℓ + r)/2⌋
4  MergeSort(A, ℓ, m)
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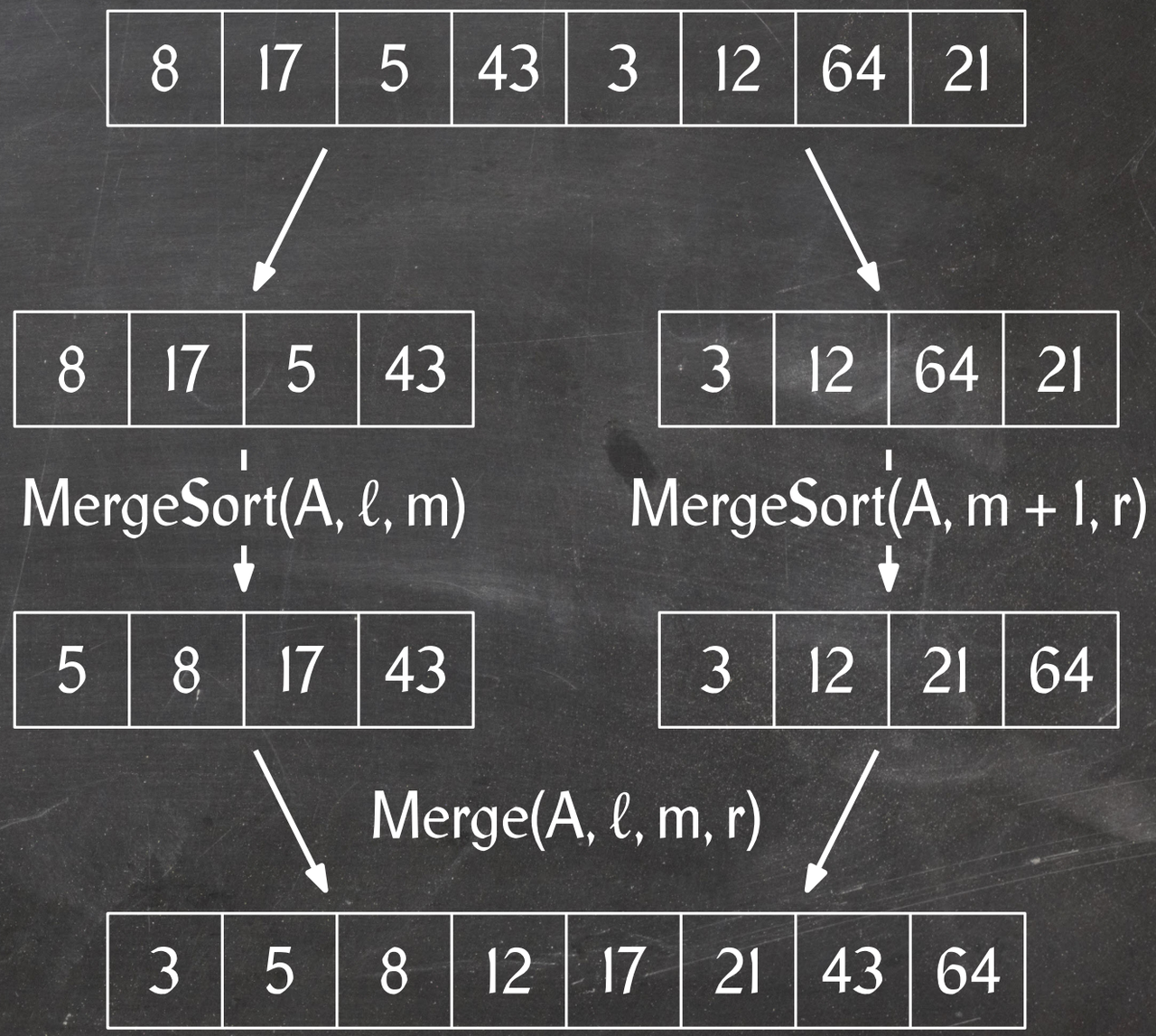

Merging Two Sorted Lists

Merge(A, ℓ, m, r)

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 2   n2 = r − m
 3   for i = 1 to n1
 4       do L[i] = A[ℓ + i − 1]
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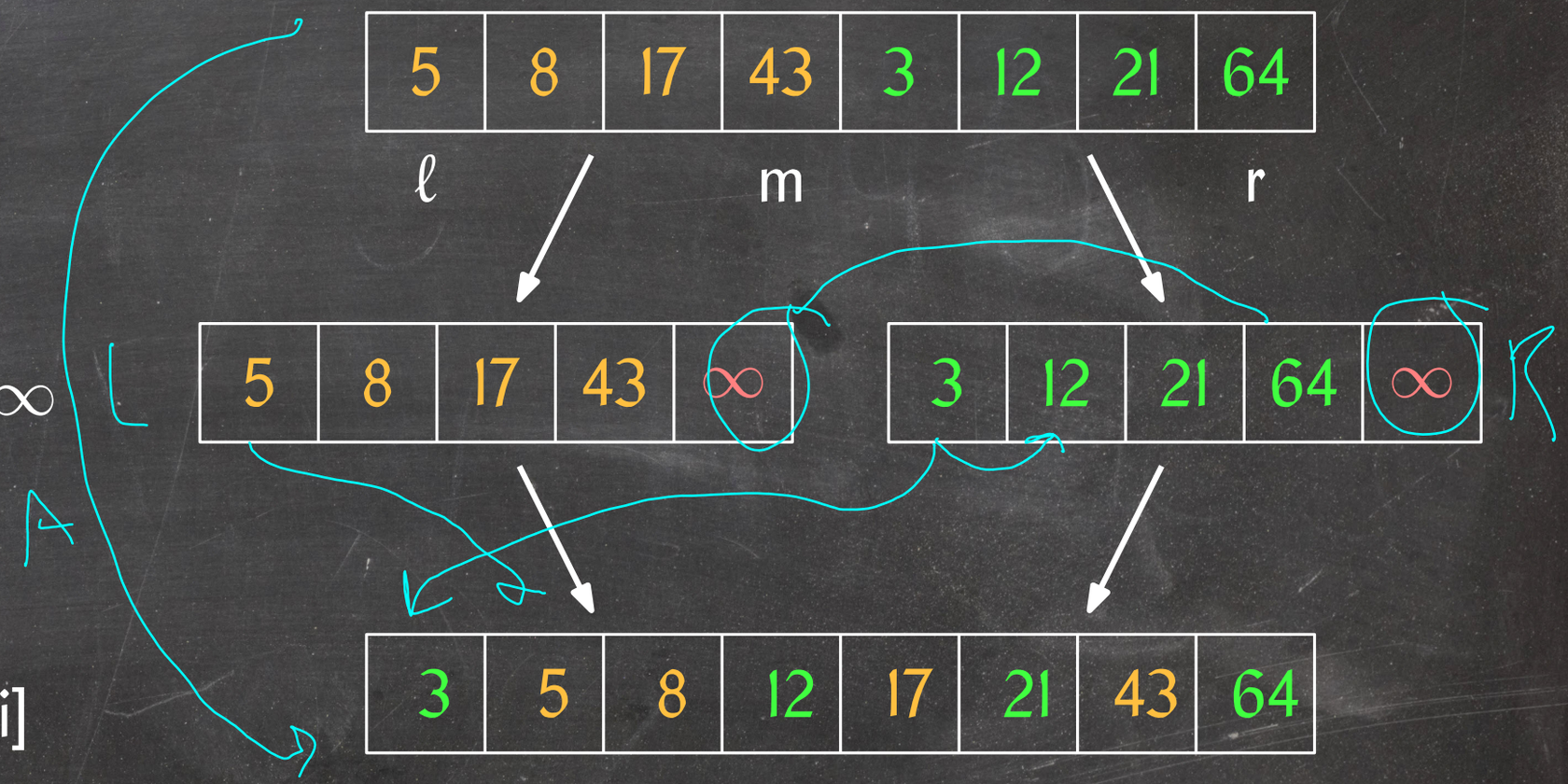
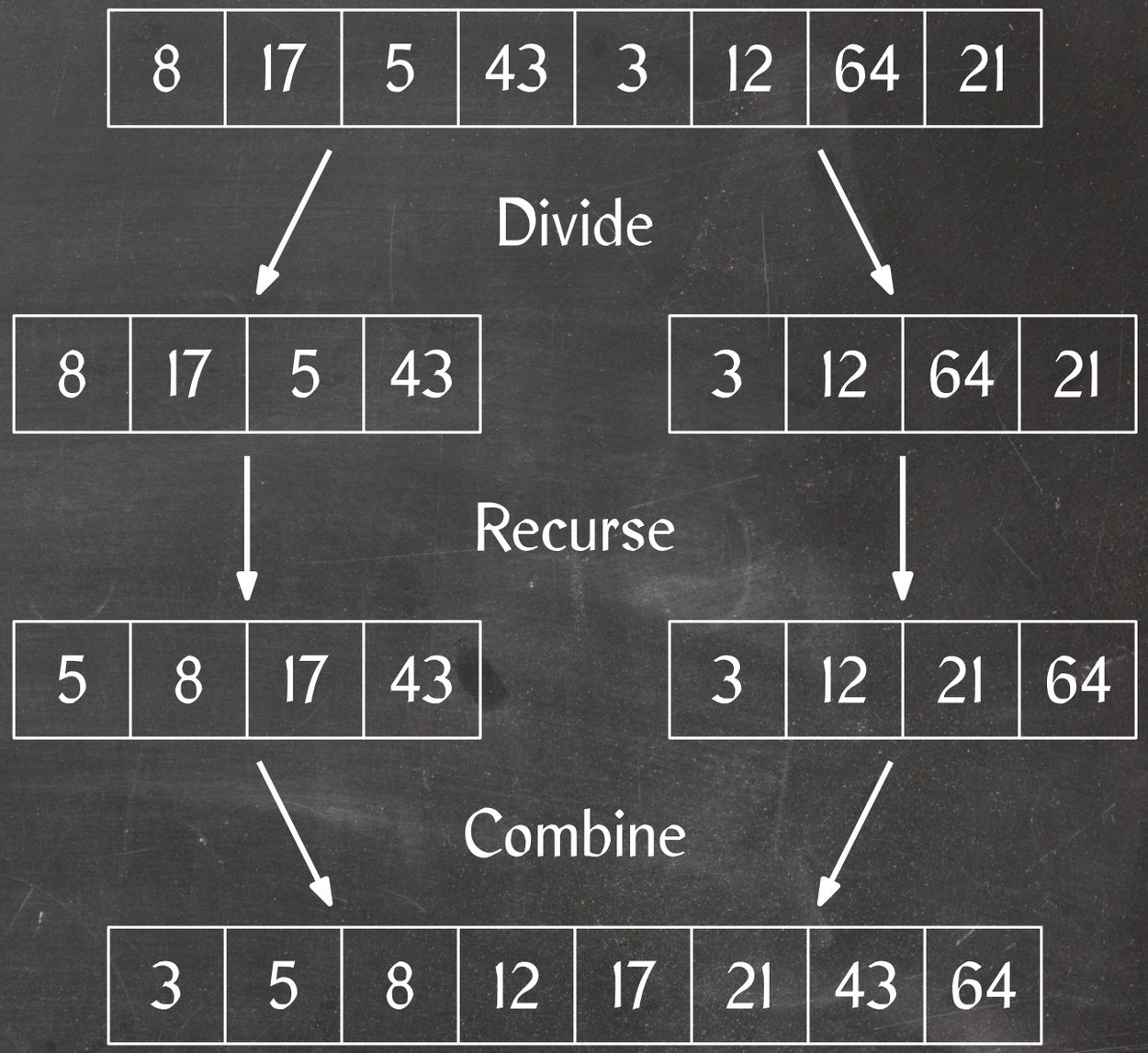


Divide and Conquer

Three steps:

- Divide the input into smaller parts.
- Recursively solve the same problem on these smaller parts.
- Combine the solutions computed by the recursive calls to obtain the final solution.



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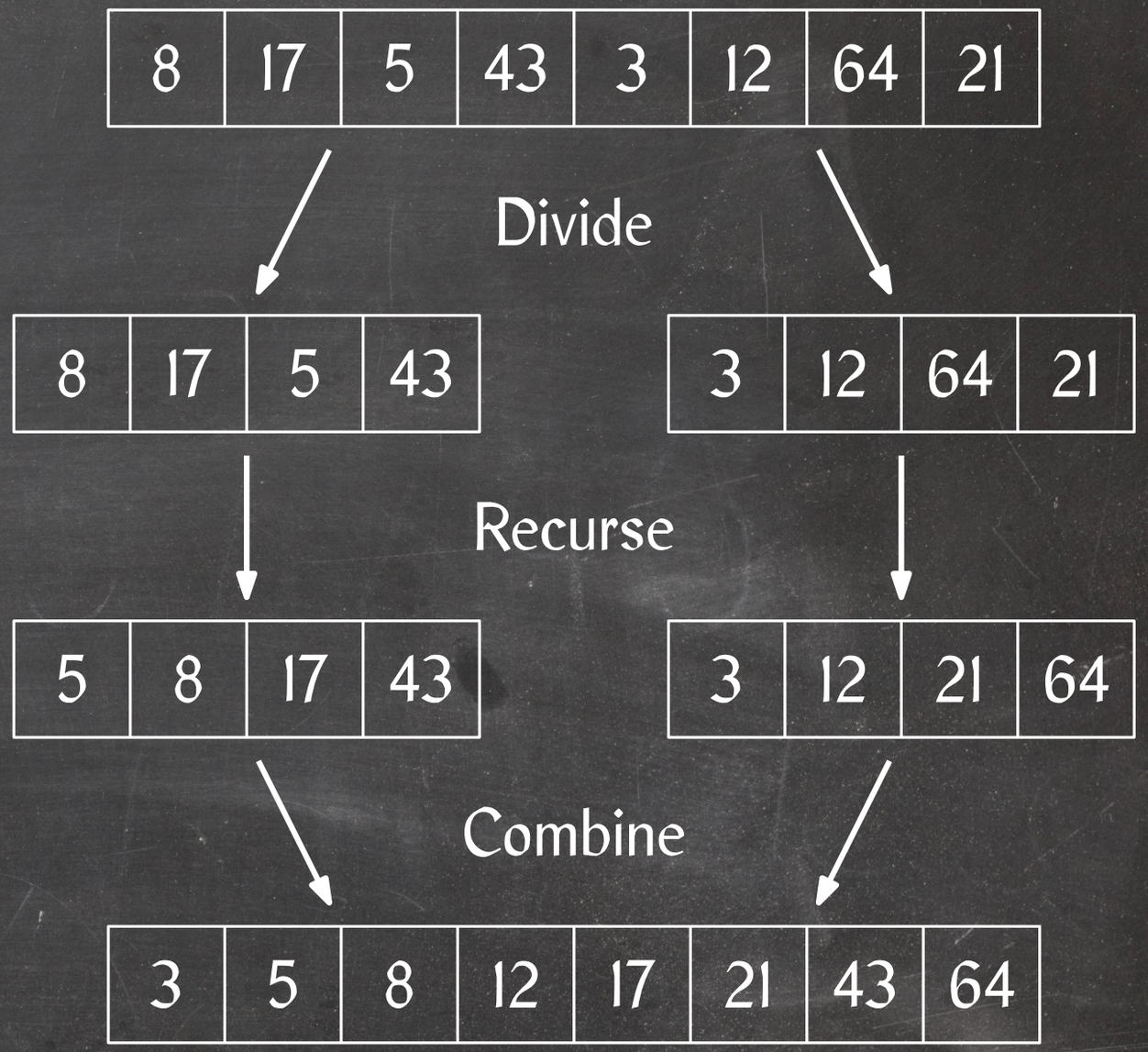

This can be viewed as a reduction technique, reducing the original problem to simpler problems to be solved in the divide and/or combine steps.

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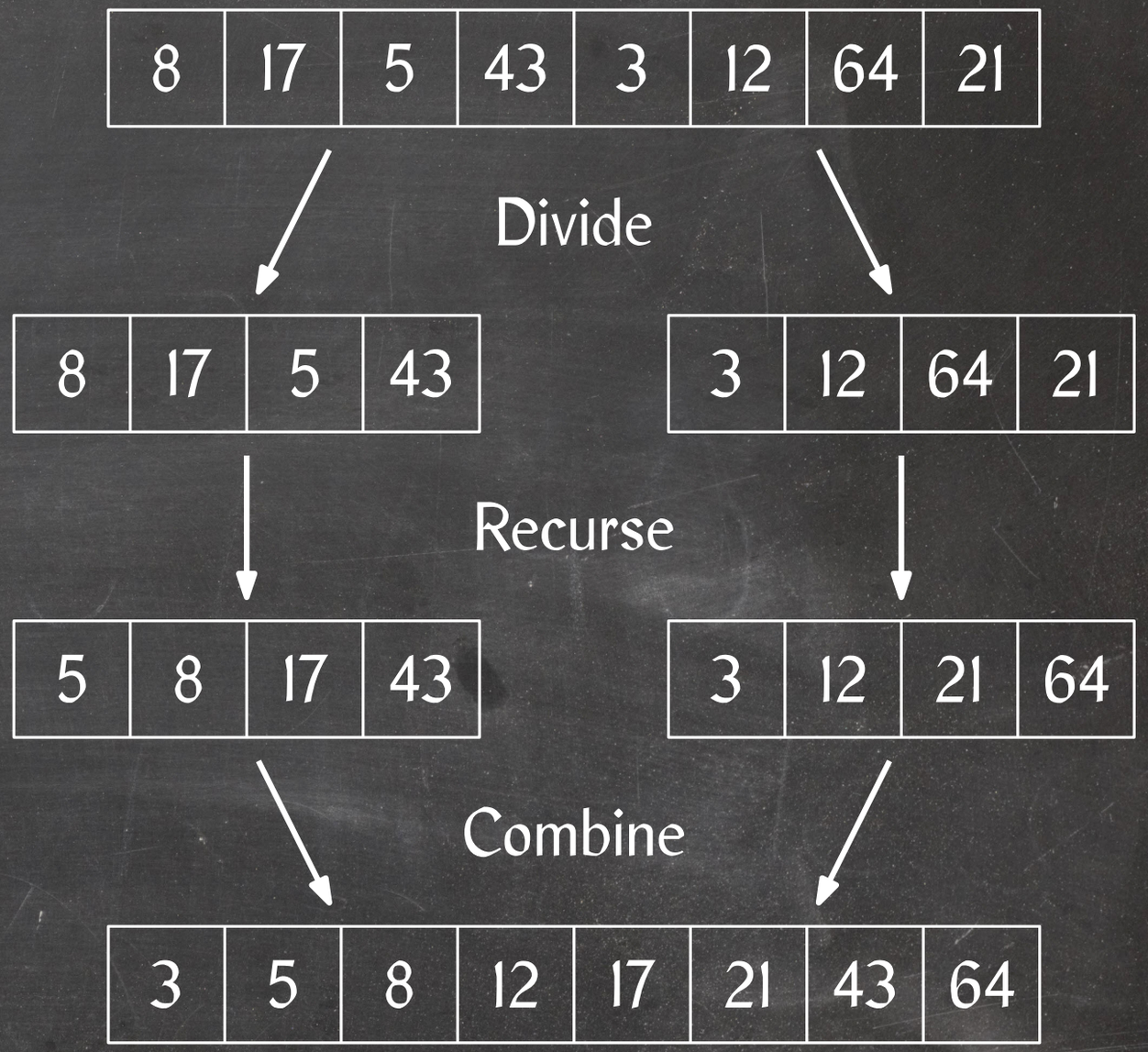

This can be viewed as a reduction technique, reducing the original problem to simpler problems to be solved in the divide and/or combine steps.

Example: Once we unfold the recursion of Merge Sort, we're left with nothing but Merge steps. Thus, we reduce sorting to the simpler problem of merging sorted lists.

Loop Invariants

... are a technique for proving the correctness of an iterative algorithm.

The invariant states conditions that should hold before and after each iteration.

Correctness proof using a loop invariant:

Initialization: Prove the invariant holds before the first iteration.

Maintenance: Prove each iteration maintains the invariant.

Termination: Prove that the correctness of the invariant after the last iteration implies correctness of the algorithm.

Correctness of Merge

Merge(A, ℓ, m, r)

```
 1   n1 = m − ℓ + 1
 2   n2 = r − m
 3   for i = 1 to n1
 4       do L[i] = A[ℓ + i − 1]
 5   for i = 1 to n2
 6       do R[i] = A[m + i]
 7   L[n1 + 1] = R[n2 + 1] = +∞
 8   i = j = 1
 9   for k = ℓ to r
10       do if L[i] ≤ R[j]
11             then A[k] = L[i]
12                  i = i + 1
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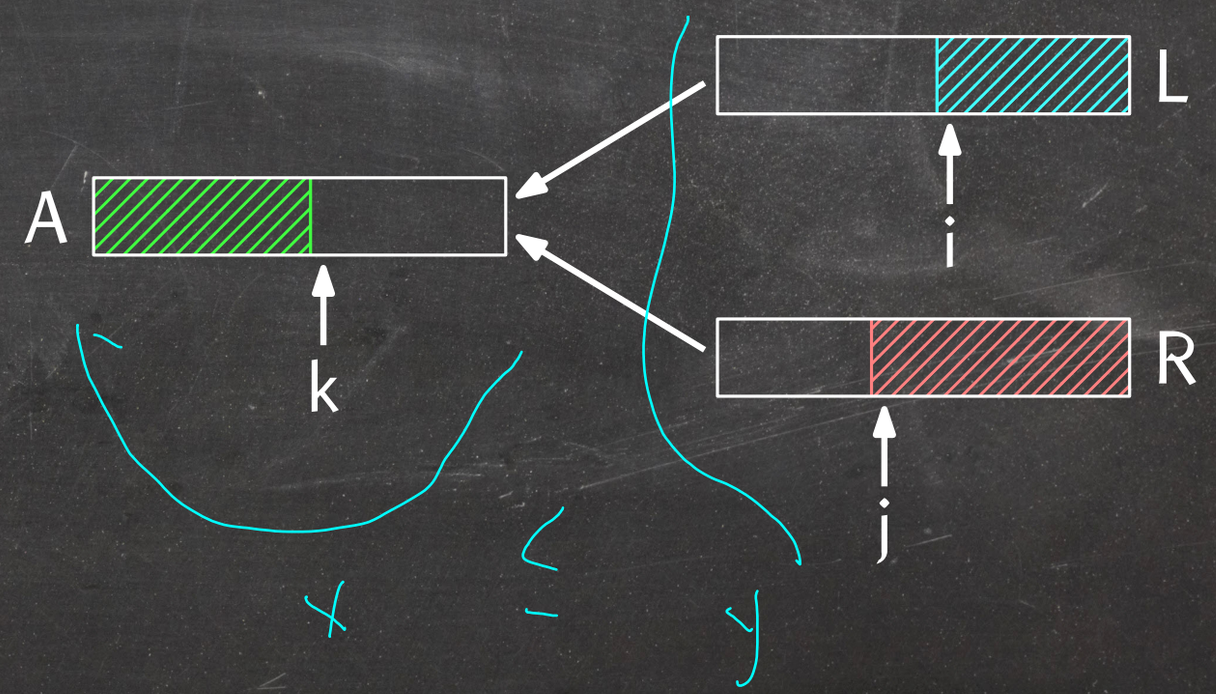
```

Loop invariant:

- already merged
- sorted but not merged
- $A[\ell \,..\, k-1] \cup L[i \,..\, n_1] \cup R[j \,..\, n_2]$ is the set of elements originally in $A[\ell \,..\, r]$.
- $A[\ell \,..\, k-1]$, $L[i \,..\, n_1]$, and $R[j \,..\, n_2]$ are sorted.
- $x \le y$ for all $x \in A[\ell \,..\, k-1]$ and $y \in L[i \,..\, n_1] \cup R[j \,..\, n_2]$.



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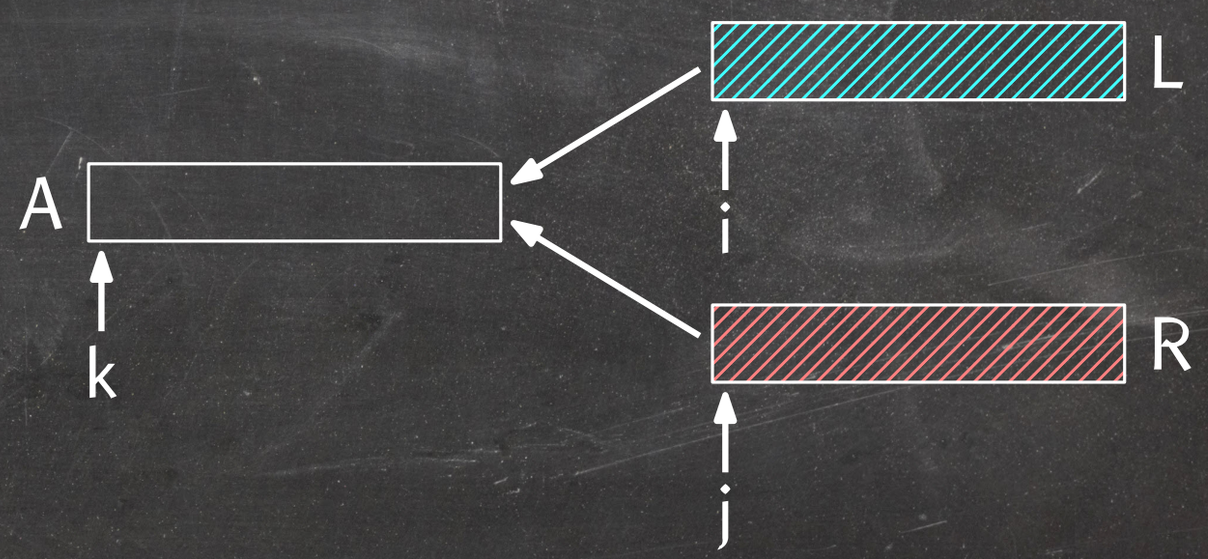
Initialization:

- $A[\ell \,..\, m]$ is copied to $L[1 \,..\, n_1]$.
- $A[m+1 \,..\, r]$ is copied to $R[1 \,..\, n_2]$.
- $i = 1$, $j = 1$, $k = 1$.



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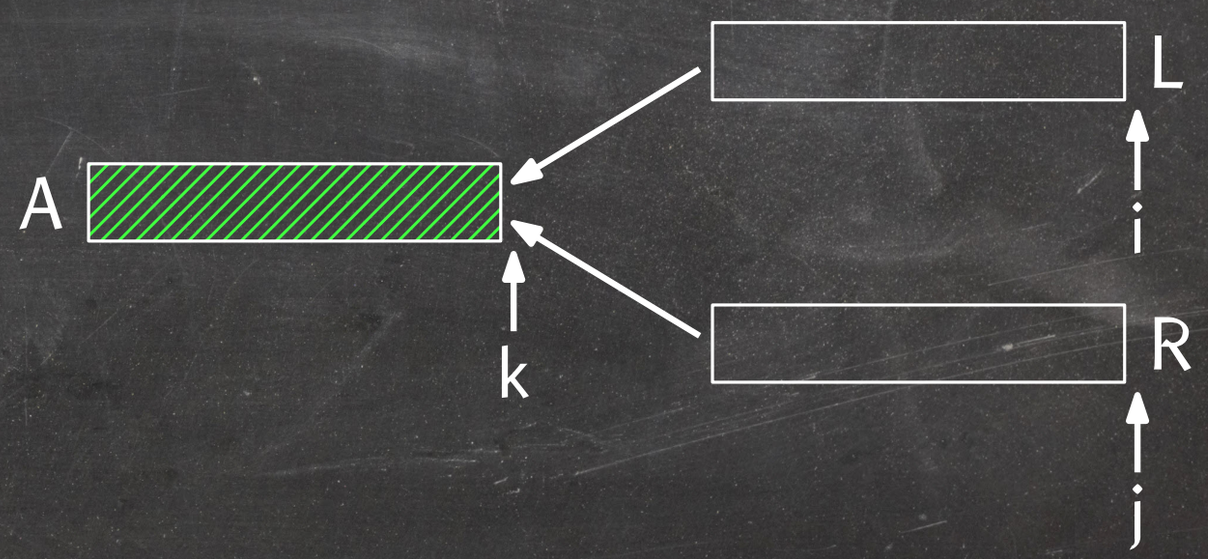
Termination:

- $k = r + 1$

$\Rightarrow$ $A[\ell \,..\, r]$ contains all items it contained initially, in sorted order.



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Maintenance:

- $A[k'] \leq L[i]$ for all $k' < k$
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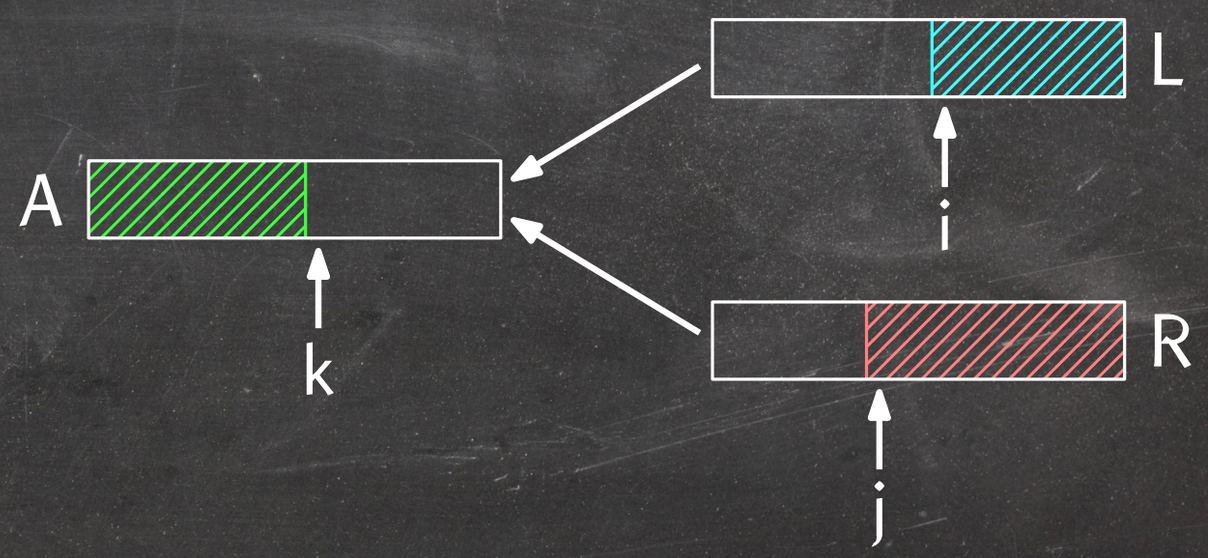

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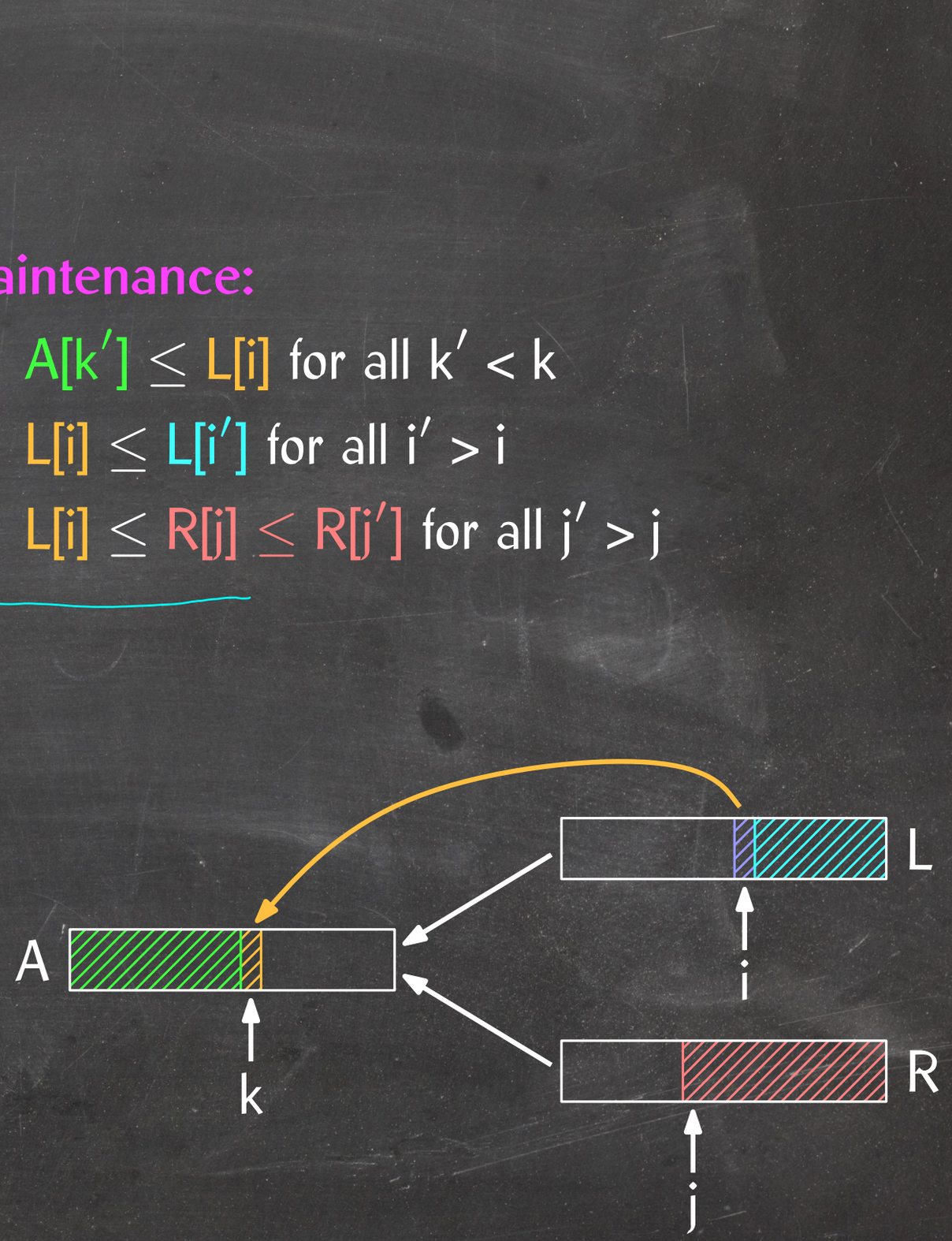

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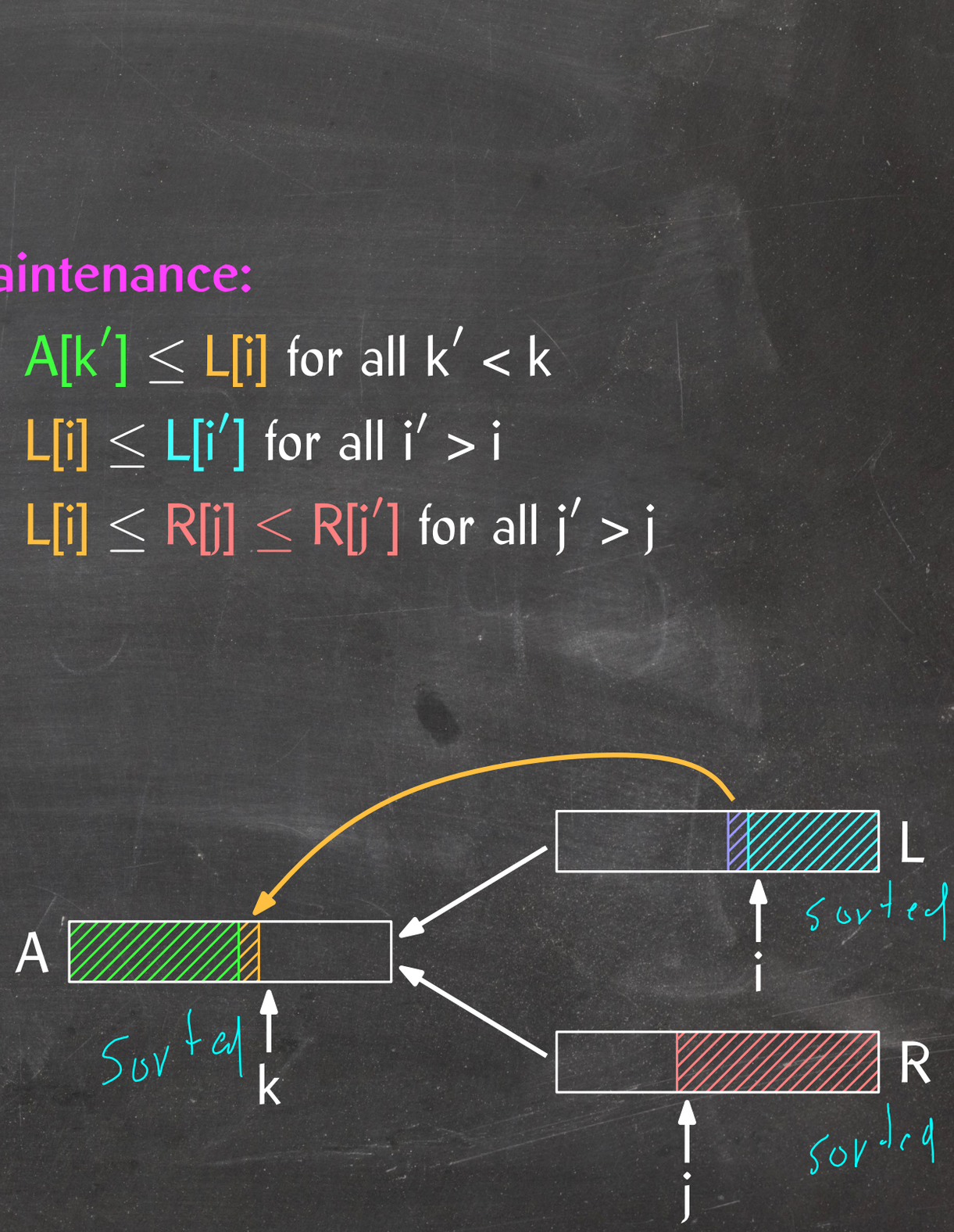

Correctness of Merge Sort

Lemma: Merge Sort correctly sorts any input array.

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MergeSort(A, ℓ, r)
1  if r ≤ ℓ
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Inductive step: (n > 1)

- The left and right halves have size less than n each.
- By the inductive hypothesis, the recursive calls sort them correctly.
- Merge correctly merges the two sorted sequences.

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Correctness of Divide and Conquer Algorithms

Divide and conquer algorithms are the algorithmic incarnation of induction:

Base case: Solve trivial instances directly, without recursing.

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Correctness of Divide and Conquer Algorithms

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$\Rightarrow$ Induction is the natural proof method for divide and conquer algorithms.

Recurrence Relations

A recurrence relation defines the value f(n) of a function f(·) for argument n in terms of the values of f(·) for arguments smaller than n.

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Examples:

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$$F_n = \begin{cases} 1 & n = 0 \text{ or } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

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Binomial coefficients

$$B(n, k) = \begin{cases} 1 & k = 1 \text{ or } k = n \\ B(n-1, k-1) + B(n-1, k) & \text{otherwise} \end{cases}$$

A Recurrence for Merge Sort

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Analysis:

If n = 0 or n = 1, we spend constant time to figure out that there is nothing to do and then exit.

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- Spend linear time to merge the two resulting sorted sequences.

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recursive calls work local to function

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A Recurrence for Binary Search

```
BinarySearch(A, ℓ, r, x)
1   if r < ℓ
2       then return "no"
3   m = ⌊(ℓ + r)/2⌋
4   if x = A[m]
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6   if x < A[m] –
7       then return BinarySearch(A, ℓ, m – 1, x)
8       else return BinarySearch(A, m + 1, r, x)
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Constant local work

Recurrence:

$$T(n) = T(\lfloor n/2 \rfloor) + \Theta(1)$$

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Simplified Recurrence Notation

The recurrences we use to analyze algorithms all have a base case of the form

$$T(n) \leq c \; \forall n \leq n_0.$$

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Floors and ceilings usually affect the value of T(n) by only a constant factor.

Simplified Recurrence Notation

The recurrences we use to analyze algorithms all have a base case of the form

$$T(n) \leq c \; \forall n \leq n_0.$$

The exact choices of c and n_0 affect the value of T(n) for any n by only a constant factor.

Floors and ceilings usually affect the value of T(n) by only a constant factor.

So we are lazy and write

- Merge Sort: $T(n) = 2T(n/2) + \Theta(n)$
- Binary search: $T(n) = T(n/2) + \Theta(1)$

"Solving" Recurrences

Given two algorithms A and B with running times

$$T_A(n) = 2T(n/2) + \Theta(n)$$
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A recurrence for $T(n)$ precisely defines $T(n)$, but it is hard for us to look at the function and say which one grows faster.

$\Rightarrow$ We want a closed-form expression for $T(n)$, that is, one of the form $T(n) \in \Theta(n)$, $T(n) \in \Theta(n^2)$, ..., one that does not depend on $T(n')$ for any $n' < n$.

Methods for Solving Recurrences

Substitution:

- Guess the solution.
  (Intuition, experience, black magic, recursion trees, trial and error)
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Recursion trees:

- Draw a tree that visualizes how the recurrence unfolds.
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Master Theorem:

- Cook book recipe for solving common recurrences.
- Immediately tells us the solution after we verify some simple conditions to determine which case of the theorem applies.

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Recurrence:

$T(n) = 2T(n/2) + O(n)$, that is,

$T(n) \leq 2T(n/2) + an$, for some $a > 0$ and all $n \geq n_0$.

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Guess:

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Base case:

For $2 \leq n < 4$, $T(n) \leq c' \leq c'n \leq c'n \lg n$, for some $c' > 0$.

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$$T(n) \leq 2T(n/2) + an$$
$$\leq 2 \cdot \left(\frac{cn}{2} \lg \frac{n}{2}\right) + an$$

$T(n') \leq cn' \lg n' \quad \forall n' < n$

$T\left(\frac{n}{2}\right) \leq c\frac{n}{2} \lg \frac{n}{2}$

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$$\begin{aligned} T(n) &\leq 2T(n/2) + an \\ &\leq 2 \cdot \left(\frac{cn}{2} \lg \frac{n}{2}\right) + an \\ &= cn(\lg n - 1) + an \end{aligned}$$

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Notes:

- We only proved the upper bound. The lower bound, $T(n) \in \Omega(n \lg n)$ can be proven analogously.

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T(3)

T(2) + T(1) + Θ(1)

↑ base case

Notes:

- We only proved the upper bound. The lower bound, $T(n) \in \Omega(n \lg n)$ can be proven analogously.
- Since the base case is valid only for $n \geq 2$ and we use the inductive hypothesis for $n/2$ in the inductive step, the inductive step is valid only for $n \geq 4$. Hence, a base case for $2 \leq n < 4$.

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Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

$$T(n) = 2T(n/2) + O(n) \quad \text{to} \quad T(n) \leq 2T(n/2) + an$$

and the claim

$$T(n) \in O(n \lg n) \quad \text{to} \quad T(n) \leq cn \lg n?$$

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If we're not careful, we may "prove" nonsensical results:

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Claim: $T(n) \in O(n)$

Note that $T(n) \in \Theta(n^2)$!

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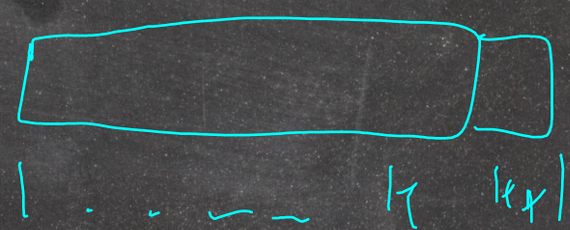

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$O(n) + n = O(n)$

$T(n) = T(n-1) + n \leq c(n-1) + n =$
$cn + (n - c) > cn!$ not a factorial

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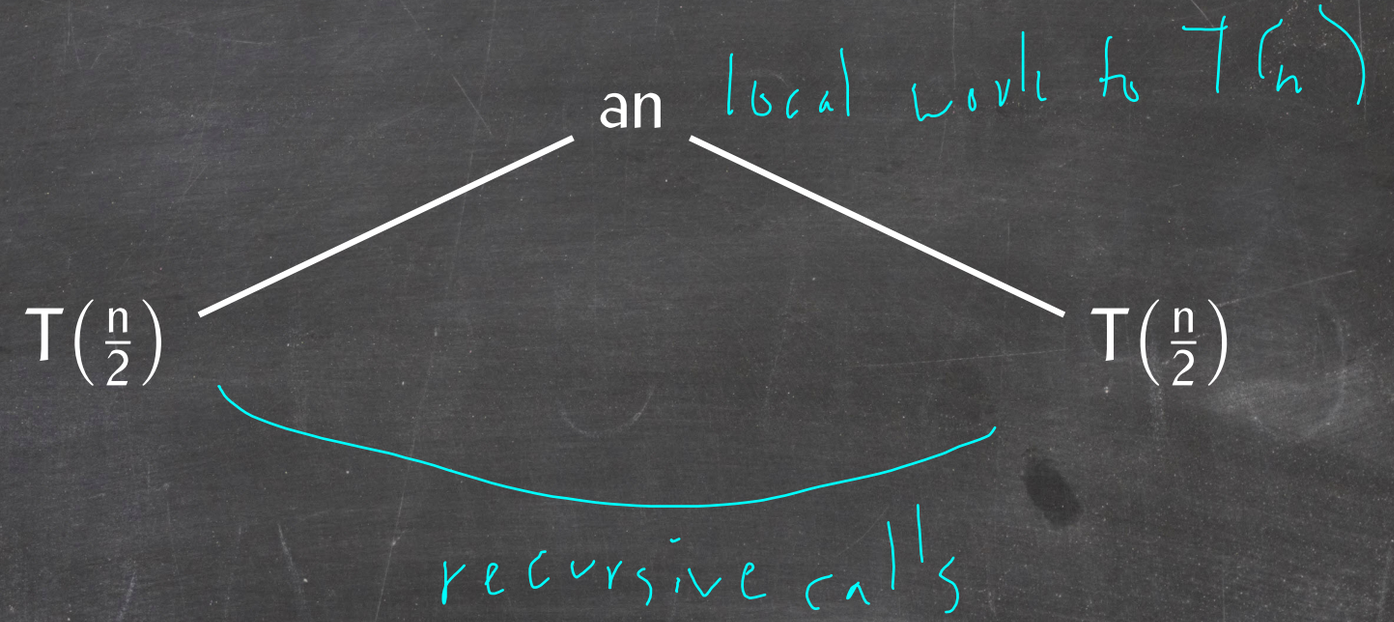

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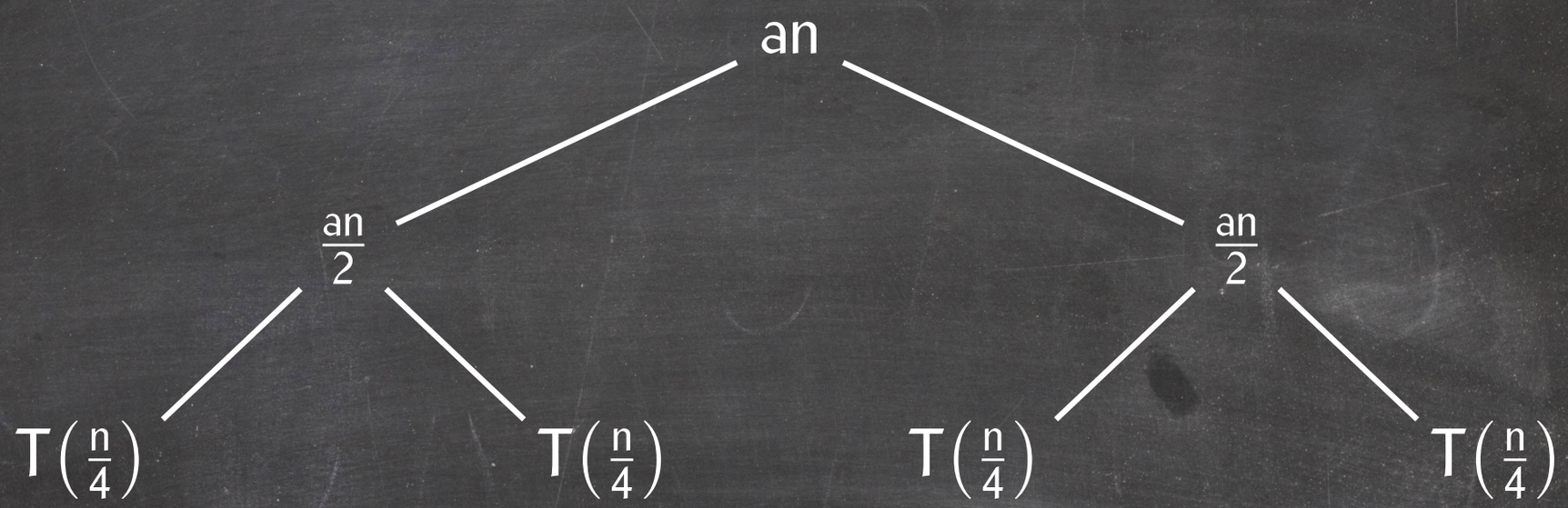

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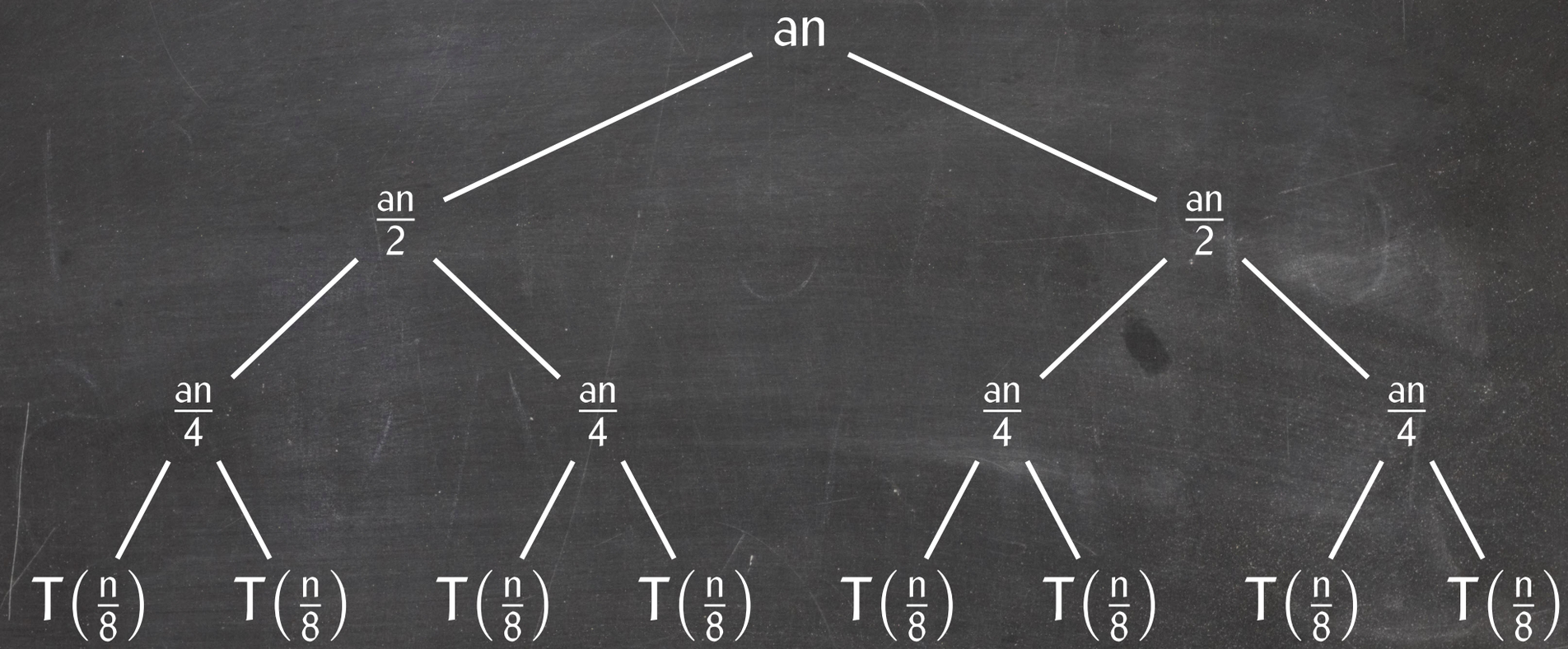

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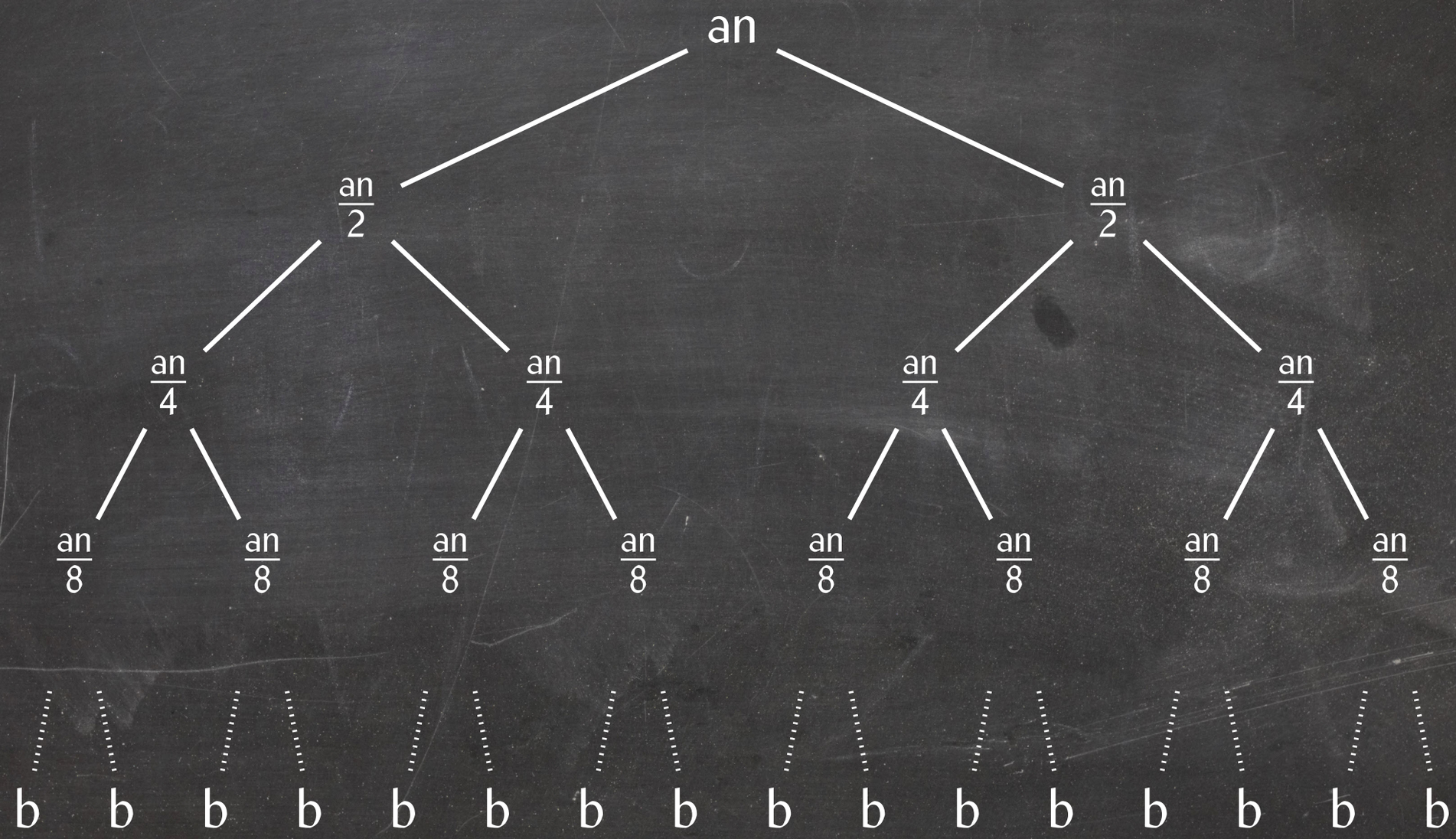

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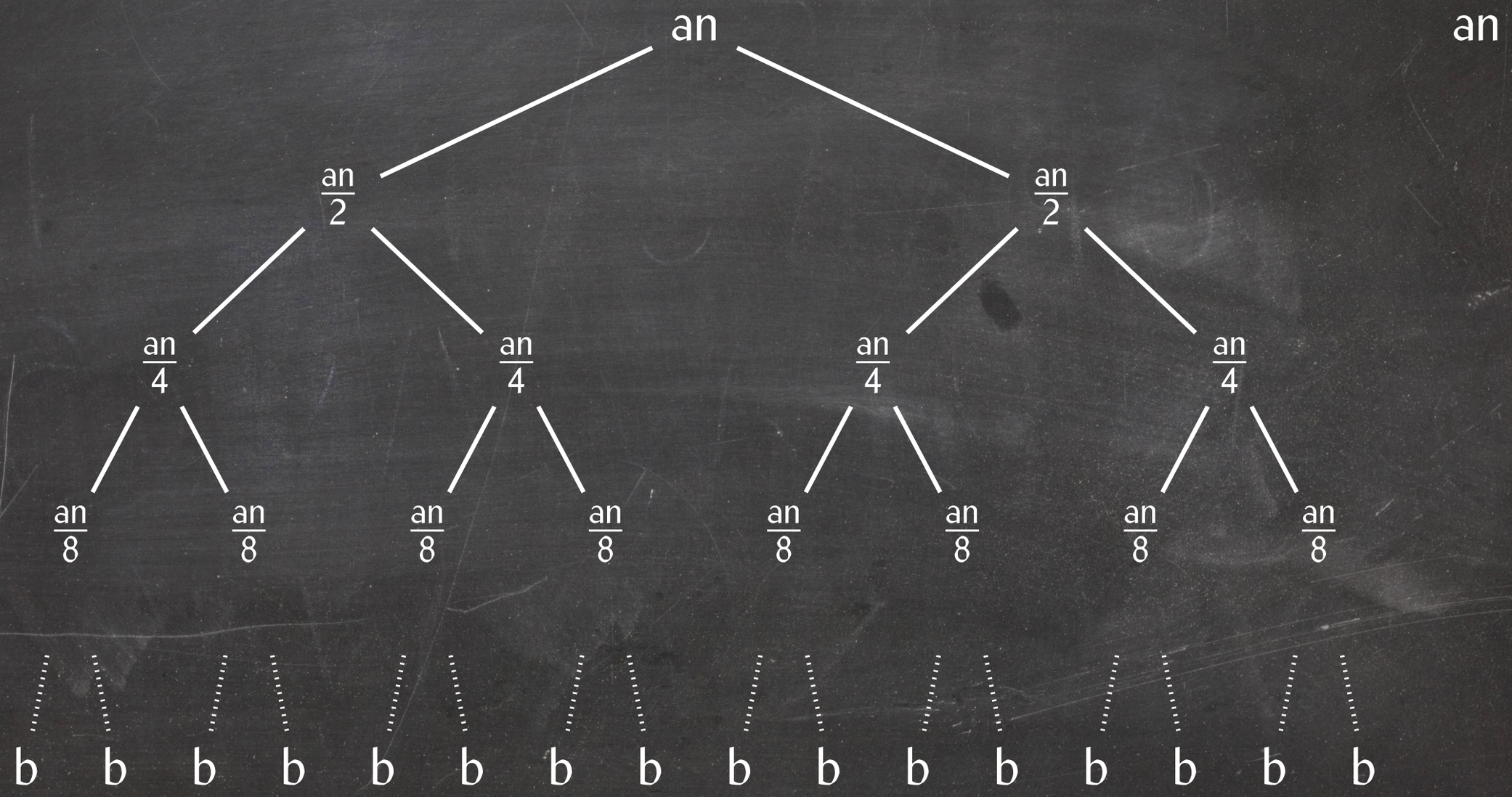

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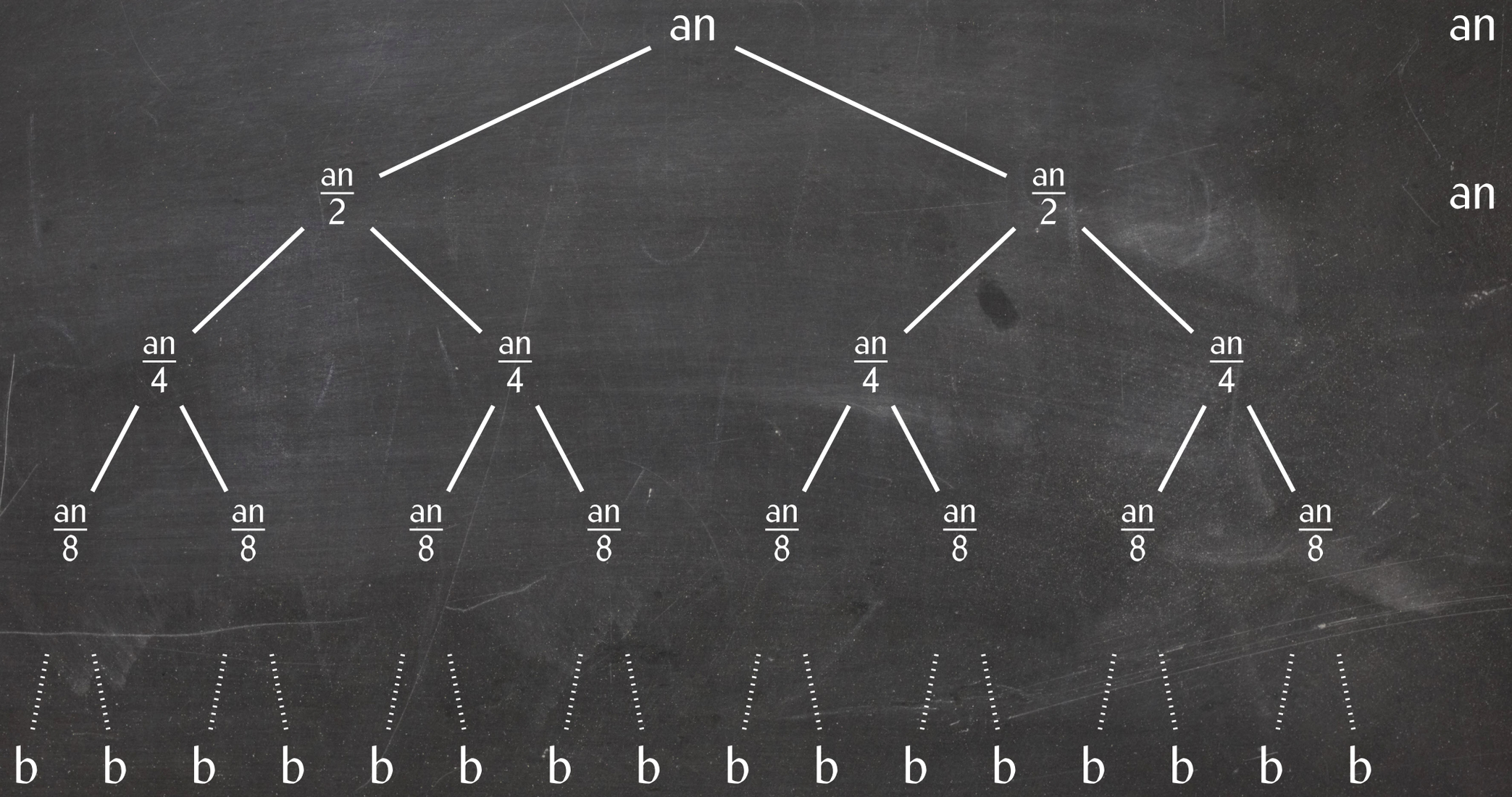

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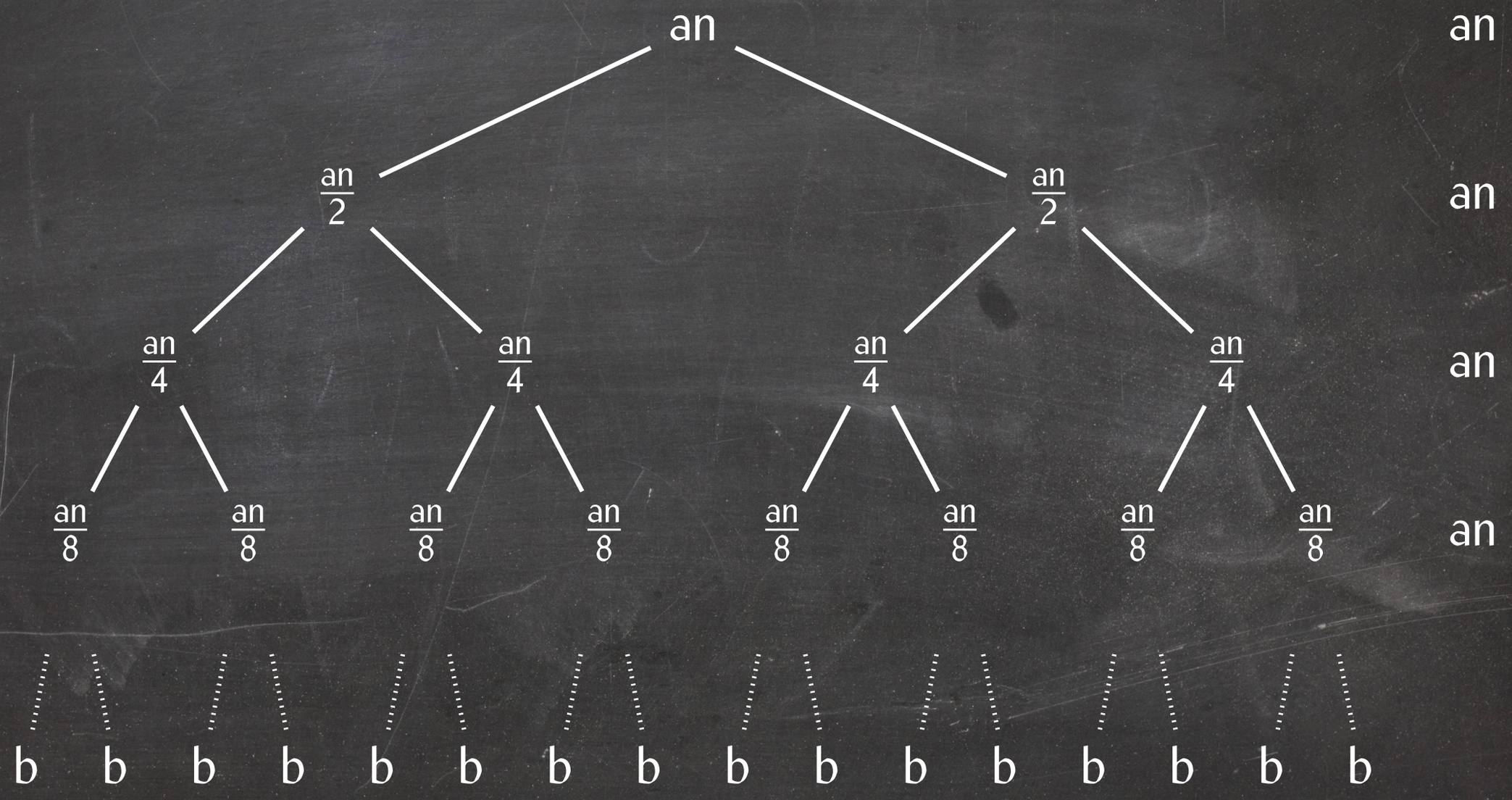

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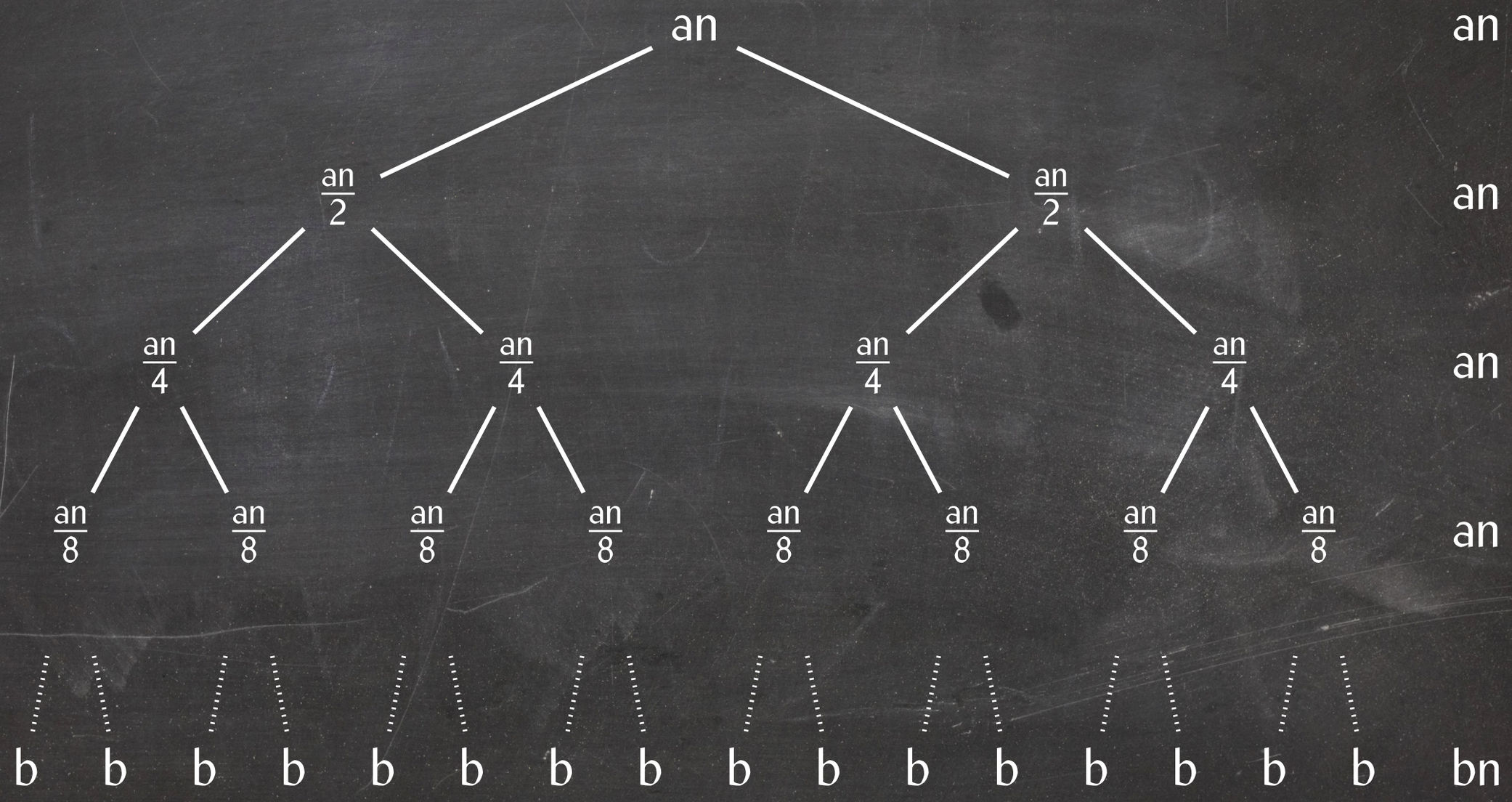

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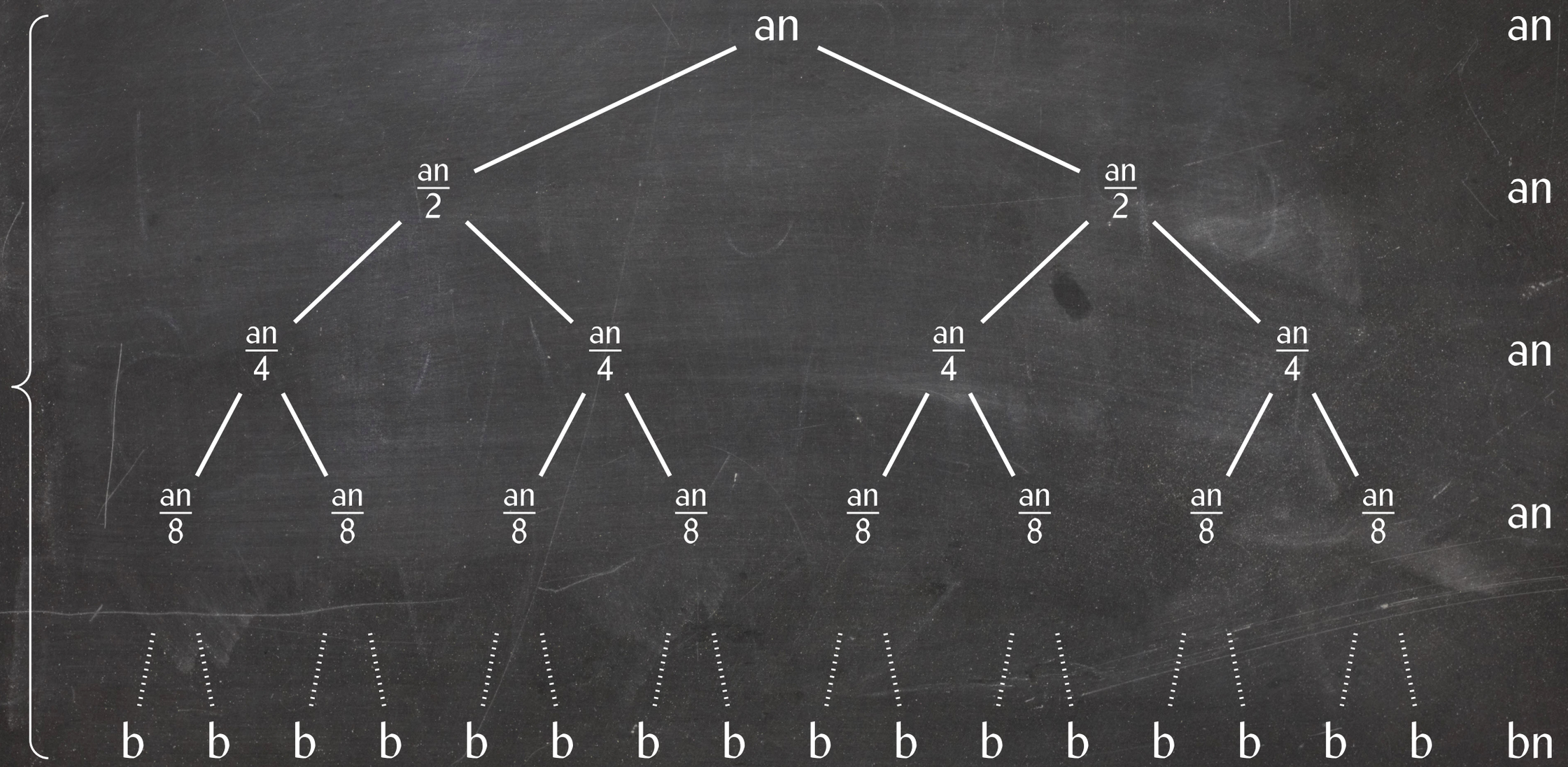

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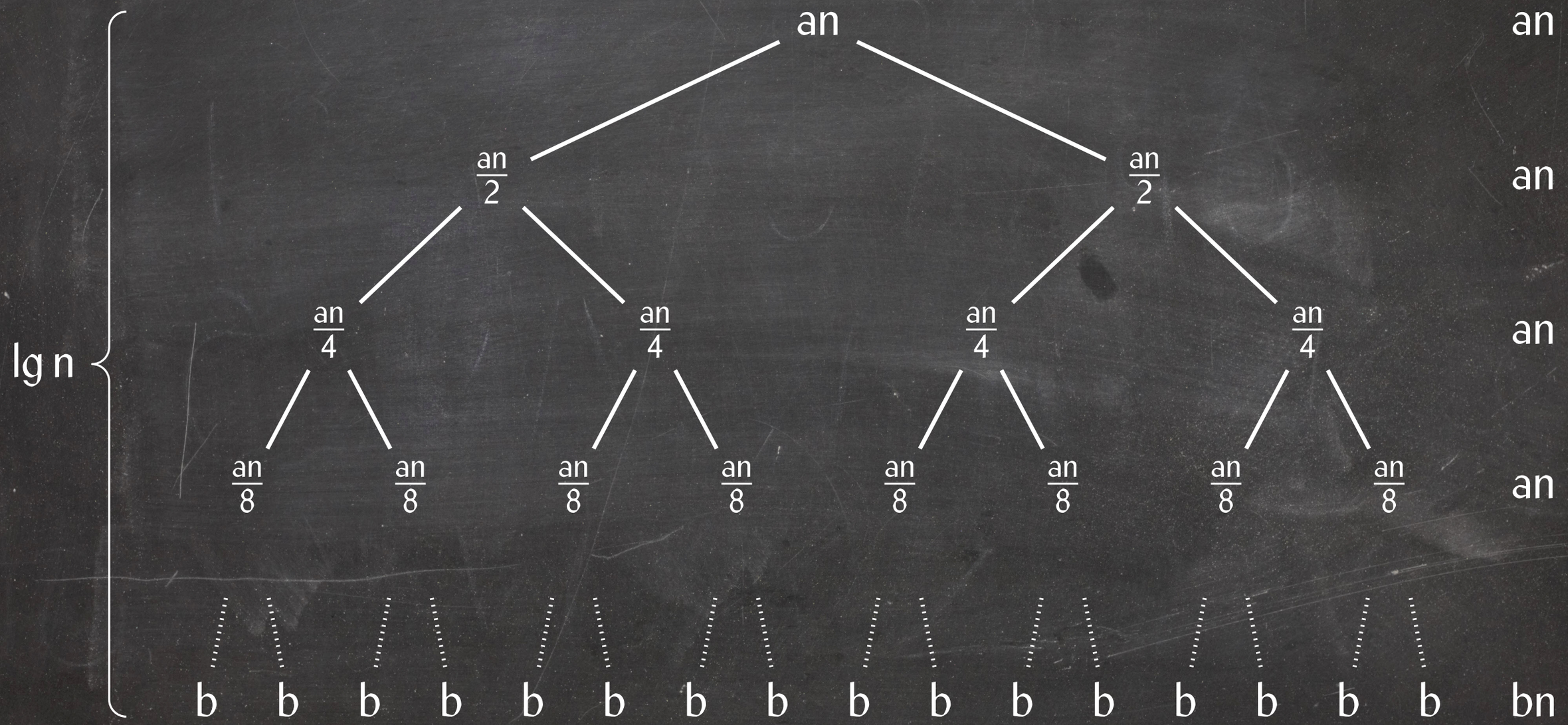

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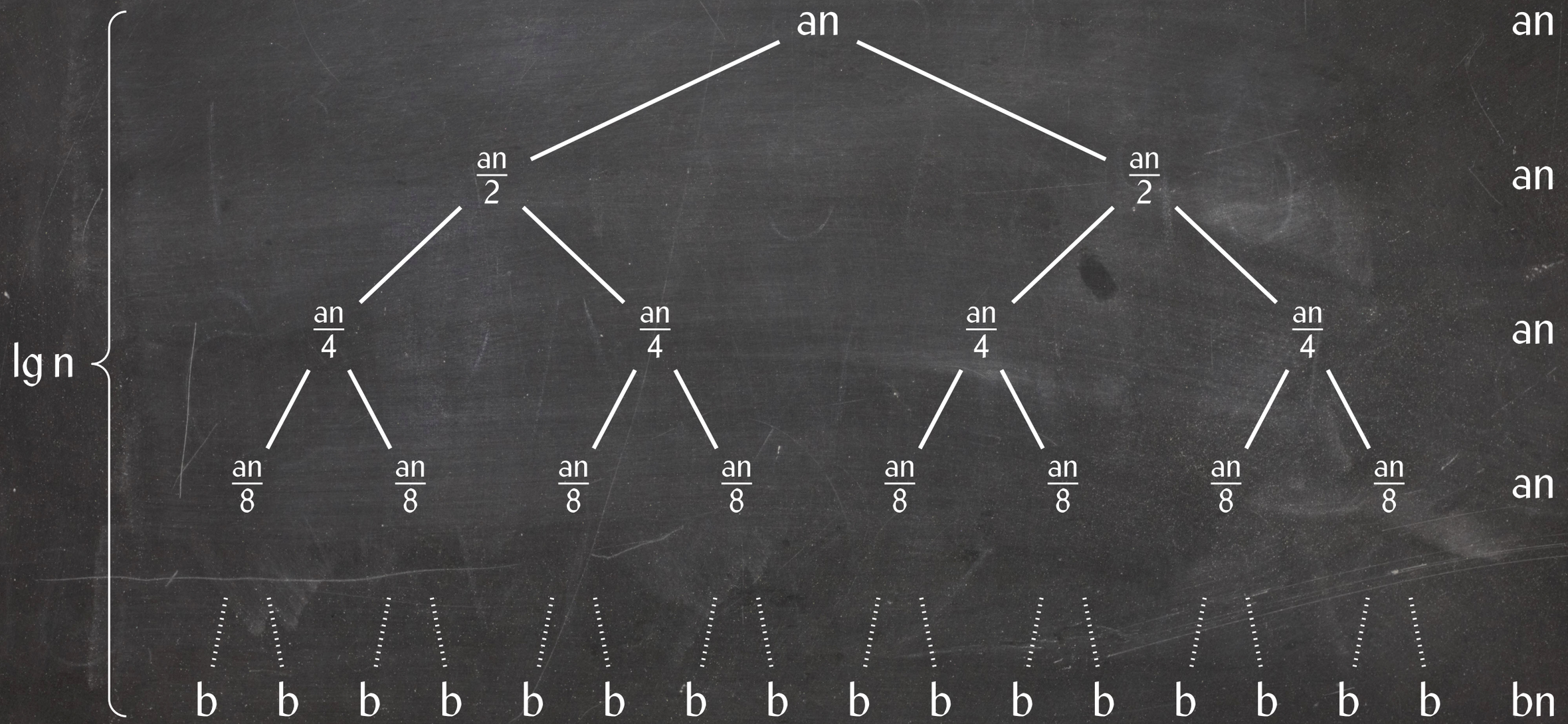

Solution: $T(n) \in \Theta(n \lg n)$

lgn · an + bn
height · work per level + base case

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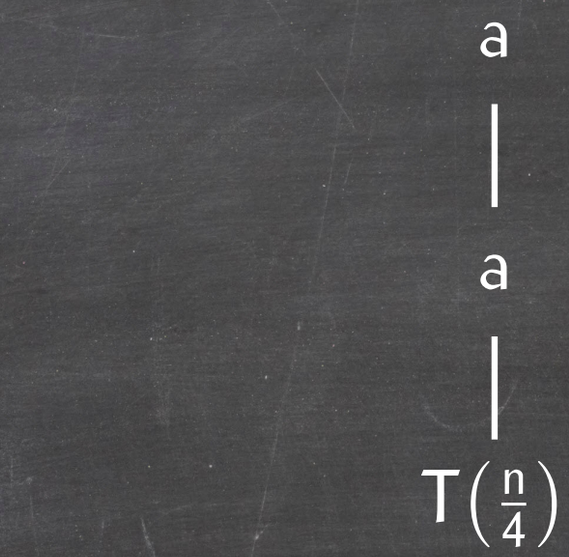

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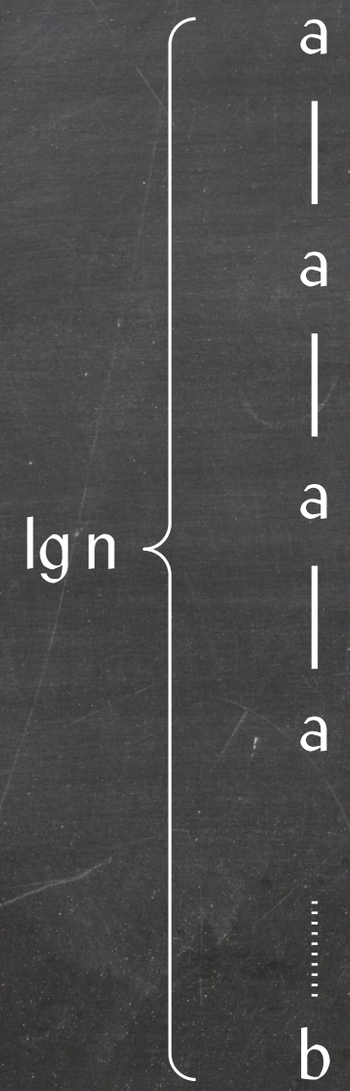

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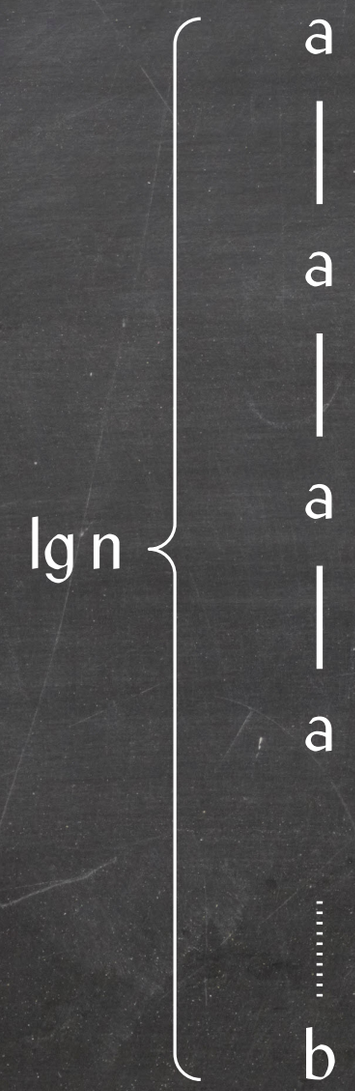

Solution: $T(n) \in \Theta(\lg n)$

A Less Obvious Recursion Tree

Recurrence: $T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an$

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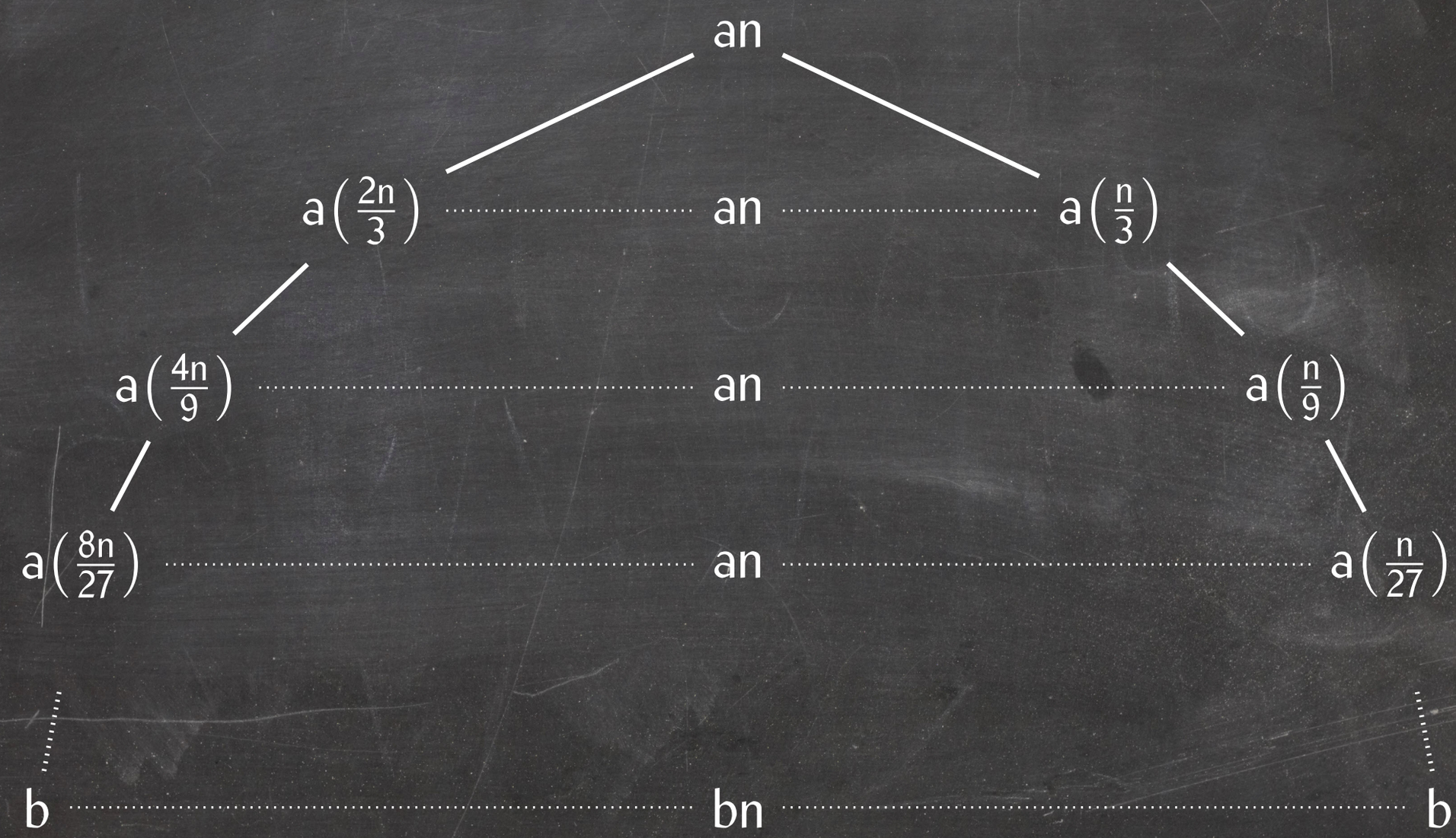

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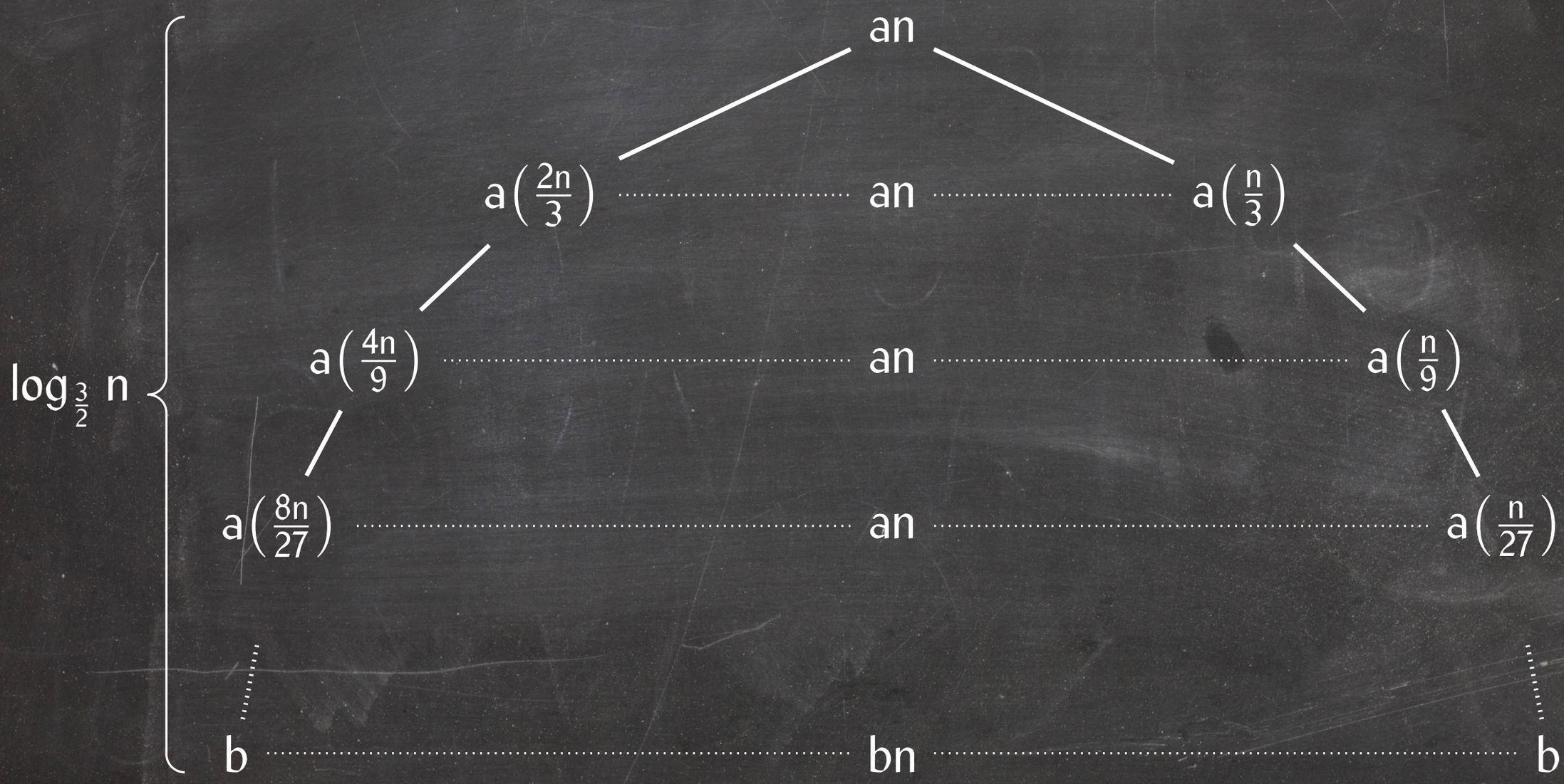

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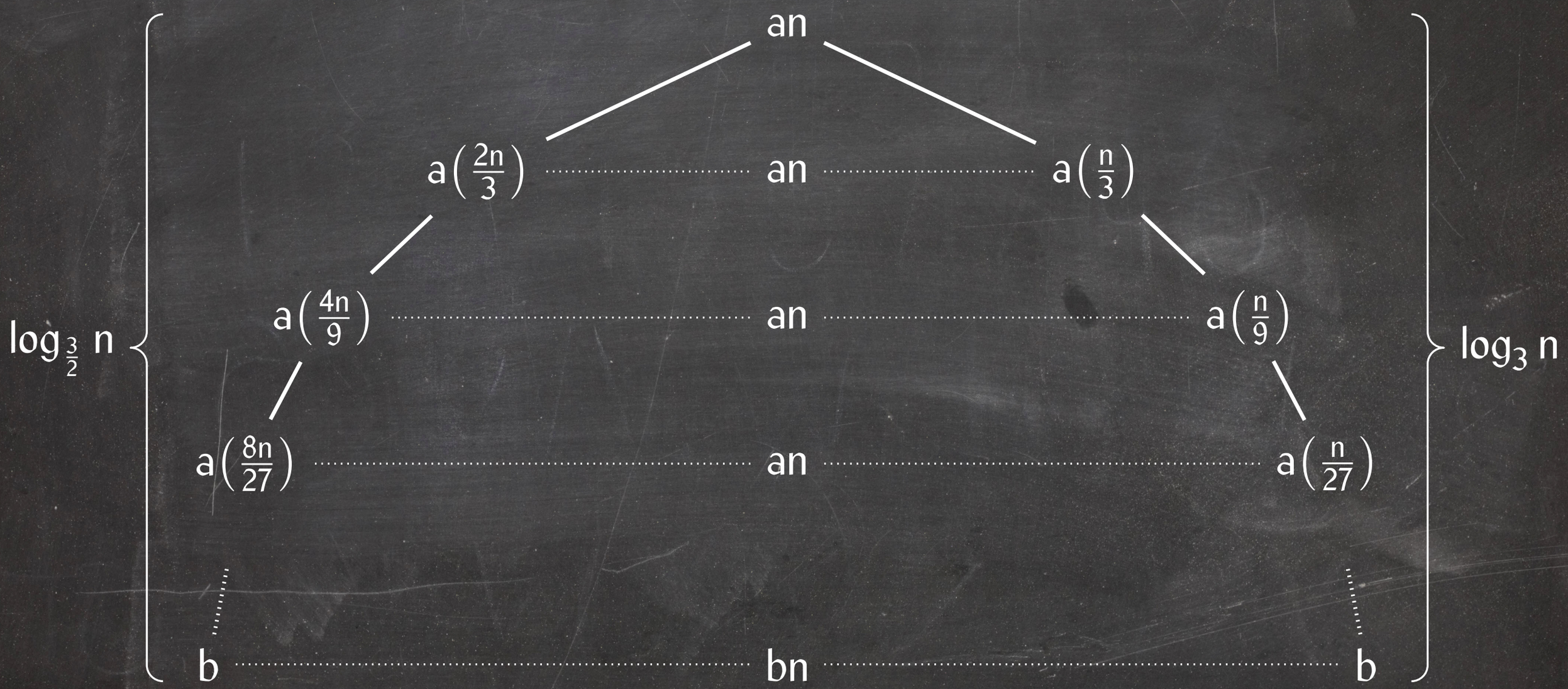

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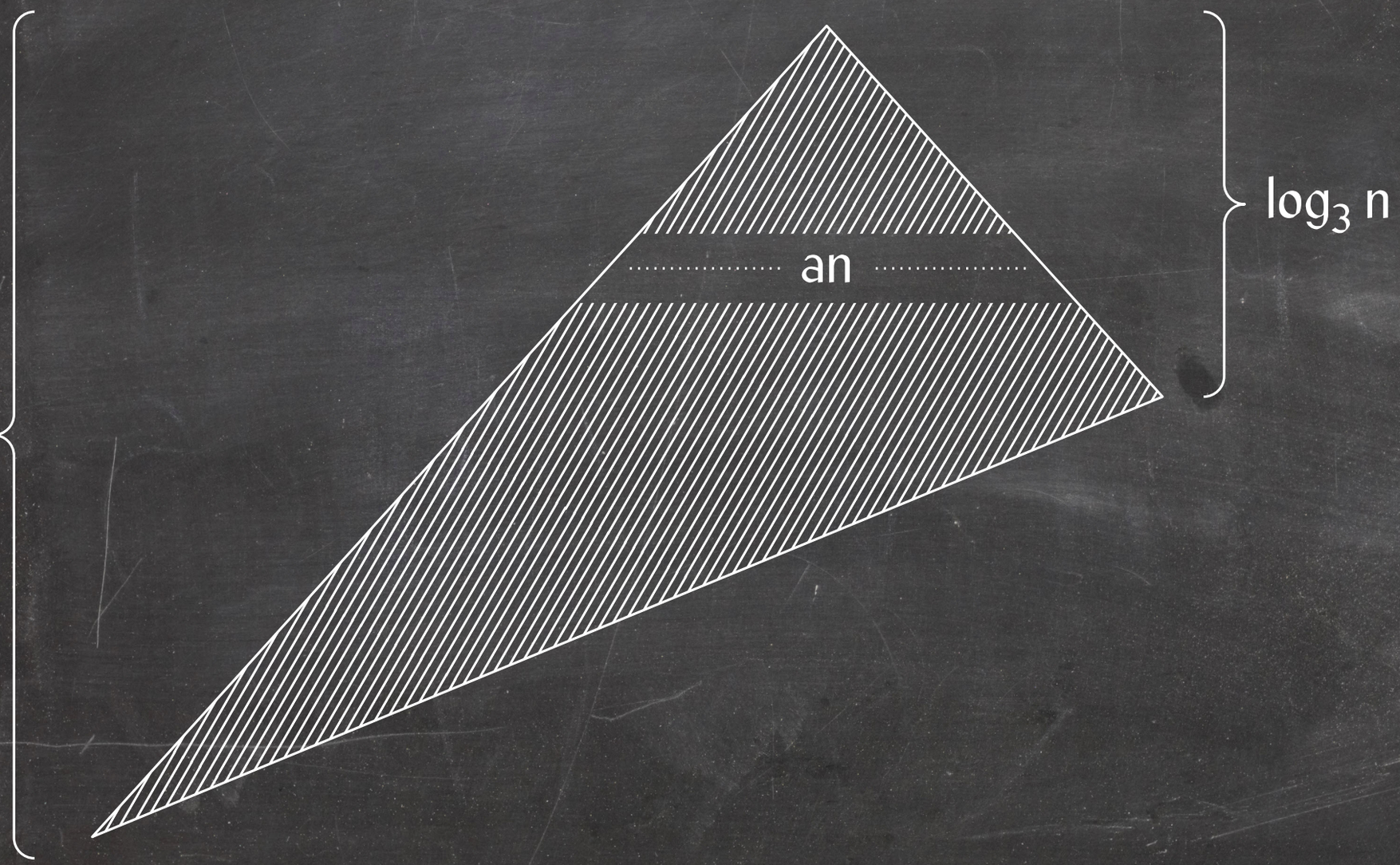

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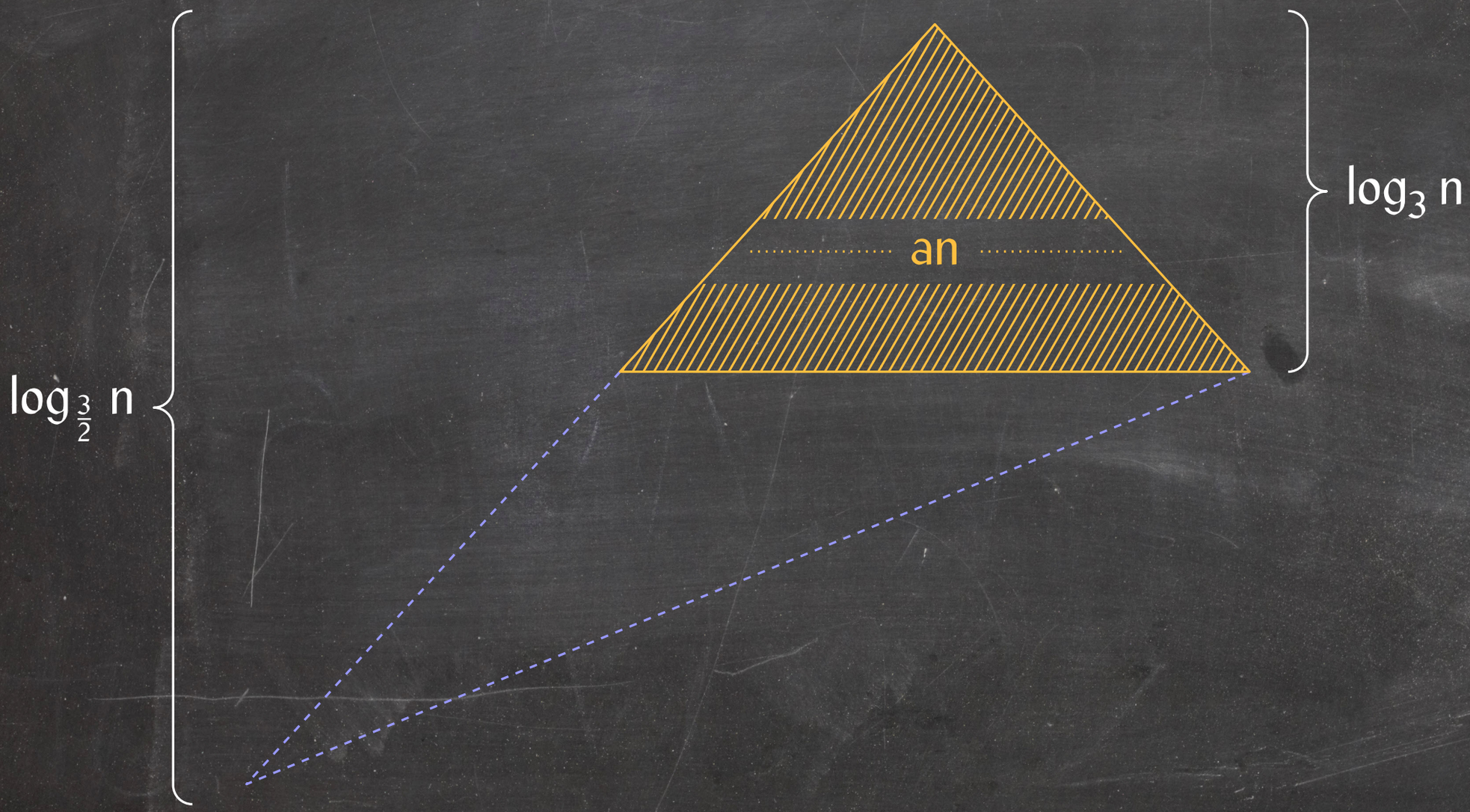

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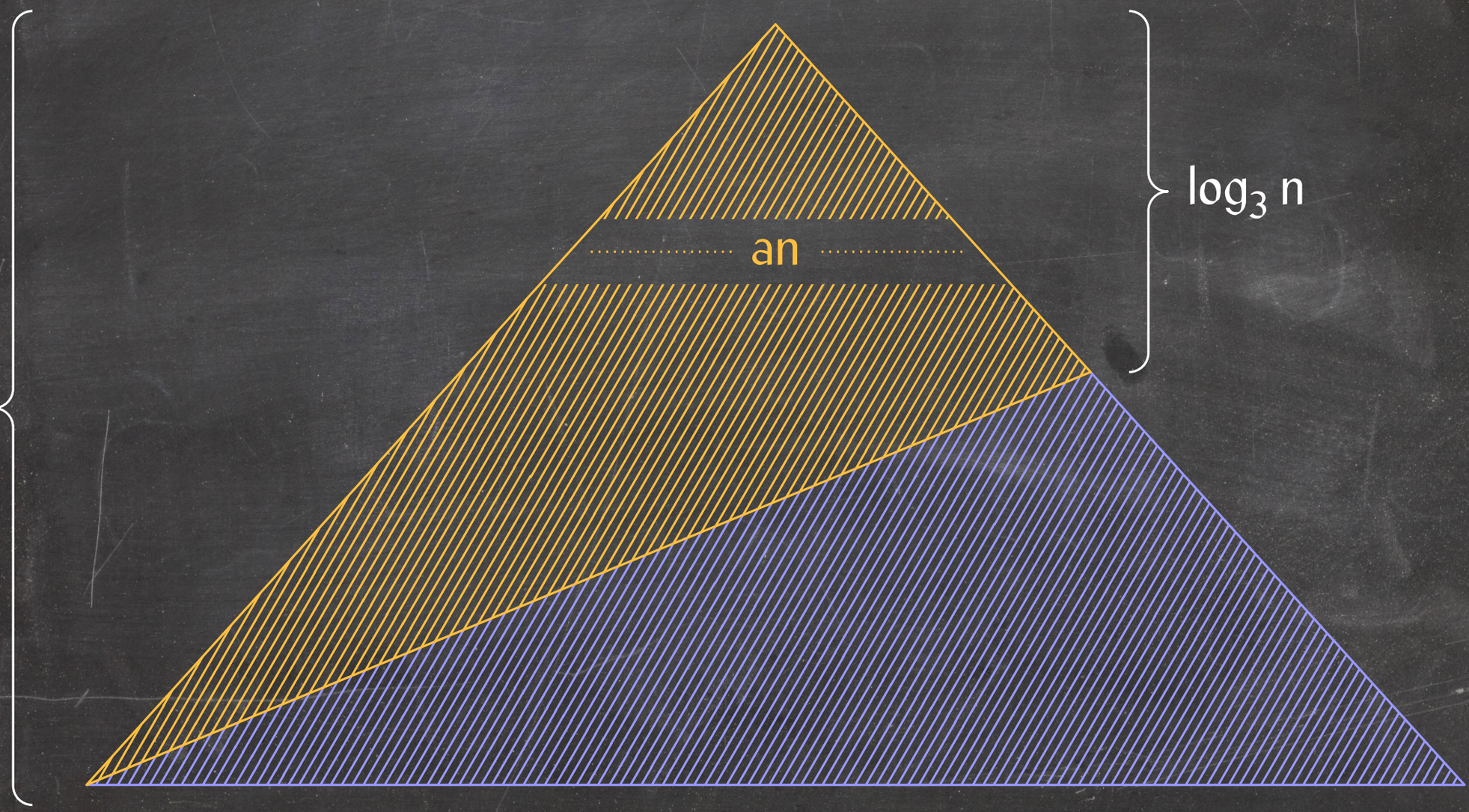

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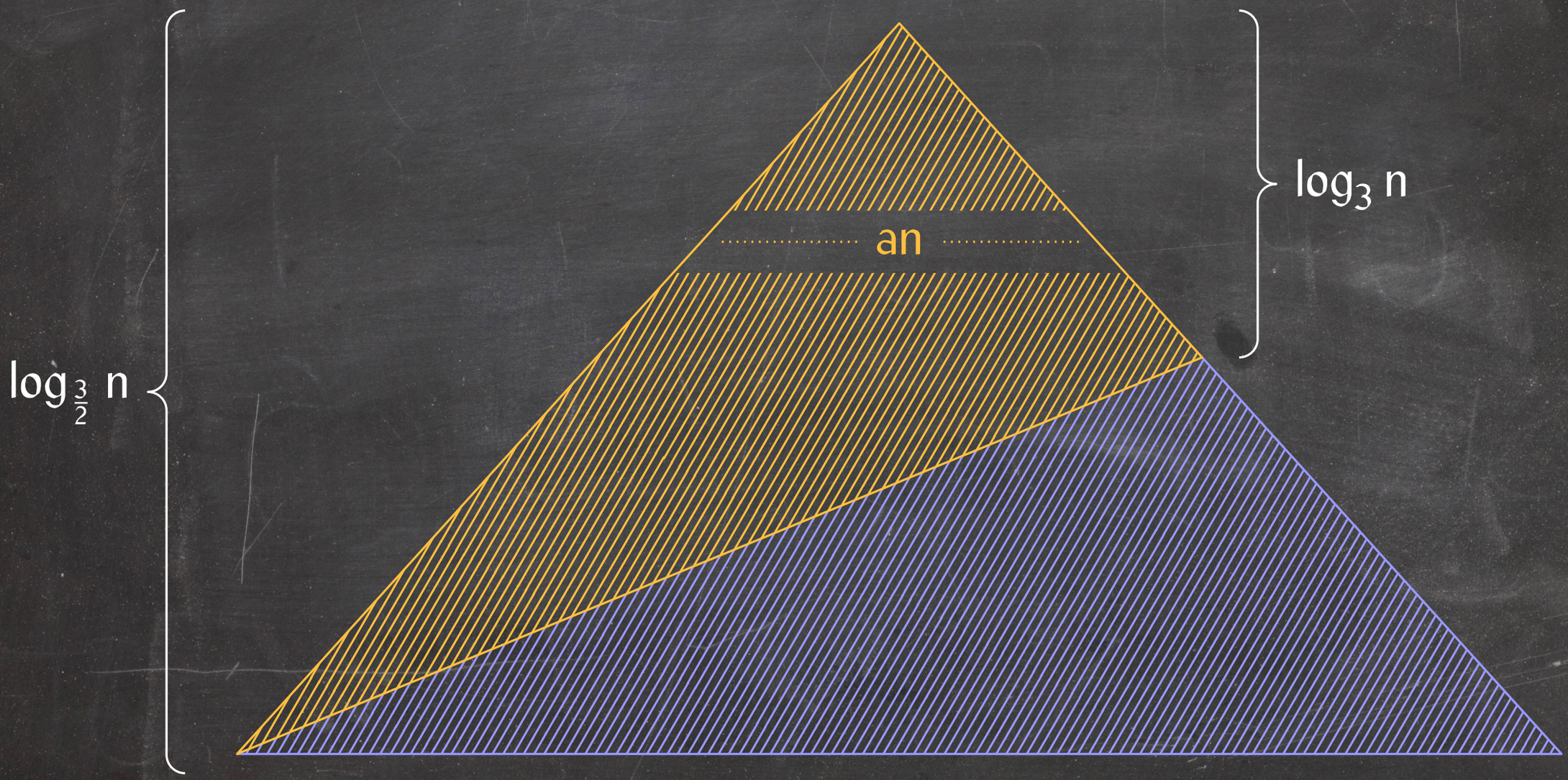
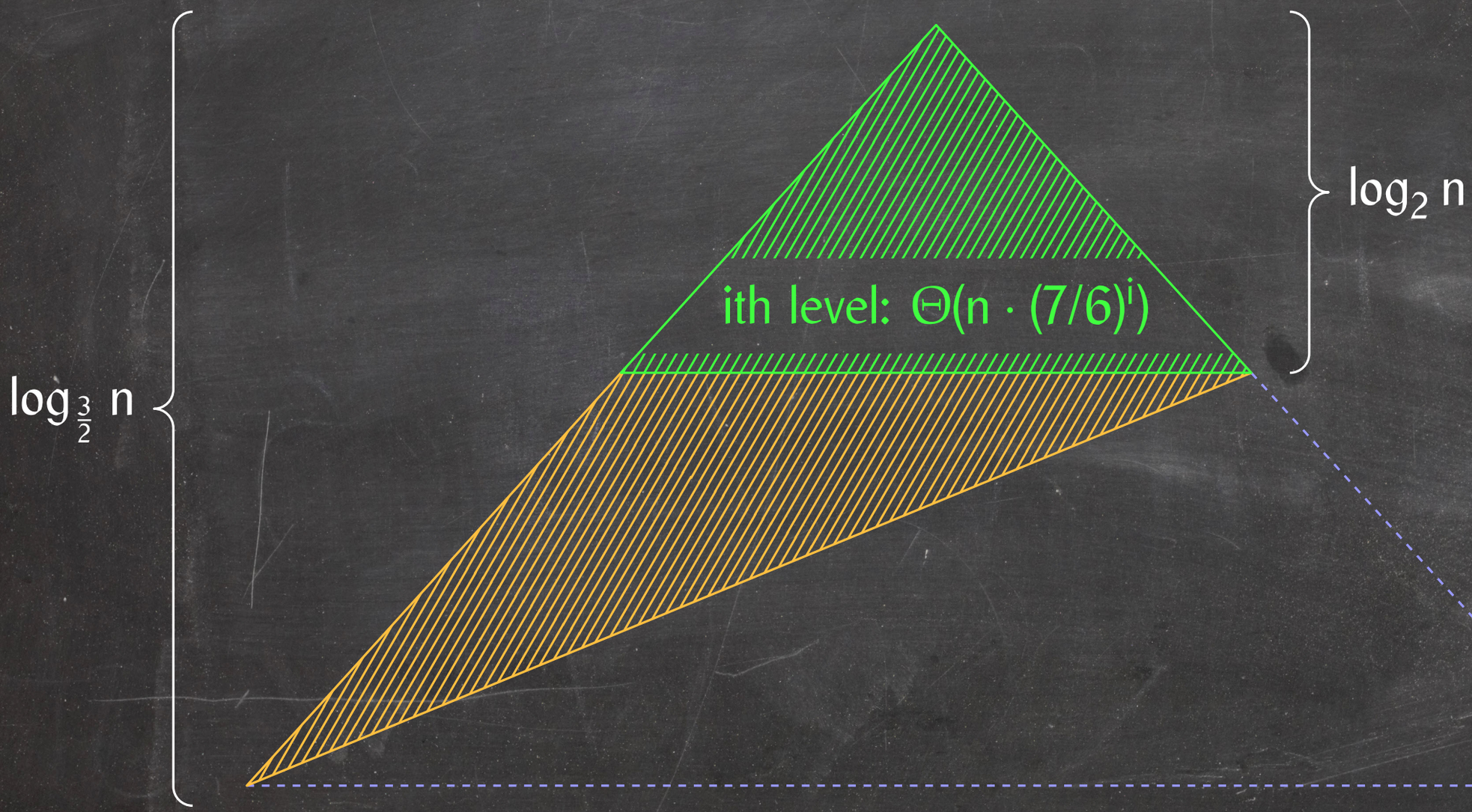


Solution: $T(n) \in \Theta(n \lg n)$

Sometimes Only Substitution Will Do

Recurrence: $T(n) = T(2n/3) + T(n/2) + \Theta(n)$



Lower bound: $T(n) \in \Omega(n^{1+\log_2(7/6)}) \approx \Omega(n^{1.22})$ triangle at top

Upper bound: $T(n) \in O(n^{1+\log_{3/2}(7/6)}) \approx O(n^{1.38})$ full triangle

Master Theorem

Master Theorem: Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).$$

(i) If $f(n) \in O(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

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$$T(n) = 2T(n/2) + \Theta(n)$$

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Example 2: Matrix multiplication

$$T(n) = 7T(n/2) + \Theta(n^2)$$

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$$T(n) \in \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81})$$

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$$a = 1 \quad b = 2$$

$$f(n) = n$$

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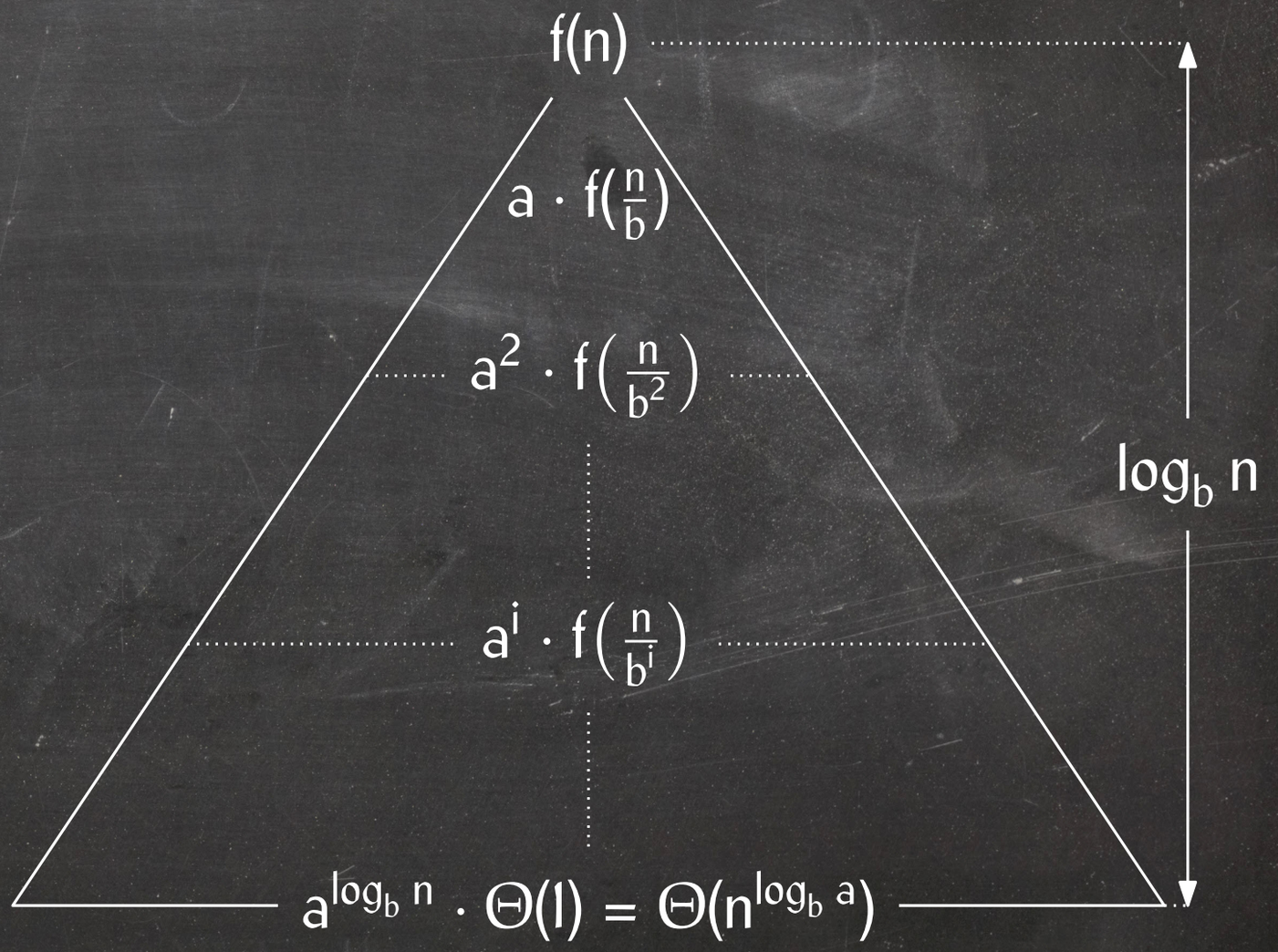
$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$



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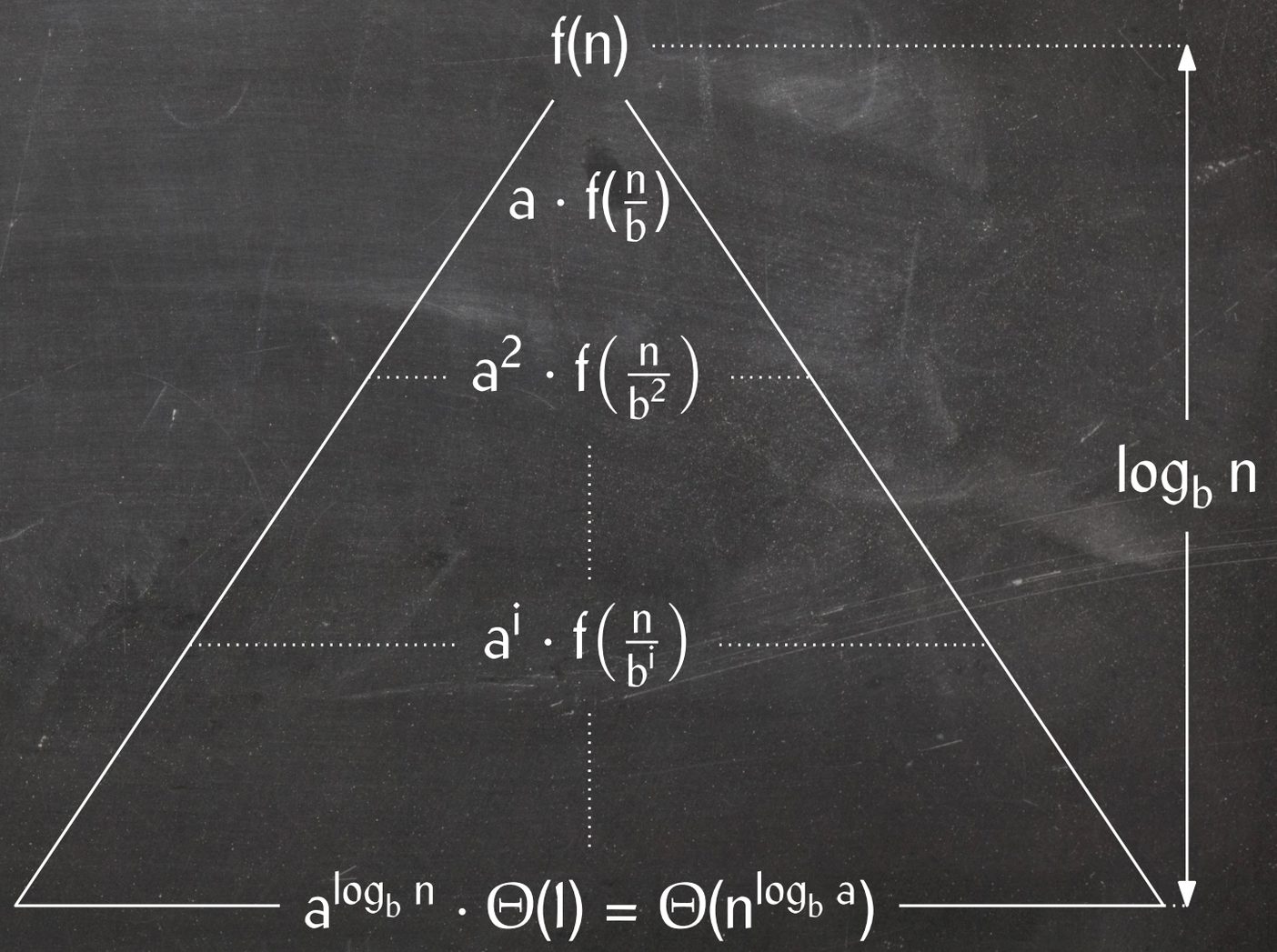

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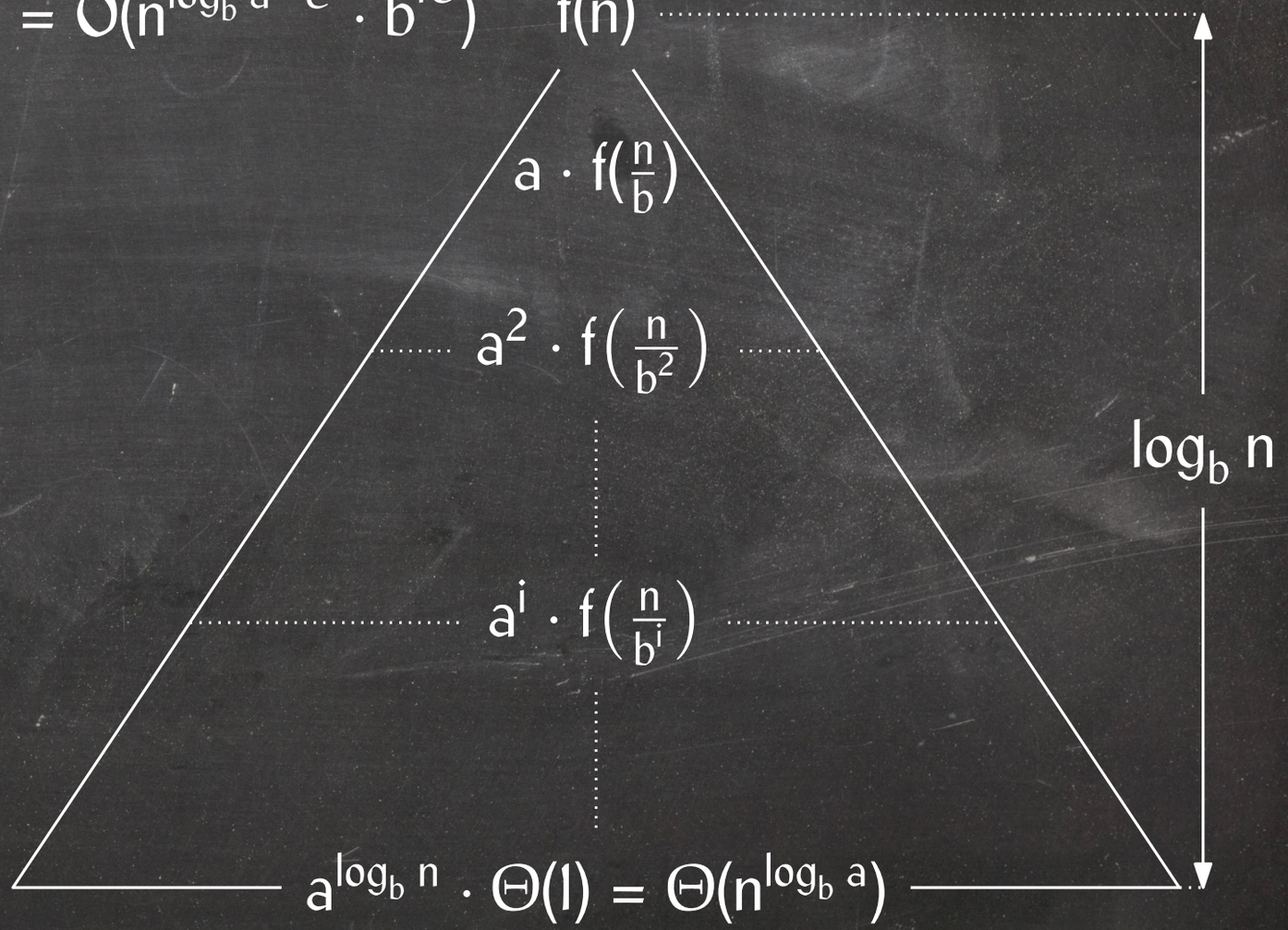

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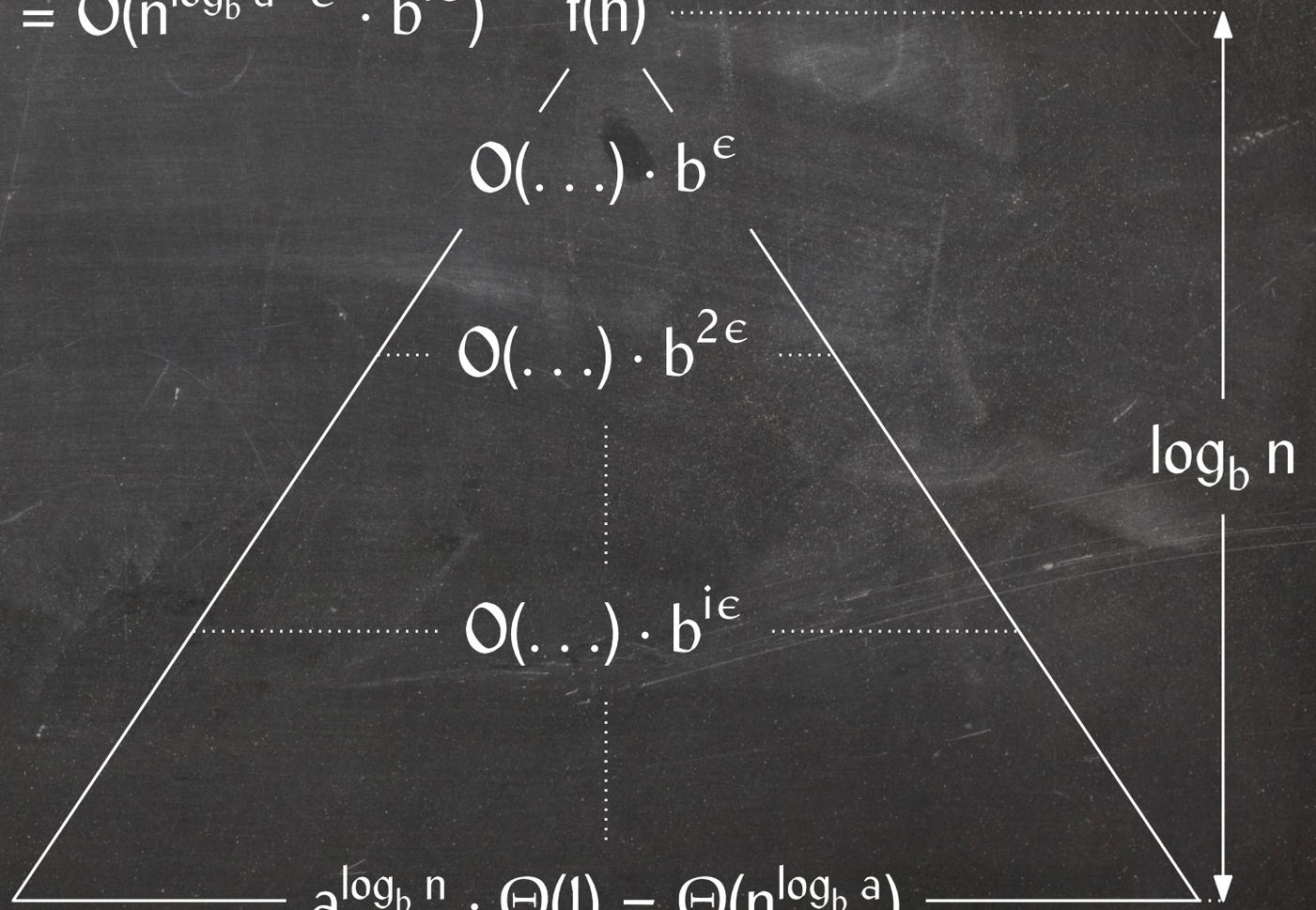

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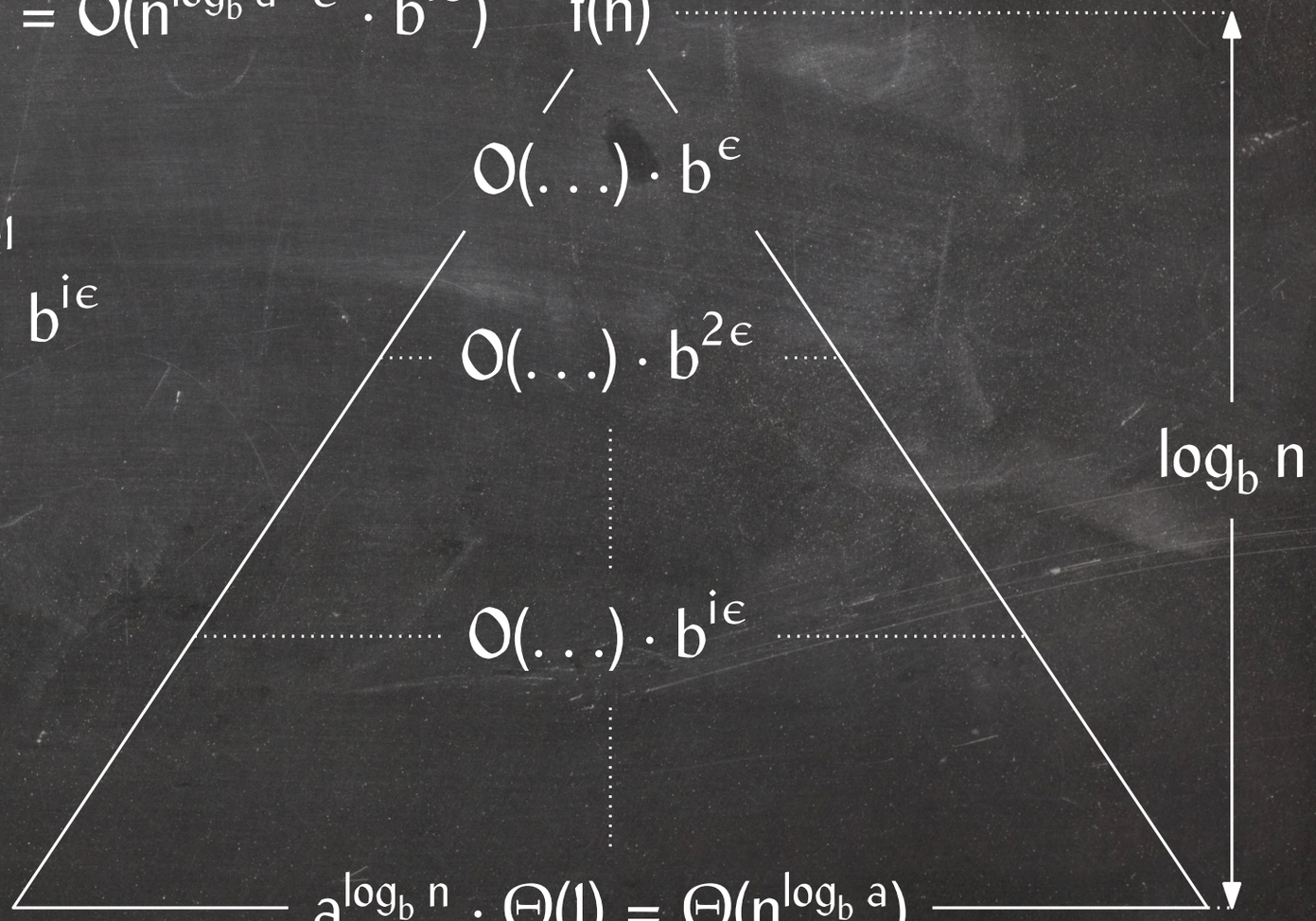

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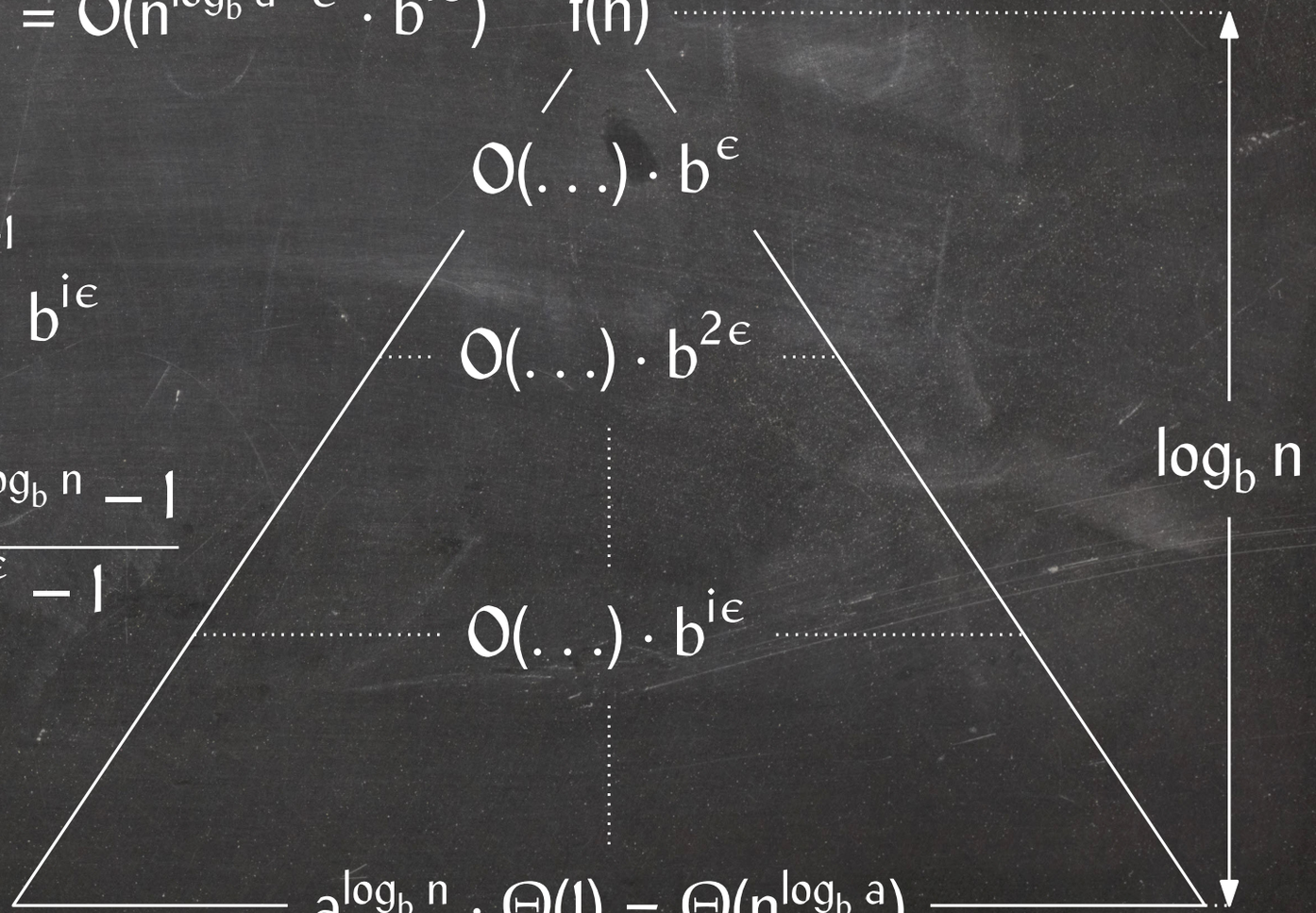
$$= \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \frac{(b^\epsilon)^{\log_b n} - 1}{b^\epsilon - 1}$$



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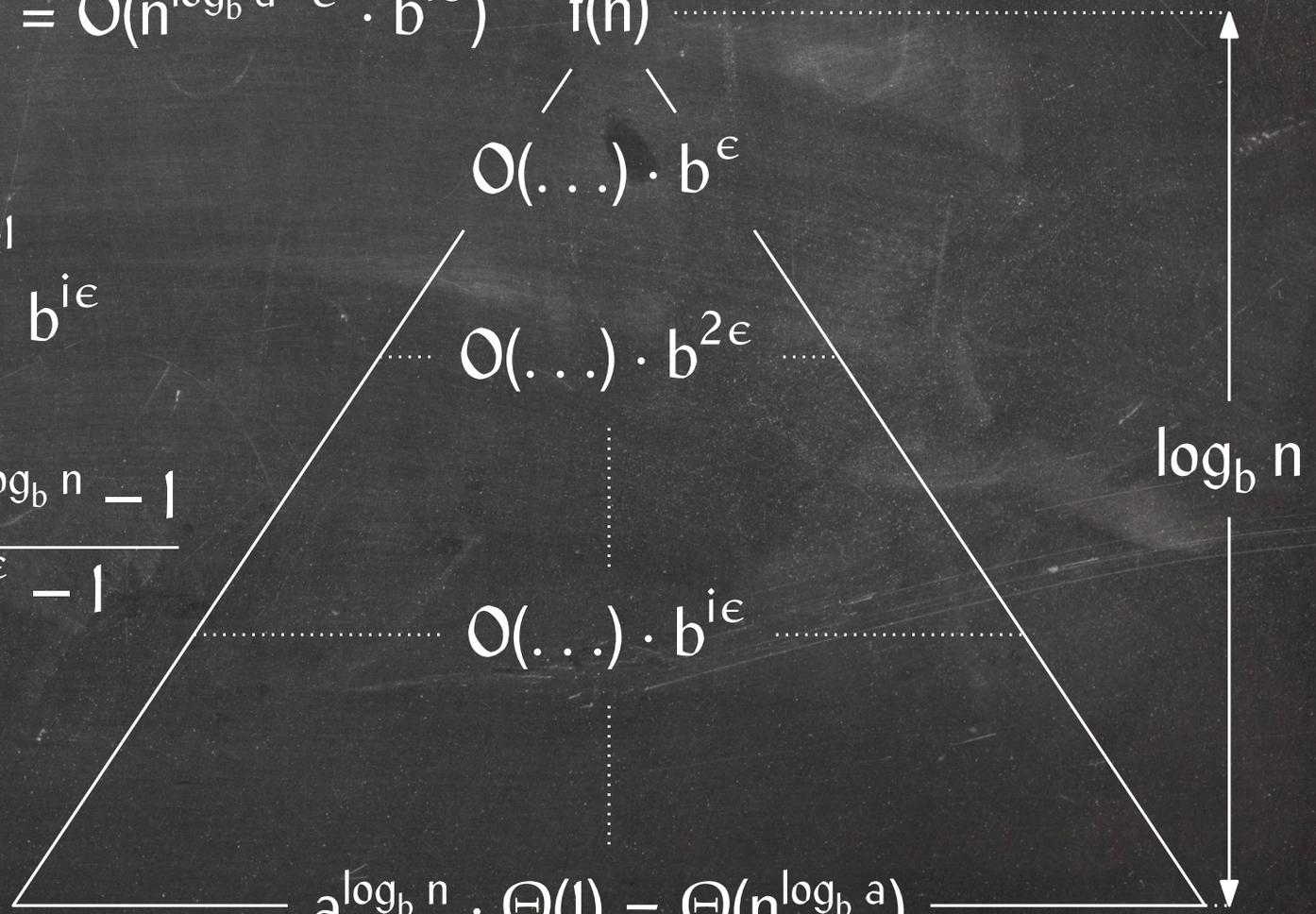
$$T(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \sum_{i=1}^{\log_b n - 1} b^{i\epsilon}$$

$$= \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot \frac{(b^\epsilon)^{\log_b n} - 1}{b^\epsilon - 1}$$

$$= \Theta(n^{\log_b a}) + O(n^{\log_b a - \epsilon}) \cdot n^\epsilon$$



Master Theorem: Proof

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

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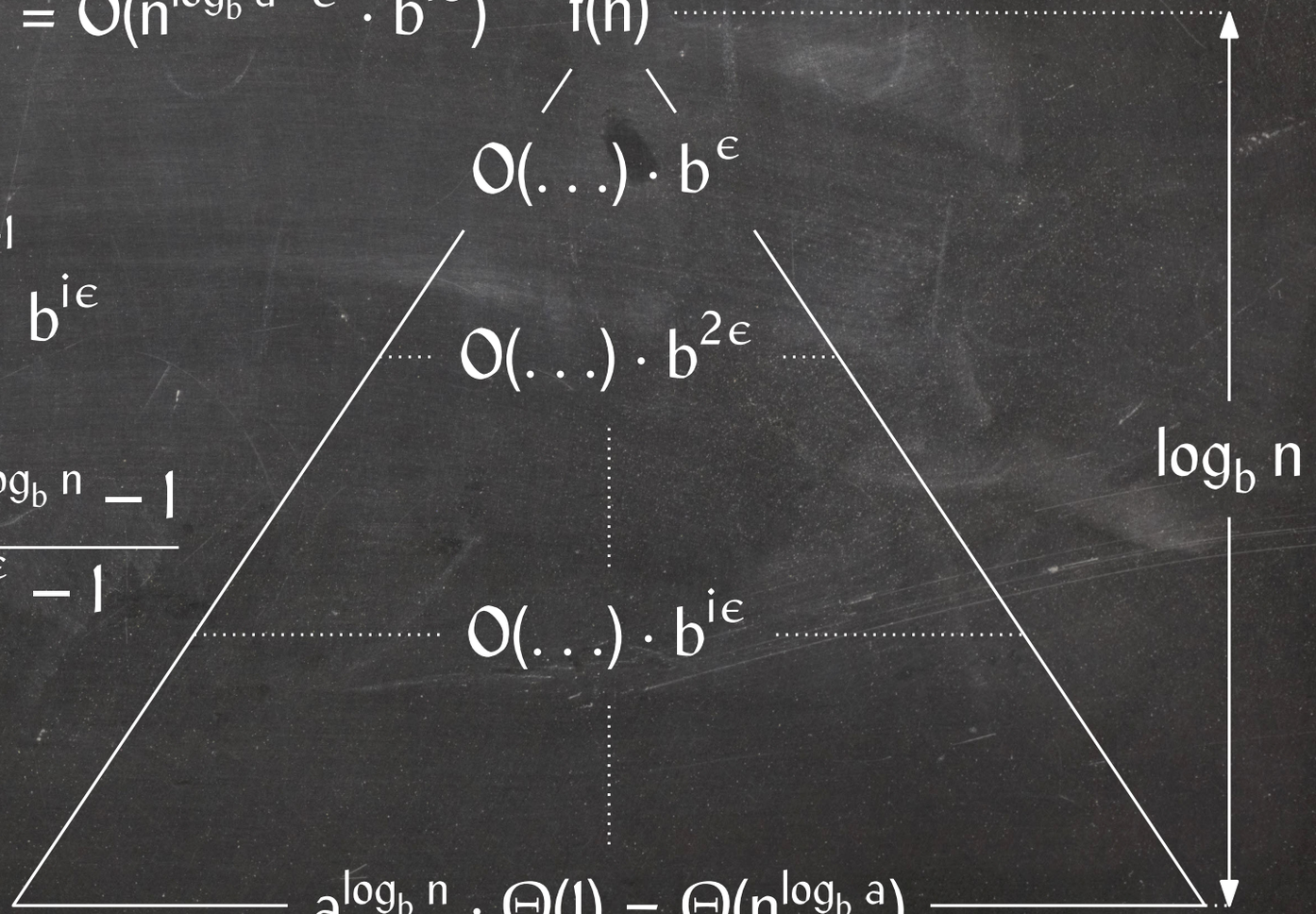

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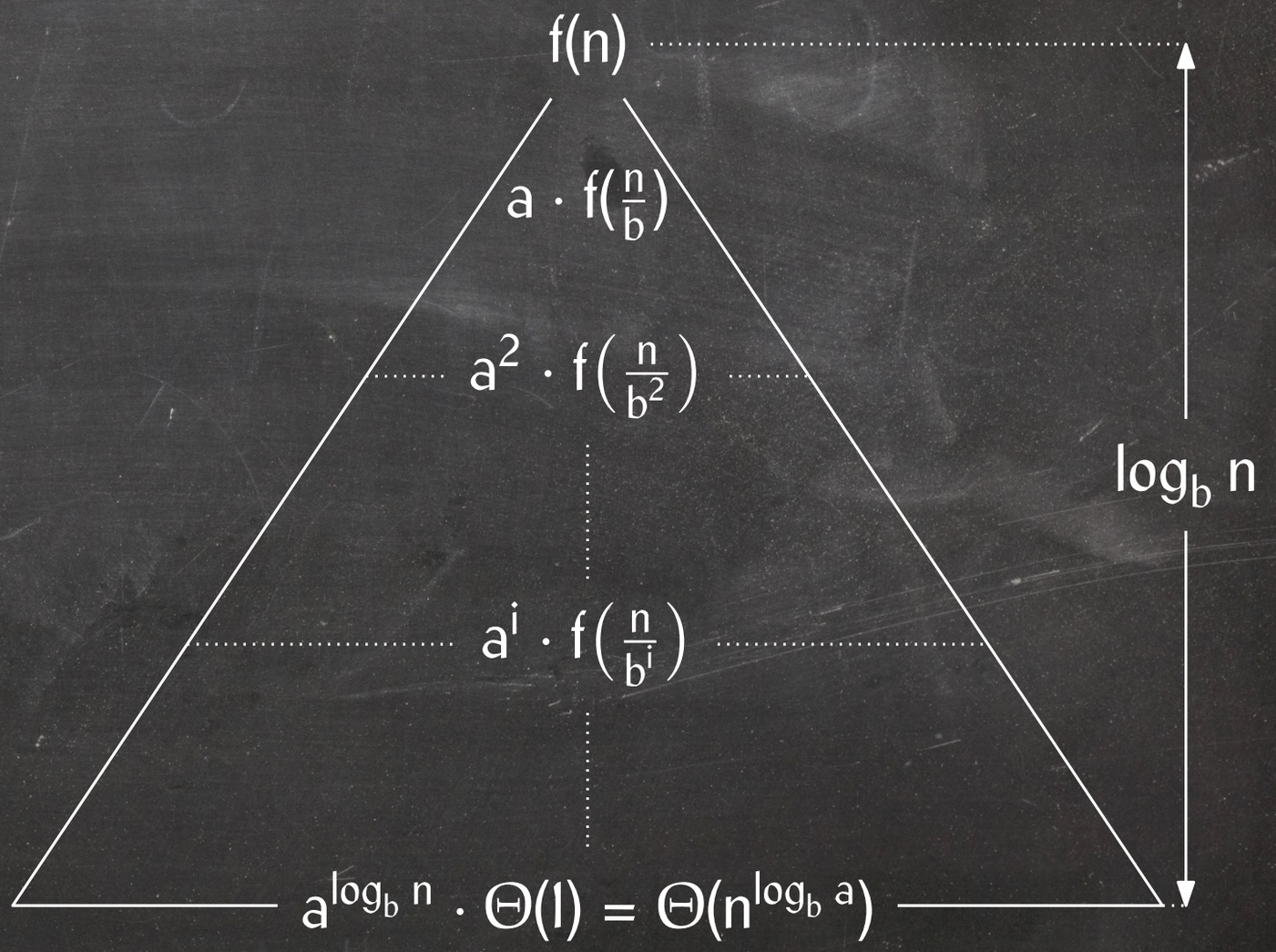

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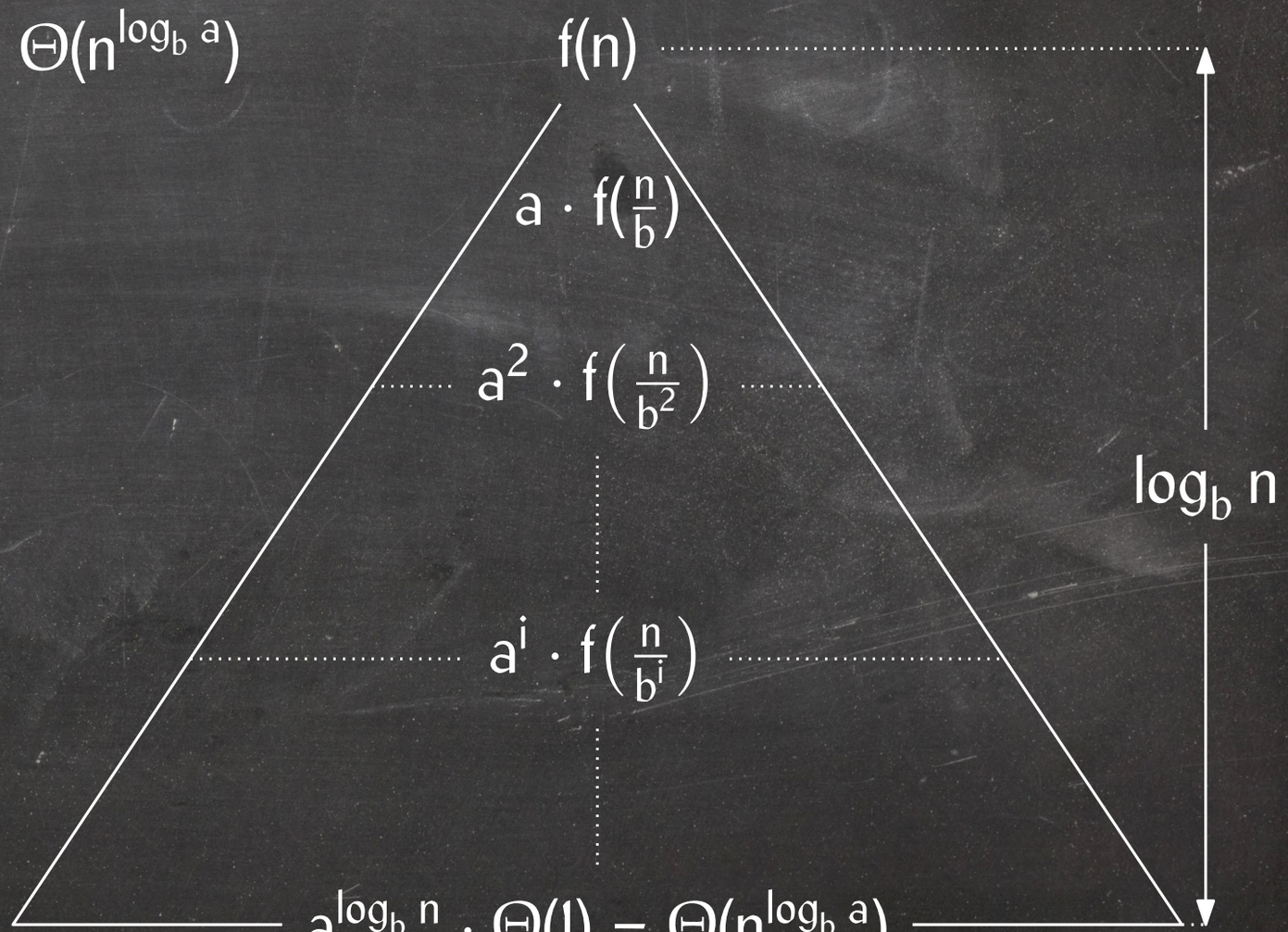

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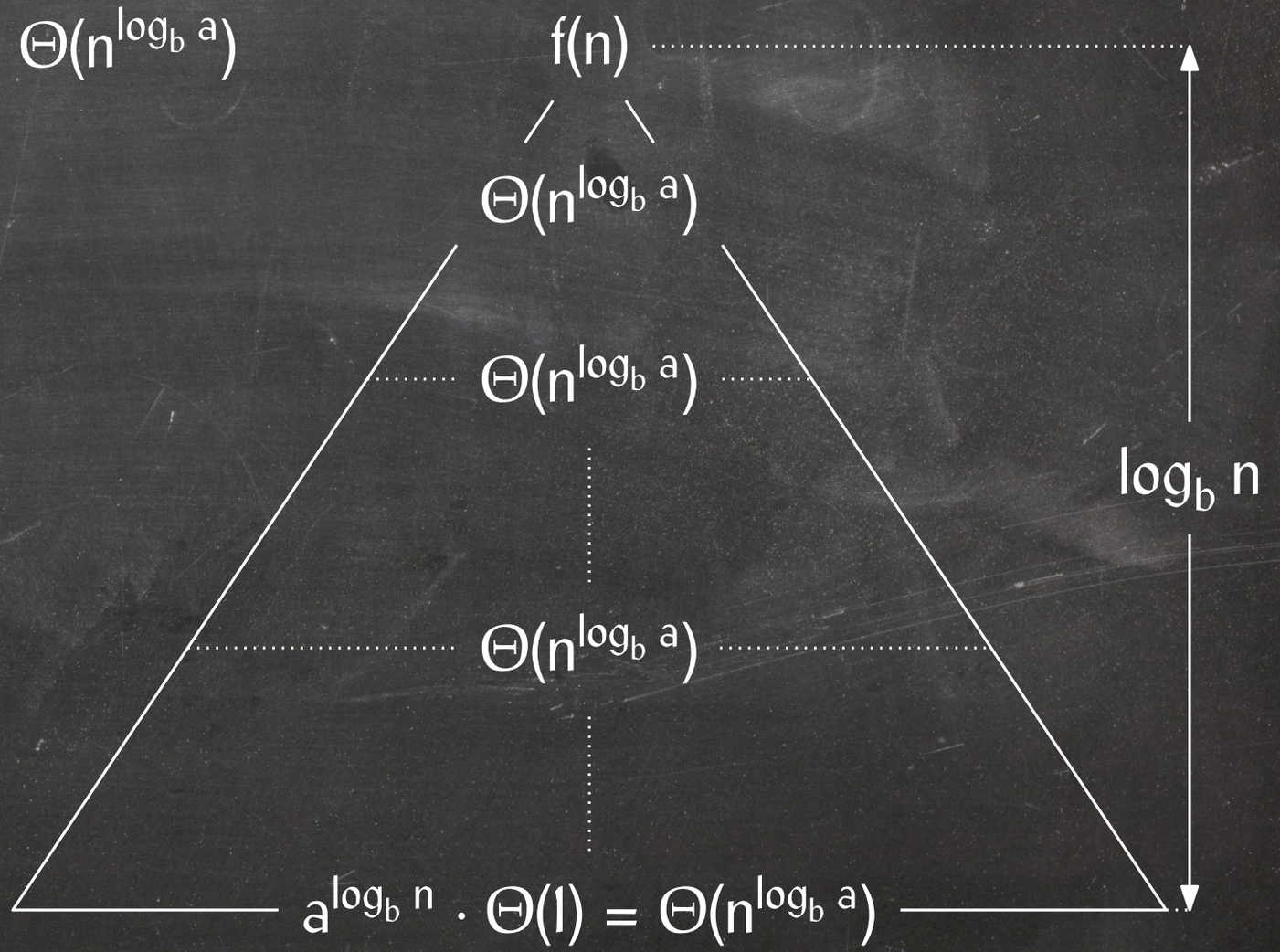

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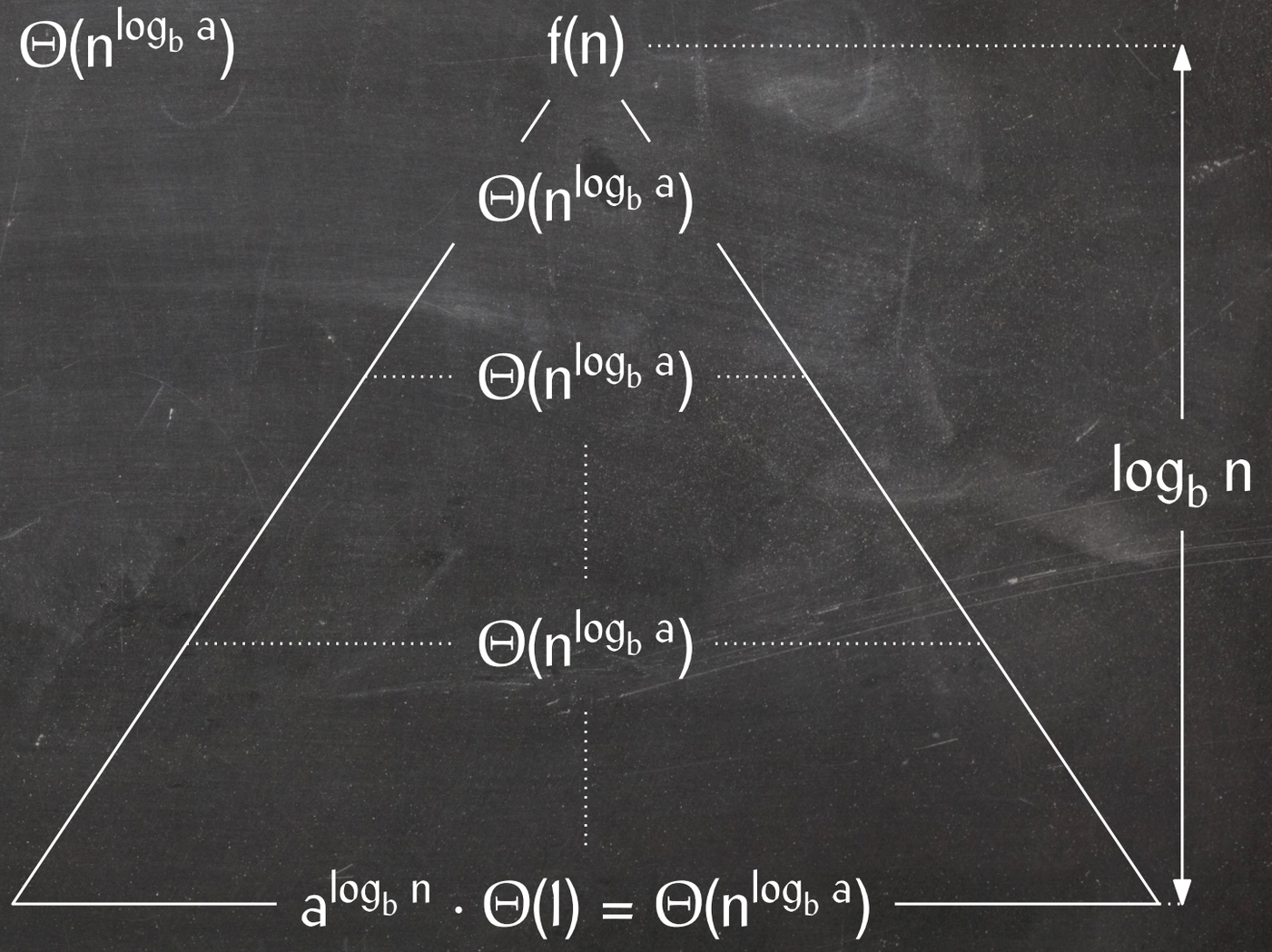

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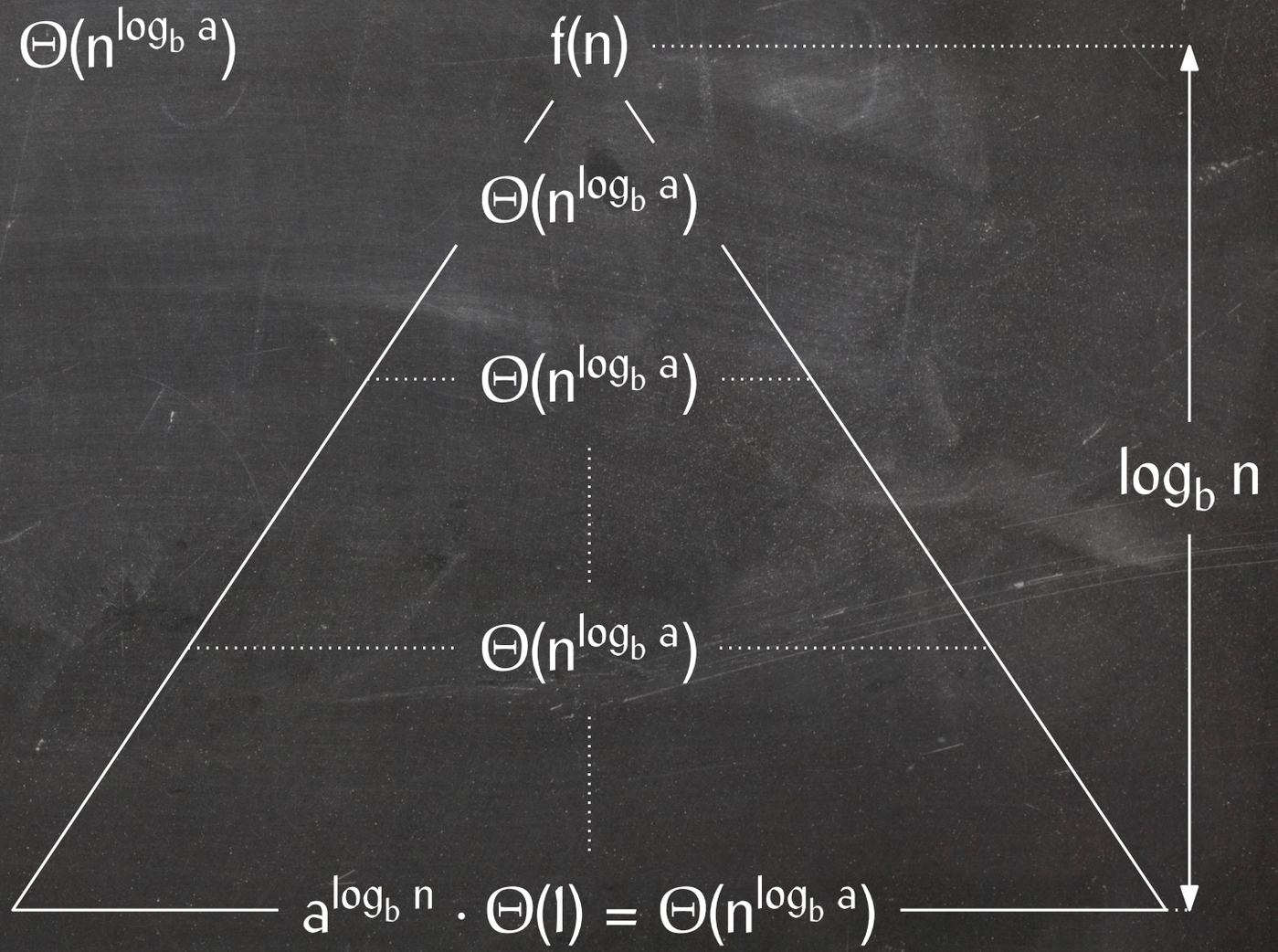
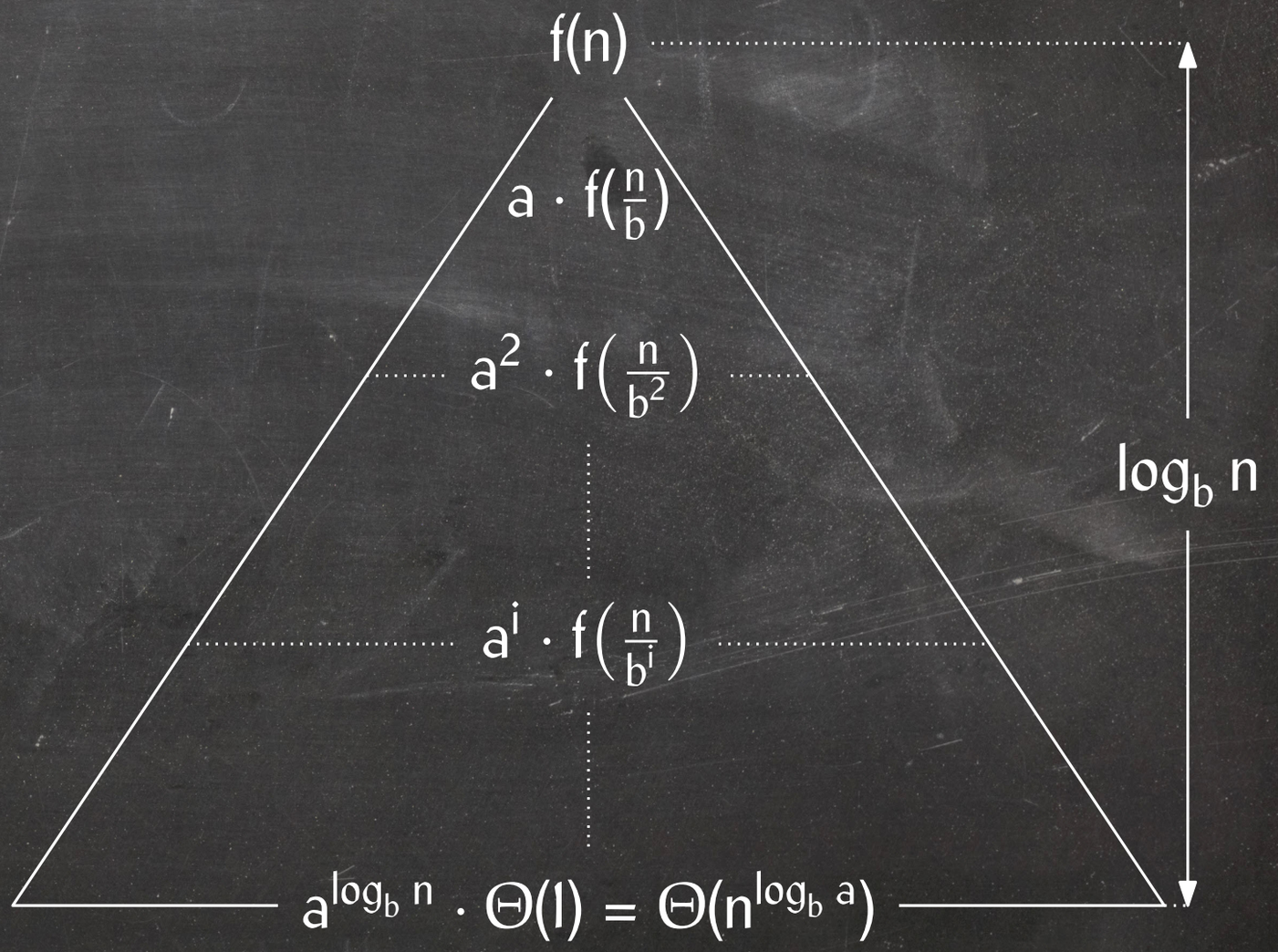


Master Theorem: Proof

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

Case 3: $f(n) \in \Omega(n^{\log_b a+\epsilon})$ and $a \cdot f(n/b) \leq c \cdot f(n)$ for some $c < 1$



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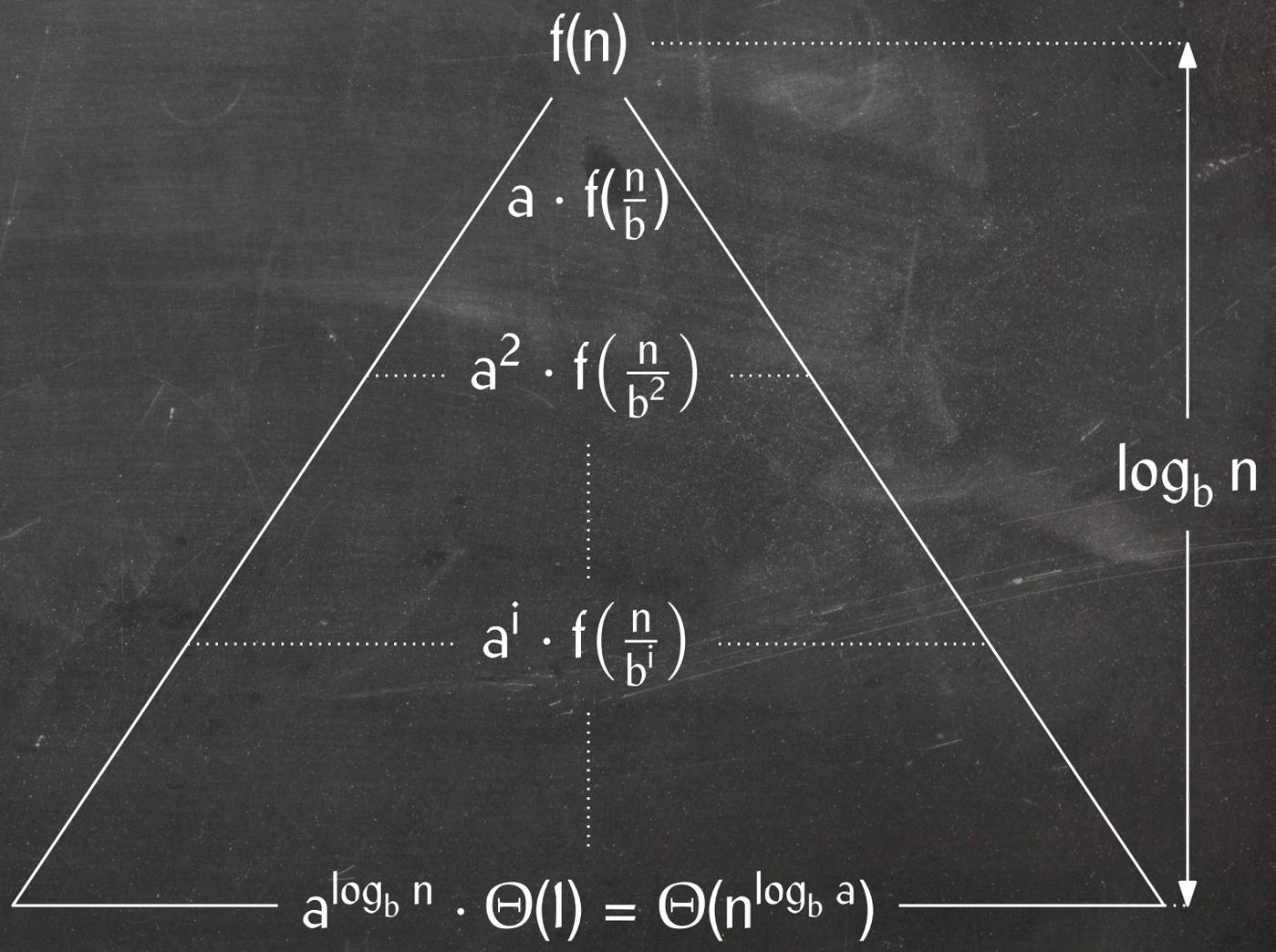

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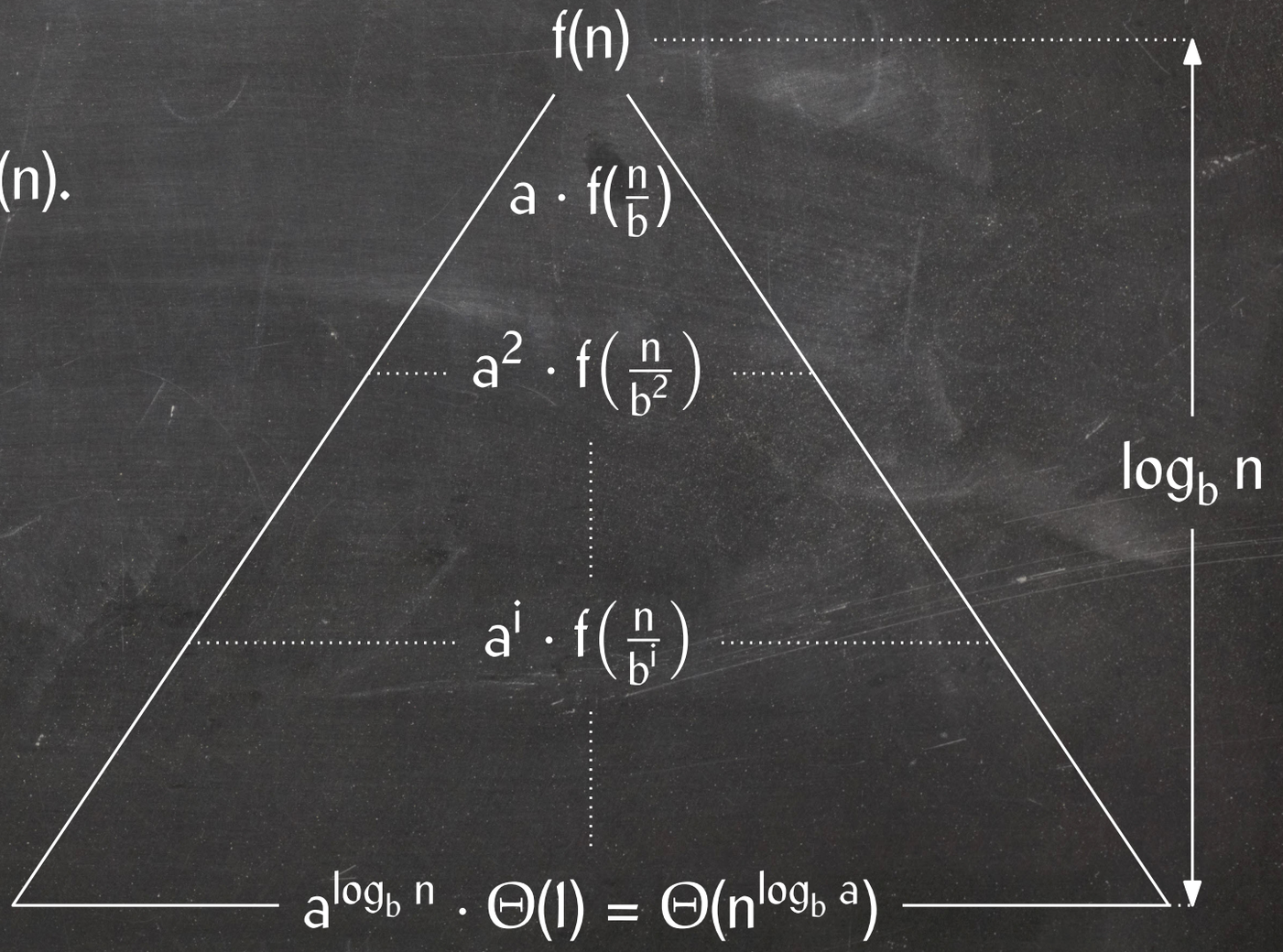
Claim: $a^i \cdot f\left(\frac{n}{b^i}\right) \leq c^i \cdot f(n)$

For $i = 0$, $a^0 \cdot f\left(\frac{n}{b^0}\right) = f(n) = c^0 \cdot f(n)$.



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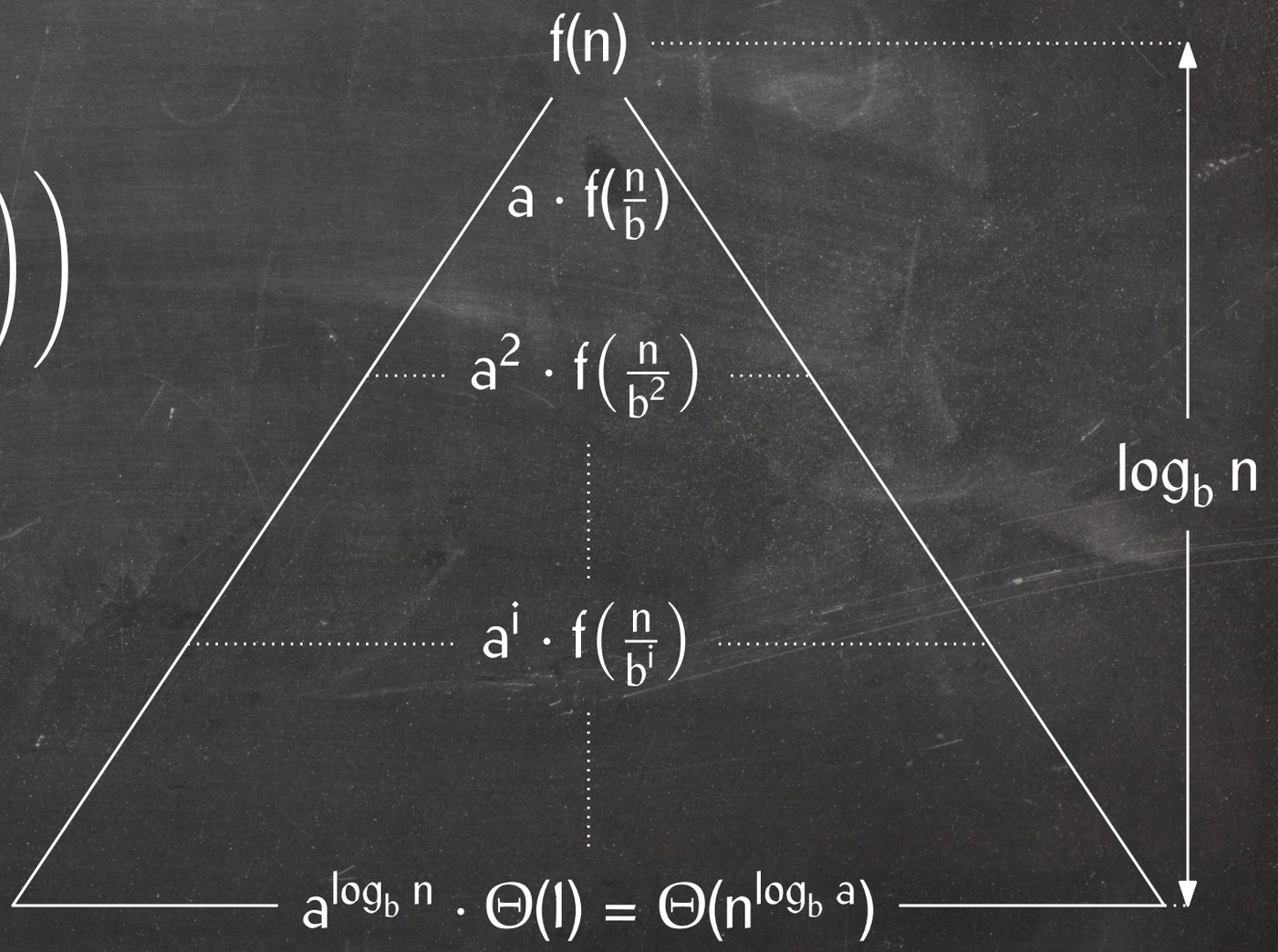
Claim: $a^i \cdot f\left(\frac{n}{b^i}\right) \leq c^i \cdot f(n)$

For $i > 0$,

$$a^i \cdot f\left(\frac{n}{b^i}\right) = a^{i-1} \cdot \left(a \cdot f\left(\frac{n/b^{i-1}}{b}\right)\right)$$



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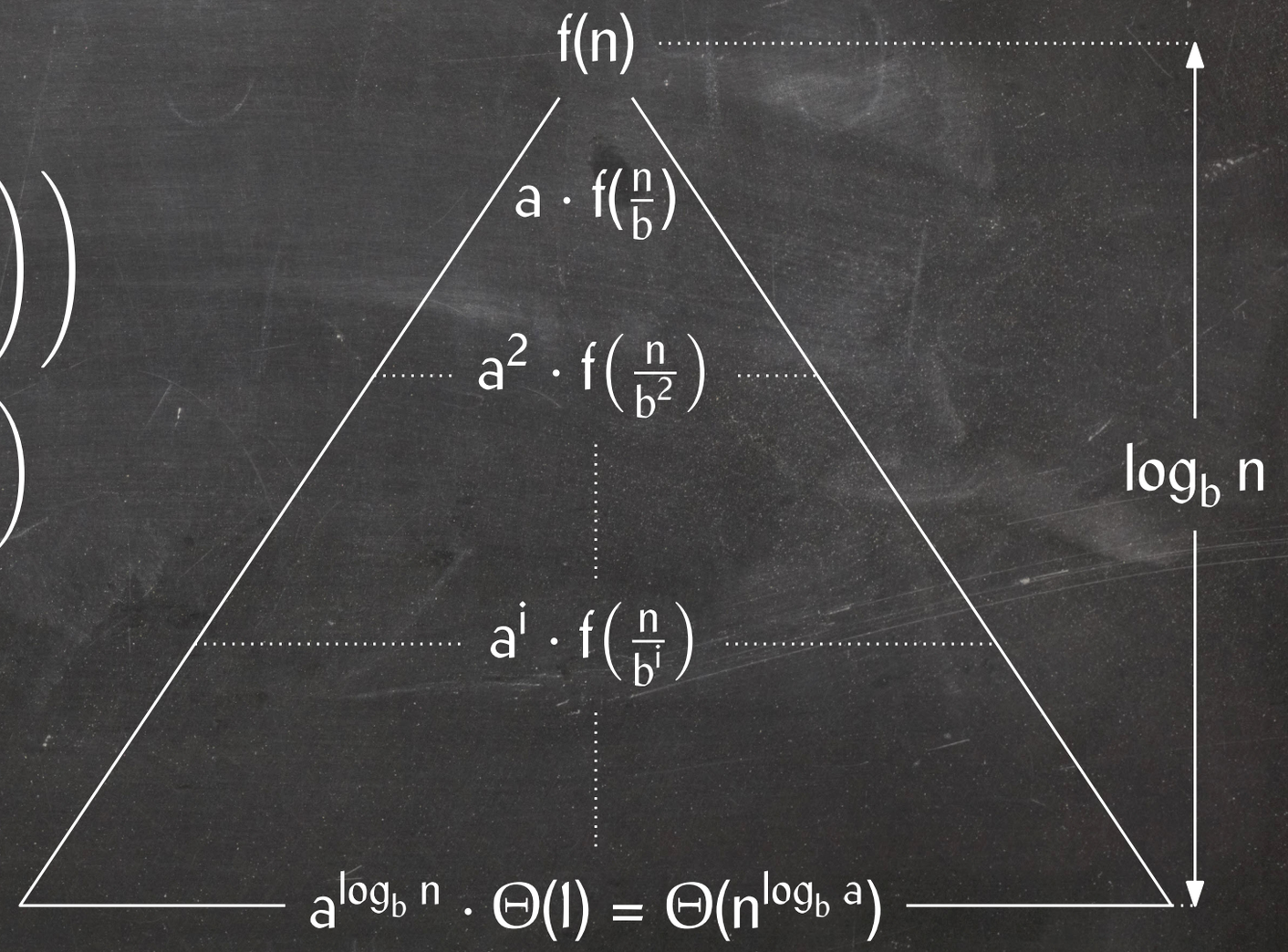

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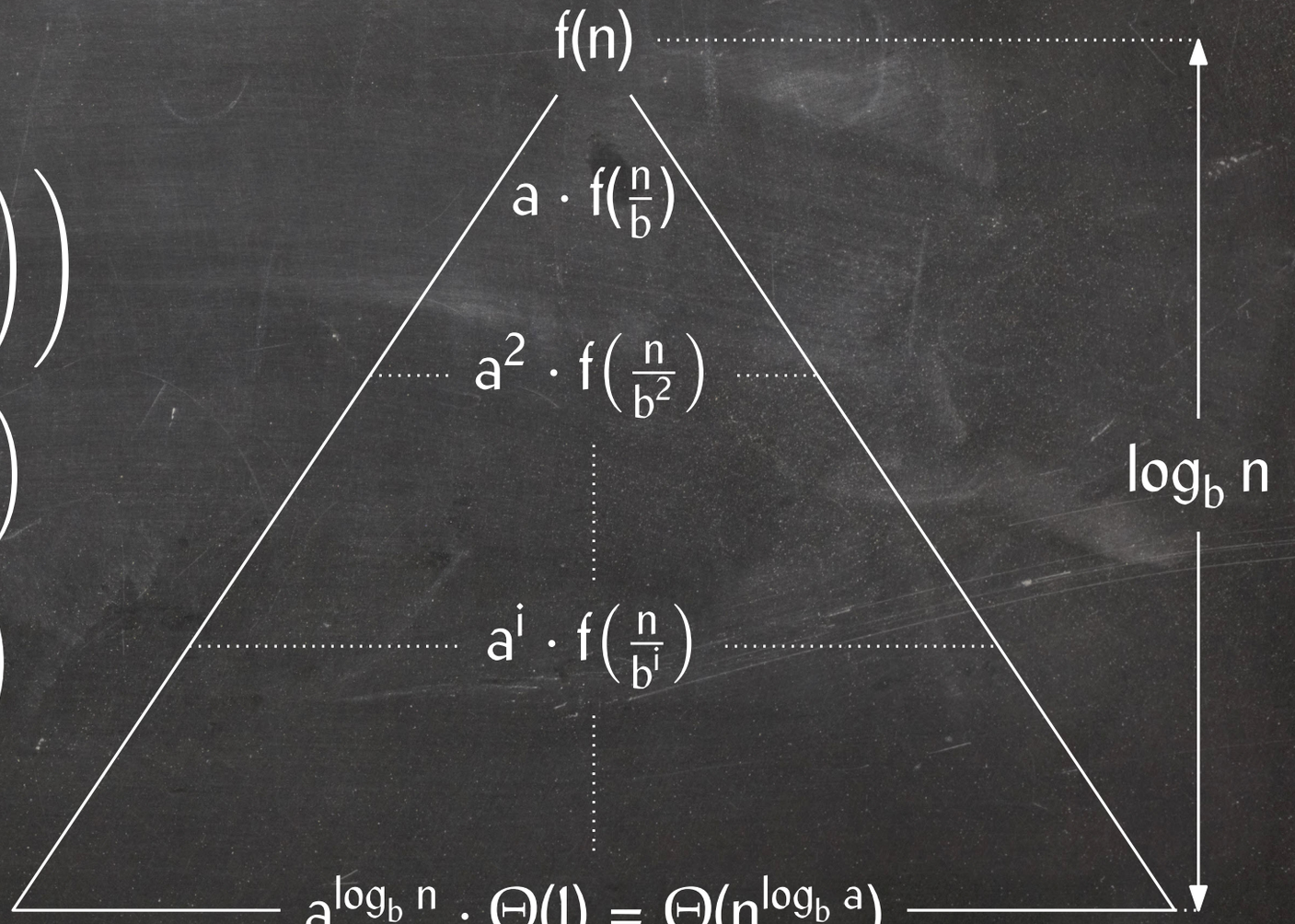

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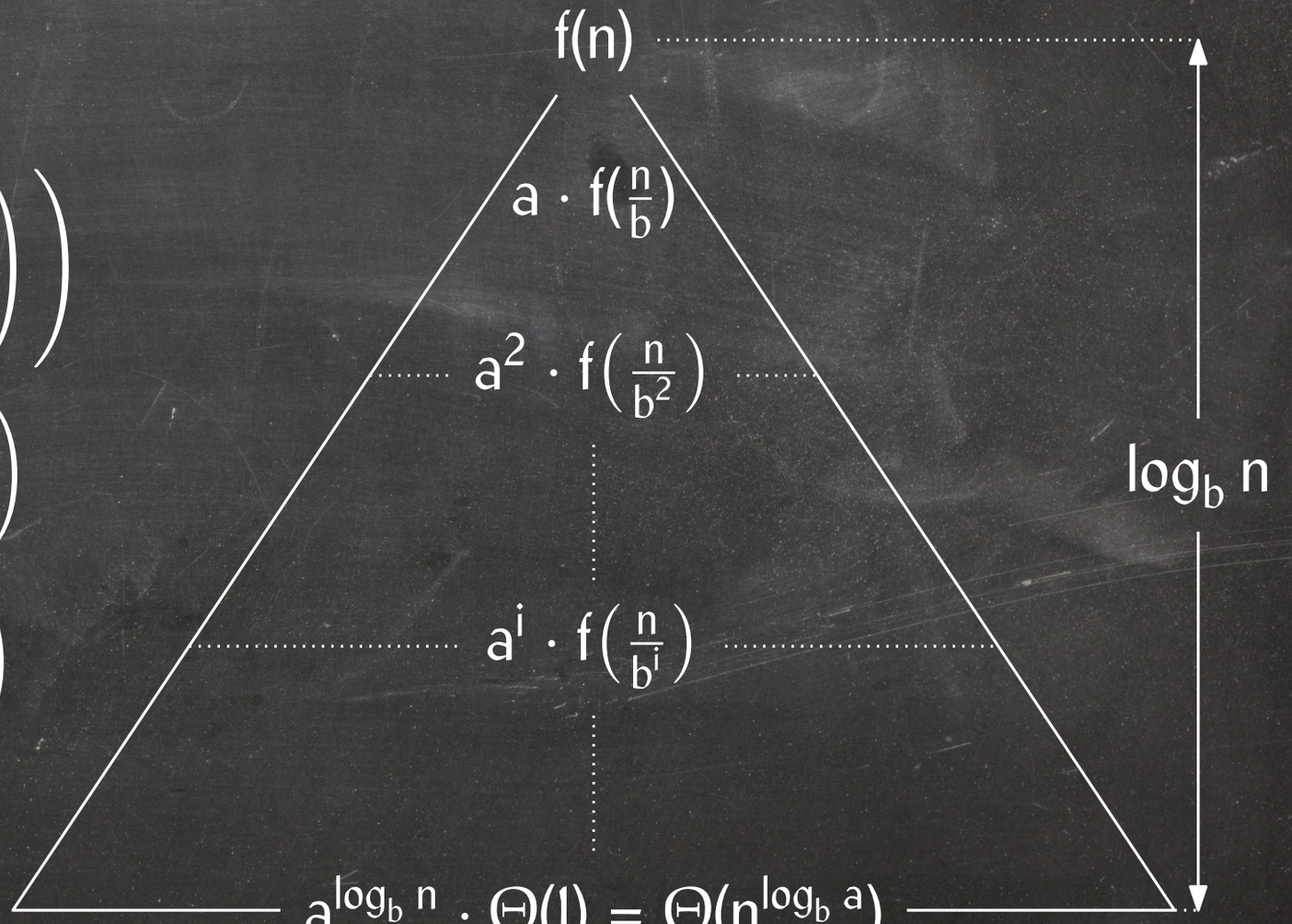

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&= c \cdot \left(a^{i-1} \cdot f\left(\frac{n}{b^{i-1}}\right)\right) \\
&\leq c \cdot \left(c^{i-1} \cdot f(n)\right) \\
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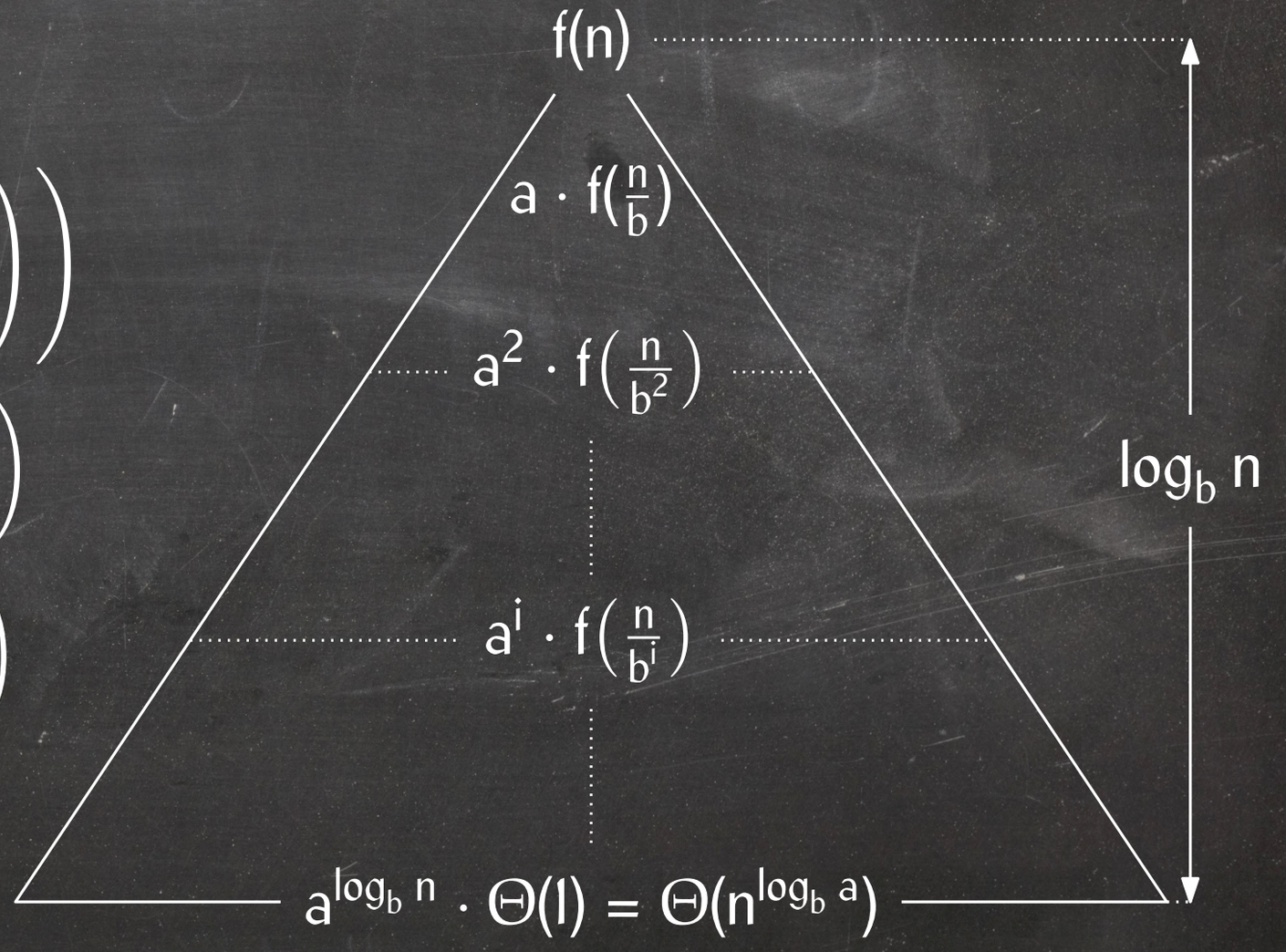
\end{aligned}$$



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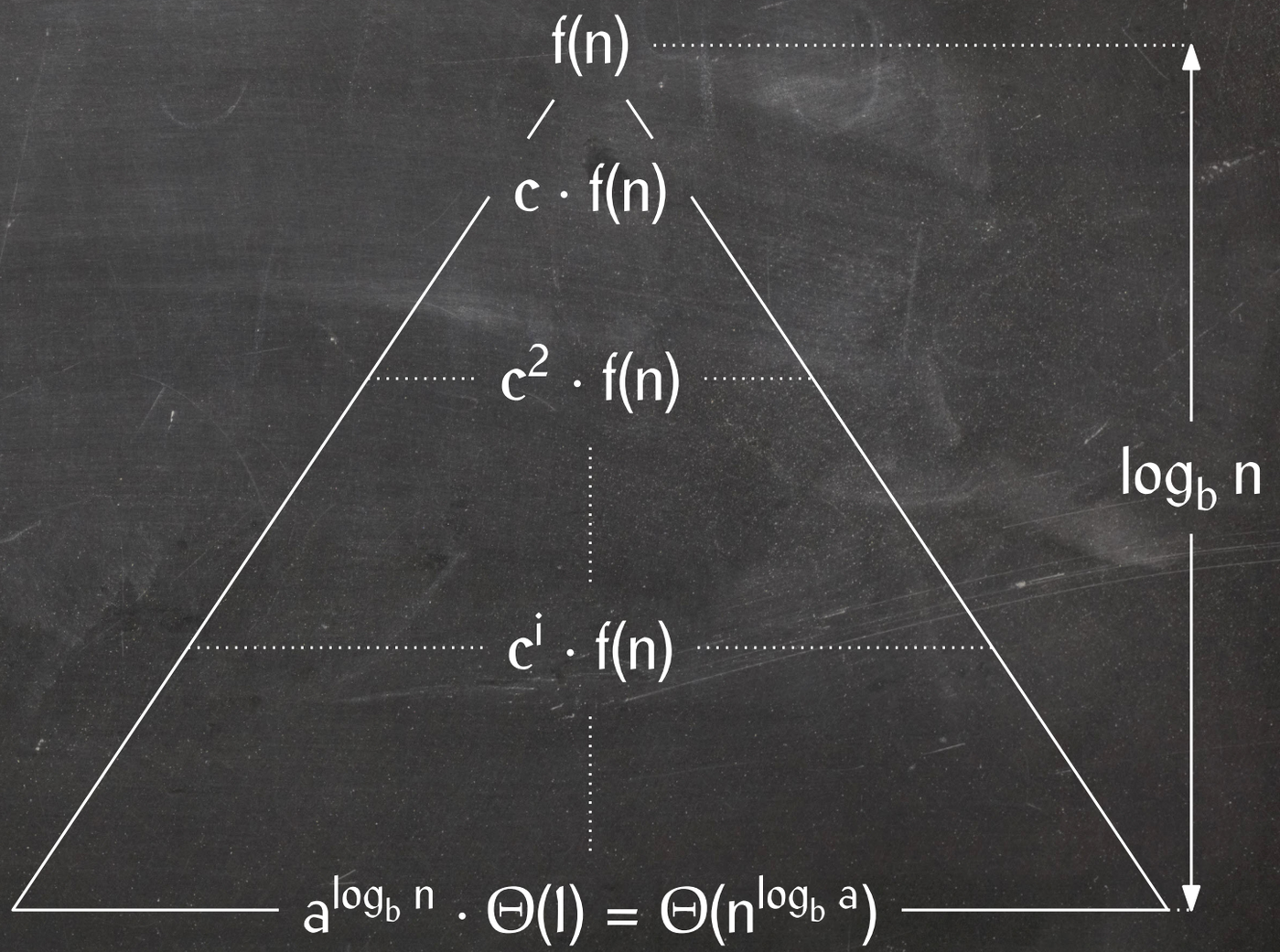

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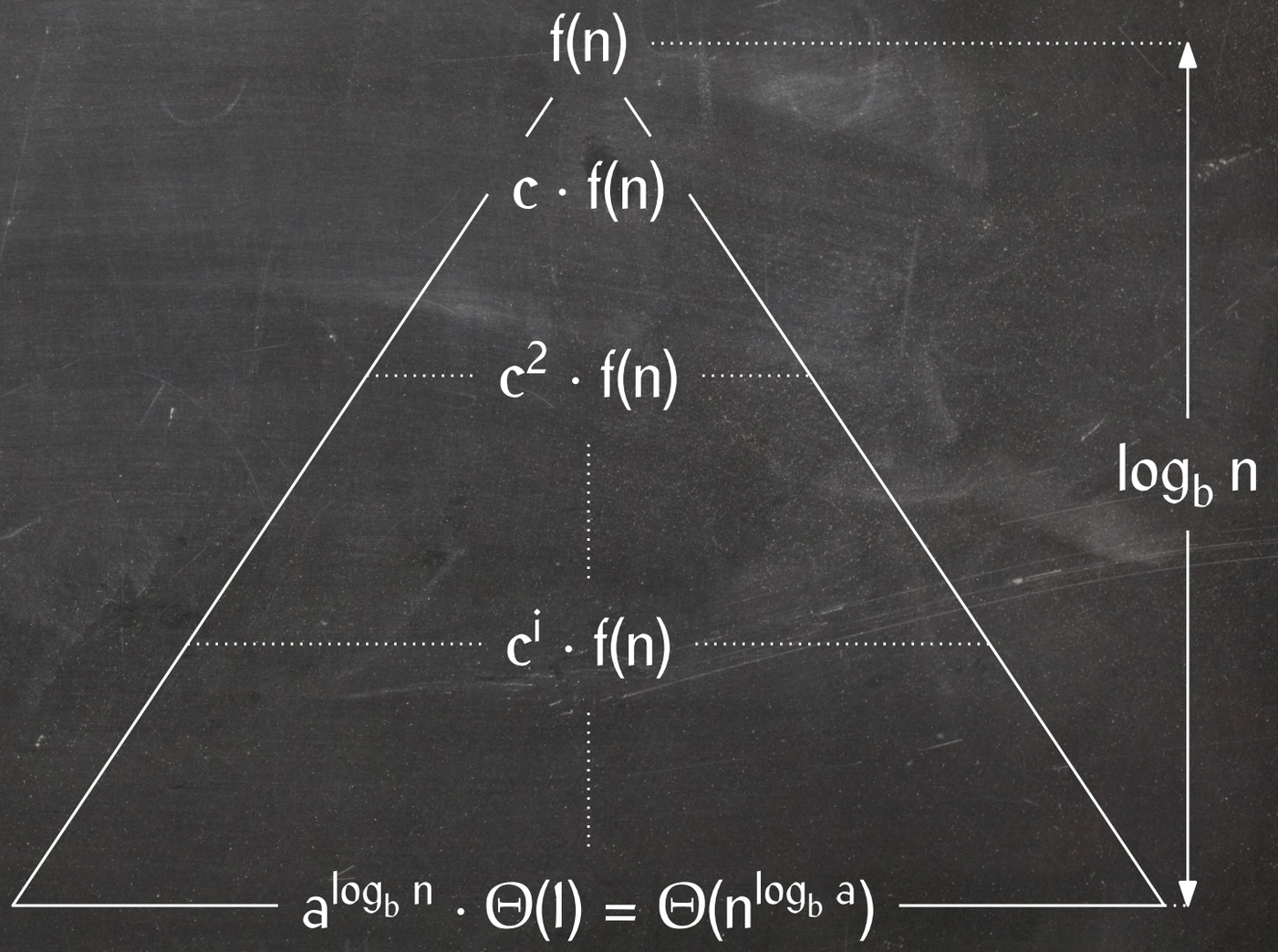

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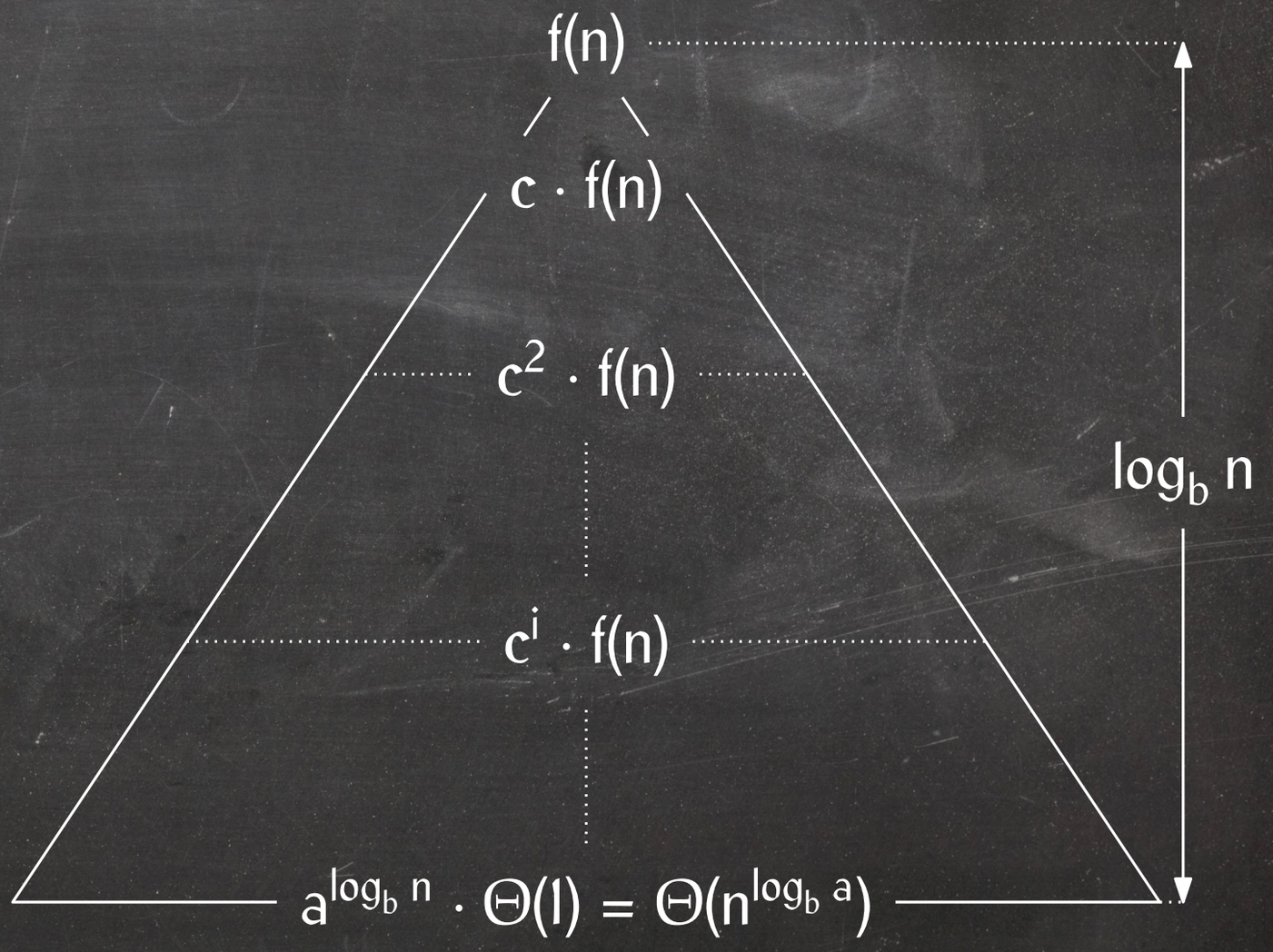

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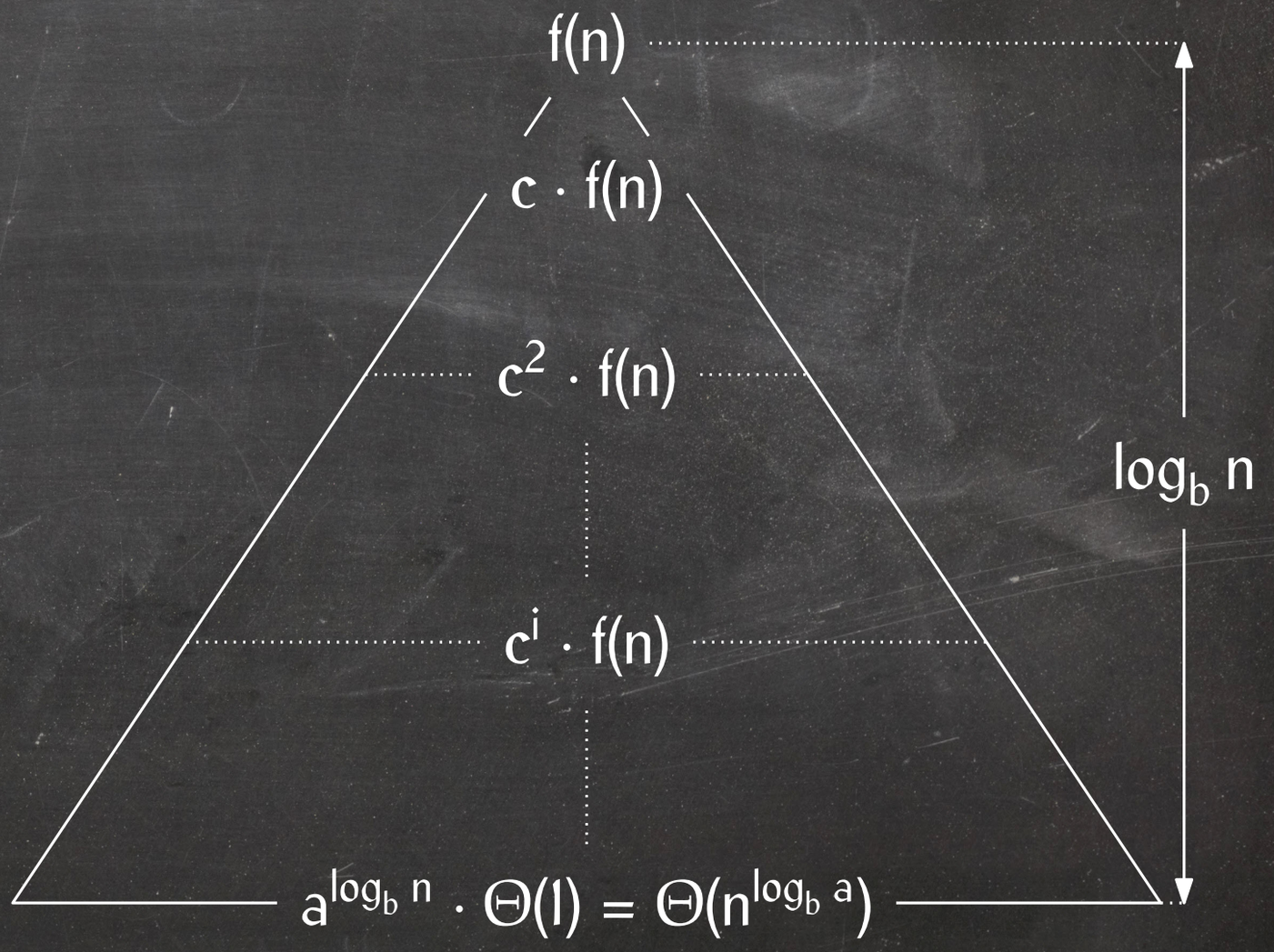

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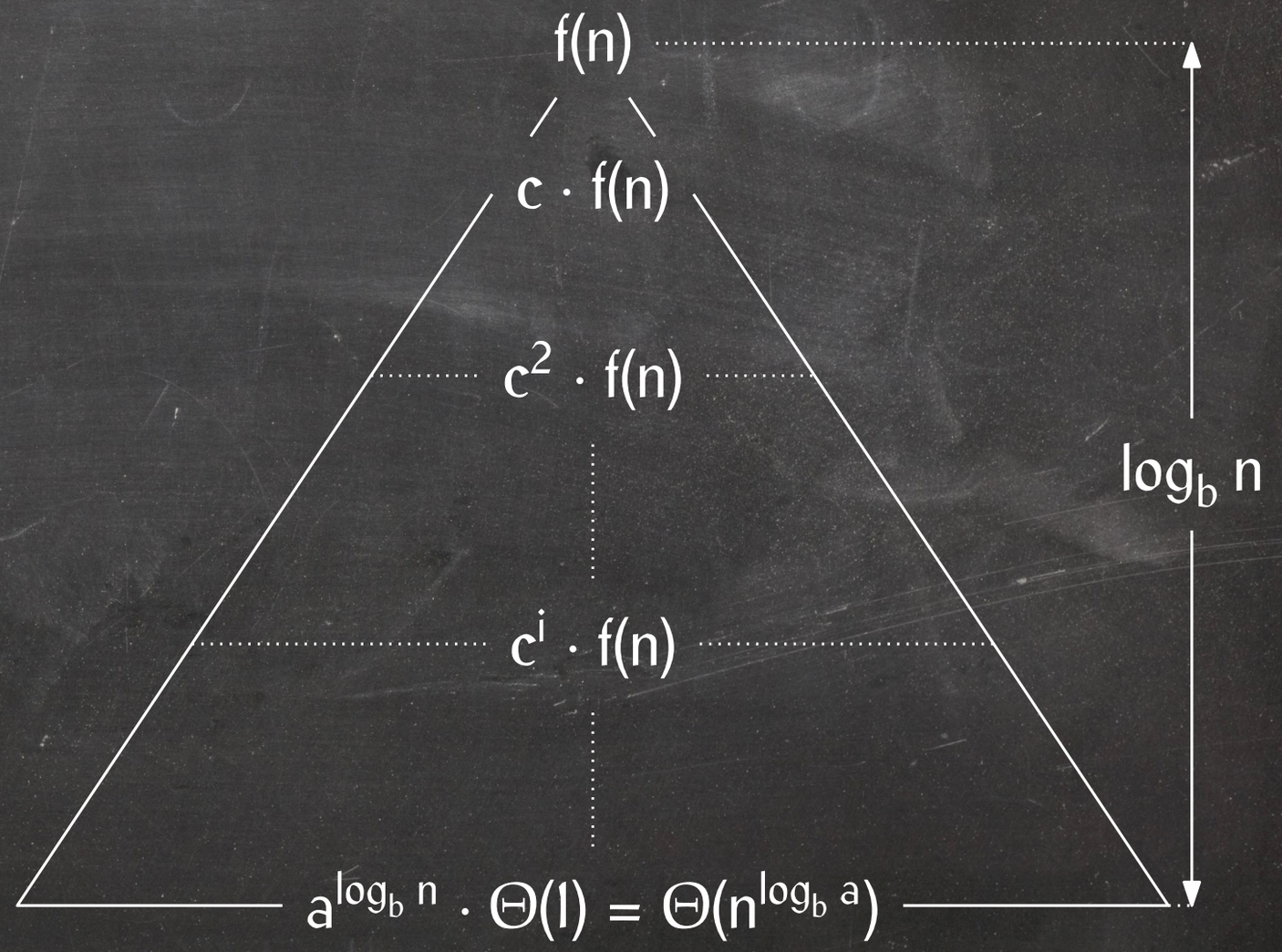

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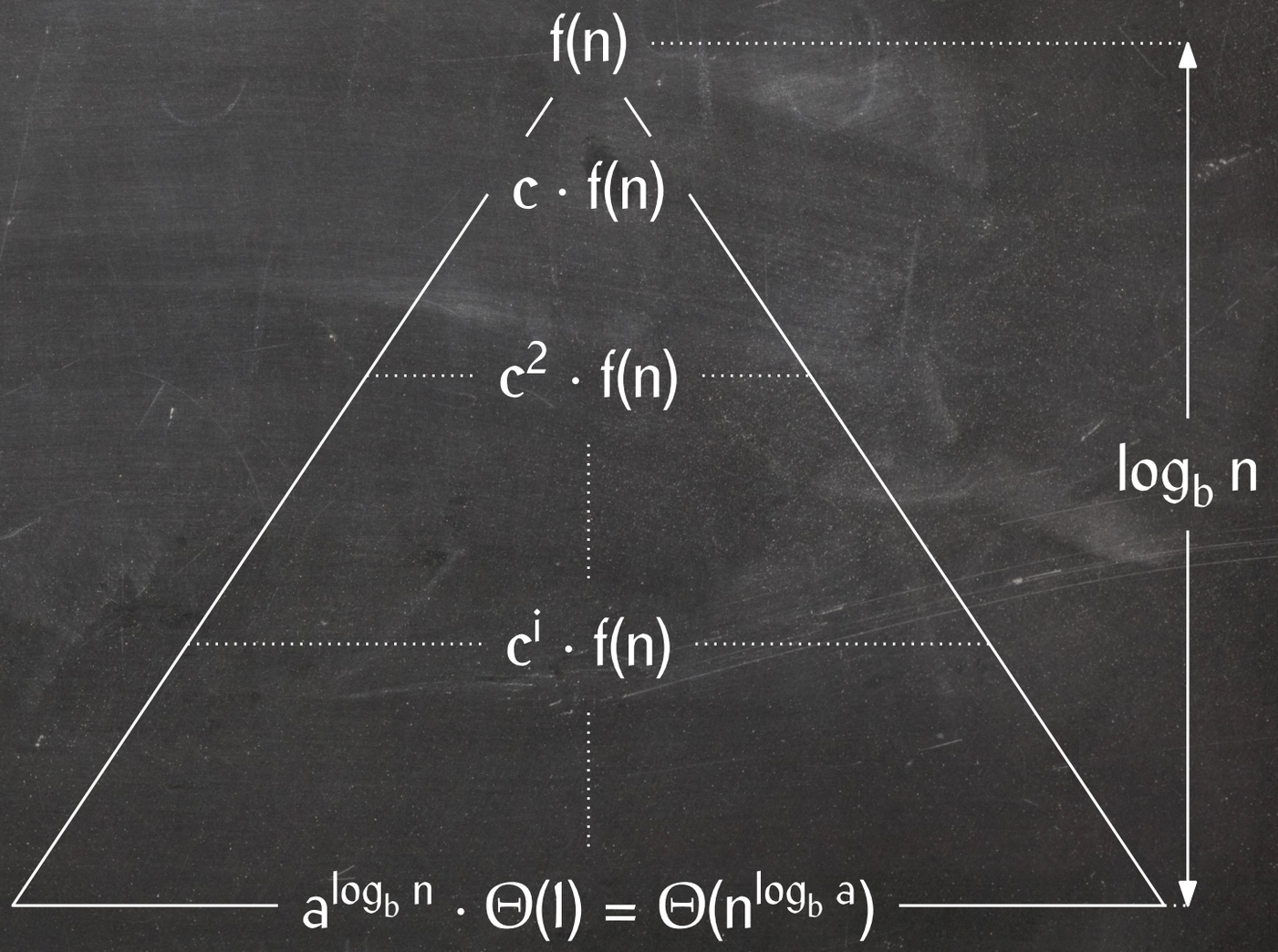

Quick Sort

QuickSort(A, ℓ, r)

```
1   if r ≤ ℓ
2       then return
3   m = Partition(A, ℓ, r)
4   QuickSort(A, ℓ, m − 1)
5   QuickSort(A, m + 1, r)
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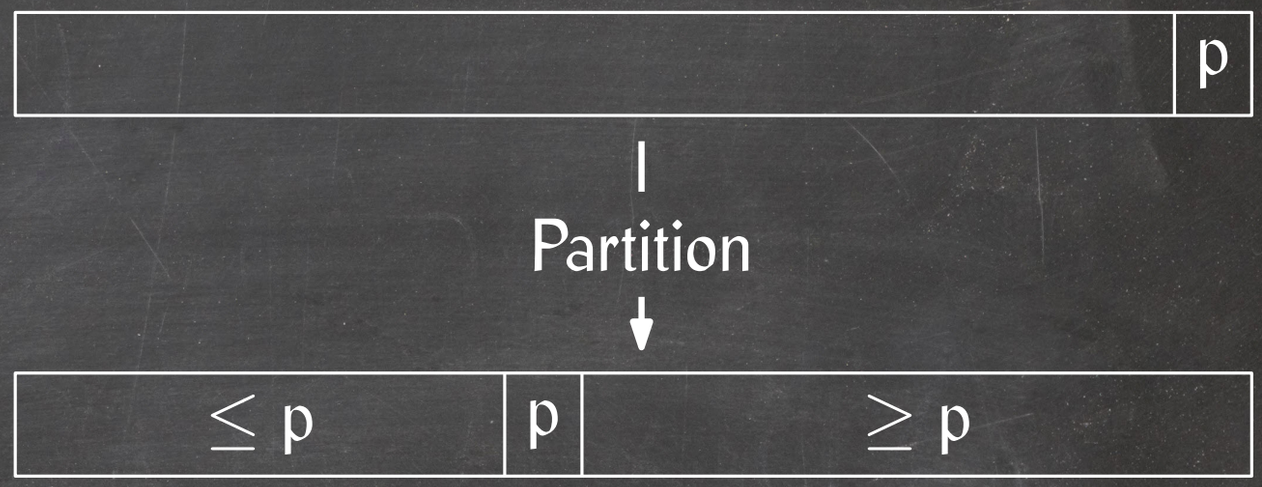

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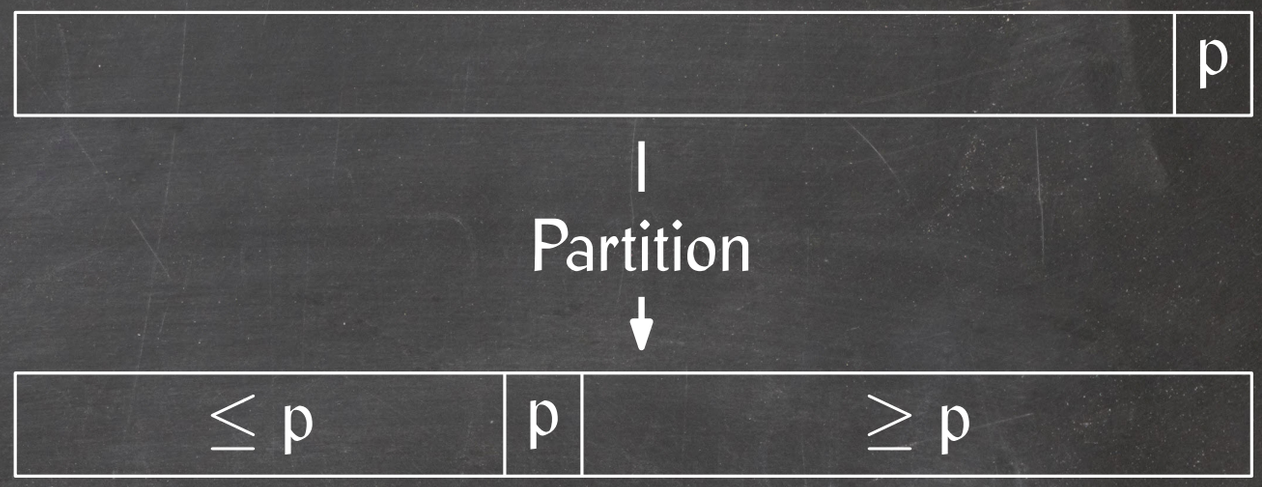

Worst case:

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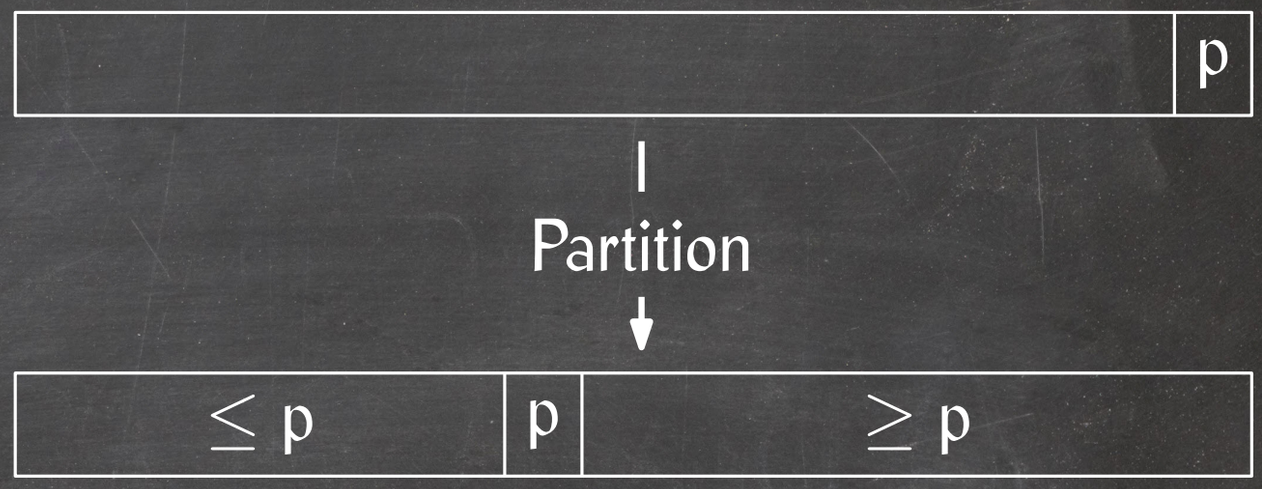

Worst case:

$$T(n) = \Theta(n) + T(n - 1)$$

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QuickSort(A, ℓ, r)

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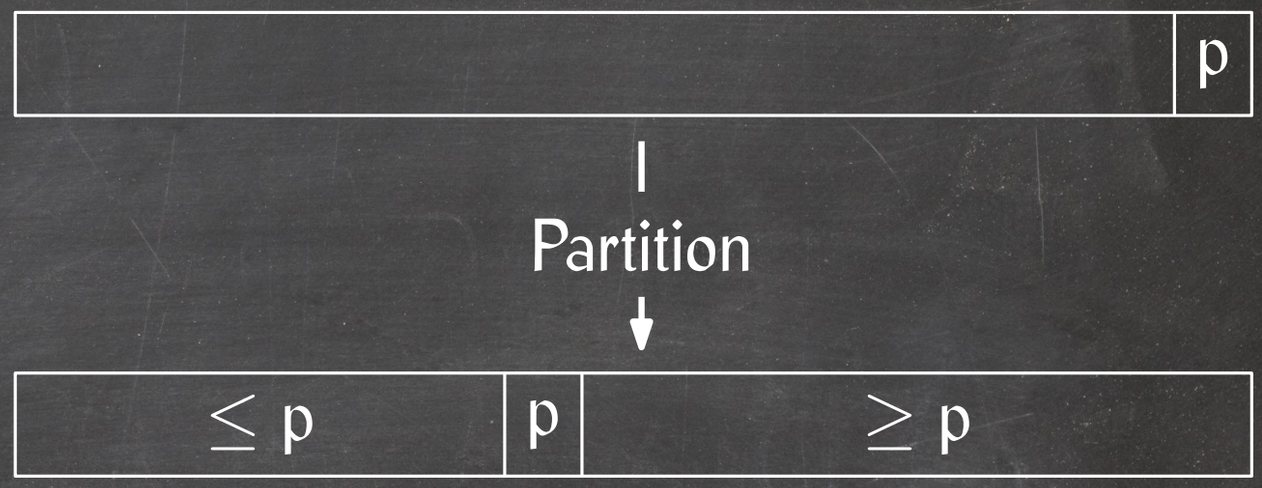

Worst case:

$$T(n) = \Theta(n) + T(n - 1) = \Theta(n^2)$$

Quick Sort

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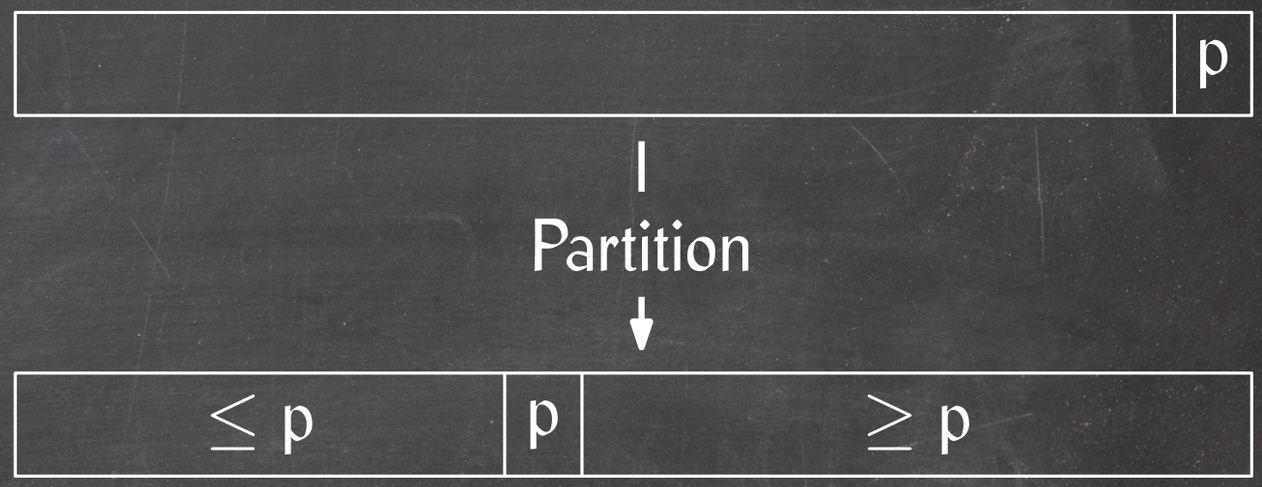


Worst case:

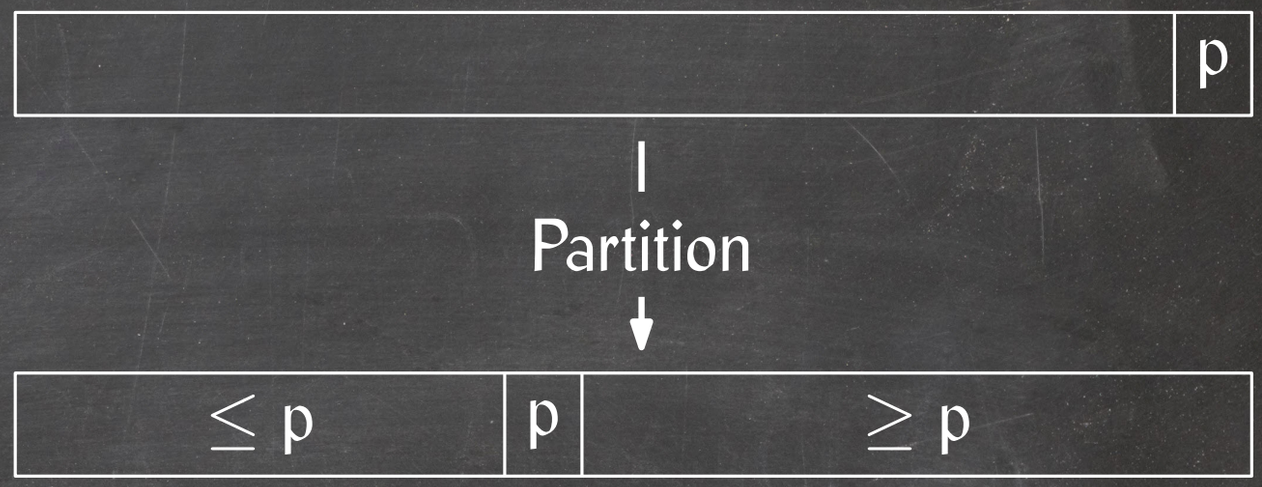
$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

Best case:

Quick Sort

QuickSort(A, ℓ, r)

```
1  if r ≤ ℓ
2       then return
3  m = Partition(A, ℓ, r)
4  QuickSort(A, ℓ, m − 1)
5  QuickSort(A, m + 1, r)
```



Worst case:

$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

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$$T(n) = \Theta(n) + 2T(\lfloor n/2 \rfloor)$$

Quick Sort

QuickSort(A, ℓ, r)

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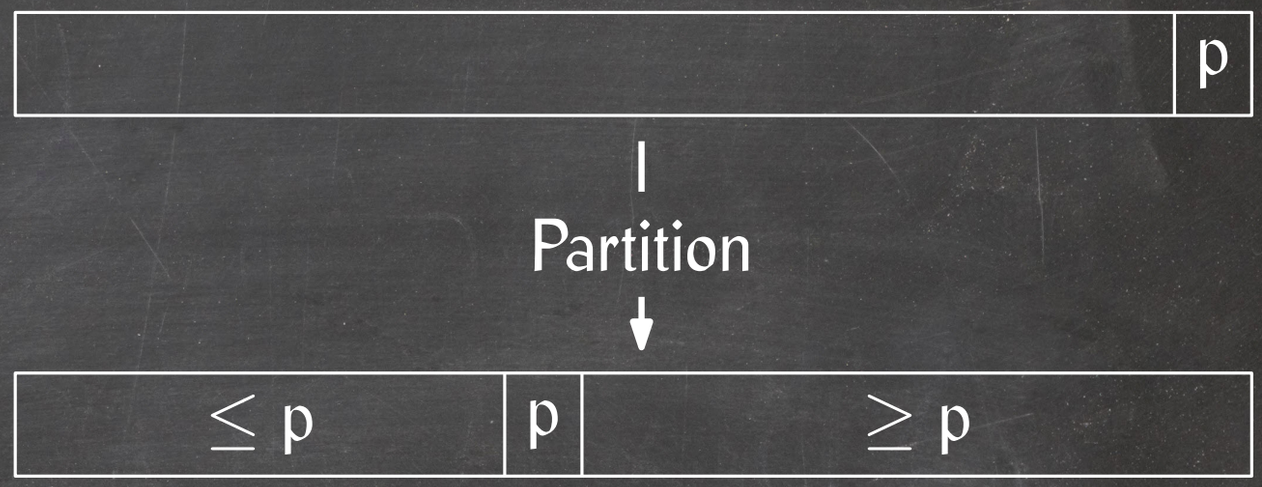

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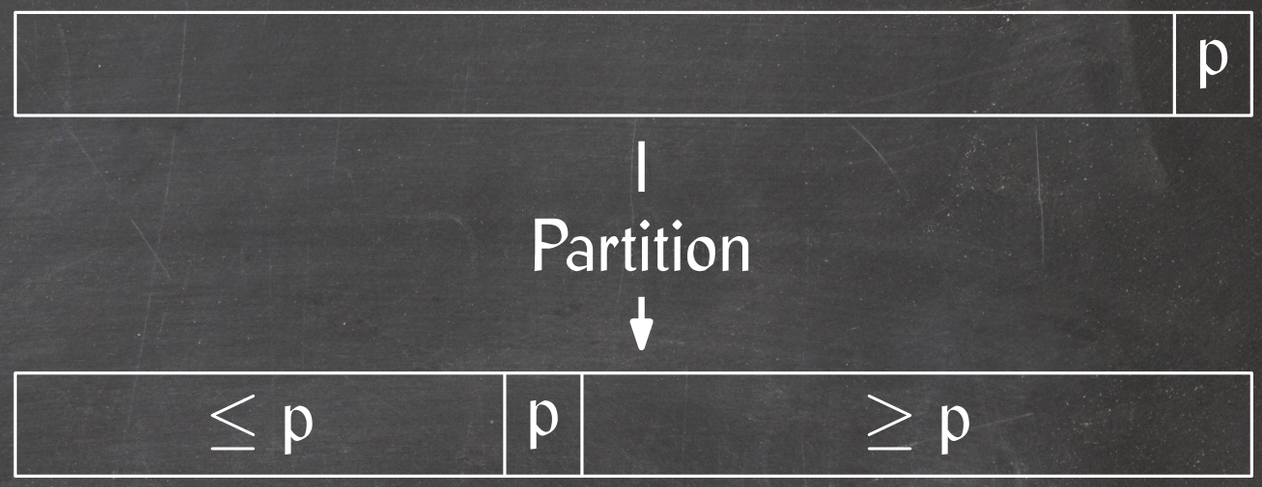

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Average case:

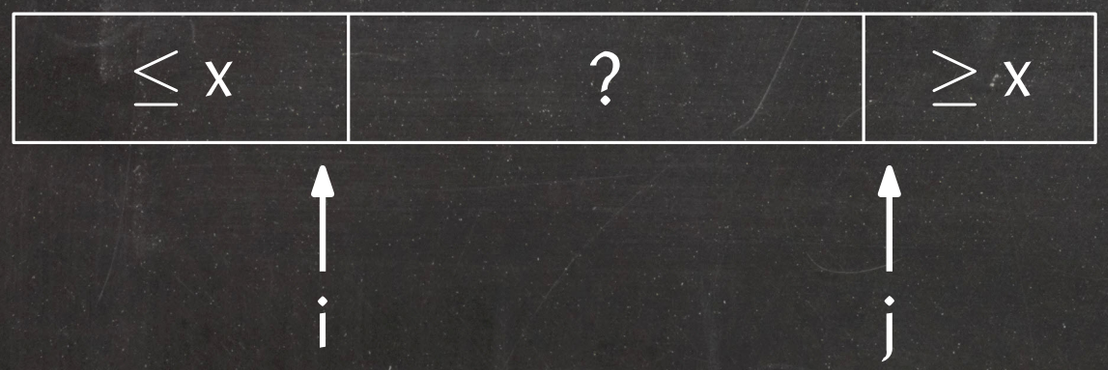
$$T(n) = \Theta(n \lg n)$$

Two Partitioning Algorithms

HoarePartition(A, l, r)

```
 1   x = A[r]
 2   i = l − 1
 3   j = r + 1
 4   while True
 5       do repeat i = i + 1
 6              until A[i] ≥ x
 7              repeat j = j − 1
 8              until A[j] ≤ x
 9              if i < j
10                  then swap A[i] and A[j]
11                  else return j
```

Loop invariants:



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LomutoPartition(A, l, r)

```
 1   i = l − 1
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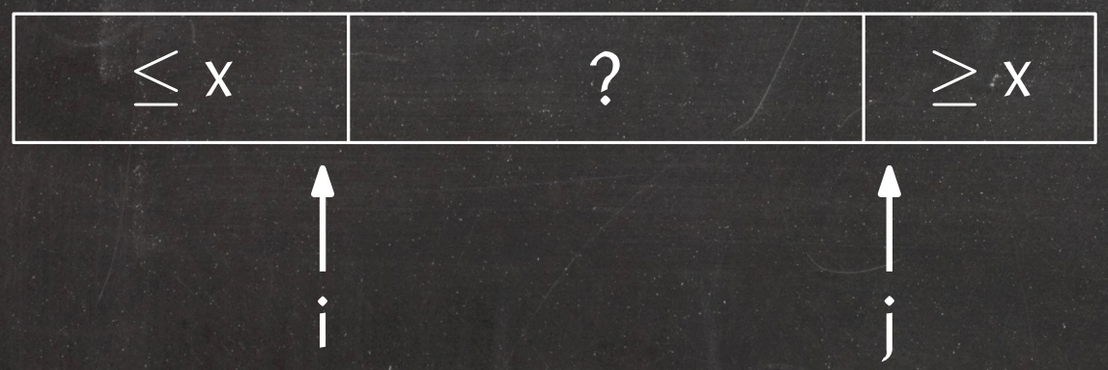
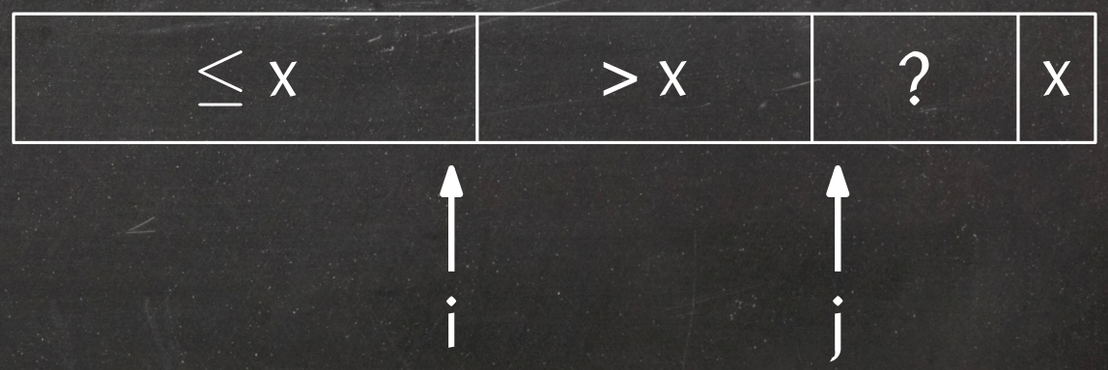
 4                  then i = i + 1
 5                       swap A[i] and A[j]
 6   swap A[i + 1] and A[r]
 7   return i + 1
```

Loop invariants:





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HoarePartition is more efficient in practice.

LomutoPartition has some properties that make average-case analysis easier.

HoarePartition is more convenient for worst-case Quick Sort.

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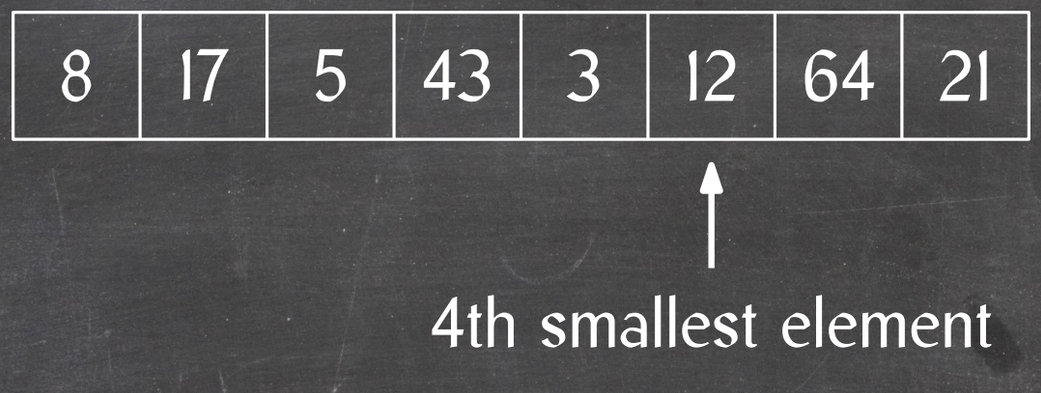
Selection

Goal: Given an unsorted array A and an integer $1 \leq k \leq n$, find the kth smallest element in A.



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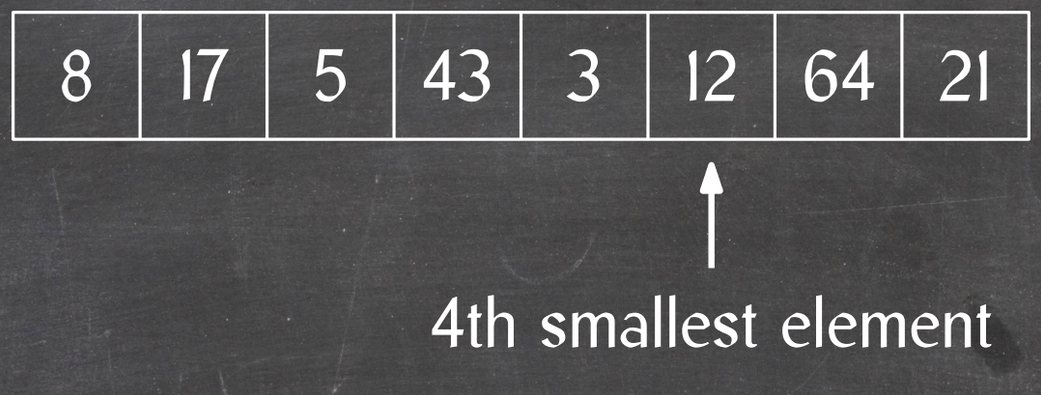

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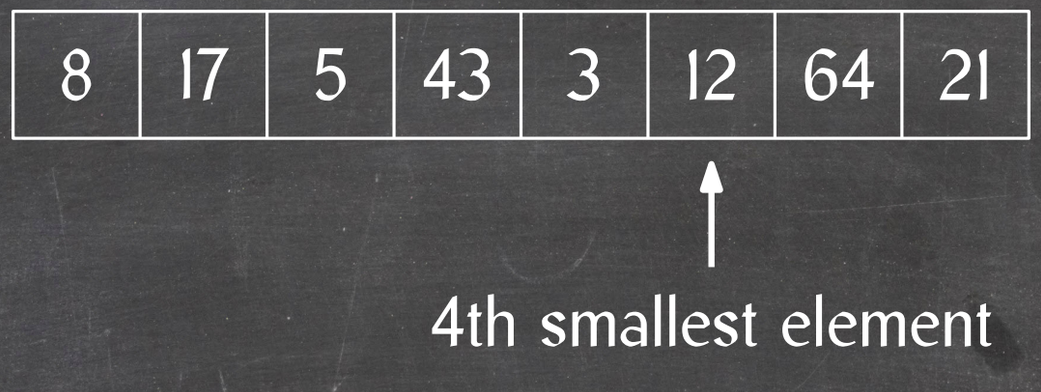

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We can find the minimum ($k = 1$) or the maximum ($k = n$) in $O(n)$ time!

It would be nice if we were able to find the kth smallest element, for any k, in $O(n)$ time.

The Sorting Idea Isn't Half Bad, But . . .

To find the kth smallest element, we don't need to sort the input completely.

We only need to verify that there are exactly k – 1 elements smaller than the element we return.

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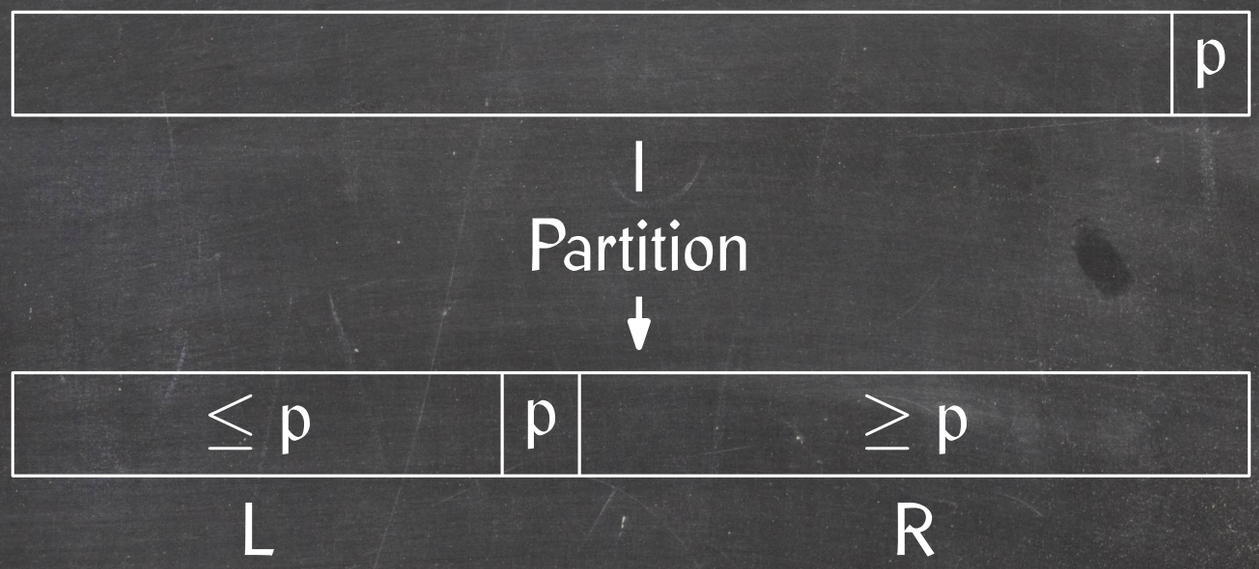

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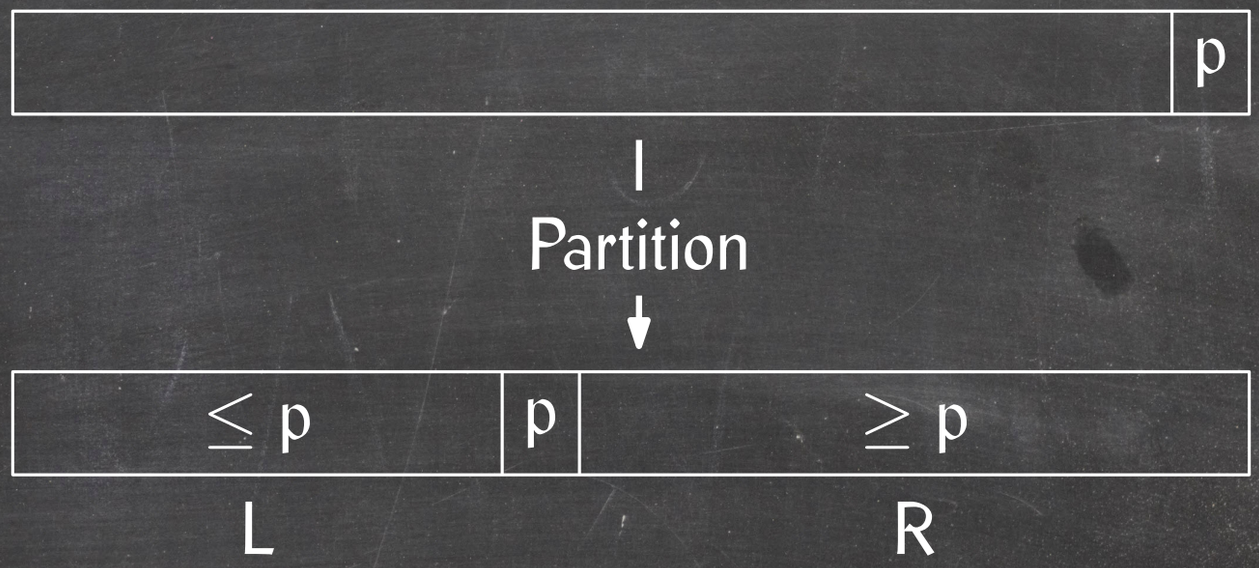

If |L| = k – 1, then p is the kth smallest element.

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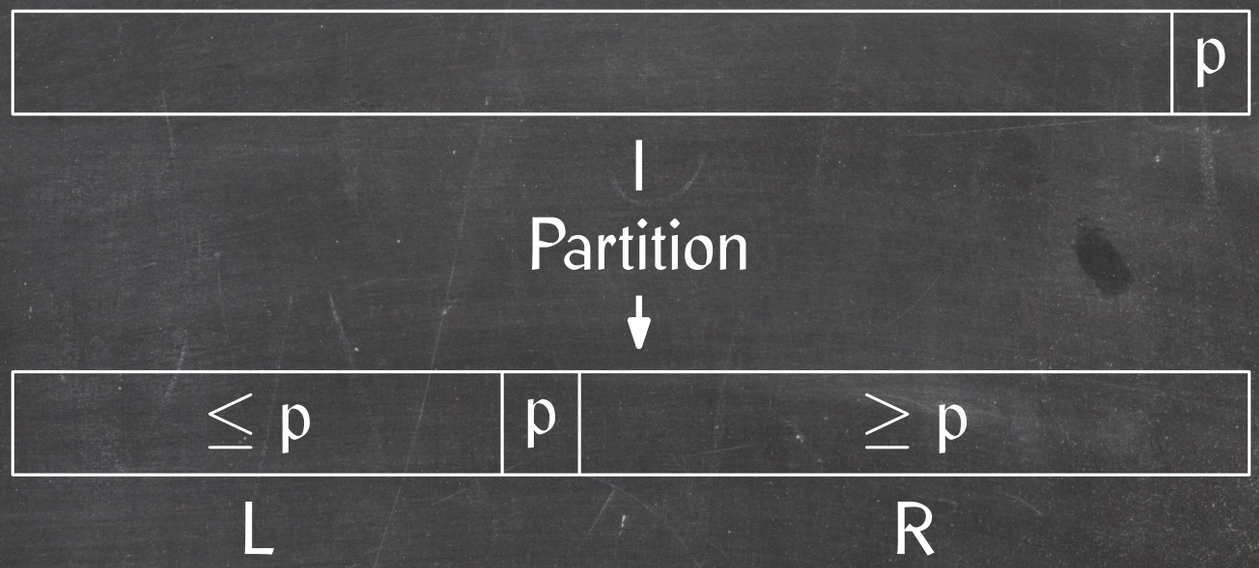

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If |L| ≥ k, then the kth smallest element in L is the kth smallest element in A.

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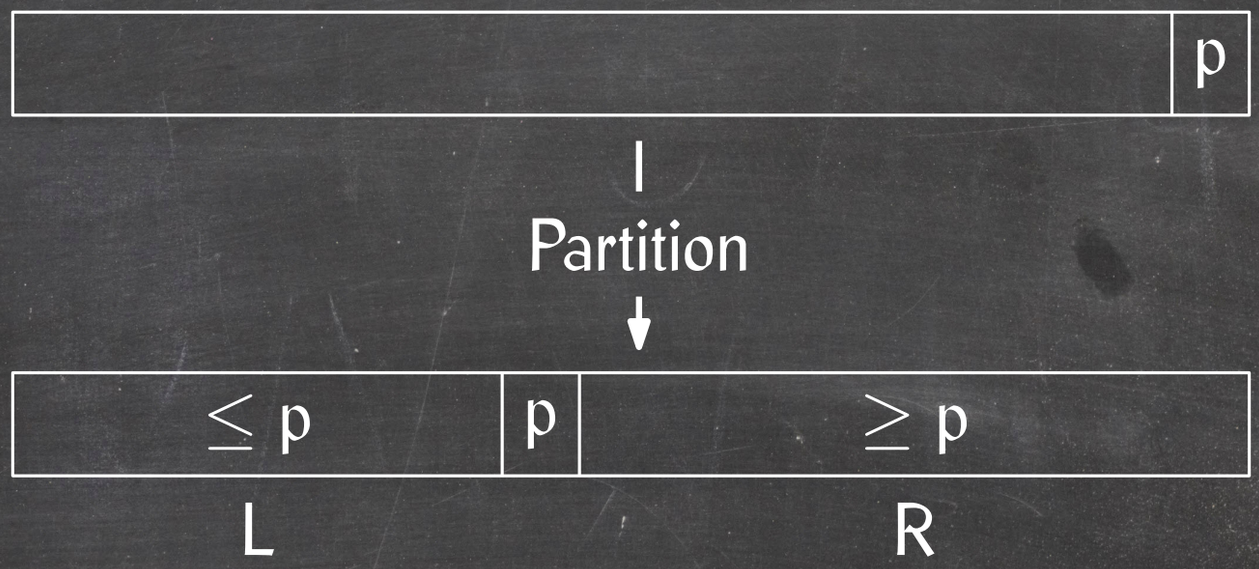

If |L| = k – 1, then p is the kth smallest element.

If |L| ≥ k, then the kth smallest element in L is the kth smallest element in A.

If |L| < k – 1, then the (k – |L| + 1)st element in R is the kth smallest element in A.

Quick Select

```
QuickSelect(A, ℓ, r, k)
 1  if r ≤ ℓ
 2      then return A[ℓ]
 3  m = Partition(A, ℓ, r)
 4  if m − ℓ = k − 1
 5      then return A[m]
 6      else if m − ℓ ≥ k
 7          then return QuickSelect(A, ℓ, m − 1, k)
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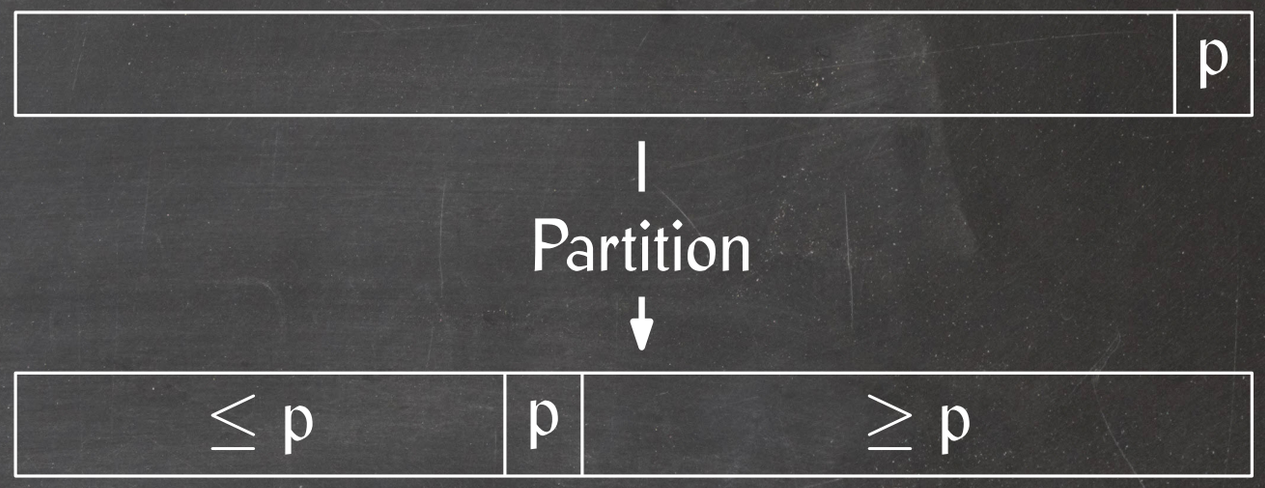

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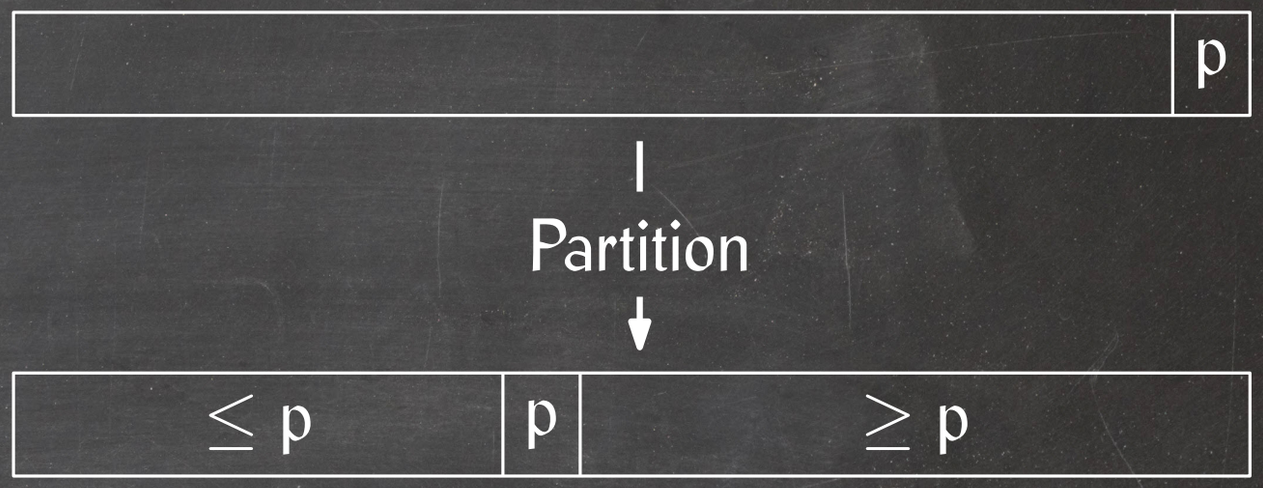

Worst case:

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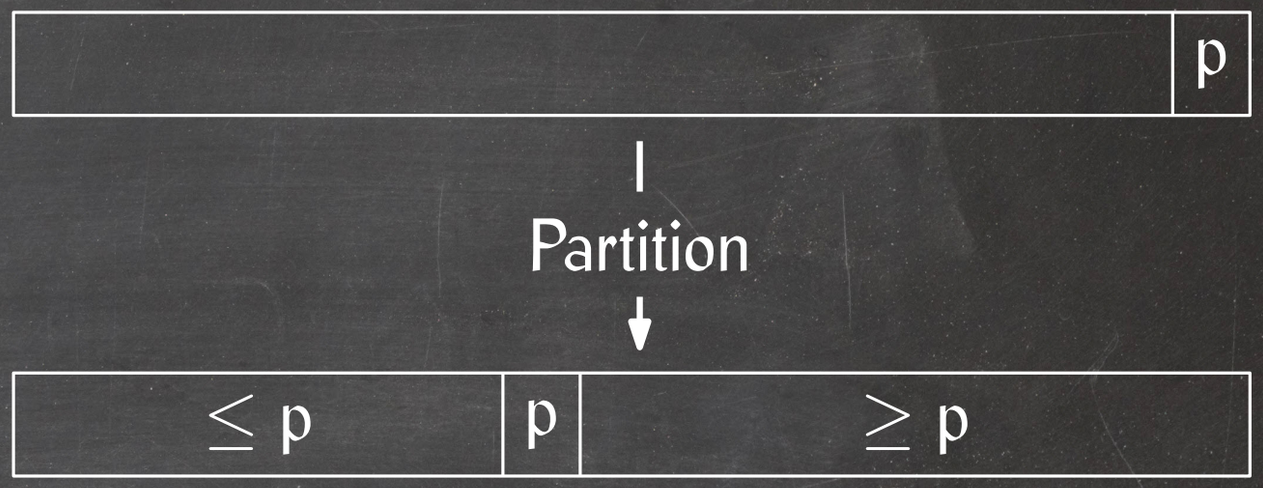

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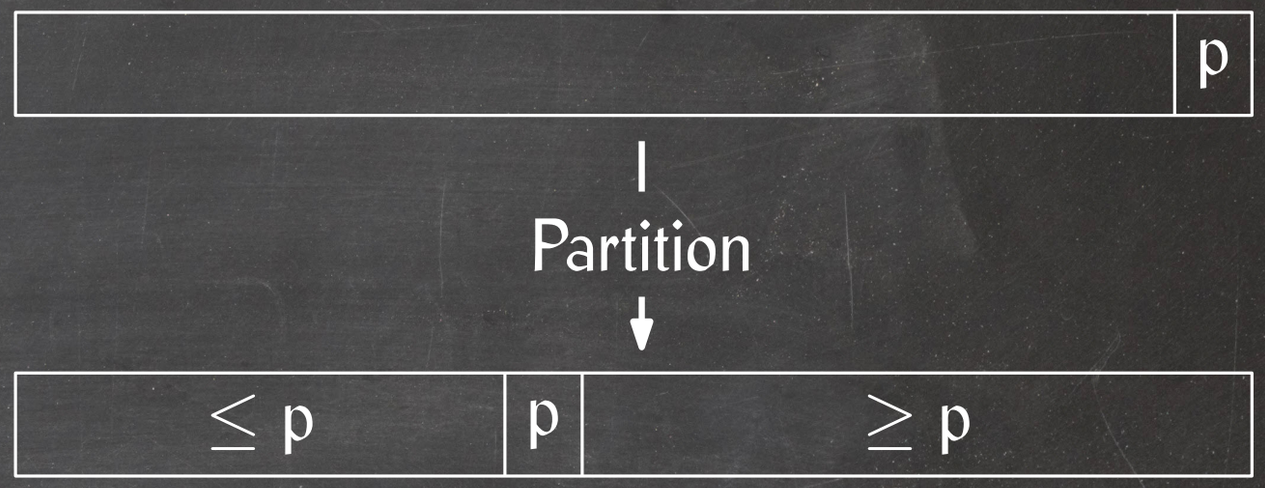

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Best case:

Quick Select

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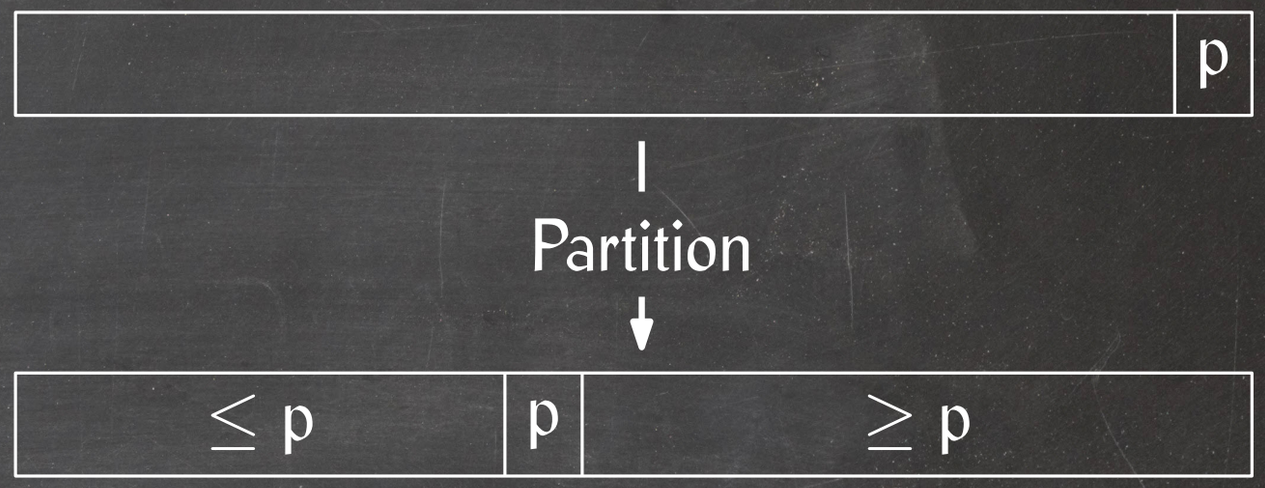

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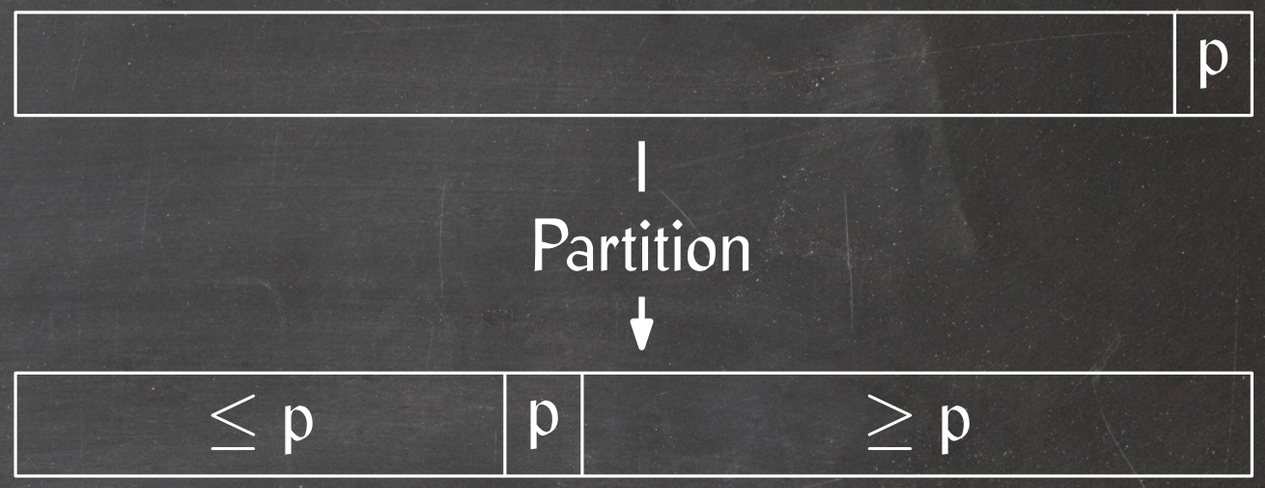

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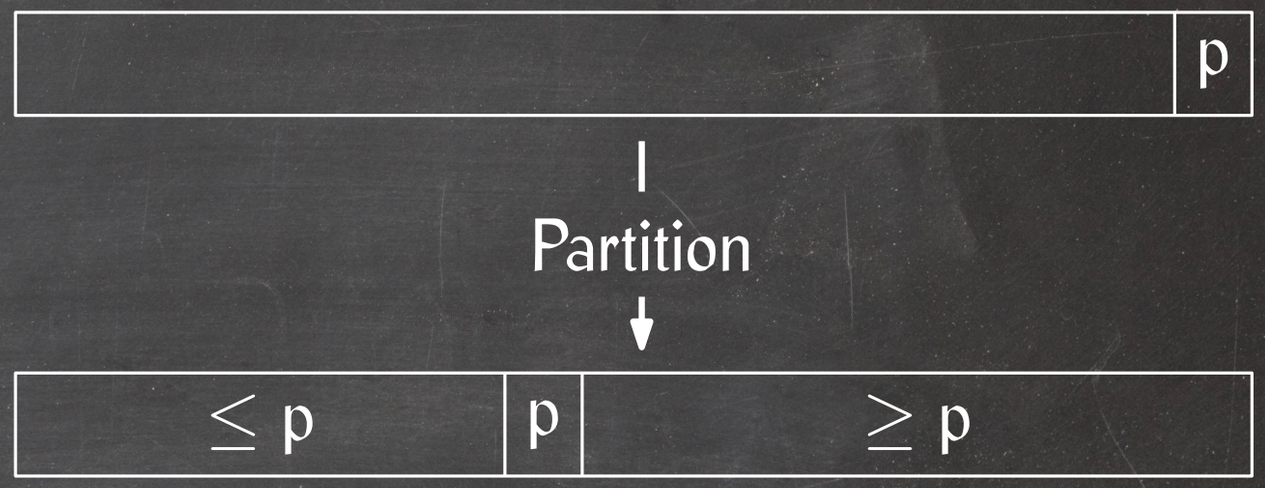

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Average case:

$$T(n) = \Theta(n)$$

Worst-Case Selection

QuickSelect(A, ℓ, r, k)

```
1  if r ≤ ℓ
2     then return A[ℓ]
3  p = FindPivot(A, ℓ, r)
4  m = HoarePartition(A, ℓ, r, p)
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If we could guarantee that p is the median of A[ℓ .. r], then we'd recurse on at most n/2 elements.

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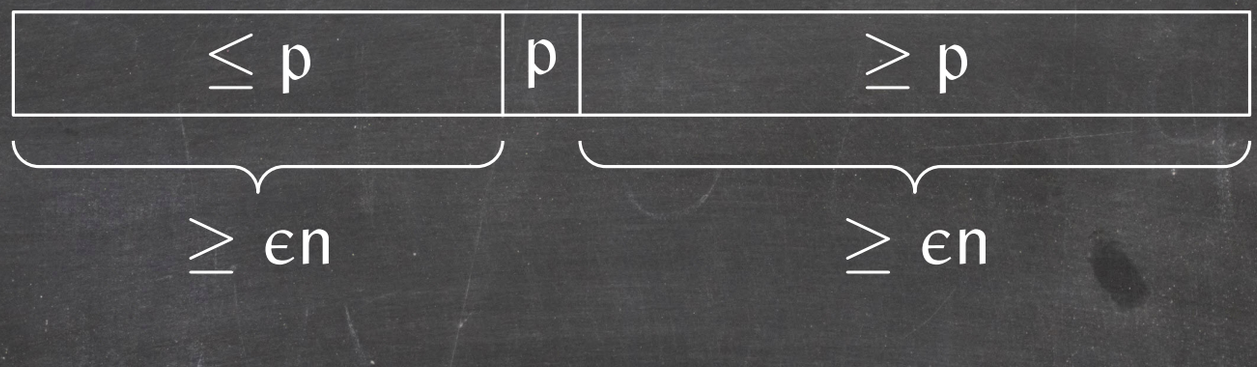
$\Rightarrow\ T(n) = \Theta(n) + T(n/2) = \Theta(n)$.

Alas, finding the median is selection!

Making Do With An Approximate Median



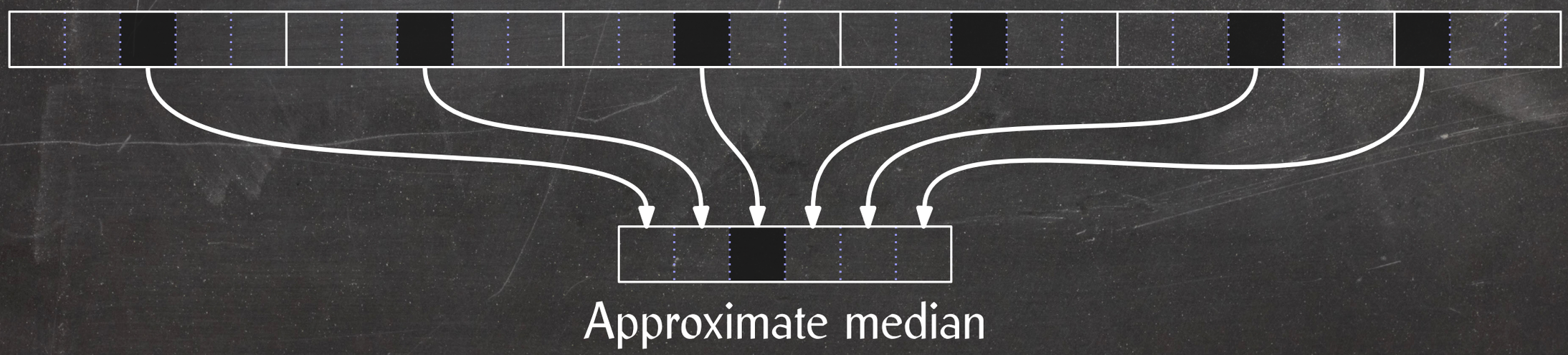
If there are at least ϵn elements smaller than p and at least ϵn elements greater than p, then

$$T(n) \leq \Theta(n) + T((1 - \epsilon)n) = \Theta(n).$$

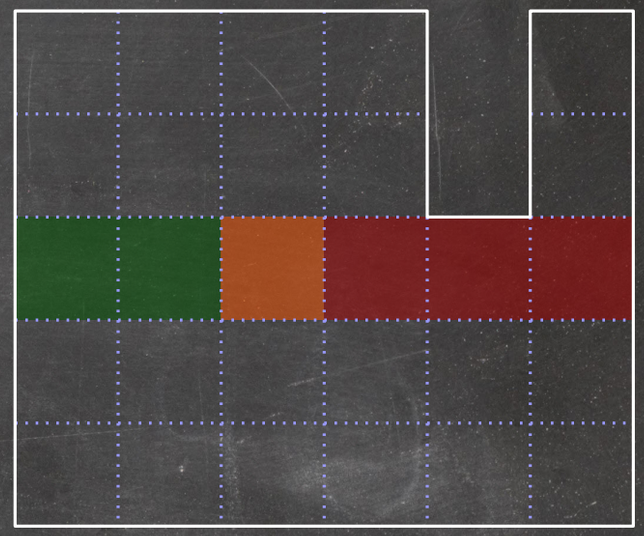
Finding An Approximate Median

FindPivot(A, ℓ, r)

```
1  n' = ⌊(r − ℓ)/5⌋ + 1
2  for i = 0 to n' − 1
3      do InsertionSort(A, ℓ + 5 · i, min(ℓ + 5 · i + 4, r))
4          if ℓ + 5i + 4 ≤ r
5              then B[i + 1] = A[ℓ + 5 · i + 2]
6              else B[i + 1] = A[ℓ + 5 · i]
7  return QuickSelect(B, 1, n', ⌈n'/2⌉)
```

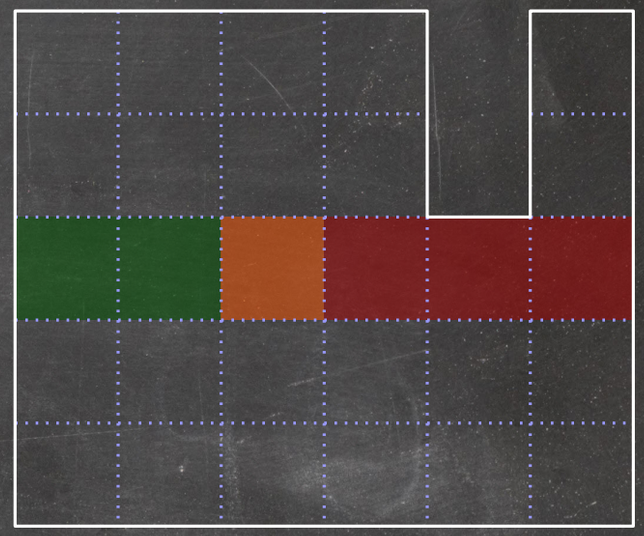


Finding An Approximate Median



Finding An Approximate Median

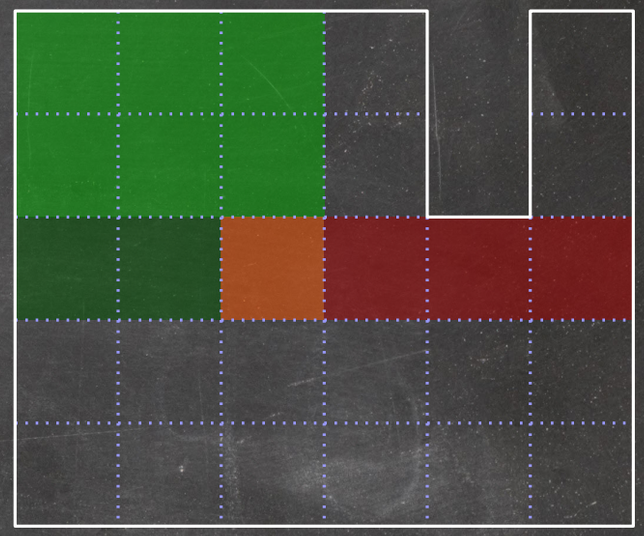
There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.



Finding An Approximate Median

There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.

For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.



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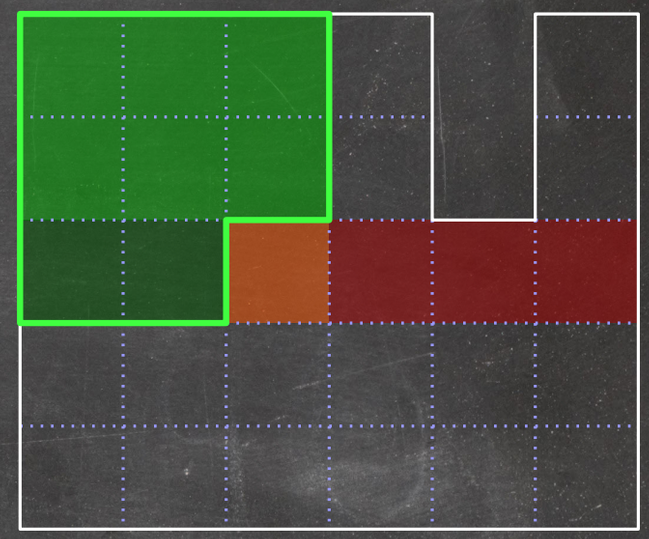

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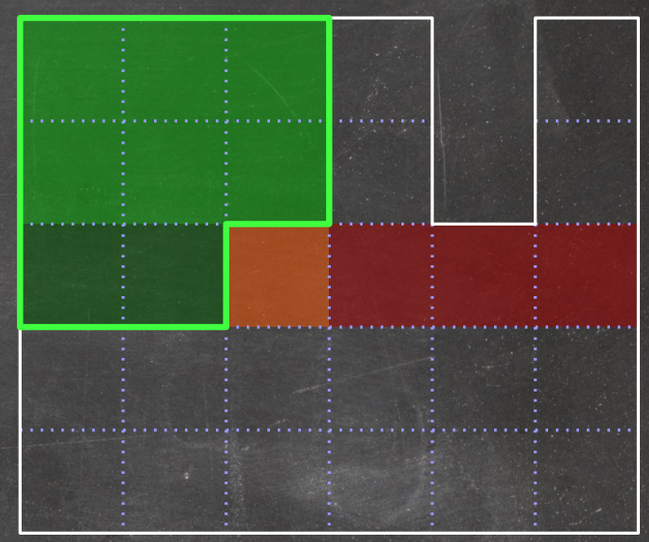
$n' = \lceil n/5 \rceil$



Finding An Approximate Median

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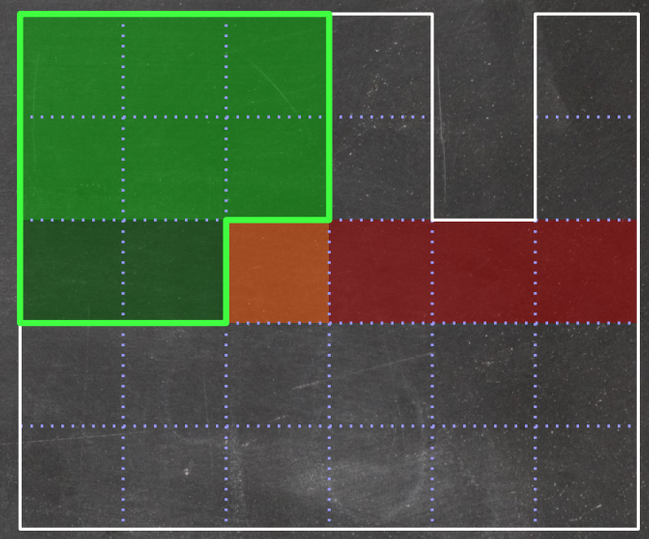

Total number of elements smaller than the median of medians:

$$3\left(\left\lceil \frac{n'}{2} \right\rceil - 1\right) = 3\left\lceil \frac{\lceil n/5 \rceil}{2} \right\rceil - 3 \geq \frac{3n}{10} - 3$$

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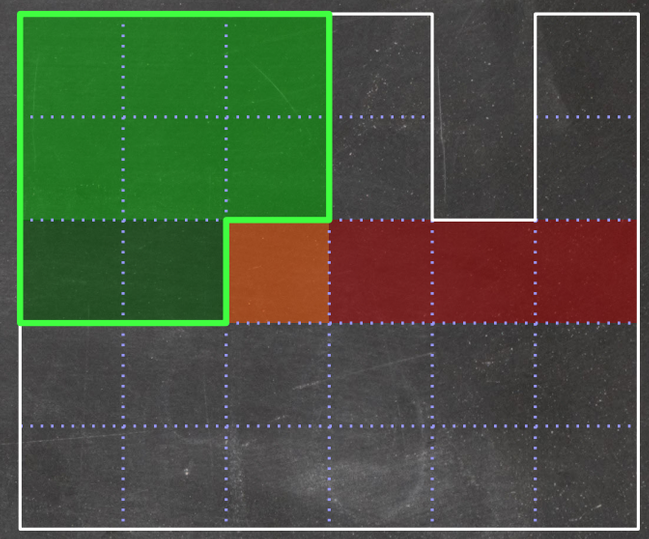

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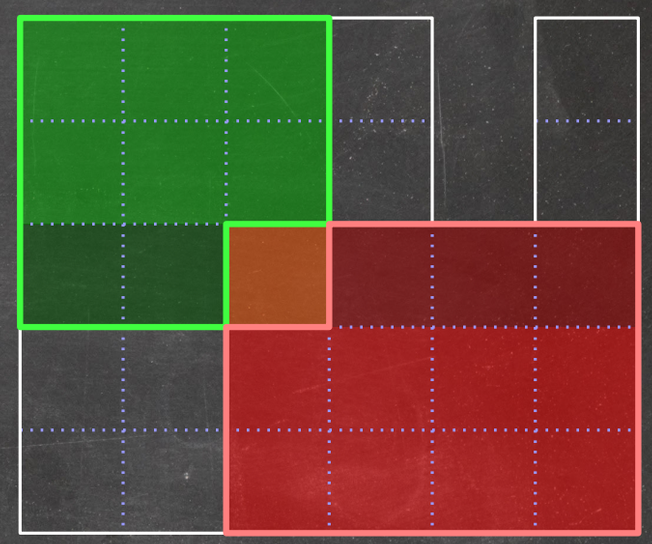

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Worst-Case Selection: Analysis

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$$T(n) \leq T(\tfrac{n}{5} + 1) + T(\tfrac{7n}{10} + 3) + O(n)$$

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Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

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$$
\begin{aligned}
T(n) &\leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an \\
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&\leq c\left(\frac{4n}{20} + \frac{14n}{20} + \frac{n}{20}\right) + an \\
&= \left(\frac{19c}{20} + a\right) n
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&= \left(\frac{19c}{20} + a\right) n \\
&\leq cn \quad \forall c \geq 20a
\end{aligned}
$$

Worst-Case Quick Sort

QuickSort(A, ℓ, r)

```
1   if r ≤ ℓ
2       then return
3   p = FindPivot(A, ℓ, r)
4   m = HoarePartition(A, ℓ, r, p)
5   return QuickSort(A, ℓ, m)
6   return QuickSort(A, m + 1, r)
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$T(n) = \Theta(n) + T(n_1) + T(n_2)$, where $n_1 + n_2 = n$ and $n_1, n_2 \leq \frac{7n}{10} + 3$

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Worst-Case Quick Sort

Inductive Step: ($n \geq 30$)

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$$\begin{aligned} T(n) &\leq T(n_1) + T(n_2) + an \\ &\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\ &\leq cn_1 \lg \left(\frac{7n}{10} + 3 \right) + cn_2 \lg \left(\frac{7n}{10} + 3 \right) + an \end{aligned}$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

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\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg \left(\frac{7n}{10} + 3 \right) + cn_2 \lg \left(\frac{7n}{10} + 3 \right) + an \\
&= cn \lg \left(\frac{7n}{10} + 3 \right) + an
\end{aligned}
$$

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Inductive Step: $(n \geq 30)$

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T(n) &\leq T(n_1) + T(n_2) + an \\
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&\leq cn \lg \left(\frac{7n}{10} + \frac{n}{10}\right) + an
\end{aligned}
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T(n) &\leq T(n_1) + T(n_2) + an \\
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$$
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&= cn \left(\lg n - \lg \frac{5}{4} \right) + an
\end{aligned}
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&= cn \left(\lg n - \lg \frac{5}{4} \right) + an \\
&= cn \lg n + \left(a - c \lg \frac{5}{4} \right) n
\end{aligned}
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&= cn \left(\lg n - \lg \frac{5}{4} \right) + an \\
&= cn \lg n + \left(a - c \lg \frac{5}{4} \right) n \\
&\leq cn \lg n \quad \forall c \geq \frac{a}{\lg(5/4)} \approx 3.1a
\end{aligned}
$$

Matrix Multiplication

We want to compute $C = A \times B$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices and hence $C = (c_{ij})$ is too.

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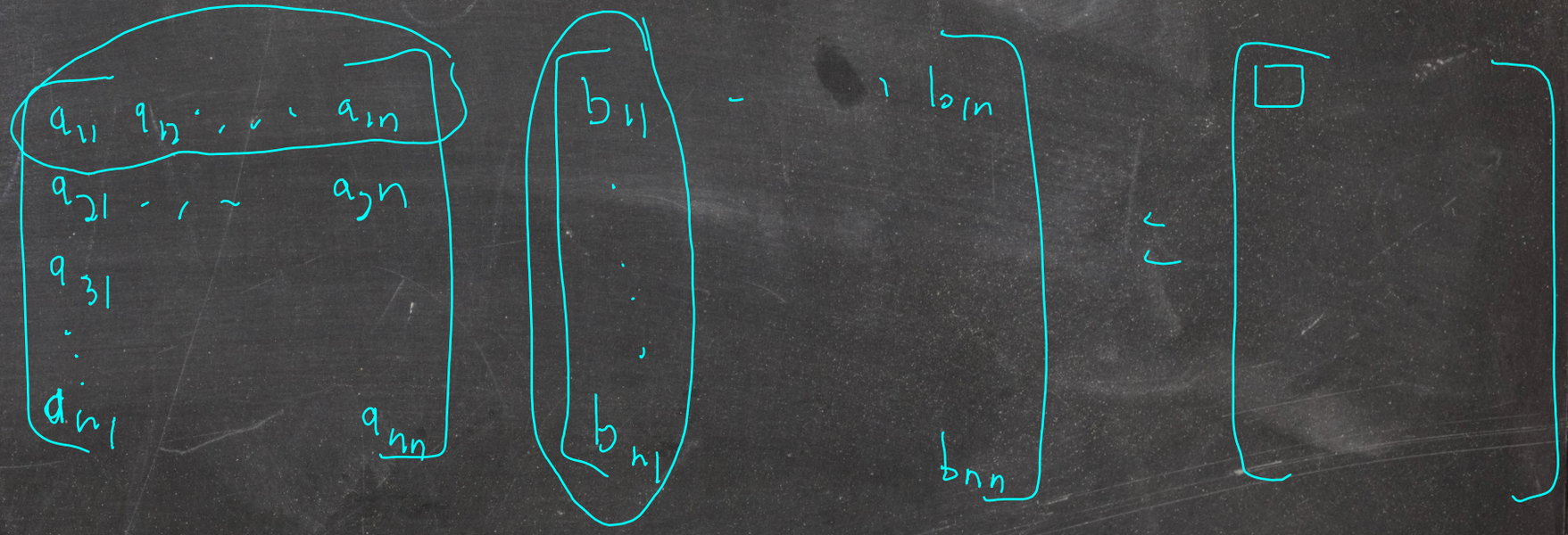

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The naïve algorithm implementing the definition:

MatrixProduct(A, B)

```
1  C = an n × n array
2  for i = 1 to n
3      do for k = 1 to n
4             do C[i, k] = 0
5                for j = 1 to n
6                   do C[i, k] = C[i, k] + A[i, j] · B[j, k]
7  return C
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Cost: $\Theta(n^3)$

Matrix Multiplication: Divide and Conquer

For simplicity, assume $n = 2^t$ for some integer t.

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Inductive step: t > 0

$$\begin{array}{|c|c|} \hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline \end{array} = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \times \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}$$

For $1 \le i, k \le 2$,

$$C_{ik} = \sum_{j=1}^{2} (A_{ij} \times B_{jk}).$$

$$C_{11} = \sum_{j=1}^{2} A_{1j} \times B_{j1}$$

$$= A_{11} B_{11} + A_{12} B_{21}$$

Matrix Multiplication: Divide and Conquer

MatrixProductDNC($A, B, C, i_l, i_u, j_l, j_u, k_l, k_u$)

```
 1  if i_l = i_u
 2     then C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] × B[j_l, k_l]
 3     else i_m = (i_l + i_u − 1)/2
 4          j_m = (j_l + j_u − 1)/2
 5          k_m = (k_l + k_u − 1)/2
 6          MatrixProductDNC(A, B, C, i_l    , i_m, j_l    , j_m, k_l    , k_m)
 7          MatrixProductDNC(A, B, C, i_l    , i_m, j_m + 1, j_u, k_l    , k_m)
 8          MatrixProductDNC(A, B, C, i_l    , i_m, j_l    , j_m, k_m + 1, k_u)
 9          MatrixProductDNC(A, B, C, i_l    , i_m, j_m + 1, j_u, k_m + 1, k_u)
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```

Matrix Multiplication: Divide and Conquer

MatrixProductDNC($A, B, C, i_l, i_u, j_l, j_u, k_l, k_u$)

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 1  if i_l = i_u
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 6          MatrixProductDNC(A, B, C, i_l     , i_m, j_l     , j_m, k_l     , k_m)
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13          MatrixProductDNC(A, B, C, i_m + 1, i_u , j_m + 1, j_u , k_m + 1, k_u )
```

Cost: $T(n) = 8T(n/2) + \Theta(1)$

Matrix Multiplication: Divide and Conquer

MatrixProductDNC$(A, B, C, i_l, i_u, j_l, j_u, k_l, k_u)$

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 2     then C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] × B[j_l, k_l]
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 6          MatrixProductDNC(A, B, C, i_l    , i_m, j_l    , j_m, k_l    , k_m)
 7          MatrixProductDNC(A, B, C, i_l    , i_m, j_m + 1, j_u, k_l    , k_m)
 8          MatrixProductDNC(A, B, C, i_l    , i_m, j_l    , j_m, k_m + 1, k_u)
 9          MatrixProductDNC(A, B, C, i_l    , i_m, j_m + 1, j_u, k_m + 1, k_u)
10          MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l    , j_m, k_l    , k_m)
11          MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l    , k_m)
12          MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l    , j_m, k_m + 1, k_u)
13          MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_m + 1, k_u)
```

Cost: $T(n) = 8T(n/2) + \Theta(1) \in \Theta(n^3)$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

$$= \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$= (A_{11} + A_{21}) \times (B_{11} + B_{12})$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

$$= \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

2 inconveniently placed ones
= 2 multiplications

$$= (A_{11} + A_{21}) \times (B_{11} + B_{12})$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

2 inconveniently placed ones
= 2 multiplications

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

$$= \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

4 conveniently placed ones
= 1 multiplication

$$= (A_{11} + A_{21}) \times (B_{11} + B_{12})$$

Matrix Multiplication: Strassen's Alg.

$$M_1 = (A_{11} + A_{22}) \times (B_{11} + B_{22})$$

$$M_2 = A_{11} \times (B_{12} + B_{22})$$

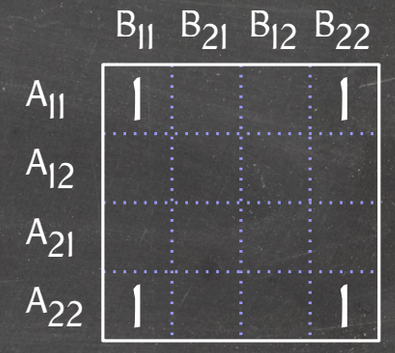
$$M_3 = A_{22} \times (B_{11} + B_{21})$$

$$M_4 = (A_{11} - A_{12}) \times B_{22}$$

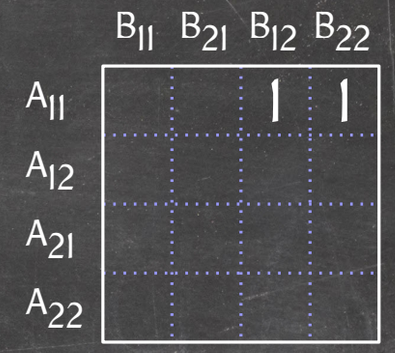
$$M_5 = (A_{22} - A_{21}) \times B_{11}$$

$$M_6 = (A_{11} + A_{21}) \times (B_{11} - B_{12})$$

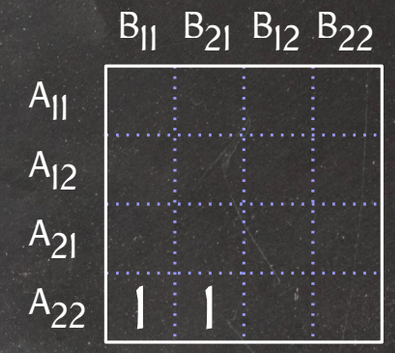
$$M_7 = (A_{12} + A_{22}) \times (B_{21} - B_{22})$$



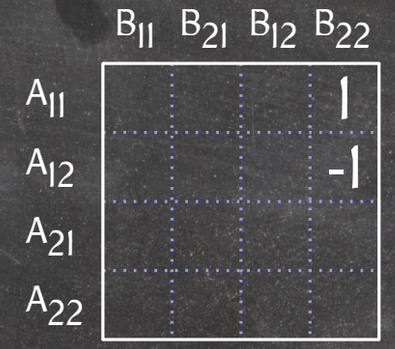
M_1



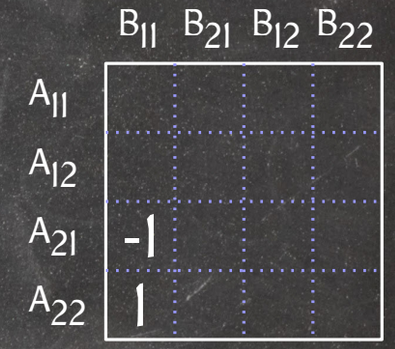
M_2



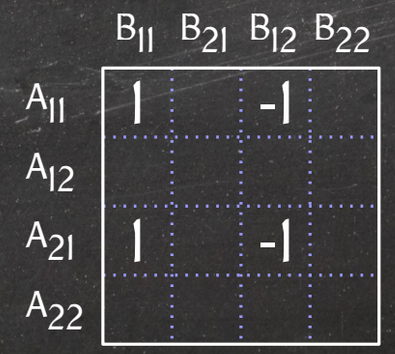
M_3



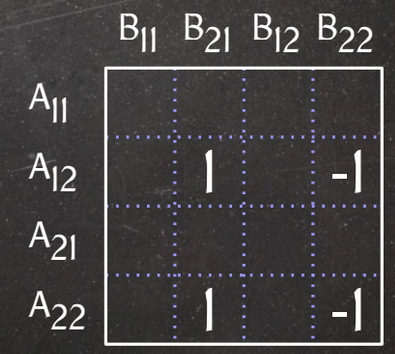
M_4



M_5



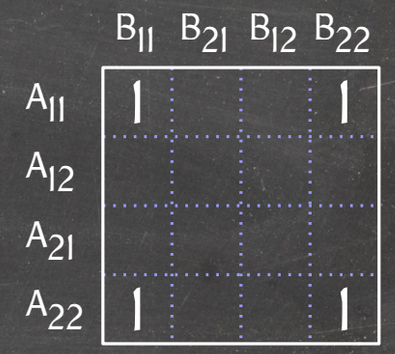
M_6



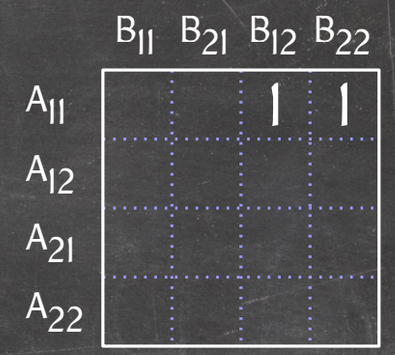
M_7

Matrix Multiplication: Strassen's Alg.

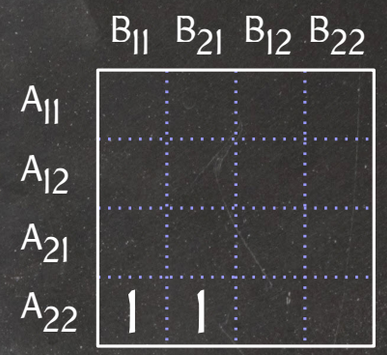
C_{11}



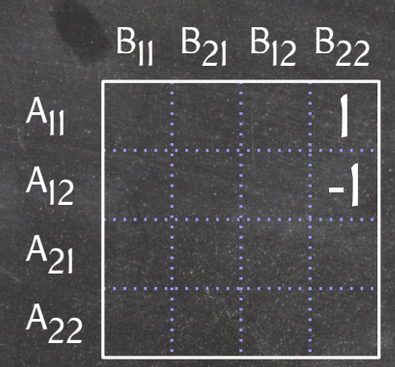
M_1



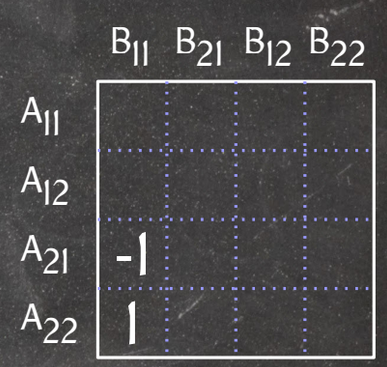
M_2



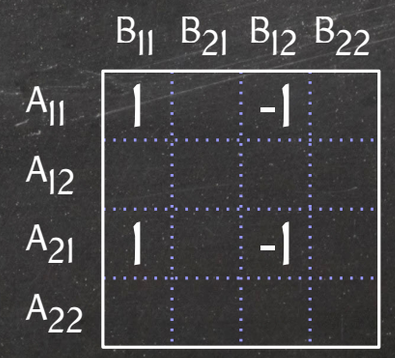
M_3



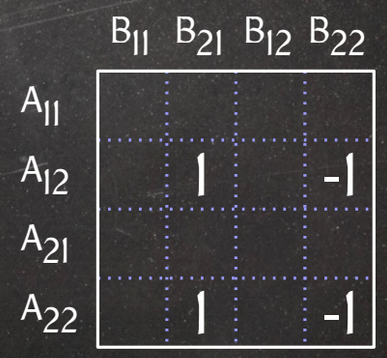
M_4



M_5



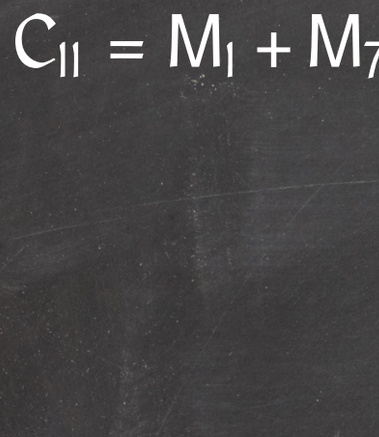
M_6



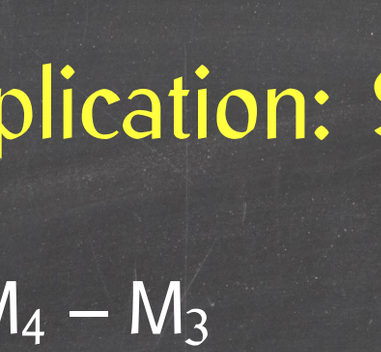
M_7

Matrix Multiplication: Strassen's Alg.

$C_{11} = M_1 + M_7 - M_4 - M_3$

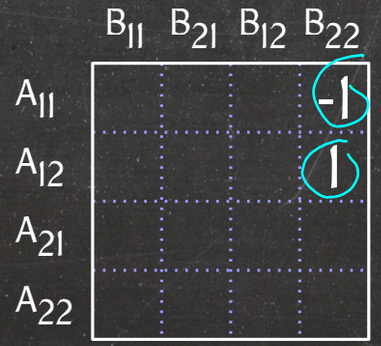


C_{11}

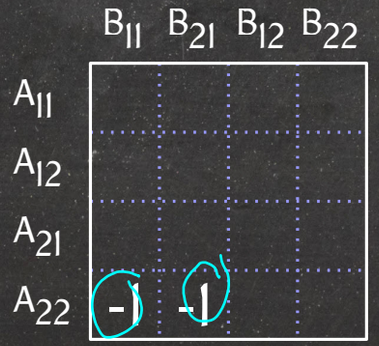


M_1

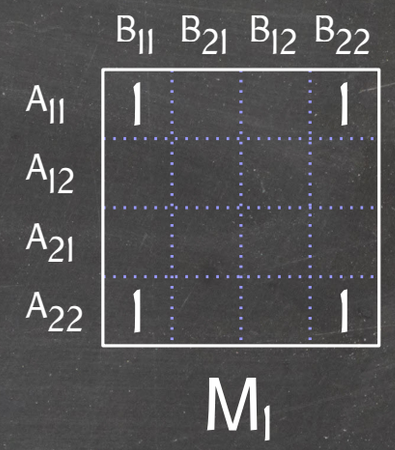
M_7



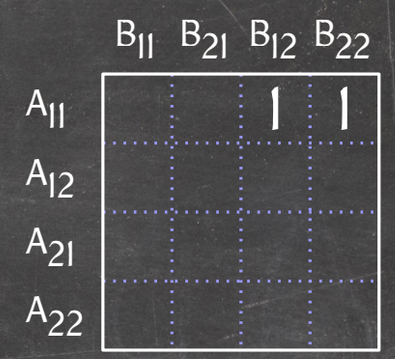
$-M_4$



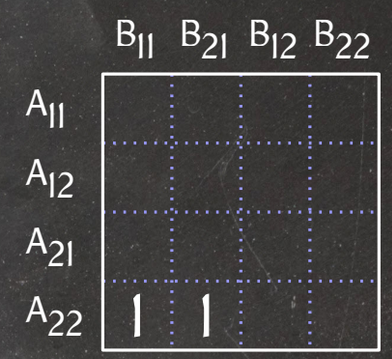
$-M_3$



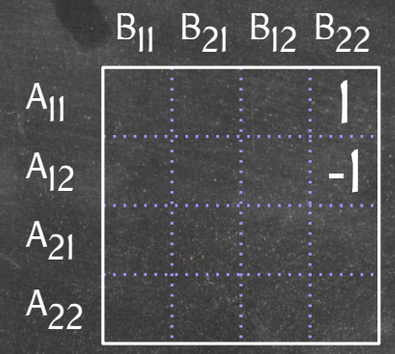
M_1



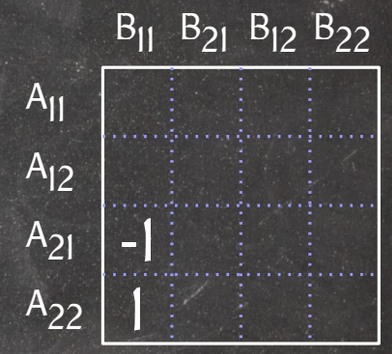
M_2



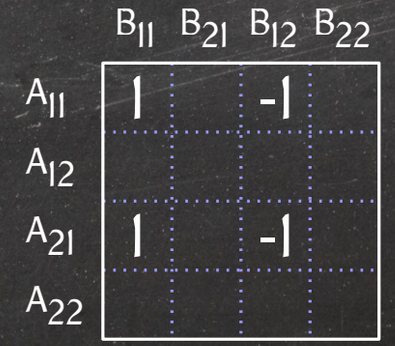
M_3



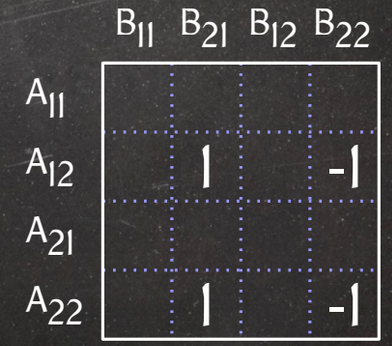
M_4



M_5



M_6

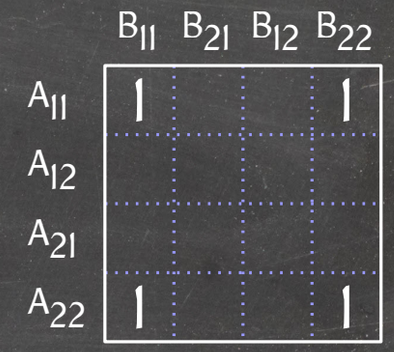


M_7

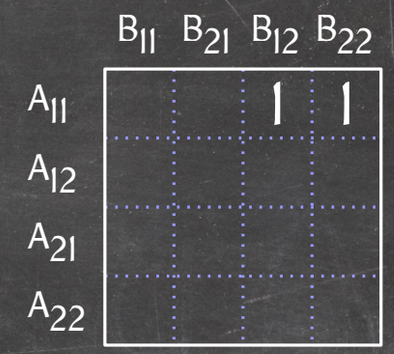
Matrix Multiplication: Strassen's Alg.

$C_{11} = M_1 + M_7 - M_4 - M_3$

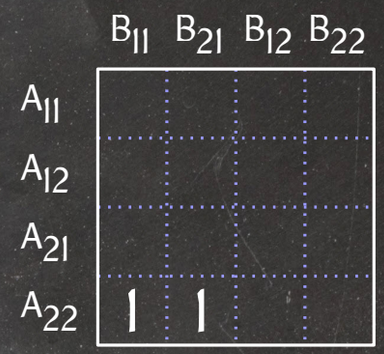
$C_{12} = M_2 - M_4$



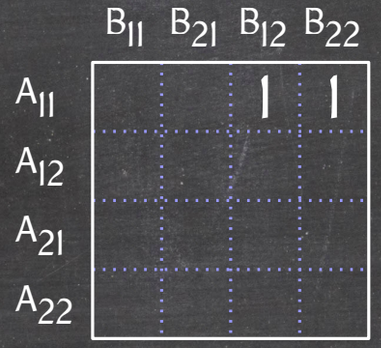
M_1



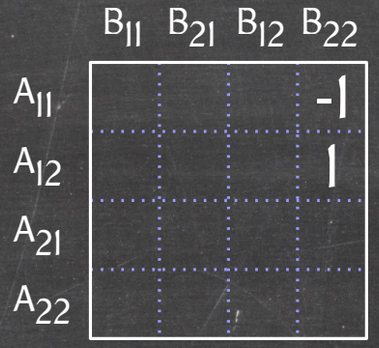
M_2



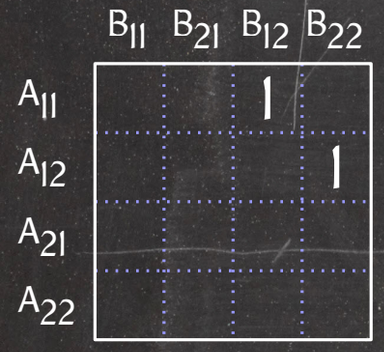
M_3



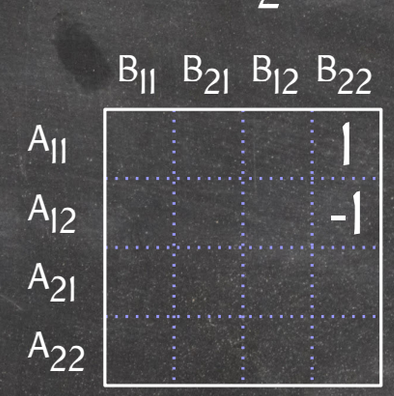
M_2



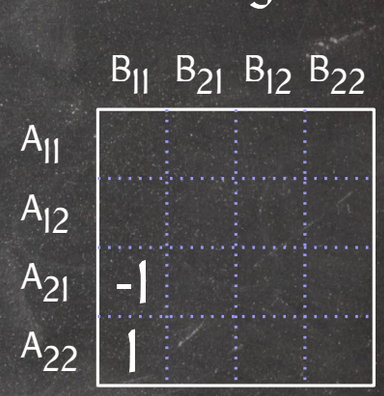
$-M_4$



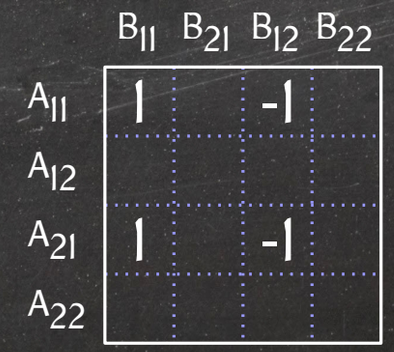
C_{12}



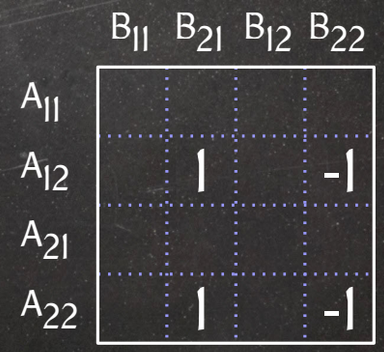
M_4



M_5



M_6



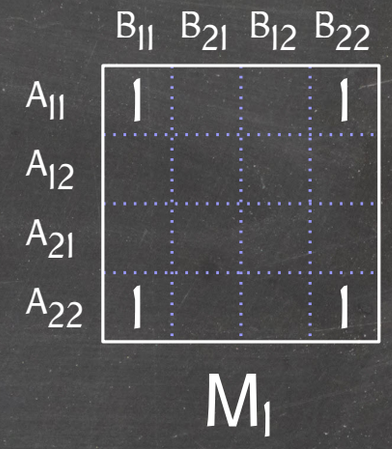
M_7

Matrix Multiplication: Strassen's Alg.

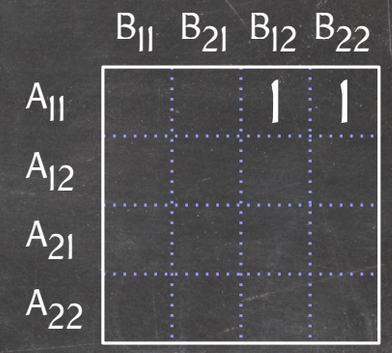
$C_{11} = M_1 + M_7 - M_4 - M_3$

$C_{12} = M_2 - M_4$

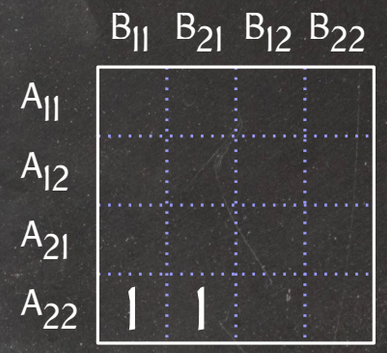
$C_{21} = M_3 - M_5$



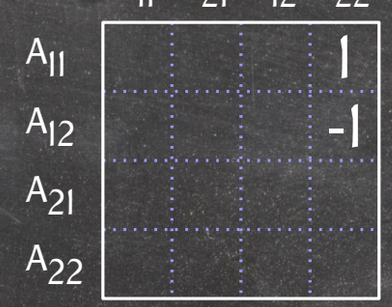
M_1



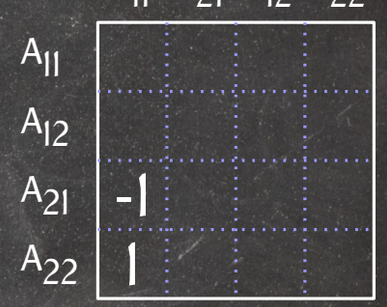
M_2



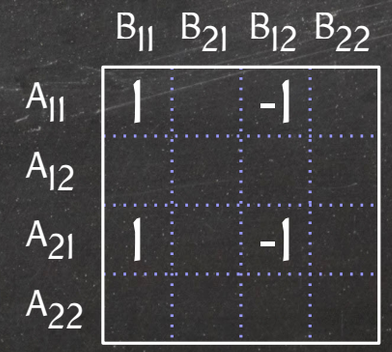
M_3



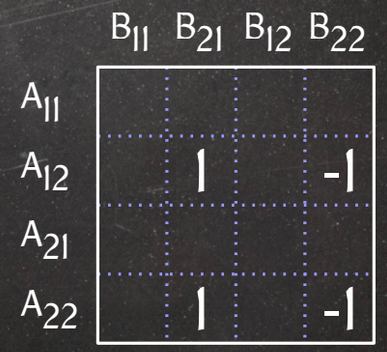
M_4



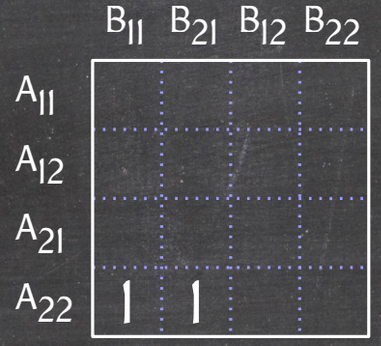
M_5



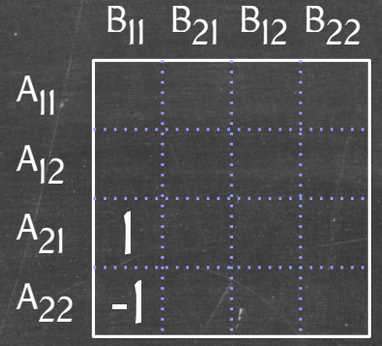
M_6



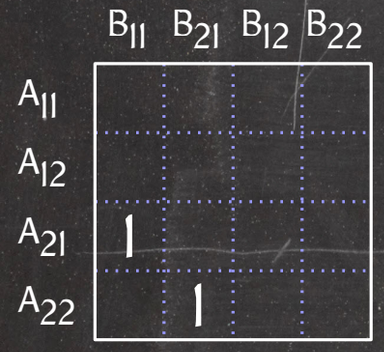
M_7



M_3



$-M_5$



C_{12}

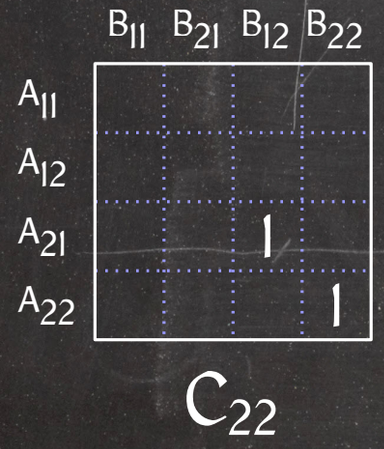
Matrix Multiplication: Strassen's Alg.

$C_{11} = M_1 + M_7 - M_4 - M_3$

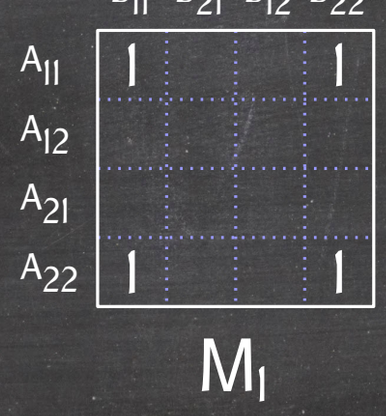
$C_{12} = M_2 - M_4$

$C_{21} = M_3 - M_5$

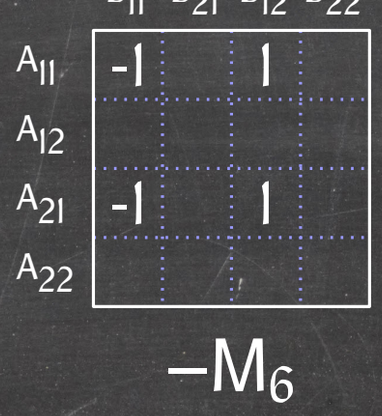
$C_{22} = M_1 - M_6 - M_2 - M_5$



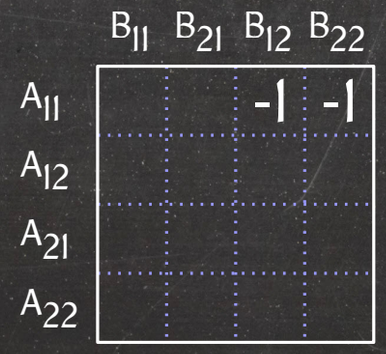
C_{22}



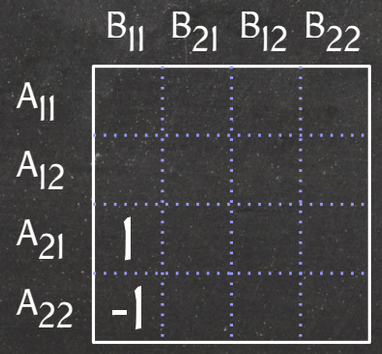
M_1



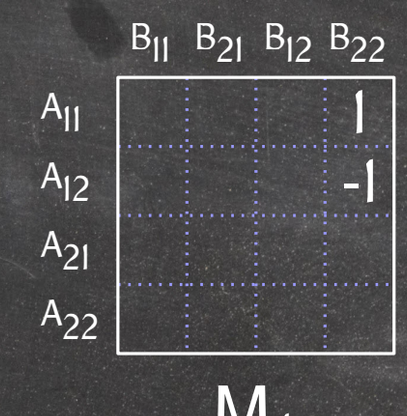
$-M_6$



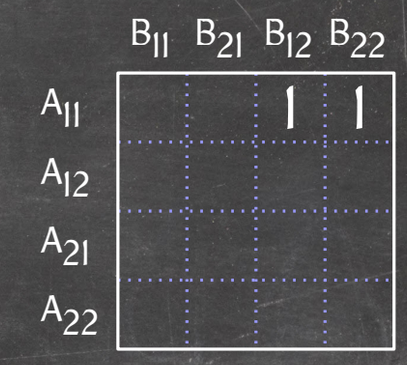
$-M_2$



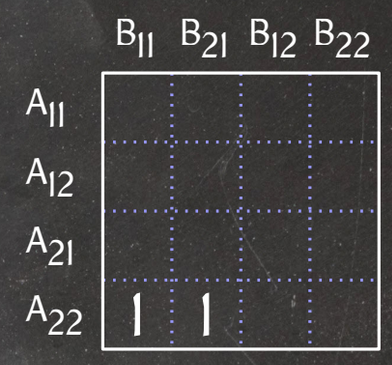
$-M_5$



M_1

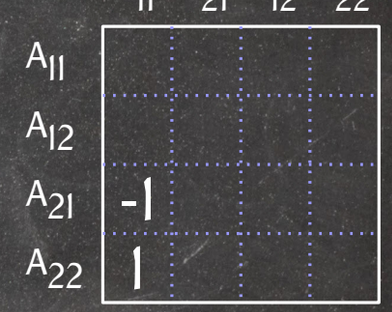


M_2

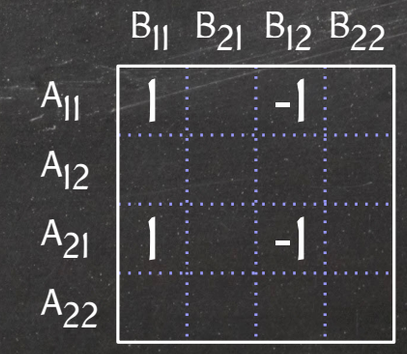


M_3

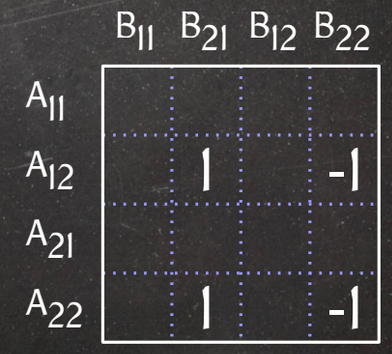
M_4



M_5



M_6



M_7

Matrix Multiplication: Strassen's Algorithm

Strassen(A, B)

```
1   let n × n be the dimension of A and B
2   if n = 1
3       then return A[1, 1] · B[1, 1]
4       else  partition A into submatrices A11, A12, A21, A22
5             partition B into submatrices B11, B12, B21, B22
6             M1 = Strassen(A11 + A22, B11 + B22)
7             M2 = Strassen(A11, B12 + B22)
8             M3 = Strassen(A22, B11 + B21)
9             M4 = Strassen(A11 − A12, B22)
10            M5 = Strassen(A22 − A21, B11)
11            M6 = Strassen(A11 + A21, B11 − B12)
12            M7 = Strassen(A12 + A22, B21 − B22)
13            C11 = M1 + M7 − M4 − M3
14            C12 = M2 − M4
15            C21 = M3 − M5
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17            assemble C from C11, C12, C21, C22
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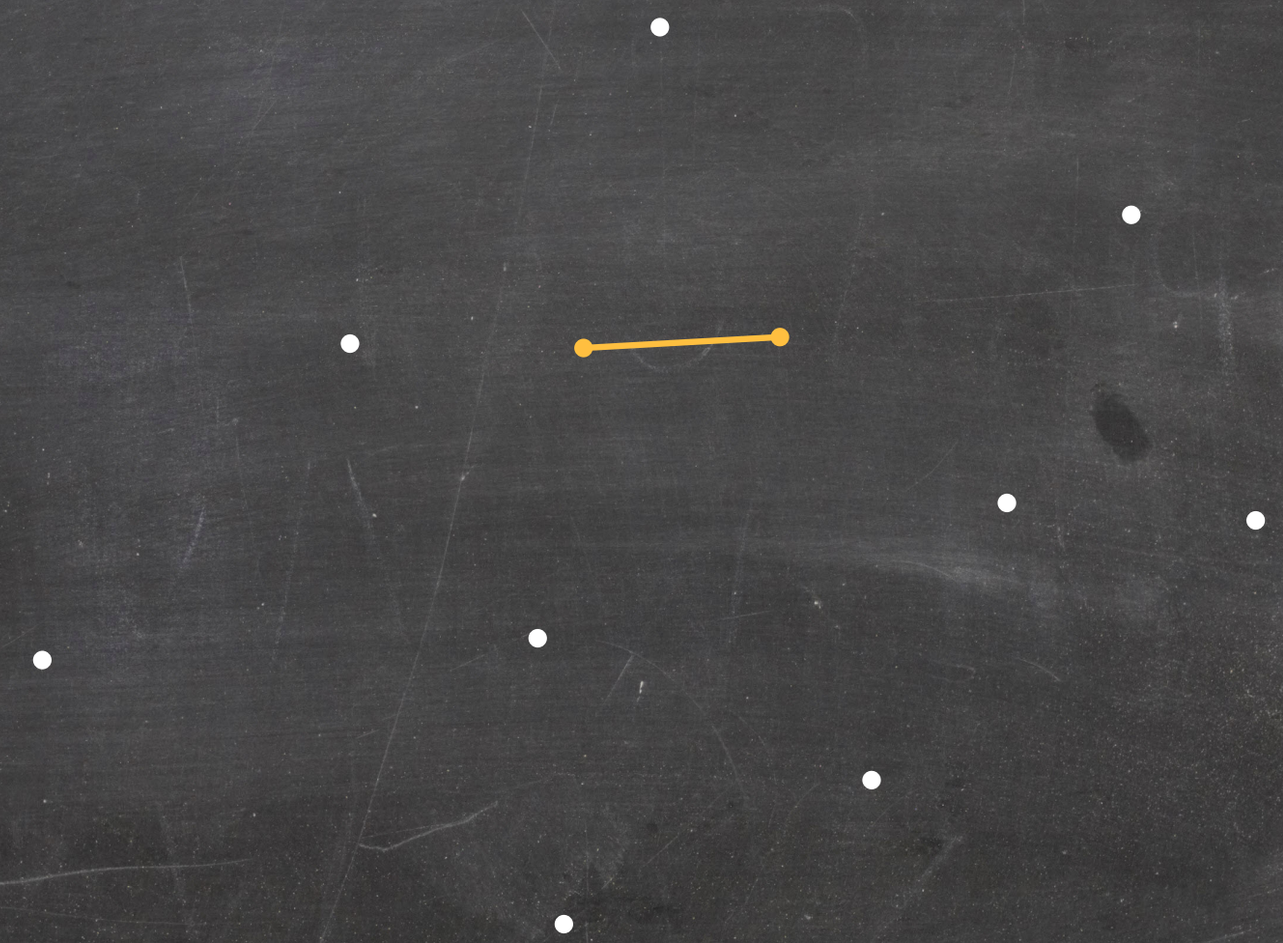
```

Cost: $T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right)$

Closest Pair

Given a point set P in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $||p - q||_2$ (the Euclidean distance from p to q).



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Can be computed in $O(n^2)$ time. How?

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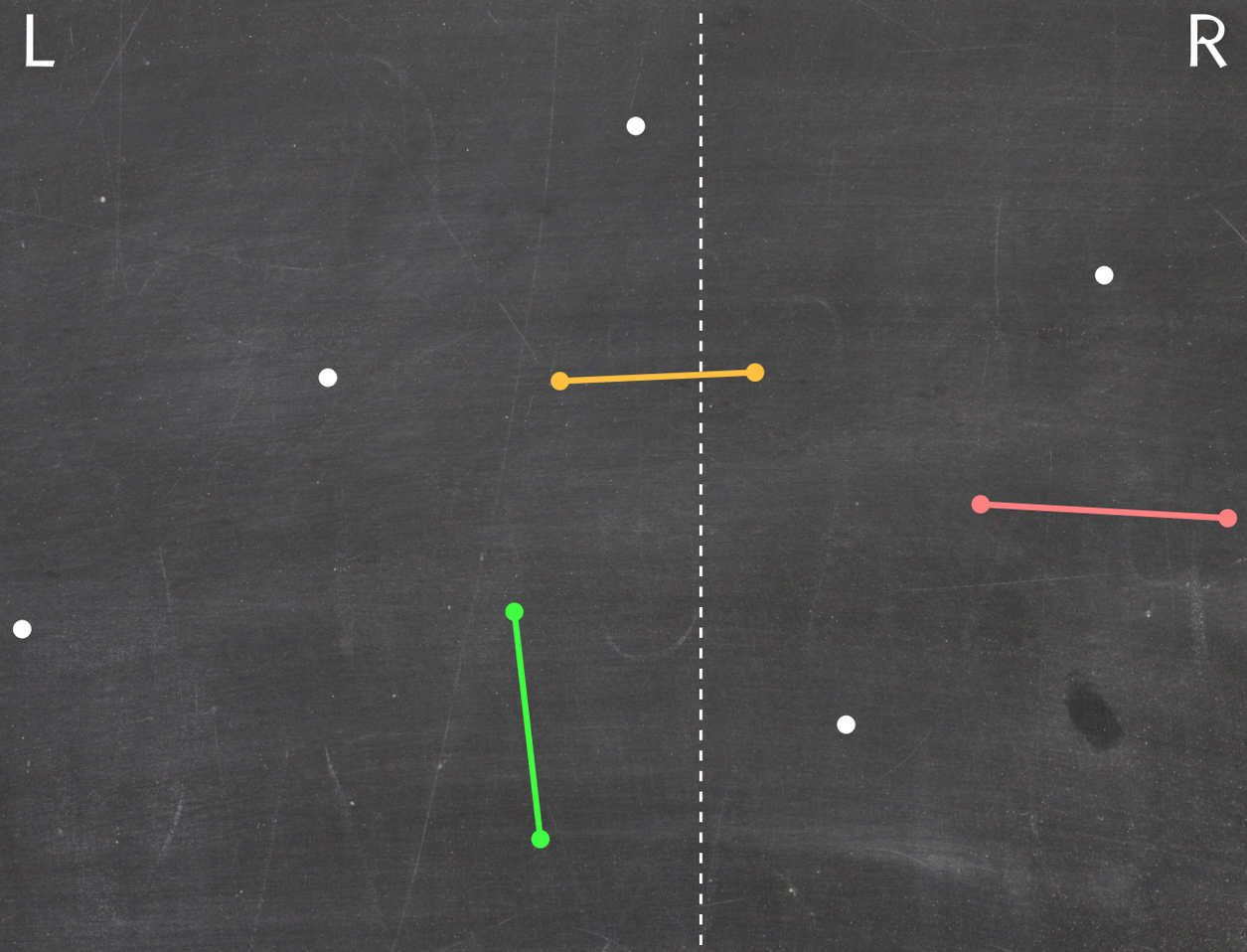
Can be computed in $O(n^2)$ time. How?

Can we do better?

Closest Pair: Divide and Conquer



If we divide the point set into the leftmost $\lceil n/2 \rceil$ points (L) and the rightmost $\lfloor n/2 \rfloor$ points (R), then the closest pair has

- both points in L,
- both points in R or
- one point in L and one point in R.

Closest Pair: Divide and Conquer

ClosestPair(P)

1 if $|P| \leq 1$
2 then return Nothing
3 Split P into two sets L and R containing the leftmost $\lceil n/2 \rceil$ and the rightmost $\lfloor n/2 \rfloor$ points, respectively.
4 (p_ℓ, q_ℓ) = ClosestPair(L)
5 (p_r, q_r) = ClosestPair(R)
6 (p_m, q_m) = ClosestPairLR(L, R)
7 return the pair (p_i, q_i), $i \in \{\ell, r, m\}$, that minimizes $\|p_i - q_i\|_2$

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Naïve implementation of ClosestPairLR: $\Theta(n^2)$ time

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Naïve implementation of ClosestPairLR: $\Theta(n^2)$ time

Running time: $T(n) = 2T(n/2) + \Theta(n^2)$

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Naïve implementation of ClosestPairLR: $\Theta(n^2)$ time

Running time: $T(n) = 2T(n/2) + \Theta(n^2) \in \Theta(n^2)$

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Better implementation of ClosestPairLR: $\Theta(n)$ time

Running time: $T(n) = 2T(n/2) + \Theta(n) \in \Theta(n \lg n)$

Closest Pair: Preprocessing

ClosestPair(P)

1. Make two copies X and Y of P
2. Sort the points in X by their x-coordinates
3. Sort the points in Y by their y-coordinates
4. return ClosestPairRec(X, Y)

Closest Pair: Preprocessing

ClosestPair(P)

1. Make two copies X and Y of P
2. Sort the points in X by their x-coordinates
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4. **return** ClosestPairRec(X, Y)

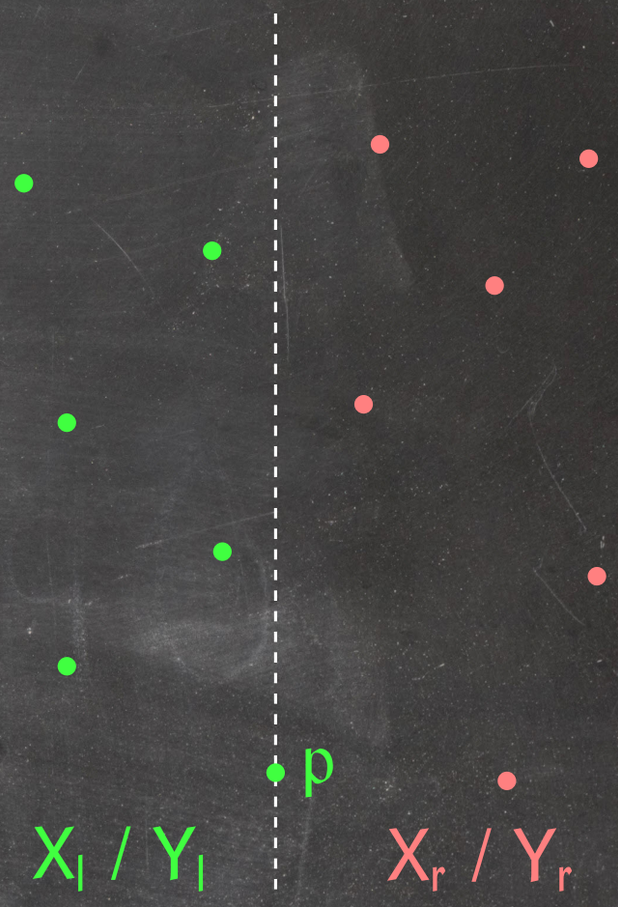
We prove that ClosestPairRec takes $O(n \lg n)$ time.

$\Rightarrow$ The whole algorithm takes $O(n \lg n)$ time.

Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

```
1   if |Y| ≤ 1
2       then return (Nothing, ∞)
3   p = the middle element in X
4   X_ℓ = the part of X up to and including p
5   X_r = the part of X after x
6   Y_ℓ = ⟨q ∈ Y | q.x ≤ p.x⟩
7   Y_r = ⟨q ∈ Y | q.x > p.x⟩
8   (pair_ℓ, δ_ℓ) = ClosestPairRec(X_ℓ, Y_ℓ)
9   (pair_r, δ_r) = ClosestPairRec(X_r, Y_r)
10  (pair_m, δ_m) = ClosestPairLR(Y_ℓ, Y_r, p.x, min(δ_ℓ, δ_r))
11  return the pair (pair_i, δ_i), i ∈ {ℓ, r, m}, that minimizes δ_i
```



Closest Pair: Divide and Conquer

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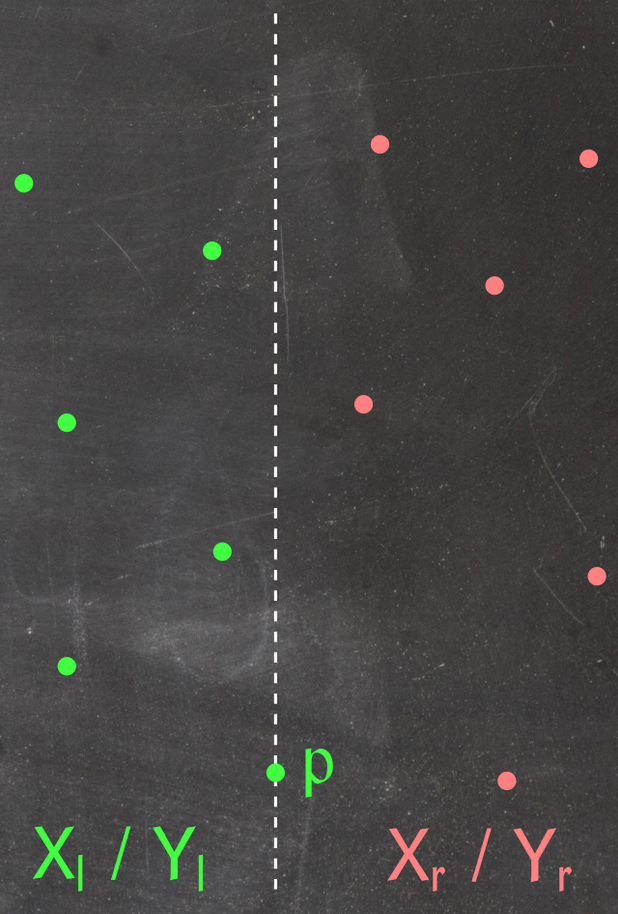

We already have a pair with distance $\delta = \min(\delta_\ell, \delta_r)$.

$\Rightarrow$ only need to look for pairs with distances $< \delta$.

Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

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1   if |Y| ≤ 1
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3   p = the middle element in X
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11  return the pair (pair_i, δ_i), i ∈ {ℓ, r, m}, that minimizes δ_i
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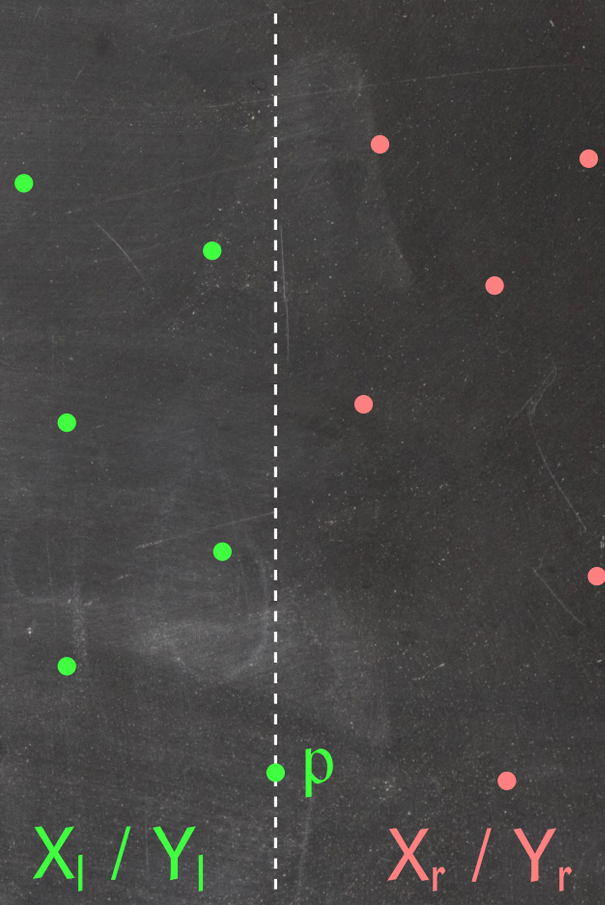


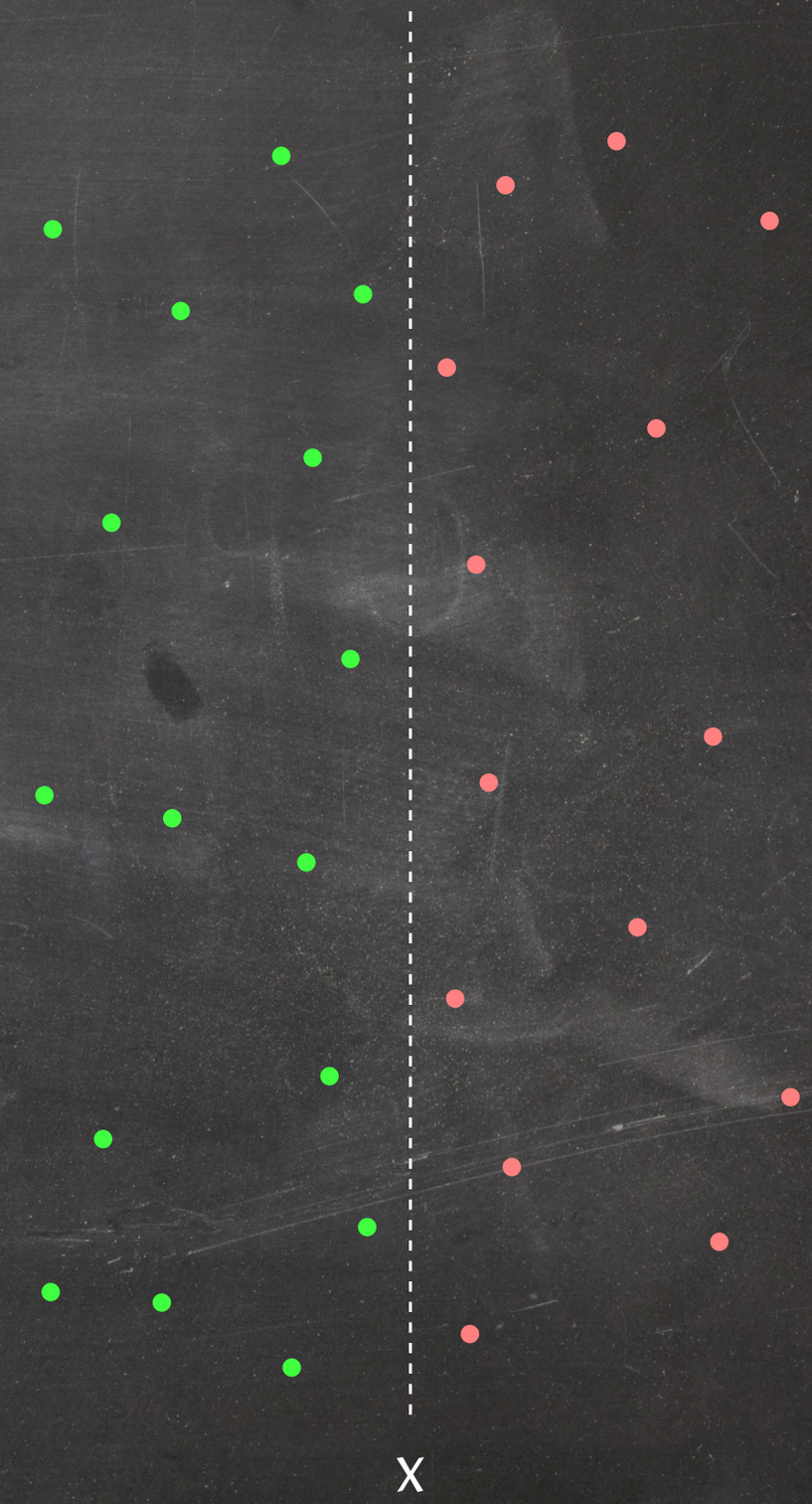
We prove that ClosestPairLR(Y_ℓ, Y_r, x, δ) takes O(n) time.

$\Rightarrow$ $T(n) = 2T(n/2) + O(n) \in O(n \lg n)$

Closest Pair: One Left, One Right

ClosestPairLR(Y_ℓ, Y_r, x, δ)

```
1   Z_ℓ = ⟨p ∈ Y_ℓ | x − p.x ≤ δ⟩
2   Z_r = ⟨p ∈ Y_r | p.x − x ≤ δ⟩
3   pair = Nothing
4   δ′ = ∞
5   j = 1
6   for i = 1 to |Z_ℓ|
7       do while j < |Z_r| and Z_r[j].y < Z_ℓ[i].y − δ
8              do j = j + 1
9          k = j
10         while k ≤ |Z_r| and Z_r[k].y ≤ Z_ℓ[i].y + δ
11             do if ||Z_ℓ[i] − Z_r[k]|| < δ′
12                   then δ′ = ||Z_ℓ[i] − Z_r[k]||
13                        pair = (Z_ℓ[i], Z_r[k])
14                k = k + 1
15  return (pair, δ′)
```



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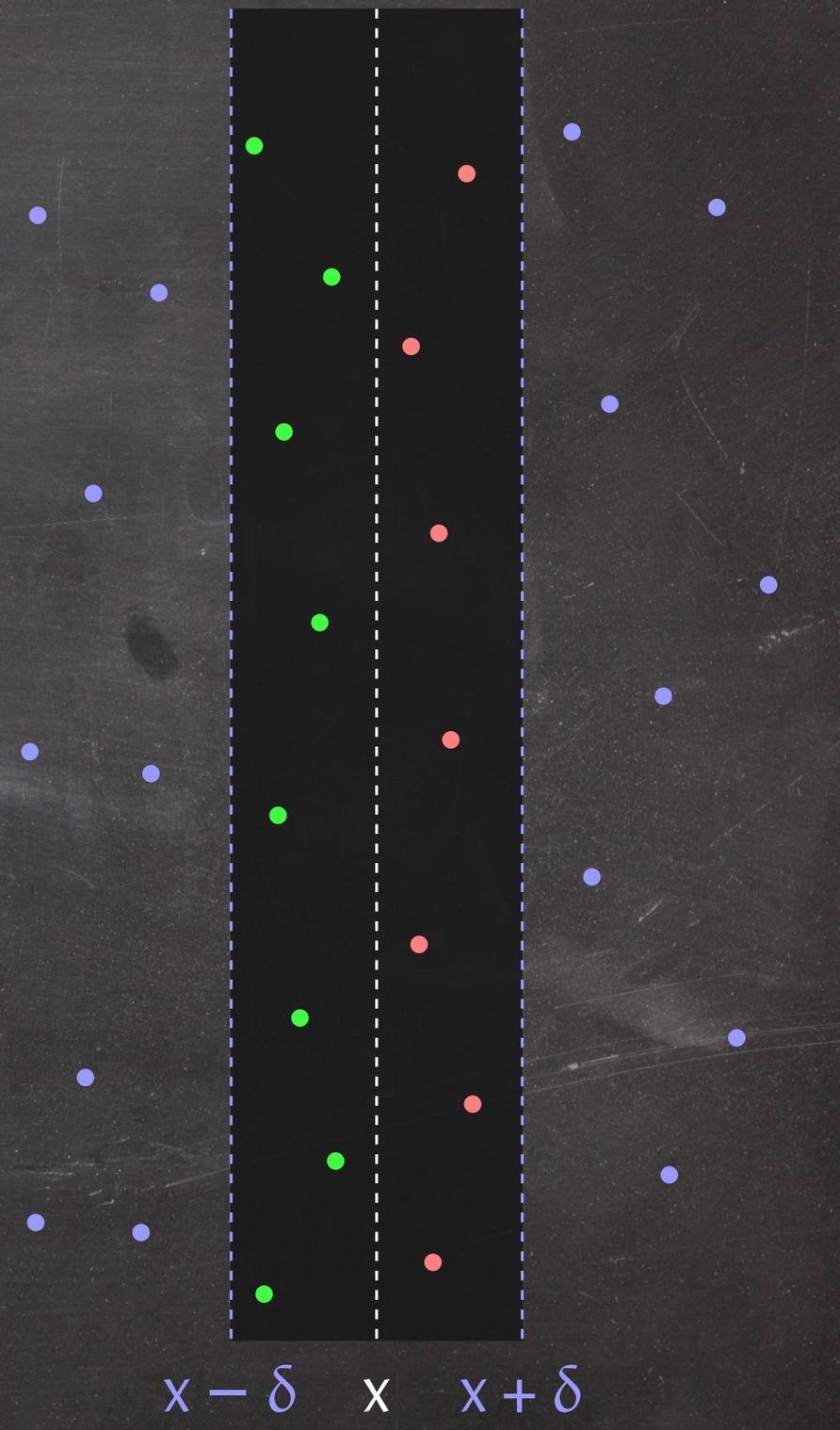

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11             do if ||Z_ℓ[i] − Z_r[k]|| < δ′
12                   then δ′ = ||Z_ℓ[i] − Z_r[k]||
13                        pair = (Z_ℓ[i], Z_r[k])
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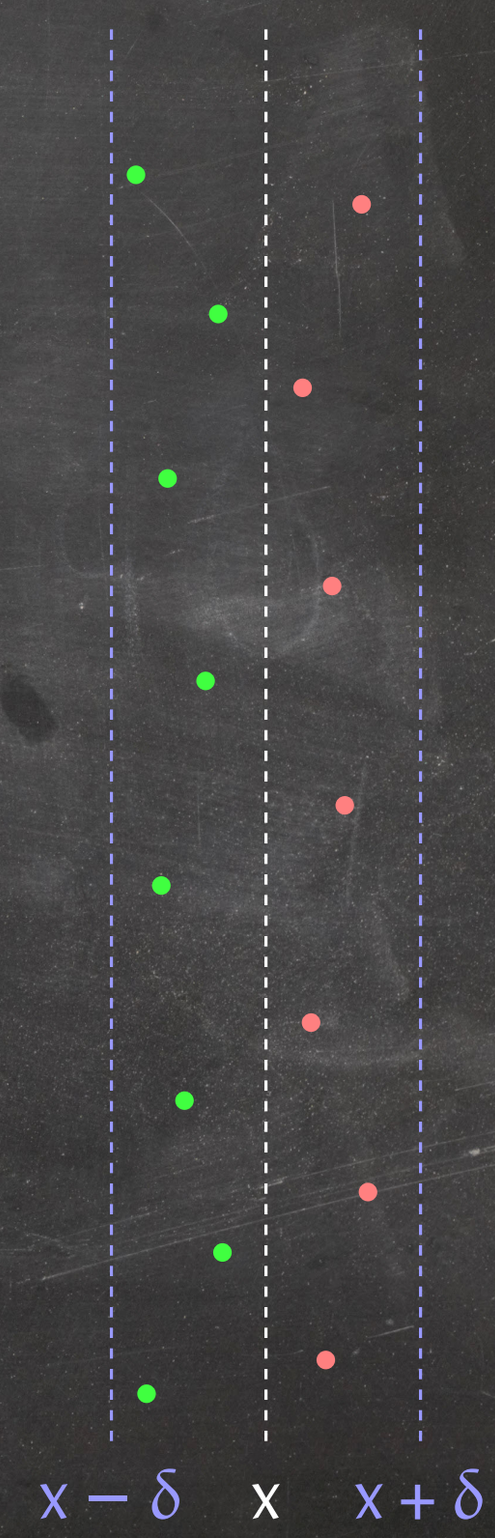

Closest Pair: One Left, One Right

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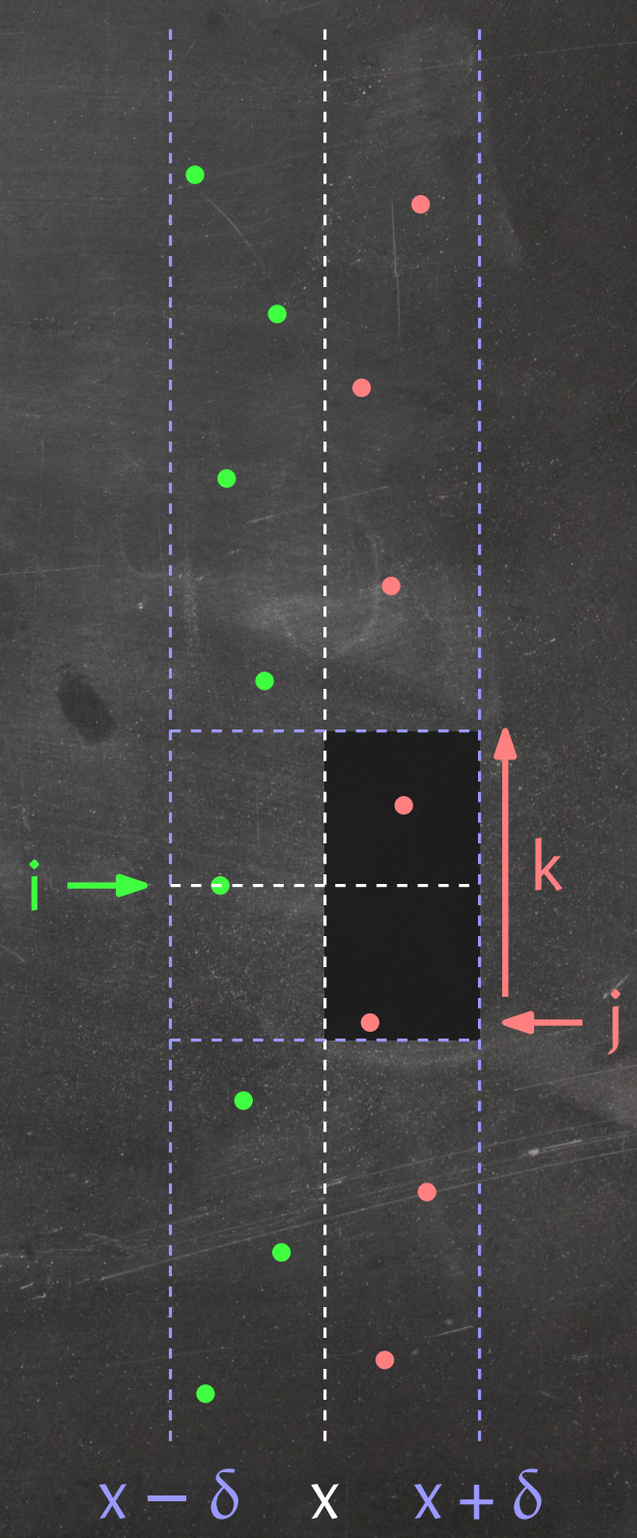

Closest Pair: One Left, One Right

Lemma: For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

Closest Pair: One Left, One Right

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Corollary: The running time of ClosestPairLR is in $O(n)$.

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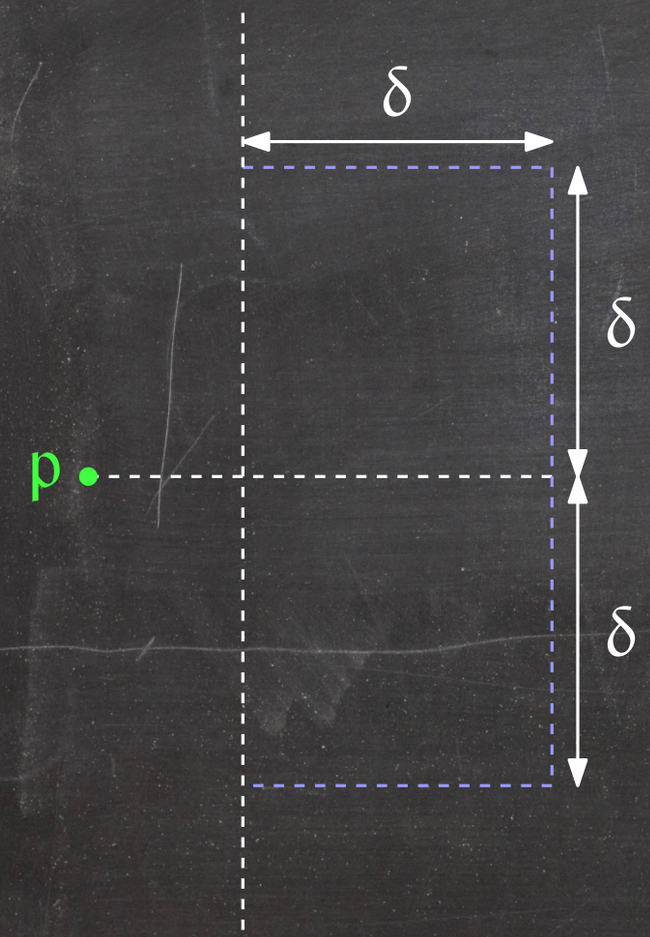
Corollary: The running time of ClosestPairLR is in O(n).



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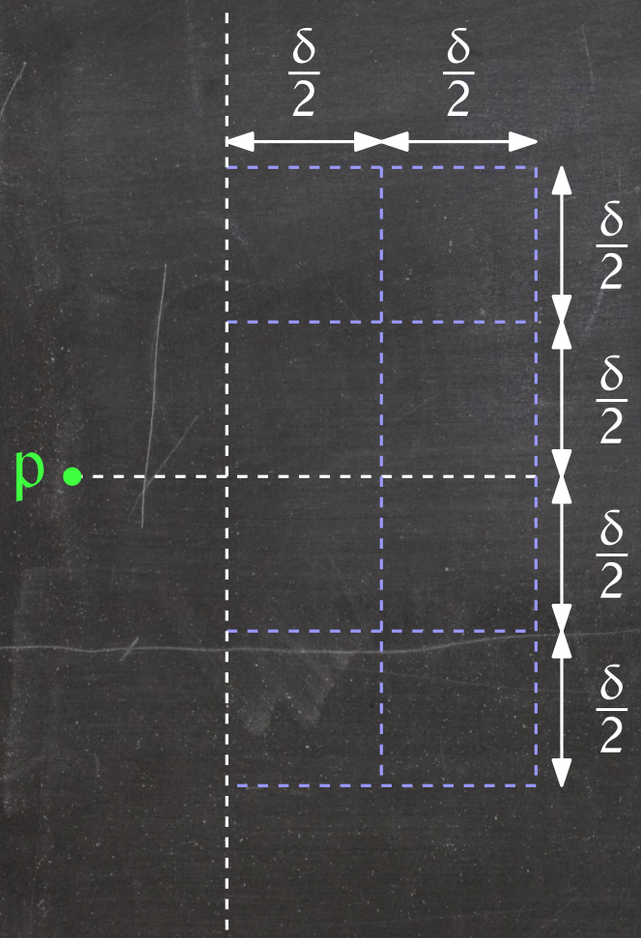

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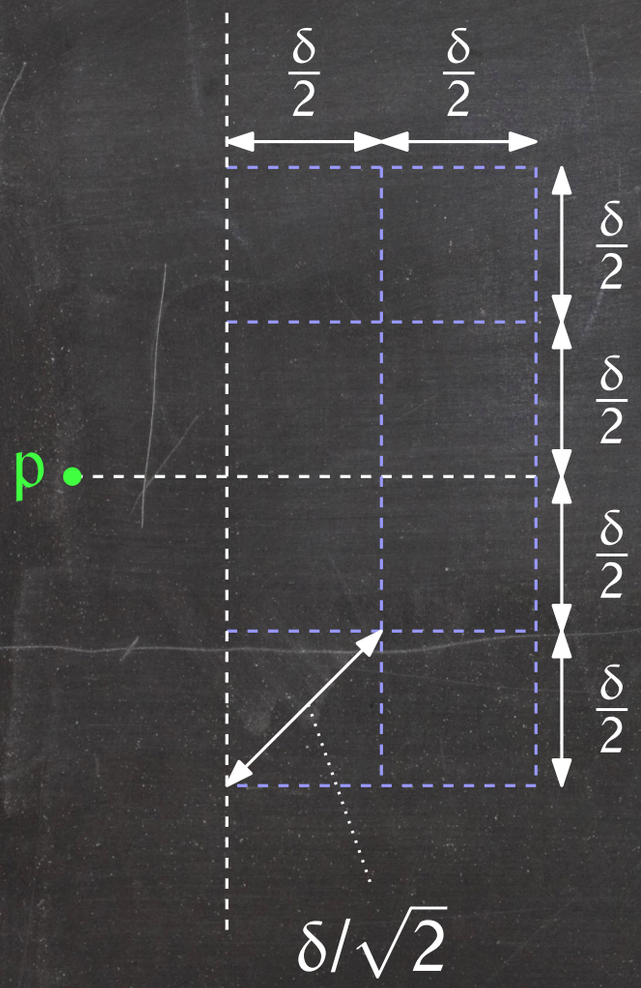

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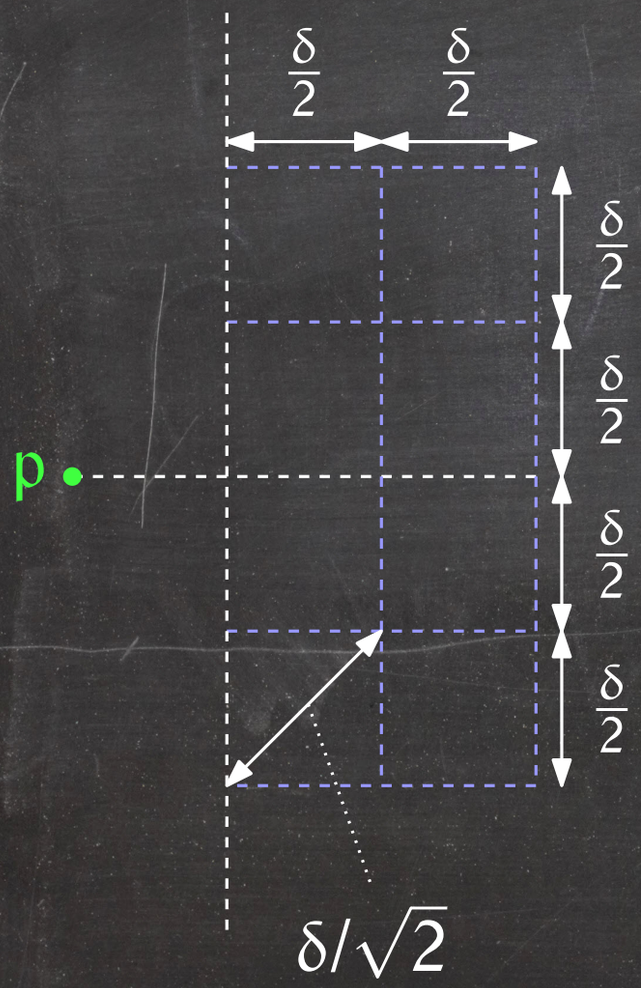

Two points in the same square have distance at most $\delta/\sqrt{2} < \delta$ from each other.

$\Rightarrow$ At most one point in each of the 8 squares.

Summary

The Divide and Conquer paradigm:

- Divide the input into smaller instances of the same problem.
- Solve these instances recursively.
- Combine the obtained solutions to obtain a solution to the original input.

Divide-and-conquer algorithms always recurse on smaller inputs.

$\Rightarrow$ Natural expression of running time using recurrence relations.

$\Rightarrow$ Natural strategy to prove correctness is induction.

Solving recurrence relations:

- Substitution
- Recursion trees
- Master Theorem