1. (14 points) Solve each of the following recurrences using the Master Theorem or, if
the Master Theorem is not applicable to the recurrence, state why. If the Master
Theorem applies, state which case is applicable and show that the recurrence satisfies
the conditions of this case.

(a) \( T(n) = 5T(n/3) + n \log n \)

**ANSWER:** We have \( a = 5, b = 3, \) and \( f(n) = n \log n. \) We begin by computing
\( n^{\log_b a} \) and comparing it to \( f(n). \) \( n^{\log_b a} = n^{\log_3 5} \approx n^{1.465} \) Thus,
\( n \log n = O(n^{\log_3 5}), \) so case (i) applies if we can find an appropriate \( \epsilon. \) It is easy
to see that \( n \log n = O(n^{\log_3 4}), \) and \( \log_3 4 = \log_3 5 - \epsilon \) where
\( \epsilon = (\log_3 5 - \log_3 4) > 0. \)

(b) \( T(n) = T(n/3) + T(2n/3) + n \)

**ANSWER:** The Master Theorem does not apply. This recurrence does not have
the correct form because there are recursive calls with different input sizes.

(c) \( T(n) = 3T(n/3) + n/\log n \)

**ANSWER:** The only case of the Master Theorem that could apply is case (i)
because \( n/\log n = O(n^{\log_3 3} = n). \) However, \( n/\log n = \Omega(n^{1-\epsilon}), \) for all \( \epsilon > 0, \) so
the Master Theorem does not apply.

(d) \( T(n) = 16T(n/4) + n^2 \)

**ANSWER:** We have \( a = 16, b = 4, \) and \( f(n) = n^2. \) Case (ii) of the Master
Theorem applies because \( n^2 = \Theta(n^{\log_4 16} = n^2). \) Thus, \( T(n) = \Theta(n^2 \log n). \)

(e) \( T(n) = 2T(n - 5) + n \log n \)

**ANSWER:** This recurrence does not have the correct form because the input
size of the recursive calls is not a fraction of the original input size. The Master
Theorem does not apply.

(f) \( T(n) = 2T(n/3) + n \)

**ANSWER:** We have \( a = 2, b = 3, \) and \( f(n) = n. \) Case (iii) of the Master
Theorem may apply because \( n = \Omega(n^{\log_3 2.5 = \log_3 2 + \epsilon}), \) for
\( \epsilon = (\log_3 2.5 - \log_3 2) > 0. \) We now check the regularity condition
\( af(n/b) \leq cf(n) \) which holds because \( 2f(n/3) = 2n/3 \leq 2n/3 = cn = cf(n), \) for
\( c = 2/3), \) and all \( n_0 \geq 1. \) Thus, \( T(n) = \Theta(n). \)
\[ T(n) = \begin{cases} 
2T(n/2) + n^3 & \text{if } n \text{ is an even integer} \\
2T(n/2) + n^2 & \text{if } n \text{ is an odd integer}
\end{cases} \]

**ANSWER:** The only case of the Master Theorem that could apply is case (iii) because \( n^3 = \Omega(n^2) = \Omega(n \log^2 n) \). However, this recurrence fails the regularity condition that \( af(n/b) \leq cf(n) \), so the Master Theorem does not apply.

2. (12 pts) Solve each of the following recurrences using substitution or a recursion tree. Do not only state your solution—Show how you obtained it. That is, if you use substitution, you must present the complete inductive proof that your solution is correct. If you use a recursion tree, show the recursion tree and discuss how you obtained the solution from the tree. *Note that you are to prove matching upper and lower bounds.*

(a) \( T(n) = 4T(n/2) + n^2 \)

**ANSWER:**

\[
T(n) = 4T(n/2) + n^2
\]

The depth of the tree is \( \log_2 n \). Thus there are \( 4^{\log_2 n} = n^{\log_2 4} = n^2 \) leaves, and
their cost is $\Theta(n^2)$. The cost of the $i$th non-leaf level is:

$$4^i c \left( \frac{n}{2^i} \right)^2$$  \hspace{1cm} (1)

$$= c \frac{4^i}{2^{2i}} n^2$$  \hspace{1cm} (2)

$$= c \frac{4^i}{4^i} n^2$$  \hspace{1cm} (3)

$$= c n^2$$  \hspace{1cm} (4)

Thus, the total cost is

$$\Theta(n^2) + \sum_{i=0}^{\log_2 n} c n^2$$  \hspace{1cm} (5)

$$= \Theta(n^2) + \log_2 n(cn^2)$$  \hspace{1cm} (6)

$$= \Theta(n^2 \log n)$$  \hspace{1cm} (7)

(b) $T(n) = T(n/2) + T(n/4) + n$

**ANSWER:** Since the first recursive call does $\Omega(n)$ work, $T(n) = \Omega(n)$. Let’s guess that $T(n) = \Theta(n)$ and prove the upper bound using substitution. So, our claim is that $T(n) \leq cn$.

For the base case, we consider $1 \leq n < 4$. We have that $T(n) = \Theta(1) \leq cn$, as long as $c$ is large enough. For $n \geq 4$, we have that $1 \leq n/2 < n$ and $1 \leq n/4 < n$, so the inductive hypothesis applies to $T(n/2)$ and $T(n/4)$. Using this we have that,

$$T(n) = T(n/2) + T(n/4) + n$$  \hspace{1cm} (8)

$$\leq c(n/2) + c(n/4) + n$$  \hspace{1cm} (9)

$$= (c/2 + c/4 + 1)n$$  \hspace{1cm} (10)

$$= (3/4c + 1)n$$  \hspace{1cm} (11)

$$\leq cn,$$  \hspace{1cm} (12)

as long as $1 \leq c/4$, that is, $c \geq 4$.

3. (Bonus: 4 pts) What is the solution to $T(n) = 2T(n-1) + T(n-2) + 1$? You do not need to prove your solution but provide some justification for it.

This is a tricky one. It is obviously exponential, so let’s suppose that $T(n) = x^n$, for some exponential base. Then $T(n) = x^n = 2x^{n-1} + x^{n-2}$. This means that $x^2 = 2x + 1$, so $x^2 - 2x - 1 = 0$ which is a quadratic! Using the quadratic formula to solve for the roots,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$
we see that the roots are

\[ x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} \]  \hspace{1cm} (13)

\[ x = \frac{2 \pm \sqrt{4 + 4}}{2} \]  \hspace{1cm} (14)

\[ x = 1 \pm \sqrt{2} \]  \hspace{1cm} (15)

Since \( 1 - \sqrt{2} < 0 \), \( T(n) = \Theta ((1 + \sqrt{2})^n \approx 2.42^n) \). We could then use substitution to prove this bound.