1 Answer to assignment 1.3

Define meaning of random variables:

$X = \{y, n\}$ is a binary random variable of the sensor being faulty. $S = \{1, 3\}$ is a binary random variable of a sensor reading less than 1m or less than 3m.

The prior knowledge (before any measurement) of the sensor being faulty is given in the text as $p(X = y) = 0.01$. The conditional probability of getting a reading of less than 1 in case of a faulty sensor is $p(S = 1|X = y) = 1$ as stated in the text. Since X is binary random variables and we know that the sum over the probability over all possible outcomes must be one, we also know that $p(X = n) = 1 - p(X = y) = 0.99$. Also, we are told to assume that the distributions of sensor reading is uniform in case of a non-faulty sensor, hence $p(S = 1|X = n) = 1/3$. We have thus all the values we need to calculate the posterior (the new believe of a faulty sensor) after an additional sensor reading using Bayes' theorem:

$$p(X = y|S = 1) = \frac{p(S = 1|X = y)p(X = y)}{p(S = 1)} \tag{1.1}$$

Note that the marginal probability of a sensor reading of less than 1 is made up of the probability to get a sensor reading of less than 1 if the sensor is working properly times the probability that the sensor is working properly plus the probability to get a sensor reading of less than one if the sensor is faulty times the probability that the sensor is faulty. This is exactly the meaning of marginalization:

$$p(S = 1) = p(S = 1|X = y)p(X = y + p(S = 1|X = n)p(X = n \tag{1.2}$$

Thus after the first reading ($n = 1$) we get

$$p(X = y|S = 1) = \frac{0.01}{0.01 + 1/3 * .99} = 0.0294 \tag{1.3}$$

and after two readings we get

$$p(X = y|S = 1) = \frac{0.0294}{0.0294 + 1/3 * .97} = 0.0833 \tag{1.4}$$

If we denote our believe (the probability) of the sensor being fault after n measurements as p_n, we can write this recursively as

$$p_n = \frac{p_{n-1}}{p_{n-1} + 1/3(1 - p_{n-1})}. \tag{1.5}$$

To write this in closed form we introduce the random variable S_n which is the event of getting a n sensor readings of less than 1 in a row. The probability of the sensor being faulty after n readings is then

$$p_n = \frac{p(S_n|X=y)p(X=y)}{p(S_n|X=y)p(X=y)+p(S_n|X=n)p(X=n)}. \tag{1.6}$$

Since the events are independent, $p(S_n|X=y) = p(S|X=y)^n = 1$ and $p(S_n|X=n) = p(S|X=n)^n = 1/3^n$. Therefore

$$p_n = \frac{0.01}{0.01 + 1/3^n * 0.99}. \tag{1.7}$$