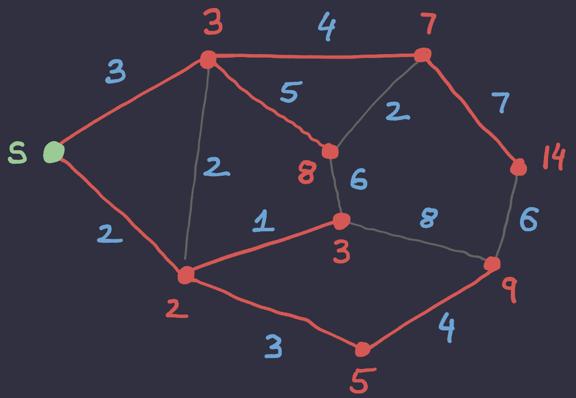
Single-Source Shortest Paths as an LP

CSCI 4113/6101

Single-Source Shortest Paths

Input:

- Graph $G = (V, E)$
- Edge weights $w : E \to \mathbb{R}$
- A source vertex $s \in V$



Output:

A distance function $d : V \to \mathbb{R}$

such that $d(v) = \mathrm{dist}_G(s, v)$

The Strings and Pearls Model

Graph G:

- Vertices = pearls
- Edge (u,v) = piece of string connecting u and v

  Length: $w(u,v)$

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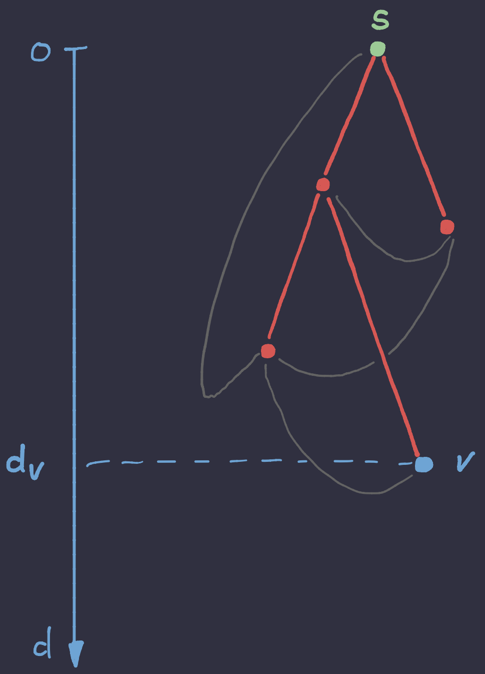
Question:

If we pick up G from s, how much lower than s does v hang?



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Key properties:

- Gravity tries to maximize $d(v)$
- $d(v) = dist_G(s,v) \quad \forall v \in V$
- $d(v) \leq d(u) + w(u,v) \quad \forall (u,v) \in E$

The LP Formulation

$$\text{Maximize} \sum_{v \in V} d_v$$

$$\text{s.t.} \quad d_s = 0$$

$$d_v - d_u \leq w(u,v) \quad \forall (u,v) \in E$$

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Lemma: $\hat{d}$ is an optimal solution of the LP if and only if $\hat{d}_v = dist_G(s,v)$.