

Tableaux and an Example of the Simplex Algorithm

CSCI 4113/6101



The LP to Be Solved

$$\text{Maximize } x_1 - 2x_2 + x_3$$

$$\text{s.t. } y_1 - x_1 - 2x_2 - 2x_3 = -3$$

$$y_2 - 3x_1 - 2x_2 - x_3 = -5$$

$$y_3 + x_1 \quad \quad + 3x_3 = 10$$

$$y_4 + x_1 + x_2 + x_3 = 9$$

$$x_1, x_2, x_3, y_1, y_2, y_3, y_4 \geq 0$$

Reminder: Tableau Representation

	Basic variables				Non-basic v.		
	y_1	y_2	y_3	y_4	x_1	x_2	x_3
-3	1				-1	-2	-2
-5		1			-3	-2	-1
10			1		1		3
9				1	1	1	1
Negated objective	0	0	0	0	1	-2	1

Constraints

Objective function

function value of basic solution

Finding the Initial BFS

	y_1	y_2	y_3	y_4	x_1	x_2	x_3	S
-3	1				-1	-2	-2	-1
-5		1			-3	-2	-1	-1
10			1		1		3	-1
9				1	1	1	1	-1
0								-1

Finding the Initial BFS



	y_1	y_2	y_3	y_4	x_1	x_2	x_3	s
-3	1				-1	-2	-2	-1
-5		1			-3	-2	-1	-1
10			1		1		3	-1
9				1	1	1	1	-1
0								-1

$$b_2 = \min \{b_1, b_2, b_3, b_4\}$$

$\Rightarrow s$ enters, y_2 leaves

Finding the Initial BFS

	y_1	S	y_3	y_4	x_1	x_2	x_3	y_2
-3	1	-1			-1	-2	-2	
-5		-1			-3	-2	-1	1
10		-1	1		1		3	
9		-1		1	1	1	1	
0		-1						

Rewrite LP

Finding the Initial BFS

	y_1	s	y_3	y_4	x_1	x_2	x_3	y_2
-3	1	-1			-1	-2	-2	
-5		-1			-3	-2	-1	1
10		-1	1		1		3	
9		-1		1	1	1	1	
0		-1						

+1

x-1

+1

+1

+1

Rewrite LP

Finding the Initial BFS

	y_1	S	y_3	y_4	x_1	x_2	x_3	y_2
2	1				2		-1	-1
5		1			3	2	1	-1
15			1		4	2	4	-1
14				1	4	3	2	-1
5					3	2	1	-1

Finding the Initial BFS

	y_1	S	y_3	y_4	x_1	x_2	x_3	y_2
2	1				2		-1	-1
5		1			3	2	1	-1
15			1		4	2	4	-1
14				1	4	3	2	-1
5					3	2	1	-1

x_1 enters

Finding the Initial BFS



	y_1	s	y_3	y_4	x_1	x_2	x_3	y_2
2	1				2		-1	-1
5		1			3	2	1	-1
15			1		4	2	4	-1
14				1	4	3	2	-1
5					3	2	1	-1

x_1 enters

$$\frac{2}{2} = \min \left\{ \frac{2}{2}, \frac{5}{3}, \frac{15}{4}, \frac{14}{4} \right\}$$

$\Rightarrow y_1$ leaves

Finding the Initial BFS

	x_1	s	y_3	y_4	y_1	x_2	x_3	y_2
2	2				1		-1	-1
5	3	1				2	1	-1
15	4		1			2	4	-1
14	4			1		3	2	-1
5	3					2	1	-1

Rewrite LP

Finding the Initial BFS

	x_1	s	y_3	y_4	y_1	x_2	x_3	y_2	
2	2				1		-1	-1	$\times \frac{1}{2}$
5	3	1				2	1	-1	-3
15	4		1			2	4	-1	-4
14	4			1		3	2	-1	-4
5	3					2	1	-1	-3

Rewrite LP

Finding the Initial BFS

	x_1	s	y_3	y_4	y_1	x_2	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
2		1			$-\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$
11			1		-2	2	6	1
10				1	-2	3	4	1
2					$-\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$

Finding the Initial BFS

	x_1	s	y_3	y_4	y_1	x_2	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
2		1			$-\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$
11			1		-2	2	6	1
10				1	-2	3	4	1
2					$-\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$

x_2 enters

Finding the Initial BFS



	x_1	s	y_3	y_4	y_1	x_2	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
2		1			$-\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$
11			1		-2	2	6	1
10				1	-2	3	4	1
2					$-\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$

x_2 enters

$$\frac{2}{2} = \min \left\{ \frac{2}{2}, \frac{11}{2}, \frac{10}{3} \right\}$$

$\Rightarrow s$ leaves

Finding the Initial BFS

	x_1	x_2	y_3	y_4	y_1	s	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
2		2			$-\frac{3}{2}$	1	$\frac{5}{2}$	$\frac{1}{2}$
11		2	1		-2		6	1
10		3		1	-2		4	1
2		2			$-\frac{3}{2}$		$\frac{5}{2}$	$\frac{1}{2}$

Rewrite LP

Finding the Initial BFS

	x_1	x_2	y_3	y_4	y_1	s	x_3	y_2	
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$	=
2		2			$-\frac{3}{2}$	1	$\frac{5}{2}$	$\frac{1}{2}$	$\times \frac{1}{2}$
11		2	1		-2		6	1	-2
10		3		1	-2		4	1	-3
2		2			$-\frac{3}{2}$		$\frac{5}{2}$	$\frac{1}{2}$	-2

Rewrite LP

Finding the Initial BFS

	x_1	x_2	y_3	y_4	y_1	s	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	-1	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$-\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
0						-1		

All coefficients ≥ 0

$\Rightarrow$ BFS is optimal

Objective function value = 0

$\Rightarrow$ original LP is feasible

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	s	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	-1	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$-\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
0						-1		

s already non-basic
 $\Rightarrow$ no need to pivot

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
0	1	-2				1	

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
0	1	-2				1	

Rewrite LP

- Row 1 + 2 · Row 2

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

Finding an Optimal Solution

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

Finding an Optimal Solution

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

x_3 enters

Finding an Optimal Solution



	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

x_3 enters

$$\frac{1}{5/4} = \min \left\{ \frac{1}{5/4}, \frac{9}{7/2}, \frac{7}{1/4} \right\}$$

$\Rightarrow x_2$ leaves

Finding an Optimal Solution

	x_1	x_3	y_3	y_4	y_1	x_2	y_2
1	1	$-\frac{1}{2}$			$\frac{1}{2}$		$-\frac{1}{2}$
1		$\frac{5}{4}$			$-\frac{3}{4}$	1	$\frac{1}{4}$
9		$\frac{7}{2}$	1		$-\frac{1}{2}$		$\frac{1}{2}$
7		$\frac{1}{4}$		1	$\frac{1}{4}$		$\frac{1}{4}$
1		4			-2		1

Rewrite LP

Finding an Optimal Solution

	x_1	x_3	y_3	y_4	y_1	x_2	y_2
1	1	$-\frac{1}{2}$			$\frac{1}{2}$		$-\frac{1}{2}$
1		$\frac{5}{4}$			$-\frac{3}{4}$	1	$\frac{1}{4}$
9		$\frac{7}{2}$	1		$-\frac{1}{2}$		$\frac{1}{2}$
7		$\frac{1}{4}$		1	$\frac{1}{4}$		$\frac{1}{4}$
1		4			-2		1

$$+ \frac{1}{2}$$

$$\times \frac{4}{5}$$

$$- \frac{7}{2}$$

$$- \frac{1}{4}$$

$$- 4$$

Rewrite LP

Finding an Optimal Solution

	x_1	x_3	y_3	y_4	y_1	x_2	y_2
$\frac{7}{5}$	1				$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$
$\frac{4}{5}$		1			$-\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$
$\frac{31}{5}$			1		$\frac{8}{5}$	$-\frac{14}{5}$	$-\frac{1}{5}$
$\frac{34}{5}$				1	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
$-\frac{11}{5}$					$\frac{2}{5}$	$-\frac{16}{5}$	$\frac{1}{5}$

Finding an Optimal Solution

	x_1	x_3	y_3	y_4	y_1	x_2	y_2
$\frac{7}{5}$	1				$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$
$\frac{4}{5}$		1			$-\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$
$\frac{31}{5}$			1		$\frac{8}{5}$	$-\frac{14}{5}$	$-\frac{1}{5}$
$\frac{34}{5}$				1	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
$-\frac{11}{5}$					$\frac{2}{5}$	$-\frac{16}{5}$	$\frac{1}{5}$

y_1 enters

Finding an Optimal Solution



	x_1	x_3	y_3	y_4	y_1	x_2	y_2
$\frac{7}{5}$	1				$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$
$\frac{4}{5}$		1			$-\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$
$\frac{31}{5}$			1		$\frac{8}{5}$	$-\frac{14}{5}$	$-\frac{1}{5}$
$\frac{34}{5}$				1	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
$-\frac{11}{5}$					$\frac{2}{5}$	$-\frac{16}{5}$	$\frac{1}{5}$

y_1 enters

$$\frac{31/5}{8/5} = \min \left\{ \frac{7/5}{1/5}, \frac{31/5}{8/5}, \frac{34/5}{2/5} \right\}$$

$\Rightarrow y_3$ leaves

Finding an Optimal Solution

	x_1	x_3	y_1	y_4	y_3	x_2	y_2
$\frac{7}{5}$	1		$\frac{1}{5}$			$\frac{2}{5}$	$-\frac{2}{5}$
$\frac{4}{5}$		1	$-\frac{3}{5}$			$\frac{4}{5}$	$\frac{1}{5}$
$\frac{31}{5}$			$\frac{8}{5}$		1	$-\frac{14}{5}$	$-\frac{1}{5}$
$\frac{34}{5}$			$\frac{2}{5}$	1		$-\frac{1}{5}$	$\frac{1}{5}$
$-\frac{11}{5}$			$\frac{2}{5}$			$-\frac{16}{5}$	$\frac{1}{5}$

Rewrite LP

Finding an Optimal Solution

	x_1	x_3	y_1	y_4	y_3	x_2	y_2	
$\frac{7}{5}$	1		$\frac{1}{5}$			$\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$
$\frac{4}{5}$		1	$-\frac{3}{5}$			$\frac{4}{5}$	$\frac{1}{5}$	$+\frac{3}{5}$
$\frac{31}{5}$			$\frac{8}{5}$		1	$-\frac{14}{5}$	$-\frac{1}{5}$	$\times \frac{8}{5}$
$\frac{34}{5}$			$\frac{2}{5}$	1		$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{2}{5}$
$-\frac{11}{5}$			$\frac{2}{5}$			$-\frac{16}{5}$	$\frac{1}{5}$	$-\frac{2}{5}$

Rewrite LP

Finding an Optimal Solution

	x_1	x_3	y_1	y_4	y_3	x_2	y_2
$\frac{5}{8}$	1				$-\frac{1}{8}$	$\frac{3}{4}$	$-\frac{3}{8}$
$\frac{25}{8}$		1			$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{8}$
$\frac{31}{8}$			1		$\frac{5}{8}$	$-\frac{7}{4}$	$-\frac{1}{8}$
$\frac{21}{4}$				1	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$-\frac{15}{4}$					$-\frac{1}{4}$	$-\frac{5}{2}$	$\frac{1}{4}$

Finding an Optimal Solution

	x_1	x_3	y_1	y_4	y_3	x_2	y_2
$\frac{5}{8}$	1				$-\frac{1}{8}$	$\frac{3}{4}$	$-\frac{3}{8}$
$\frac{25}{8}$		1			$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{8}$
$\frac{31}{8}$			1		$\frac{5}{8}$	$-\frac{1}{4}$	$-\frac{1}{8}$
$\frac{21}{4}$				1	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$-\frac{15}{4}$					$-\frac{1}{4}$	$-\frac{5}{2}$	$\frac{1}{4}$

y_2 enters

Finding an Optimal Solution



	x_1	x_3	y_1	y_4	y_3	x_2	y_2
$\frac{5}{8}$	1				$-\frac{1}{8}$	$\frac{3}{4}$	$-\frac{3}{8}$
$\frac{25}{8}$		1			$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{8}$
$\frac{31}{8}$			1		$\frac{5}{8}$	$-\frac{1}{4}$	$-\frac{1}{8}$
$\frac{21}{4}$				1	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$-\frac{15}{4}$					$-\frac{1}{4}$	$-\frac{5}{2}$	$\frac{1}{4}$

y_2 enters

$$\frac{21/4}{1/4} = \min \left\{ \frac{25/8}{1/8}, \frac{21/4}{1/4} \right\}$$

$\Rightarrow y_4$ leaves

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{5}{8}$	1			$-\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{4}$	
$\frac{25}{8}$		1		$\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$	
$\frac{31}{8}$			1	$-\frac{1}{8}$	$\frac{5}{8}$	$-\frac{1}{4}$	
$\frac{21}{4}$				$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	1
$-\frac{15}{4}$				$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{5}{2}$	

Rewrite LP

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{5}{8}$	1			$-\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{4}$	
$\frac{25}{8}$		1		$\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$	
$\frac{31}{8}$			1	$-\frac{1}{8}$	$\frac{5}{8}$	$-\frac{1}{4}$	
$\frac{21}{4}$				$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	1
$-\frac{15}{4}$				$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{5}{2}$	

$$+ \frac{3}{8}$$

$$- \frac{1}{8}$$

$$+ \frac{1}{8}$$

$$\times 4$$

$$- \frac{1}{4}$$

Rewrite LP

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{17}{2}$	1				$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$		1			$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{13}{2}$			1		$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
21				1	-1	2	4
-9						-3	-1

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{17}{2}$	1				$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$		1			$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{13}{2}$			1		$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
21				1	-1	2	4
-9						-3	-1

All coefficients ≤ 0

$\Rightarrow$ BFS is optimal

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{17}{2}$	1				$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$		1			$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{13}{2}$			1		$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
21				1	-1	2	4
-9						-3	-1

All coefficients ≤ 0

$\Rightarrow$ BFS is optimal

$$x_1 = \frac{17}{2} \quad x_2 = 0 \quad x_3 = \frac{1}{2}$$

$$y_1 = \frac{13}{2} \quad y_2 = 21 \quad y_3 = 0$$

$$y_4 = 0$$

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{17}{2}$	1				$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$		1			$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{13}{2}$			1		$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
21				1	-1	2	4
-9						-3	-1

All coefficients ≤ 0

$\Rightarrow$ BFS is optimal

$$x_1 = \frac{17}{2} \quad x_2 = 0 \quad x_3 = \frac{1}{2}$$

$$y_1 = \frac{13}{2} \quad y_2 = 21 \quad y_3 = 0$$

$$y_4 = 0$$

Objective function value = 9

Finding an Optimal Solution

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{17}{2}$	1				$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$		1			$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{13}{2}$			1		$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
21				1	-1	2	4
-9						-3	-1

All coefficients ≤ 0

$\Rightarrow$ BFS is optimal

$$x_1 = \frac{17}{2} \quad x_2 = 0 \quad x_3 = \frac{1}{2}$$

$$y_1 = \frac{13}{2} \quad y_2 = 21 \quad y_3 = 0$$

$$y_4 = 0$$

Objective function value = 9

$$1 \cdot \frac{17}{2} - 2 \cdot 0 + 1 \cdot \frac{1}{2}$$