

Simplex Algorithm

CSCI 4113/6101



Simplex Algorithm

Input: LP in standard form

Goal: Find an optimal solution

(or decide that there is no feasible solution).

Maximize $c'z$

s.t. $A'z = b$

$z \geq 0$

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Main ideas:

1. Transform LP into equivalent LP whose basic solution is a BFS.

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Main ideas:

Same set of feasible solutions

Same objective function value for every feasible sol'n.

1. Transform LP into equivalent LP whose basic solution is a BFS.

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1. Transform LP into equivalent LP whose basic solution is a BFS.
2. If $c \leq 0$, then the BFS is optimal, stop.

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s.t. $A'z = b$

Goal: Find an optimal solution

$z \geq 0$

(or decide that there is no feasible solution).

Main ideas:

1. Transform LP into equivalent LP whose basic solution is a BFS.
2. If $c \leq 0$, then the BFS is optimal, stop.
3. If $c_j > 0$, we can transform the LP into equivalent LP with better BFS.

An Equivalent LP with a BFS

	y_1	y_2	y_3	y_4	x_1	x_2	x_3
-3	1				-1	-2	-2
-5		1			-3	-2	-1
10			1		1		3
9				1	1	1	1
0	0	0	0	0	1	-2	1

LP to be solved

Basic solution infeasible

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	y_1	y_2	y_3	y_4	x_1	x_2	x_3
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LP to be solved

Basic solution infeasible

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

Equivalent LP

Basic solution is BFS

An LP Whose BFS is Optimal

	x_1	x_3	y_1	y_2	y_3	x_2	y_4
$\frac{17}{2}$	1				$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$		1			$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{13}{2}$			1		$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
21				1	-1	2	4
-9						-3	-1

Changing basic variables does not help: coefficients = 0.

Non-basic variables can only increase.

This can only hurt: coefficients ≤ 0 .

Improving the Solution

$c_j > 0 \Rightarrow$ Increasing z_j increases the objective function value.

$c_h = 0 \quad \forall z_h \in B \Rightarrow$ changing z_h does not affect objective function value.

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$\Rightarrow$ Try to increase z_j without changing other non-basic variables.
Changing basic variables is fine.

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$c_h = 0 \quad \forall z_h \in B \Rightarrow$ changing z_h does not affect objective function value.

$\Rightarrow$ Try to increase z_j without changing other non-basic variables.
Changing basic variables is fine.

How much can we increase z_j while keeping the solution feasible?

z_{ji} = basic variable with coefficient 1 in i th constraint.

z_{j_i} = basic variable with coefficient 1 in i th constraint.

$$\forall 1 \leq i \leq m: \quad b_i = z_{j_i} + \sum_{h \in B} a_{ih} z_h$$

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$$\forall 1 \leq i \leq m: \quad b_i = z_{j_i} + \sum_{h \in B} a_{ih} z_h = z_{j_i} + a_{ij} z_j$$

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$$\forall 1 \leq i \leq m: \quad b_i = z_{j_i} + \sum_{h \in B} a_{ih} z_h = z_{j_i} + a_{ij} z_j$$

$$\Rightarrow \quad z_{j_i} = b_i - a_{ij} z_j$$

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$$\forall 1 \leq i \leq m: \quad b_i = z_{ji} + \sum_{h \in B} a_{ih} z_h = z_{ji} + a_{ij} z_j$$

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Case 1: $a_{ij} \leq 0$: Increasing z_j does not decrease z_{ji} .

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$$\forall 1 \leq i \leq m: \quad b_i = z_{ji} + \sum_{h \in B} a_{ih} z_h = z_{ji} + a_{ij} z_j$$

$$\Rightarrow \quad z_{ji} = b_i - a_{ij} z_j$$

Case 1: $a_{ij} \leq 0$: Increasing z_j does not decrease z_{ji} .

$$\text{Case 2: } a_{ij} > 0: \quad z_j \leq \frac{b_i}{a_{ij}} \Leftrightarrow z_{ji} \geq 0.$$

Case 1: $a_{ij} \leq 0 \quad \forall 1 \leq i \leq m$:

- We can increase z_j as much as we want.
- The LP is **unbounded**, has a feasible solution with objective function value ∞ .

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Case 2: $\exists i: a_{ij} > 0$:

- $k = \operatorname{argmin} \left\{ \frac{b_i}{a_{ij}} \mid a_{ij} > 0 \right\}, \Delta_k = \frac{b_k}{a_{kj}}$.
- Set $z_j = \Delta_k$ and $z_{ji} = b_i - a_{ij} \Delta_k \quad \forall 1 \leq i \leq m$.

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$\Rightarrow z_{ji} \geq 0 \quad \forall 1 \leq i \leq m$, the new solution is feasible.

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$z_{jk} = 0$, can move z_j into the basis and z_{jk} out of the basis.

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

$$\frac{1}{4/5} = \min \left\{ \frac{1}{5/4}, \frac{9}{7/2}, \frac{7}{1/4} \right\}$$

$$\Rightarrow k=2$$

$$\Delta_k = \frac{1}{5/4} = \frac{4}{5}$$

x_3 enters the basis,

x_2 leaves



	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1

	x_1	x_3	y_3	y_4	y_1	x_2	y_2
1	1	$-\frac{1}{2}$			$\frac{1}{2}$		$-\frac{1}{2}$
1		$\frac{5}{4}$			$-\frac{3}{4}$	1	$\frac{1}{4}$
9		$\frac{7}{2}$	1		$-\frac{1}{2}$		$\frac{1}{2}$
7		$\frac{1}{4}$		1	$\frac{1}{4}$		$\frac{1}{4}$
1		4			-2		1

Rewriting the LP

New basis : $B \setminus \{z_{jk}\} \cup \{z_j\}$

Rewriting the LP

New basis : $B \setminus \{z_{j_k}\} \cup \{z_j\}$

- Need to ensure that z_j has coefficient 1 in k th constraint, 0 everywhere else.
- Need coefficient 0 for z_j in objective function.

Rewriting the LP

New basis : $B \setminus \{z_{j_k}\} \cup \{z_j\}$

- Need to ensure that z_j has coefficient 1 in k th constraint, 0 everywhere else.
- Need coefficient 0 for z_j in objective function.

Basic row operations:

- Multiply row (constraint) by scalar.
- Subtract one row from another.

Do not change set of solutions.

Basic row operations



	x_1	x_3	y_3	y_4	y_1	x_2	y_2
1	1	$-\frac{1}{2}$			$\frac{1}{2}$		$-\frac{1}{2}$
1		$\frac{5}{4}$			$-\frac{3}{4}$	1	$\frac{1}{4}$
9		$\frac{7}{2}$	1		$-\frac{1}{2}$		$\frac{1}{2}$
7		$\frac{1}{4}$		1	$\frac{1}{4}$		$\frac{1}{4}$
1		4			-2		1

	x_1	x_3	y_3	y_4	y_1	x_2	y_2
$\frac{7}{5}$	1				$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$
$\frac{4}{5}$		1			$-\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$
$\frac{31}{5}$			1		$\frac{8}{5}$	$-\frac{14}{5}$	$-\frac{1}{5}$
$\frac{34}{5}$				1	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
$-\frac{11}{5}$					$\frac{2}{5}$	$-\frac{16}{5}$	$\frac{1}{5}$

Pivoting

if $\exists i : a_{ij} > 0$ then

$$k = \operatorname{argmin} \left\{ \frac{b_i}{a_{ij}} \mid a_{ij} > 0 \right\}, \Delta_k = \frac{b_k}{a_{kj}}$$

$$z_j = \Delta_k, z_{ji} = b_i - a_{ij} \Delta_k \quad \forall 1 \leq i \leq m$$

$$B = B \setminus \{z_{j_k}\} \cup \{z_j\}$$

Rewrite LP to turn B into a basis.

else

Report that the LP is unbounded.

Simplex Algorithm

Transform LP so its basic solution is feasible.

while $\exists j: c_j > 0$ do

 Pivot z_j into the basis

return current BFS

Initialization

How to find the initial BFS?

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The easy case: $b \geq 0 \Rightarrow z_{ji} = b_i \geq 0 \quad \forall 1 \leq i \leq m$, we have a BFS.

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The hard case:

LP: Maximize cz
s.t. $Az = b$
 $z \geq 0$

Auxiliary LP: Maximize $-s$
s.t. $Az - Is = b$
 $z, s \geq 0$

Initialization

How to find the initial BFS?

The easy case: $b \geq 0 \Rightarrow z_{ji} = b_i \geq 0 \quad \forall 1 \leq i \leq m$, we have a BFS.

The hard case:

$$\begin{aligned} \text{LP:} \quad & \text{Maximize } cz \\ & \text{s.t. } Az = b \\ & \quad z \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Auxiliary LP:} \quad & \text{Maximize } -s \\ & \text{s.t. } Az - Is = b \\ & \quad z, s \geq 0 \end{aligned}$$

Observation: The LP is feasible if and only if the auxiliary LP has an optimal solution with objective function value $-s = 0$.

LP to be solved

	y_1	y_2	y_3	y_4	x_1	x_2	x_3
-3	1				-1	-2	-2
-5		1			-3	-2	-1
10			1		1		3
9				1	1	1	1
0	0	0	0	0	1	-2	1

Auxiliary LP

	y_1	y_2	y_3	y_4	x_1	x_2	x_3	S
-3	1				-1	-2	-2	-1
-5		1			-3	-2	-1	-1
10			1		1		3	-1
9				1	1	1	1	-1
0								-1

How to Solve the Auxiliary LP

Using the Simplex Algorithm!

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Problem: The basic solution of the auxiliary LP is infeasible.

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	y_1	y_2	y_3	y_4	x_1	x_2	x_3
-3	1				-1	-2	-2
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10			1		1		3
9				1	1	1	1
0	0	0	0	0	1	-2	1

Auxiliary LP

	y_1	y_2	y_3	y_4	x_1	x_2	x_3	S
-3	1				-1	-2	-2	-1
-5		1			-3	-2	-1	-1
10			1		1		3	-1
9				1	1	1	1	-1
0								-1

How to Solve the Auxiliary LP

Using the Simplex Algorithm!

Problem: The basic solution of the auxiliary LP is infeasible.

Solution: A single (special) pivoting operation gives an auxiliary LP whose basic solution is feasible.

Pivoting to Obtain Auxiliary LP with a BFS

$$k = \underset{1 \leq i \leq m}{\operatorname{argmin}} b_i$$

Move s into basis, z_{j_k} out of basis

Pivoting to Obtain Auxiliary LP with a BFS

$$k = \underset{1 \leq i \leq m}{\operatorname{argmin}} b_i$$

Move s into basis, z_{jk} out of basis

$$\Rightarrow \hat{z}_{jk} = 0$$

Pivoting to Obtain Auxiliary LP with a BFS

$$k = \underset{1 \leq i \leq m}{\operatorname{argmin}} b_i$$

Move s into basis, z_{jk} out of basis

$$\Rightarrow \hat{z}_{jk} = 0$$

$$\hat{z}_{jk} - \hat{s} = b_k$$

Pivoting to Obtain Auxiliary LP with a BFS

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Move s into basis, z_{jk} out of basis

$$\Rightarrow \hat{z}_{jk} = 0$$

$$\hat{z}_{jk} - \hat{s} = b_k$$

$$\Rightarrow \hat{s} = -b_k > 0$$

Pivoting to Obtain Auxiliary LP with a BFS

$$k = \underset{1 \leq i \leq m}{\operatorname{argmin}} b_i$$

Move s into basis, z_{jk} out of basis

$$\Rightarrow \hat{z}_{jk} = 0$$

$$\hat{z}_{jk} - \hat{s} = b_k$$

$$\Rightarrow \hat{s} = -b_k > 0$$

$$\hat{z}_{ji} - \hat{s} = b_i$$

Pivoting to Obtain Auxiliary LP with a BFS

$$k = \underset{1 \leq i \leq m}{\operatorname{argmin}} b_i$$

Move s into basis, z_{jk} out of basis

$$\Rightarrow \hat{z}_{jk} = 0$$

$$\hat{z}_{jk} - \hat{s} = b_k$$

$$\Rightarrow \hat{s} = -b_k > 0$$

$$\hat{z}_{ji} - \hat{s} = b_i$$

$$\begin{aligned} \Rightarrow \hat{z}_{ji} &= b_i + \hat{s} \\ &= b_i - b_k \\ &\geq 0 \end{aligned}$$

	y_1	y_2	y_3	y_4	x_1	x_2	x_3	s
-3	1				-1	-2	-2	-1
-5		1			-3	-2	-1	-1
10			1		1		3	-1
9				1	1	1	1	-1
0								-1

	y_1	s	y_3	y_4	x_1	x_2	x_3	y_2
2	1				2		-1	-1
5		1			3	2	1	-1
15			1		4	2	4	-1
14				1	4	3	2	-1
5					3	2	1	-1

The Initialization Procedure

Construct auxiliary LP

Pivot to obtain equivalent LP with a BFS

Find optimal solution $(\hat{z}, \hat{s})$ using Simplex

if $\hat{s} = 0$ then

Drop s and restore original objective function

else

Report that there is no feasible solution

Dropping s and Restoring the Objective Function

- If s is in the basis, pivot to remove it from the basis.
- Remove s from all constraints.
- Restore original objective function.
- Transform objective function using elementary row operations so basic variables have coefficient 0.

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	s	x_3	y_2
1	1				$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	-1	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$-\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
0						-1		

s already non-basic
 $\Rightarrow$ no need to pivot

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
0	1	-2				1	

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
0	1	-2				1	

Rewrite LP

- Row 1 + 2 · Row 2

Dropping s and Restoring the Objective Function

	x_1	x_2	y_3	y_4	y_1	x_3	y_2
1	1				$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
1		1			$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$
9			1		$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
7				1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1					-2	4	1