Minimum Spanning Tree as an Integer Linear Program

CSCI 4113/6101

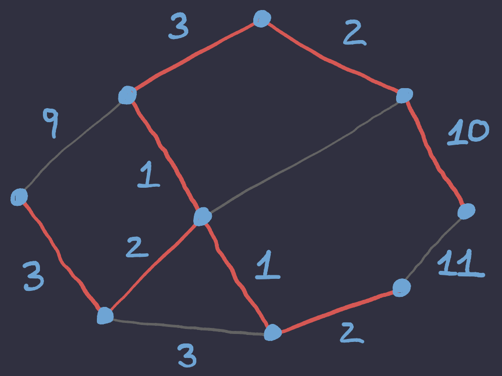
The Minimum Spanning Tree Problem

Input:

- Graph $G = (V, E)$
- Edge weights $q: E \to \mathbb{R}$

Output:

A spanning tree T of G of minimum weight



Spanning tree = tree $T \subseteq G$ containing all vertices of G.

Minimum Spanning Tree as an Integer Linear Program

Integer linear program (ILP): LP whose variables are integers

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Minimum Spanning Tree as an Integer Linear Program

Integer linear program (ILP): LP whose variables are integers

$$x_e = \begin{cases} 1 & \text{if } e \in MST \\ 0 & \text{if } e \notin MST \end{cases}$$

ILP: Minimize $\sum_{e \in E} q_e x_e$

s.t. $x_e \in \{0, 1\} \quad \forall e \in E$ ← integer ~~linear prog~~

"$T = (V, \{e \in E \mid x_e = 1\})$ is a tree"

Characterizing a Tree

Definition of a tree:

A graph is a tree if and only if

- It is connected and
- Has no cycle.

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$n = |V|$

$m = |E|$

Lemma: $T = (V, E)$ is a tree if and only if it satisfies at least two of the following conditions:

- $m = n - 1$.
- T is connected.
- T has no cycle.

Characterizing a Tree

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Lemma: If T has $n-1$ edges, then it is connected if and only if it has no cycles.

MST as an ILP

A tree has no cycles:

$$\text{Minimize} \sum_{e \in E} q_e x_e$$

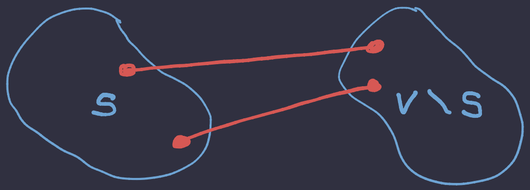
$$\text{s.t.} \sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in C} x_e \leq |C| - 1 \quad \forall \text{ cycle } C$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

Cuts and Connectivity

Cut = non-empty proper subset $\emptyset \subset S \subset V$



Edges with one endpoint in S are said to cross the cut S.

Characterizing a Tree

Lemma: A tree has $n-1$ edges $(n=|V|)$.

Lemma: If T has $n-1$ edges, then it is connected if and only if it has no cycles.

Lemma: G is connected if and only if every cut is crossed by at least one edge.

MST as an ILP

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$$\text{s.t.} \quad \sum_{e \in E} x_e = n-1$$

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$$x_e \in \{0,1\} \quad \forall e \in E$$

A tree is connected:

$$\text{Minimize} \sum_{e \in E} q_e x_e$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n-1$$

$$\sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} x_{(u,v)} \geq 1 \quad \forall \text{ cut } S$$

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Both ILPs have exponentially many constraints!

A Few Notes

- Exponentially many constraints are ok in this case!
- Constraining variables to be integers is the bigger problem.
  ILP is NP-hard.
- Prim and Kruskal are the simpler, faster choice.