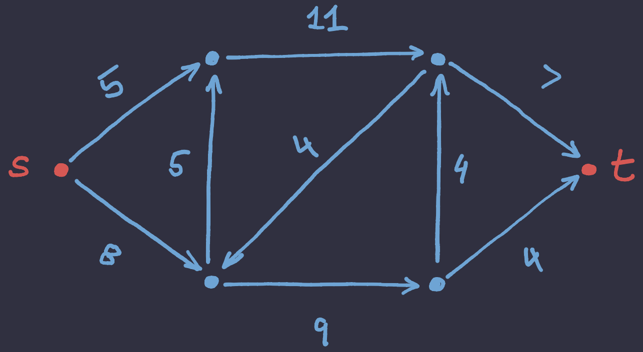
Maximum Flow

CSCI 4113 / 6101

Maximum Flow

Input:

- Graph $G = (V, E)$
- Edge capacities $c : E \to \mathbb{R}^+$
- Source $s \in V$
- Sink $t \in V$



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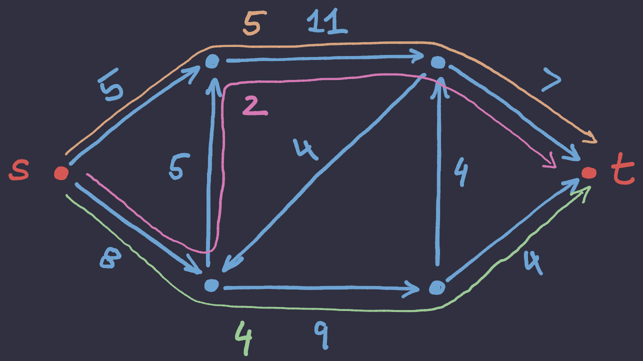

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Flow $f : E \to \mathbb{R}^+$

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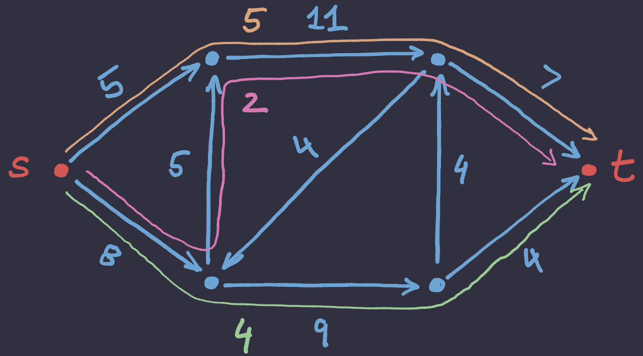

Flow:

Capacity constraints:

$$0 \leq f(e) \leq c(e) \quad \forall\, e \in E$$

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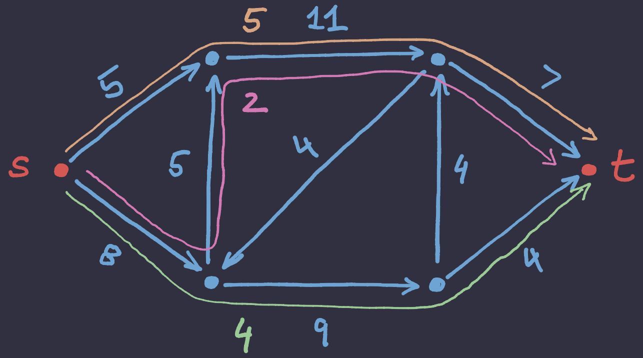

Flow:

Capacity constraints:

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

Flow conservation:

$$\sum_{v \in V} \left(f(u,v) - f(v,u)\right) = 0 \quad \forall u \in V \setminus \{s,t\}$$

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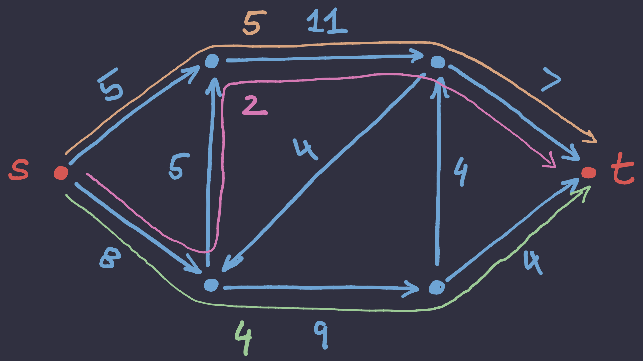

Output:

Flow $f : E \to \mathbb{R}^+$

Goal:

Maximize $\sum_{v \in V} (f(s,v) - f(v,s))$

An LP Formulation

$$\text{Maximize} \sum_{v \in V} \left(f(s,v) - f(v,s) \right)$$

$$\text{s.t.} \qquad 0 \leq f(e) \leq c(e) \qquad \forall e \in E$$

$$\sum_{v \in V} \left(f(u,v) - f(v,u) \right) = 0 \qquad \forall v \in V \setminus \{s,t\}$$

Notation: $f(u) = \sum_{v \in V} \left(f(u,v) - f(v,u) \right)$ (u's net outflow)

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$$0 = \sum_{u \in V \setminus \{s,t\}} \sum_{v \in V} (f(u,v) - f(v,u))$$

$$= \sum_{v \in V} (f(v,s) - f(s,v)) + \sum_{v \in V} (f(v,t) - f(t,v))$$

$$= -f(s) - f(t)$$

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$$= \sum_{v \in V} (f(v,s) - f(s,v)) + \sum_{v \in V} (f(v,t) - f(t,v))$$

$$= -f(s) - f(t)$$

$$\Rightarrow \quad f(s) = -f(t)$$

$$
\begin{aligned}
&\text{Maximize } f(s) \\
&\text{s.t.} \quad 0 \leq f(e) \leq c(e) \quad \forall e \in E \\
&\qquad\qquad f(v) = 0 \qquad \forall v \in V \setminus \{s,t\}
\end{aligned}
$$

Overview of Maximum Flow Algorithms

Augmenting Paths Algorithms

- Maintain flow
- Increase flow along paths from s to t

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Ford-Fulkerson Arbitrary paths	May not terminate
Edmonds-Karp Shortest paths	$O(nm^2)$
Dinitz Multiple shortest paths	$O(n^2m)$

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Preflow-Push Algorithms

- Saturate out-edges of s
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Generic $O(n^2m)$

Relabel to front $O(n^3)$

"Eliminate excess first"

Highest-vertex $O(n^2\sqrt{m})$

"Drain vertex farthest from the sink"