The Max-Flow Min-Cut Theorem

CSCI 4113/6101

Cuts

Cut C $= \emptyset \subset C \subset V$

Cuts

Cut C $= \emptyset \subset C \subset V$

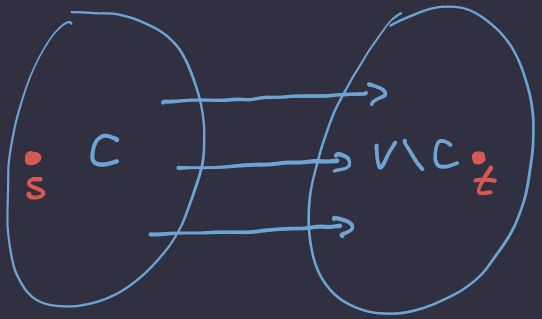
st-cut C $= s \in C,\ t \notin C$

Cuts

Cut C = $\emptyset \subset C \subset V$

st-cut C = $s \in C,\ t \notin C$

Capacity $c(C) = \sum_{u \in C} \sum_{v \notin C} c(u,v)$

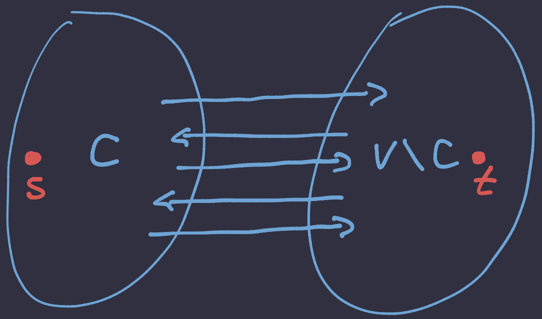


Cuts

Cut C $= \emptyset \subset C \subset V$

st-cut C $= s \in C,\ t \notin C$

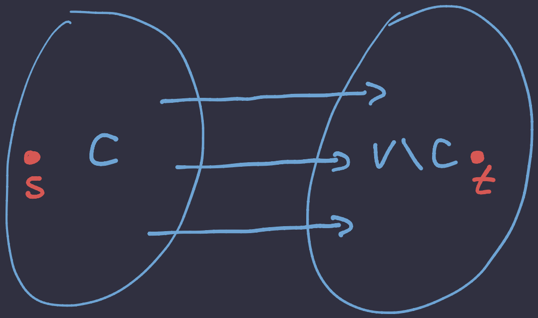
Capacity $c(C) = \sum_{u \in C} \sum_{v \notin C} c(u,v)$



Flow across cut

$$f(C) = \sum_{u \in C} \sum_{v \notin C} \left(f(u,v) - f(v,u)\right) \leq c(C)$$

Max-Flow Min-Cut Theorem



Minimum st-cut = st-cut of minimum capacity

Min-cost max-flow theorem: The value of a maximum st-flow in G equals the capacity of a minimum st-cut C in G:

$$f(s) = c(C).$$

Proof Outline

f = st-flow $\quad f^*$ = max st. flow $\quad C$ = st-cut $\quad C^*$ = min st-cut

$$\boxed{f(s) = f(C) \leq c(C)}$$

$$\downarrow$$

$$\boxed{f^*(s) \leq c(C^*)}$$

There is no augmenting path in G_{f^*}

$$\downarrow$$

There exists a cut C s.t.

$$f^*(s) = f^*(C) = c(C) \geq c(C^*)$$

$$\boxed{f^*(s) = c(C^*)}$$

Correctness of Ford-Fulkerson

Corollary: Once G_f has no augmenting path, f is a maximum st-flow.

Corollary: If Ford-Fulkerson terminates, it computes a maximum st-flow.

Proof Outline

f = st-flow $\quad$ f^* = max st. flow $\quad$ C = st-cut $\quad$ C^* = min st-cut

$$\boxed{f(s) = f(C) \leq c(C)}$$

$$\boxed{f^*(s) \leq c(C^*)}$$

There is no augmenting path in G_{f^*}

There exists a cut C s.t. $f^*(s) = f^*(C) = c(C) \geq c(C^*)$

$$\boxed{f^*(s) = c(C^*)}$$

Equal Flow Across All Cuts

$$f(s) = \sum_{v \in V} \left(f(s,v) - f(v,s) \right)$$

Equal Flow Across All Cuts

$$f(s) = \sum_{v \in V} \left(f(s,v) - f(v,s)\right)$$

$$= \sum_{v \in V} \left(f(s,v) - f(v,s)\right) + \sum_{u \in C \setminus \{s\}} \sum_{v \in V} \left(f(u,v) - f(v,u)\right) \quad \text{(flow conservation)}$$

Equal Flow Across All Cuts

$$f(s) = \sum_{v \in V} (f(s,v) - f(v,s))$$

$$= \sum_{v \in V} (f(s,v) - f(v,s)) + \sum_{u \in C \setminus \{s\}} \sum_{v \in V} (f(u,v) - f(v,u))$$ (flow conservation)

$$= \sum_{u \in C} \sum_{v \in V} (f(u,v) - f(v,u))$$

Equal Flow Across All Cuts

$$
\begin{aligned}
f(s) &= \sum_{v \in V} \left(f(s,v) - f(v,s)\right) \\
&= \sum_{v \in V} \left(f(s,v) - f(v,s)\right) + \sum_{u \in C \setminus \{s\}} \sum_{v \in V} \left(f(u,v) - f(v,u)\right) \quad \text{(flow conservation)} \\
&= \sum_{u \in C} \sum_{v \in V} \left(f(u,v) - f(v,u)\right) \\
&= \sum_{u \in C} \sum_{v \in C} \left(f(u,v) - f(v,u)\right) + \sum_{u \in C} \sum_{v \notin C} \left(f(u,v) - f(v,u)\right)
\end{aligned}
$$

Equal Flow Across All Cuts

$$f(s) = \sum_{v \in V} \left(f(s,v) - f(v,s) \right)$$

$$= \sum_{v \in V} \left(f(s,v) - f(v,s) \right) + \sum_{u \in C \setminus \{s\}} \sum_{v \in V} \left(f(u,v) - f(v,u) \right) \quad \text{(flow conservation)}$$

$$= \sum_{u \in C} \sum_{v \in V} \left(f(u,v) - f(v,u) \right)$$

$$= \sum_{u \in C} \sum_{v \in C} \left(f(u,v) - f(v,u) \right) + \sum_{u \in C} \sum_{v \notin C} \left(f(u,v) - f(v,u) \right)$$

$$= \quad 0 \quad + \quad f(C)$$

Proof Outline

f = st-flow $\quad f^*$ = max st. flow $\quad C$ = st-cut $\quad C^*$ = min st-cut

$$\boxed{f(s) = f(C) \leq c(C)}$$

$$\downarrow$$

$$\boxed{f^*(s) \leq c(C^*)}$$

There is no augmenting path in G_{f^*}

$$\downarrow$$

There exists a cut C s.t.

$$f^*(s) = f^*(C) = c(C) \geq c(C^*)$$

$$\boxed{f^*(s) = c(C^*)}$$

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C \Rightarrow C$ is an st-cut

Let $u \in C$, $v \notin C$

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

Let $u \in C$, $v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

Let $u \in C$, $v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ $\lightning$

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

Let $u \in C$, $v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ $\lightning$

$\Rightarrow f(u,v) = c(u,v)$

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

Let $u \in C$, $v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ $\lightning$

$\Rightarrow f(u,v) = c(u,v)$

If $f(v,u) > 0$, then $(u,v) \in G_f$

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

Let $u \in C$, $v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ ↯

$\Rightarrow f(u,v) = c(u,v)$

If $f(v,u) > 0$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ ↯

A Flow Without Augmenting Paths Matches Some Cut

C = all vertices reachable from $s \in G_f$

$s \in C$, $t \notin C$ $\Rightarrow$ C is an st-cut

Let $u \in C$, $v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ ↯

$\Rightarrow f(u,v) = c(u,v)$

If $f(v,u) > 0$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ ↯

$\Rightarrow f(v,u) = 0$

A Flow Without Augmenting Paths Matches Some Cut

$C =$ all vertices reachable from $s \in G_f$

$s \in C,\ t \notin C \Rightarrow C$ is an st-cut

Let $u \in C,\ v \notin C$

If $f(u,v) < c(u,v)$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ ↯

$\Rightarrow f(u,v) = c(u,v)$

If $f(v,u) > 0$, then $(u,v) \in G_f$

$\Rightarrow v \in C$ ↯

$\Rightarrow f(v,u) = 0$

$$\Rightarrow f(s) = f(C) = \sum_{u \in C} \sum_{v \notin C} \left(f(u,v) - f(v,u) \right) = \sum_{u \in C} \sum_{v \notin C} c(u,v) = c(C)$$