Canonical Form and Standard Form

CSCI 4113/6101

Canonical Form and Standard Form

(a.k.a. standard form and slack form)

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Canonical Form

Maximize cx

s.t. $Ax \leq b$

$x \geq 0$

$$c = (c_1, \cdots, c_n)$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Canonical Form and Standard Form

(a.k.a. standard form and slack form)

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Canonical form

Maximize cx

s.t. $Ax \leq b$

$x \geq 0$

Standard form

Maximize cx

s.t. $y = b - Ax$

$x, y \geq 0$

Canonical Form and Standard Form

(a.k.a. standard form and slack form)

$c = (c_1, \cdots, c_n)$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Canonical Form

Maximize cx

s.t. $Ax \leq b$

$x \geq 0$

Important in LP duality.

Standard Form

Maximize cx

s.t. $y = b - Ax$

$x, y \geq 0$

Important as input to Simplex Algorithm.

Transforming LPs into Canonical Form

Minimize $2x_1 - 3x_2 + x_3$

s.t. $x_1 + x_2 - x_3 \geq 0$

$2x_1 - x_2 + 5x_3 = 4$

$x_3 \leq 7$

$x_1 \leq 0$

$x_3 \geq 0$

Transforming LPs into Canonical Form

Minimize $2x_1 - 3x_2 + x_3$

s.t. $x_1 + x_2 - x_3 \geq 0$

$2x_1 - x_2 + 5x_3 = 4$

$x_3 \leq 7$

$x_1 \leq 0$

$x_3 \geq 0$

$\Longleftrightarrow$

$x_2 = x_2' - x_2''$

$x_1 = -x_1'$

Maximize $2x_1' + 3x_2' - 3x_2'' - x_3$

s.t. $x_1' - x_2' + x_2'' + x_3 \leq 0$

$-2x_1' - x_2' + x_2'' + 5x_3 \leq 4$

$2x_1' + x_2' - x_2'' - 5x_3 \leq -4$

$x_3 \leq 7$

$x_1', x_2', x_2'', x_3 \geq 0$

Transforming LPs into Standard Form

$$
\begin{aligned}
\text{Maximize} \quad & 2x_1' + 3x_2' - 3x_2'' - x_3 \\
\text{s.t.} \quad & x_1' - x_2' + x_2'' + x_3 \leq 0 \\
& -2x_1' - x_2' + x_2'' + 5x_3 \leq 4 \\
& 2x_1' + x_2' - x_2'' - 5x_3 \leq -4 \\
& x_3 \leq 7 \\
& x_1', x_2', x_2'', x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & 2x_1' + 3x_2' - 3x_2'' - x_3 \\
\text{s.t.} \quad & y_1 = 0 - x_1' + x_2' - x_2'' - x_3 \\
& y_2 = 4 + 2x_1' + x_2' - x_2'' - 5x_3 \\
& y_3 = -4 - 2x_1' - x_2' + x_2'' + 5x_3 \\
& y_4 = 7 - x_3 \\
& x_1', x_2', x_2'', x_3 \geq 0 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

An Alternative Standard Form

$$\text{Maximize} \quad 2x_1' + 3x_2' - 3x_2'' - x_3$$

$$\begin{aligned}
\text{s.t.} \quad Y_1 \quad & + x_1' - x_2' + x_2'' + x_3 = 0 \\
Y_2 \quad & - 2x_1' - x_2' + x_2'' + 5x_3 = 4 \\
Y_3 \quad & + 2x_1' + x_2' - x_2'' - 5x_3 = -4 \\
Y_4 \quad & + x_3 = 7 \\
& x_1', x_2', x_2'', x_3 \geq 0 \\
& Y_1, Y_2, Y_3, Y_4 \geq 0
\end{aligned}$$

An Alternative Standard Form

$$\text{Maximize} \quad 2z_5 + 3z_6 - 3z_7 - z_8$$

$$\begin{aligned}
\text{s.t.} \quad z_1 &+ z_5 - z_6 + z_7 + z_8 = 0 \\
z_2 &- 2z_5 - z_6 + z_7 + 5z_8 = 4 \\
z_3 &+ 2z_5 + z_6 - z_7 - 5z_8 = -4 \\
z_4 &+ z_8 = 7 \\
& z_i \geq 0 \quad \forall\, 1 \leq i \leq 8
\end{aligned}$$

Basis: $\{z_1, z_2, z_3, z_4\}$

Non-basic variables: z_5, z_6, z_7, z_8

An Alternative Standard Form

$$\text{Maximize} \quad 2z_5 + 3z_6 - 3z_7 - z_8$$

$$\begin{aligned}
\text{s.t.} \quad z_1 &+ z_5 - z_6 + z_7 + z_8 = 0 \\
z_2 &- 2z_5 - z_6 + z_7 + 5z_8 = 4 \\
z_3 &+ 2z_5 + z_6 - z_7 - 5z_8 = -4 \\
z_4 &+ z_8 = 7
\end{aligned}$$

$$z_i \geq 0 \quad \forall\, 1 \leq i \leq 8$$

Every basic variable:

- Has coefficient 1 in exactly one constraint.
- Has coefficient 0 in objective function.

Basic Solution

Basic solution:

- $z_{j_i} = b_i \quad \forall\, 1 \le i \le m$ (z_{j_i} = ith basic variable)
- $z_j = 0 \quad \forall\, j \notin B$ (non-basic variables are 0)

Basic Solution

Basic solution:

- $z_{j_i} = b_i \quad \forall\, 1 \le i \le m$ (z_{j_i} = ith basic variable)
- $z_j = 0 \quad \forall\, j \notin B$ (non-basic variables are 0)

Basic feasible solution (BFS):

What we call the basic solution if it is feasible — i.e., $z_j \ge 0 \ \forall j$.

Tableau

$$
\begin{aligned}
&\text{Maximize} && 2z_5 + 3z_6 - 3z_7 - z_8 \\
&\text{s.t.} && z_1 = z_5 - z_6 + z_7 + z_8 \\
& && z_2 = -2z_5 - z_6 + z_7 + 5z_8 \\
& && z_3 = 2z_5 + z_6 - z_7 - 5z_8 \\
& && z_4 = x_3 \\
& && z_i \geq 0 \quad \forall\, 1 \leq i \leq 8
\end{aligned}
$$

Tableau

	Basic variables				Non-basic vars			
	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
0	1				1	-1	1	1
4		1			-2	-1	1	5
-4			1		2	1	-1	-5
7				1				1
0	0	0	0	0	2	3	-3	-1

Constraints

Negated objective → Objective function

Function value of basic solution