LP Relaxations

CSCI 4113/6101

LP Relaxation

LP relaxation of an ILP: Drop constraint that variables are integers

$$\text{Minimize} \sum_{e \in E} q_e x_e$$

$$\text{s.t.} \sum_{e \in E} x_e = n-1$$

$$\sum_{e \in C} x_e \leq |C| - 1 \quad \forall \text{ cycle } C$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

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$$\cancel{x_e \in \{0,1\}} \quad \forall e \in E$$

$$0 \leq x_e \leq 1$$

Integrality Gap

Observation: Let P be a minimization ILP with objective function f, let $\hat{x}$ be an optimal solution of P, and let $\tilde{x}$ be an optimal solution of its LP relaxation. Then $f(\tilde{x}) \leq f(\hat{x})$.

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Fundamental question: By how much do $f(\tilde{x})$ and $f(\hat{x})$ differ?

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Fundamental question: By how much do $f(\tilde{x})$ and $f(\hat{x})$ differ?

$$\text{Integrality gap of } P = \frac{f(\hat{x})}{f(\tilde{x})}$$

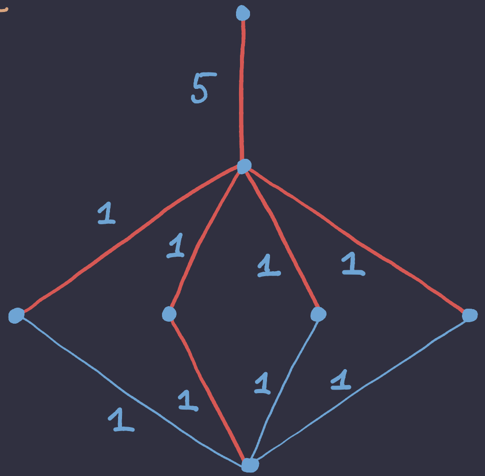
Integrality Gap of "No Cycles" MST ILP

Minimize $\sum_{e \in E} w_e x_e$

s.t. $\sum_{e \in E} x_e = n-1$

$\sum_{e \in C} x_e \leq |C| - 1 \quad \forall \text{ cycle } C$

$x_e \in \{0,1\} \quad \forall e \in E$



$w(MST) = 10$

Integrality Gap of "No Cycles" MST ILP

$$\text{Minimize} \quad \sum_{e\in E} w_e x_e$$

$$\text{s.t.} \quad \sum_{e\in E} x_e = n-1$$

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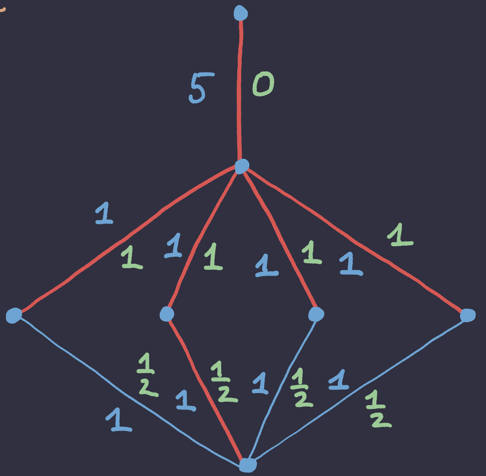


$w(MST) = 10$ $\quad w(\hat{x}) = 6$

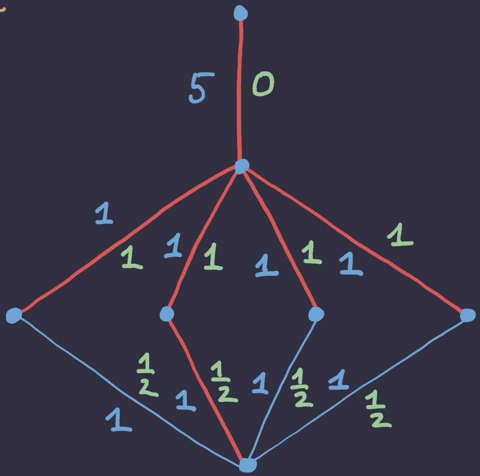
Integrality Gap of "No Cycles" MST ILP

$$\text{Minimize} \sum_{e \in E} w_e x_e$$

$$\text{s.t.} \sum_{e \in E} x_e = n-1$$

$$\sum_{e \in C} x_e \le |C|-1 \quad \forall \text{ cycle } C$$

$$x_e \in \{0,1\} \quad \forall e \in E$$



Integrality gap ≥ 1.67

$w(MST) = 10 \qquad w(\hat{x}) = 6$

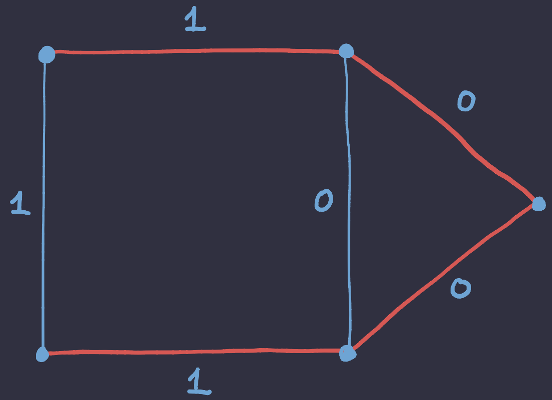
Integrality Gap of "Connectivity" MST ILP

$$\text{Minimize} \sum_{e \in E} w_e x_e$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n-1$$

$$\sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} x_e \geq 1 \qquad \forall \text{cut } S$$

$$x_e \in \{0,1\} \qquad \forall e \in E$$



$w(MST) = 2$

Integrality Gap of "Connectivity" MST ILP

$$\text{Minimize} \sum_{e \in E} w_e x_e$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n - 1$$

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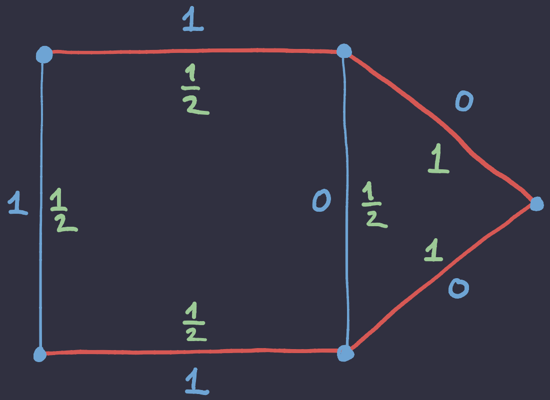

$w(MST) = 2 \qquad w(\hat{x}) = \frac{3}{2}$

Integrality Gap of "Connectivity" MST ILP

$$\text{Minimize } \sum_{e \in E} w_e x_e$$

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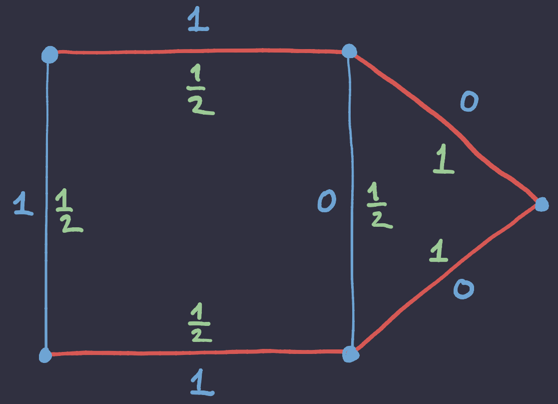


Integrality gap ≥ 1.33

$w(MST) = 2$ $\qquad$ $w(\hat{x}) = \frac{3}{2}$

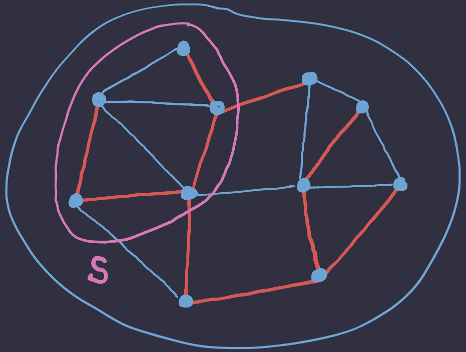
An MST ILP with Integrality Gap 1

Minimize $\sum_{e \in E} w_e x_e$

s.t. $\sum_{e \in E} x_e \geq n-1$

$$\sum_{\substack{(u,v) \in E \\ u,v \in S}} x_{(u,v)} \leq |S|-1 \quad \forall \emptyset \subset S \subseteq V$$

$$x_e \in \{0,1\} \quad \forall e \in E$$



An MST ILP with Integrality Gap 1

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$$\text{s.t.} \quad \sum_{e \in E} x_e \geq n-1$$

$$\sum_{\substack{(u,v) \in E \\ u,v \in S}} x_{(u,v)} \leq |S|-1 \quad \forall \emptyset \subset S \subseteq V$$

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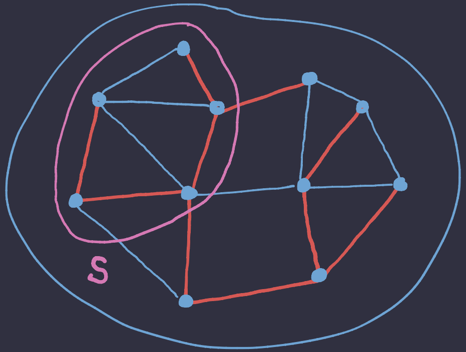

Observation: The solutions of this ILP correspond exactly to spanning trees of G.

An MST ILP with Integrality Gap 1

Proof in the notes.

$$\text{Minimize} \quad \sum_{e\in E} w_e x_e$$

$$\text{s.t.} \quad \sum_{e\in E} x_e \geq n-1$$

$$\sum_{\substack{(u,v)\in E\\ u,v\in S}} x_{(u,v)} \leq |S|-1 \quad \forall \emptyset \subset S \subseteq V$$

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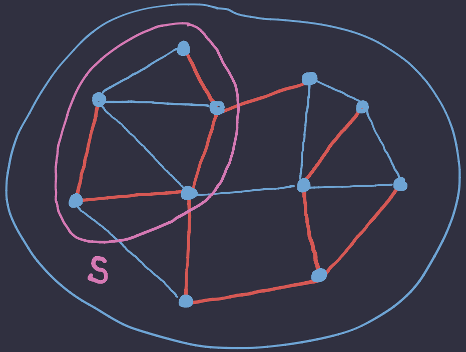

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