LP Duality

CSCI 4113/6101

LP Duality

Primal LP

$$\begin{aligned} \text{Maximize} \quad & cx \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{Minimize} \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c^T \\ & y \geq 0 \end{aligned}$$

LP Duality

Primal LP

Maximize cx

s.t. $Ax \leq b$

$x \geq 0$

Dual LP

Minimize $b^T y$

s.t. $A^T y \geq c^T$

$y \geq 0$

Each variable x_j corresponds to jth constraint in dual.

Each variable y_i corresponds to ith constraint in primal.

Example

Primal

$$
\begin{aligned}
\text{Maximize} \quad & x_1 - 2x_2 + x_3 \\
\text{s.t.} \quad & -x_1 - 2x_2 - 2x_3 \leq -3 \\
& -3x_1 - 2x_2 - x_3 \leq -5 \\
& x_1 + 3x_3 \leq 10 \\
& x_1 + x_3 + x_3 \leq 9 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Example

Primal

$$
\begin{array}{llr}
\text{Maximize} & x_1 - 2x_2 + x_3 & \\
\text{s.t.} & -x_1 - 2x_2 - 2x_3 \leq -3 & y_1 \\
 & -3x_1 - 2x_2 - x_3 \leq -5 & y_2 \\
 & x_1 + 3x_3 \leq 10 & y_3 \\
 & x_1 + x_3 + x_3 \leq 9 & y_4 \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Example

Primal

$$
\begin{aligned}
\text{Maximize} \quad & x_1 - 2x_2 + x_3 \\
\text{s.t.} \quad & -x_1 - 2x_2 - 2x_3 \leqslant -3 && y_1 \\
& -3x_1 - 2x_2 - x_3 \leqslant -5 && y_2 \\
& x_1 + 3x_3 \leqslant 10 && y_3 \\
& x_1 + x_3 + x_3 \leqslant 9 && y_4 \\
& x_1, x_2, x_3 \geqslant 0
\end{aligned}
$$

Dual

$$
\text{Minimize} \quad -3y_1 - 5y_2 + 10y_3 + 9y_4
$$

Example

Primal

$$
\begin{array}{llr}
\text{Maximize} & x_1 - 2x_2 + x_3 & \\
\text{s.t.} & -x_1 - 2x_2 - 2x_3 \leq -3 & y_1 \\
 & -3x_1 - 2x_2 - x_3 \leq -5 & y_2 \\
 & x_1 + 3x_3 \leq 10 & y_3 \\
 & x_1 + x_3 + x_3 \leq 9 & y_4 \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{Minimize} & -3y_1 - 5y_2 + 10y_3 + 9y_4 \\
\text{s.t.} & -y_1 - 3y_2 + y_3 + y_4 \geq 1
\end{array}
$$

Example

Primal

$$
\begin{array}{llll}
\text{Maximize} & x_1 - 2x_2 + x_3 & & \\
\text{s.t.} & -x_1 - 2x_2 - 2x_3 \leq -3 & & y_1 \\
 & -3x_1 - 2x_2 - x_3 \leq -5 & & y_2 \\
 & x_1 + 3x_3 \leq 10 & & y_3 \\
 & x_1 + x_3 + x_3 \leq 9 & & y_4 \\
 & x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{Minimize} & -3y_1 - 5y_2 + 10y_3 + 9y_4 \\
\text{s.t.} & -y_1 - 3y_2 + y_3 + y_4 \geq 1 \\
 & -2y_1 - 2y_2 + y_4 \geq -2
\end{array}
$$

Example

Primal

$$
\begin{array}{lrl}
\text{Maximize} & x_1 - 2x_2 + x_3 & \\
\text{s.t.} & -x_1 - 2x_2 - 2x_3 \leq -3 & y_1 \\
 & -3x_1 - 2x_2 - x_3 \leq -5 & y_2 \\
 & x_1 \qquad\quad + 3x_3 \leq 10 & y_3 \\
 & x_1 + x_3 + x_3 \leq 9 & y_4 \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{Minimize} & -3y_1 - 5y_2 + 10y_3 + 9y_4 \\
\text{s.t.} & -y_1 - 3y_2 + y_3 + y_4 \geq 1 \\
 & -2y_1 - 2y_2 \qquad + y_4 \geq -2 \\
 & -2y_1 - y_2 + 3y_3 + y_4 \geq 1
\end{array}
$$

Example

Primal

$$
\begin{array}{llr}
\text{Maximize} & x_1 - 2x_2 + x_3 & \\
\text{s.t.} & -x_1 - 2x_2 - 2x_3 \leq -3 & y_1 \\
 & -3x_1 - 2x_2 - x_3 \leq -5 & y_2 \\
 & x_1 + 3x_3 \leq 10 & y_3 \\
 & x_1 + x_3 + x_3 \leq 9 & y_4 \\
 & x_1, x_2, x_3 \geq 0 &
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$$

Dual

$$
\begin{array}{ll}
\text{Minimize} & -3y_1 - 5y_2 + 10y_3 + 9y_4 \\
\text{s.t.} & -y_1 - 3y_2 + y_3 + y_4 \geq 1 \\
 & -2y_1 - 2y_2 + y_4 \geq -2 \\
 & -2y_1 - y_2 + 3y_3 + y_4 \geq 1 \\
 & y_1, y_2, y_3, y_4 \geq 0
\end{array}
$$

Example

Primal

$$
\begin{array}{llr}
\text{Maximize} & x_1 - 2x_2 + x_3 & \\
\text{s.t.} & -x_1 - 2x_2 - 2x_3 \leq -3 & y_1 \\
 & -3x_1 - 2x_2 - x_3 \leq -5 & y_2 \\
 & x_1 + 3x_3 \leq 10 & y_3 \\
 & x_1 + x_3 + x_3 \leq 9 & y_4 \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Dual

$$
\begin{array}{llr}
\text{Minimize} & -3y_1 - 5y_2 + 10y_3 + 9y_4 & \\
\text{s.t.} & -y_1 - 3y_2 + y_3 + y_4 \geq 1 & x_1 \\
 & -2y_1 - 2y_2 + y_4 \geq -2 & x_2 \\
 & -2y_1 - y_2 + 3y_3 + y_4 \geq 1 & x_3 \\
 & y_1, y_2, y_3, y_4 \geq 0 &
\end{array}
$$

Weak Duality

Lemma: If $\hat{x}$ is a feasible primal solution and $\hat{y}$ is a feasible dual solution, then $c\hat{x} \leq b^T\hat{y}$.

Weak Duality

Lemma: If $\hat{x}$ is a feasible primal solution and $\hat{y}$ is a feasible dual solution, then $c\hat{x} \leq b^T\hat{y}$.

Proof:

$$c\hat{x} \leq (A^T\hat{y})^T\hat{x} = (\hat{y}^TA)\hat{x} = \hat{y}^T(A\hat{x}) \leq \hat{y}^Tb = b^T\hat{y}$$

$\square$

Weak Duality

Lemma: If $\hat{x}$ is a feasible primal solution and $\hat{y}$ is a feasible dual solution, then $c\hat{x} \leq b^T\hat{y}$.

Proof:

$$c\hat{x} \leq (A^T\hat{y})^T\hat{x} = (\hat{y}^T A)\hat{x} = \hat{y}^T(A\hat{x}) \leq \hat{y}^T b = b^T\hat{y}$$

$\square$

Corollary: If $c\hat{x} = b^T\hat{y}$, then both $\hat{x}$ and $\hat{y}$ are optimal solutions.

Strong LP Duality

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Reminder: From Canonical to Standard Form

$$\text{Maximize} \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall\, 1 \leq i \leq m$$

$$x_j \geq 0 \quad \forall\, 1 \leq j \leq n$$

$$\text{Maximize} \sum_{j=1}^{m+n} c'_j z_j$$

$$\text{s.t.} \quad z_i = b_i - \sum_{j=1}^{n} a_{ij} z_{j+m} \quad \forall\, 1 \leq i \leq m$$

$$z_j \geq 0 \quad \forall\, 1 \leq j \leq m+n$$

- Slack variables $z_1, \dots, z_m$
- $z_{m+1}, \dots, z_{m+n} = x_1, \dots, x_n$

- $c'_1, \dots, c'_m = 0$
- $c'_{m+1}, \dots, c'_{m+n} = c_1, \dots, c_n$

Correctness of Simplex Algorithm

$$\sum_{j=1}^{m+n} c_j'' z_j + d'' = \text{objective function in final iteration of Simplex Algorithm}$$

$$\hat{z} = \text{BFS of final LP} = \text{computed solution}$$

Correctness of Simplex Algorithm

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$\hat{z}$ = BFS of final LP = computed solution

Let $\hat{x}_j = \hat{z}_{j+m} \quad \forall\, 1 \le j \le n$

Correctness of Simplex Algorithm

$$\sum_{j=1}^{m+n} c_j'' z_j + d'' = \text{objective function in final iteration of Simplex Algorithm}$$

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Let $\hat{x}_j = \hat{z}_{j+m} \quad \forall\, 1 \le j \le n$

$\hat{y}_i = -c_i'' \quad \forall\, 1 \le i \le m$

Correctness of Simplex Algorithm

$$\sum_{j=1}^{m+n} c_j'' z_j + d'' \;=\; \text{objective function in final iteration of Simplex Algorithm}$$

$\hat{z}$ = BFS of final LP = computed solution

Let $\quad \hat{x}_j = \hat{z}_{j+m} \quad \forall\, 1 \le j \le n$

$\qquad\;\; \hat{y}_i = -c_i'' \quad \forall\, 1 \le i \le m$

Claims:

- $\hat{x}$ is a feasible primal solution with objective function value d''.
- $\hat{y}$ is a feasible dual solution with objective function value d''.

Feasibility and Objective Function Value of $\hat{x}$

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- $\hat{x}$ is feasible because $\hat{z}$ is.

Feasibility and Objective Function Value of $\hat{x}$

Claim: $\hat{x}$ is a feasible primal solution with objective function value d''.

- $\hat{x}$ is feasible because $\hat{z}$ is.

- $$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{j=1}^{m+n} c_j' \hat{z}_j = \sum_{j=1}^{m+n} c_j'' \hat{z}_j + d'' = d''.$$

Objective Function Value of $\hat{y}$

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$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{j=1}^{m+n} c_j' \hat{z}_j$$

Objective Function Value of $\hat{y}$

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{j=1}^{m+n} c_j' \hat{z}_j$$

$$= \sum_{j=1}^{m+n} c_j'' \hat{z}_j + d''$$

Objective Function Value of $\hat{y}$

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{j=1}^{m+n} c_j' \hat{z}_j$$

$$= \sum_{j=1}^{m+n} c_j'' \hat{z}_j + d''$$

$$= d'' + \sum_{i=1}^{m} c_i'' \hat{z}_i + \sum_{j=1}^{n} c_{j+m}'' \hat{z}_{j+m}$$

Objective Function Value of $\hat{y}$

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{j=1}^{m+n} c_j' \hat{z}_j$$

$$= \sum_{j=1}^{m+n} c_j'' \hat{z}_j + d''$$

$$= d'' + \sum_{i=1}^{m} c_i'' \hat{z}_i + \sum_{j=1}^{n} c_{j+m}'' \hat{z}_{j+m}$$

$$= d'' + \sum_{i=1}^{m} (-\hat{y}_i) \left(b_i - \sum_{j=1}^{n} a_{i,j+m}' \hat{z}_{j+m} \right) + \sum_{j=1}^{n} c_{j+m}'' \hat{z}_{j+m}$$

$$\sum_{j=1}^{n} c_j \hat{x}_j = d'' + \sum_{i=1}^{m} (-\hat{y}_i)\left(b_i - \sum_{j=1}^{n} a'_{i,j+m} \hat{z}_{j+m}\right) + \sum_{j=1}^{n} c''_{j+m} \hat{z}_{j+m}$$

$$\sum_{j=1}^{n} c_j \hat{x}_j = d'' + \sum_{i=1}^{m} (-\hat{y}_i)\left(b_i - \sum_{j=1}^{n} a'_{i,j+m}\,\hat{z}_{j+m}\right) + \sum_{j=1}^{n} c''_{j+m}\,\hat{z}_{j+m}$$

$$= d'' + \sum_{i=1}^{m} (-\hat{y}_i)\left(b_i - \sum_{j=1}^{n} a_{ij}\,\hat{x}_j\right) + \sum_{j=1}^{n} c'_{j+m}\,\hat{x}_j$$

$$\sum_{j=1}^{n} c_j \hat{x}_j = d'' + \sum_{i=1}^{m} (-\hat{y}_i)\left(b_i - \sum_{j=1}^{n} a'_{i,j+m}\, \hat{z}_{j+m}\right) + \sum_{j=1}^{n} c''_{j+m}\, \hat{z}_{j+m}$$

$$= d'' + \sum_{i=1}^{m} (-\hat{y}_i)\left(b_i - \sum_{j=1}^{n} a_{ij}\, \hat{x}_j\right) + \sum_{j=1}^{n} c''_{j+m}\, \hat{x}_j$$

$$= \left(d'' - \sum_{i=1}^{m} b_i\, \hat{y}_i\right) + \sum_{j=1}^{n}\left(c''_{j+m} + \sum_{i=1}^{m} a_{ij}\, \hat{y}_i\right)\hat{x}_j$$

Let $1 \le k \le n$ and $\tilde{\hat{z}}_j = \begin{cases} \hat{z}_j - a_{jk}\delta & \text{if } 1 \le j \le m \\ \hat{z}_j + \delta & \text{if } j = k+m \\ \hat{z}_j & \text{otherwise} \end{cases}$

Let $1 \leq k \leq n$ and $\tilde{z}_j = \begin{cases} \hat{z}_j - a_{jk}\delta & \text{if } 1 \leq j \leq m \\ \hat{z}_j + \delta & \text{if } j = k+m \\ \hat{z}_j & \text{otherwise} \end{cases}$

$\tilde{z}$ is a solution of standard form primal (not necessarily feasible).

Let $1 \leq k \leq n$ and $\tilde{z}_j = \begin{cases} \hat{z}_j - a_{jk}\delta & \text{if } 1 \leq j \leq m \\ \hat{z}_j + \delta & \text{if } j = k+m \\ \hat{z}_j & \text{otherwise} \end{cases}$

$\tilde{z}$ is a solution of standard form primal (not necessarily feasible).

$$\Rightarrow \sum_{j=1}^{m+n} c_j' \tilde{z}_j = \sum_{j=1}^{m+n} c_j'' \tilde{z}_j + d''$$

Let $1 \le k \le n$ and $\tilde{z}_j = \begin{cases} \hat{z}_j - a_{jk}\delta & \text{if } 1 \le j \le m \\ \hat{z}_j + \delta & \text{if } j = k+m \\ \hat{z}_j & \text{otherwise} \end{cases}$

$\tilde{z}$ is a solution of standard form primal (not necessarily feasible).

$$\Rightarrow \sum_{j=1}^{m+n} c_j' \tilde{z}_j = \sum_{j=1}^{m+n} c_j'' \tilde{z}_j + d''$$

$\tilde{x}$ = corresponding solution of canonical form primal:

$$\tilde{x}_j = \tilde{z}_{j+m} = \begin{cases} \hat{x}_j + \delta & \text{if } j = k \\ \hat{x}_j & \text{if } j \neq k \end{cases}$$

$$\sum_{j=1}^{n} c_j \tilde{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i \right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i \right) \tilde{x}_j$$

$$\sum_{j=1}^{n} c_j \tilde{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i \right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i \right) \tilde{x}_j$$

$$\sum_{j=1}^{n} c_j \hat{x}_j + c_k \delta = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i \right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i \right) \hat{x}_j$$

$$+ \left(c''_{k+m} + \sum_{i=1}^{m} a_{ik} \hat{y}_i \right) \delta$$

$$\sum_{j=1}^{n} c_j \tilde{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i\right) + \sum_{j=1}^{n}\left(c''_{j+m} + \sum_{i=1}^{m} a_{ij}\hat{y}_i\right)\tilde{x}_j$$

$$\sum_{j=1}^{n} c_j \hat{x}_j + c_k\delta = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i\right) + \sum_{j=1}^{n}\left(c''_{j+m} + \sum_{i=1}^{m} a_{ij}\hat{y}_i\right)\hat{x}_j$$

$$+\left(c''_{k+m} + \sum_{i=1}^{m} a_{ik}\hat{y}_i\right)\delta$$

$$\Rightarrow \quad c_k\delta = \left(c''_{k+m} + \sum_{i=1}^{m} a_{ik}\hat{y}_i\right)\delta$$

$$\sum_{j=1}^{n} c_j \tilde{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i\right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i\right) \tilde{x}_j$$

$$\sum_{j=1}^{n} c_j \hat{x}_j + c_k \delta = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i\right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i\right) \hat{x}_j$$

$$+ \left(c''_{k+m} + \sum_{i=1}^{m} a_{ik} \hat{y}_i\right) \delta$$

$$\Rightarrow \quad c_k \delta = \left(c''_{k+m} + \sum_{i=1}^{m} a_{ik} \hat{y}_i\right) \delta \quad \Rightarrow \quad c_k = c''_{k+m} + \sum_{i=1}^{m} a_{ik} \hat{y}_i$$

$$c_k = c''_{k+m} + \sum_{i=1}^{m} a_{ik} \hat{y}_i \qquad \forall\, 1 \le k \le m$$

$$\sum_{j=1}^{n} c_j \hat{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i \right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i \right) \hat{x}_j$$

$$c_k = c''_{k+m} + \sum_{i=1}^{m} a_{ik} \hat{y}_i \qquad \forall\ 1 \le k \le m$$

$$\sum_{j=1}^{n} c_j \hat{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i \right) + \sum_{j=1}^{n} \left(c''_{j+m} + \sum_{i=1}^{m} a_{ij} \hat{y}_i \right) \hat{x}_j$$

$$= \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i \right) + \sum_{j=1}^{n} c_j \hat{x}_j$$

$$c_k = c''_{k+m} + \sum_{i=1}^{m} a_{ik}\,\hat{y}_i \qquad \forall\ 1 \le k \le m$$

$$\sum_{j=1}^{n} c_j\,\hat{x}_j = \left(d'' - \sum_{i=1}^{m} b_i\,\hat{y}_i\right) + \sum_{j=1}^{n}\left(c''_{j+m} + \sum_{i=1}^{m} a_{ij}\,\hat{y}_i\right)\hat{x}_j$$

$$= \left(d'' - \sum_{i=1}^{m} b_i\,\hat{y}_i\right) + \sum_{j=1}^{n} c_j\,\hat{x}_j$$

$$0 = d'' - \sum_{i=1}^{m} b_i\,\hat{y}_i$$

$$c_k = c''_{k+m} + \sum_{i=1}^{m} a_{ik}\hat{y}_i \qquad \forall\ 1 \le k \le m$$

$$\sum_{j=1}^{n} c_j \hat{x}_j = \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i\right) + \sum_{j=1}^{n}\left(c''_{j+m} + \sum_{i=1}^{m} a_{ij}\hat{y}_i\right)\hat{x}_j$$

$$= \left(d'' - \sum_{i=1}^{m} b_i \hat{y}_i\right) + \sum_{j=1}^{n} c_j \hat{x}_j$$

$$0 = d'' - \sum_{i=1}^{m} b_i \hat{y}_i \implies \sum_{i=1}^{m} b_i \hat{y}_i = d''$$

Claim: $\hat{y}$ has objective function value d^*.

Feasibility of $\hat{y}$

Claim: $\hat{y}$ is a feasible dual solution.

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- $\hat{y}_i = -c_i'' \geq 0 \quad \forall\ 1 \leq i \leq m$

Feasibility of $\hat{y}$

Claim: $\hat{y}$ is a feasible dual solution.

- $\hat{y}_i = -c_i'' \geq 0 \quad \forall\ 1 \leq i \leq m$

- $c_k = c_{k+m}'' + \sum_{i=1}^{m} a_{ik} \hat{y}_i \implies c_k \leq \sum_{i=1}^{m} a_{ik} \hat{y}_i \quad \forall\ 1 \leq k \leq n.$

Strong Duality

Theorem: Let $\hat{x}$ be a feasible primal solution and let $\hat{y}$ be a feasible dual solution. They are both optimal if and only if

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i .$$