An Augmenting Paths Algorithm That Terminates

CSCI 4113/6101

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- Uses BFS to find a shortest augmenting path in each iteration.

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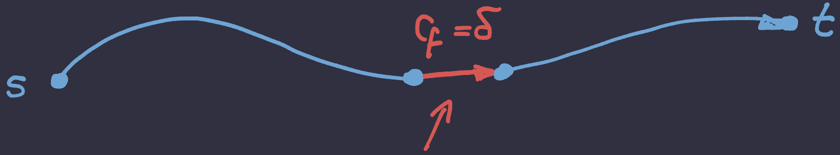
Corollary: Edmonds-Karp finds a maximum st-flow in $O(nm^2)$ time.

Critical Edges

Augmenting path in G_f



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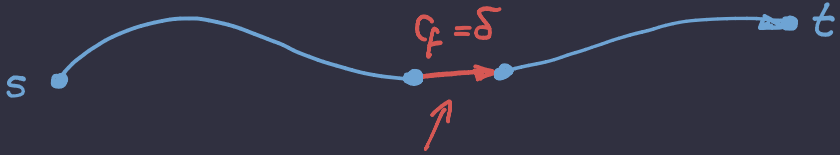

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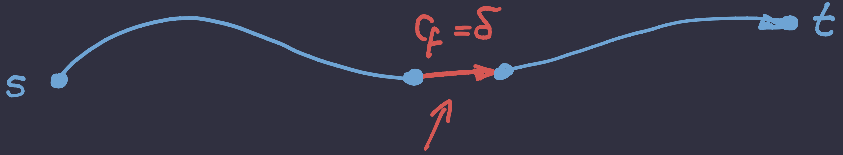

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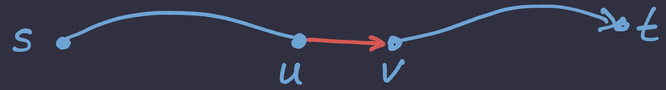
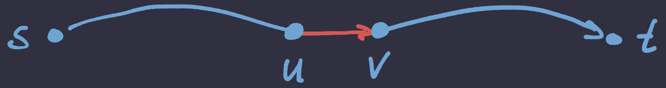
G_f



$G_{f''}$



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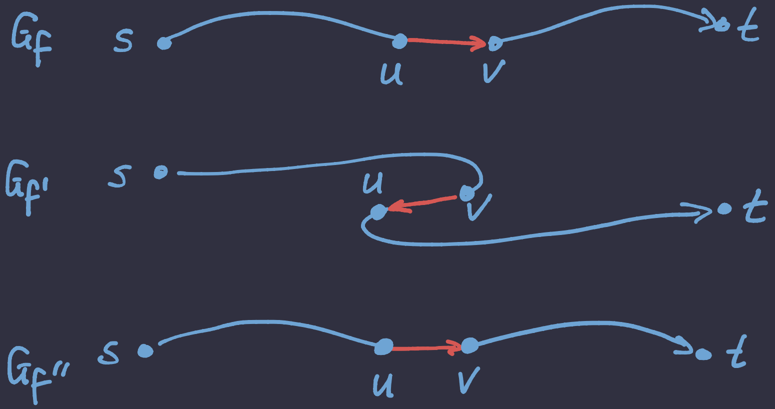

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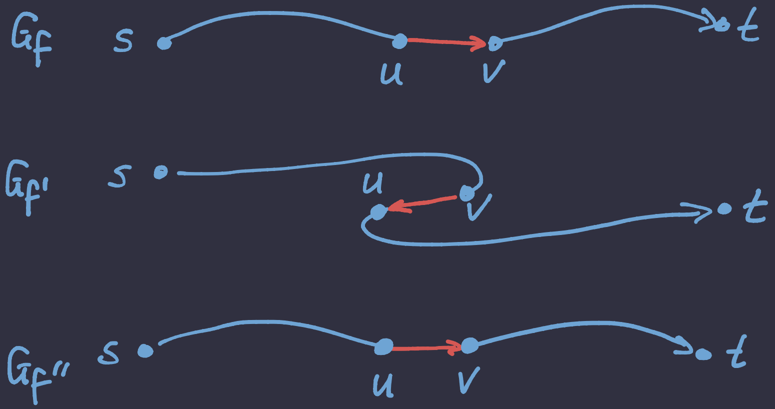

$$d_{G_{f'}}(s,u) = d_{G_{f'}}(s,v) + \underline{1}$$
$$\geq d_{G_f}(s,v) + 1$$

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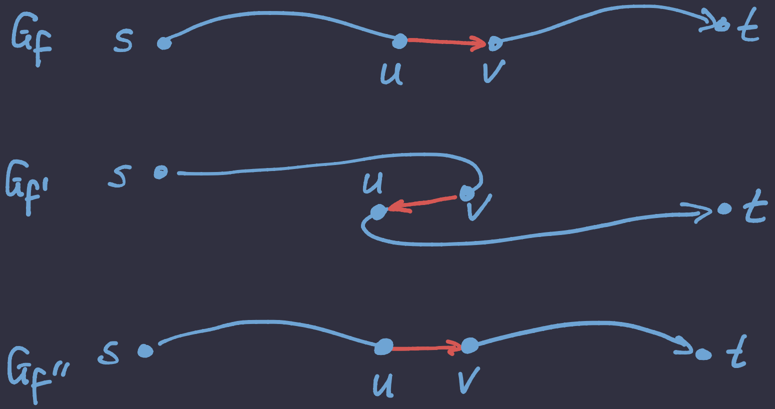

$$
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d_{G_{f'}}(s,u) &= d_{G_{f'}}(s,v) + 1 \\
&\geq d_{G_f}(s,v) + 1
\end{aligned}
$$

$$
\begin{aligned}
d_{G_{f''}}(s,v) &= d_{G_{f''}}(s,u) + 1 \\
&\geq d_{G_{f'}}(s,u) + 1 \\
&\geq d_{G_f}(s,v) + 2
\end{aligned}
$$

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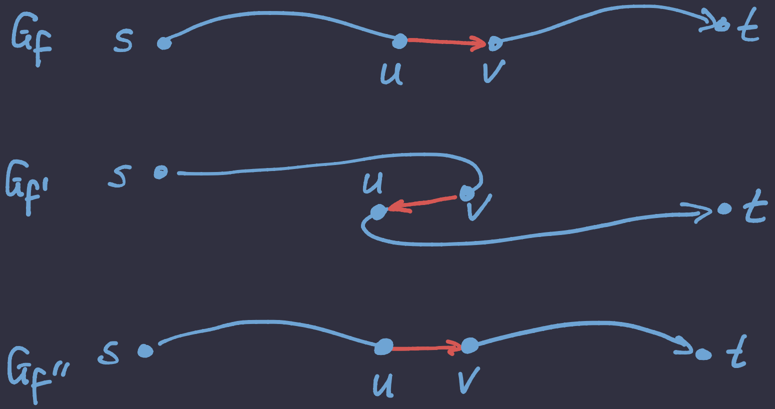

$$d_{G_{f'}}(s,u) = d_{G_{f'}}(s,v)+1 \geq d_{G_f}(s,v)+1$$

$$d_{G_{f''}}(s,v) = d_{G_{f''}}(s,u)+1 \geq d_{G_{f'}}(s,u)+1 \geq d_{G_f}(s,v)+2$$

$\Rightarrow (u,v)$ critical for $\leq \frac{n}{2}$ paths

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Assume $\mathrm{dist}_{G_{f'}}(s,v) < \mathrm{dist}_{G_f}(s,v)$.

Choose v with minimum $\mathrm{dist}_{G_{f'}}(s,v)$.

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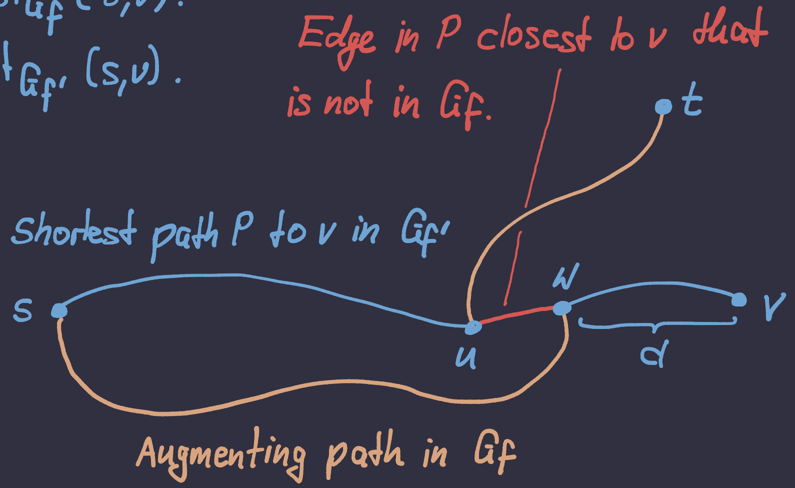
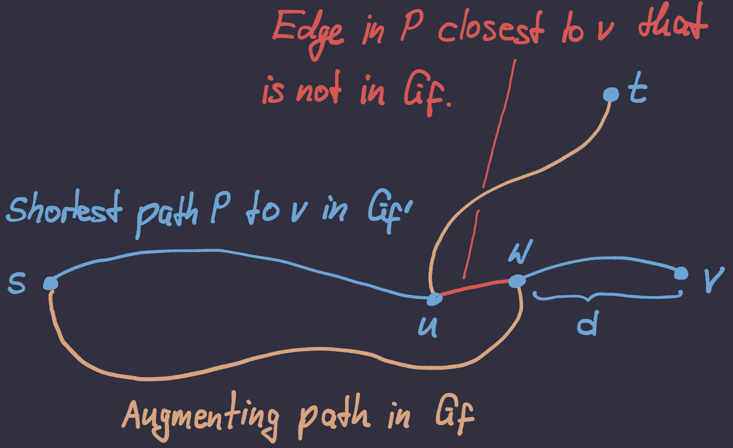
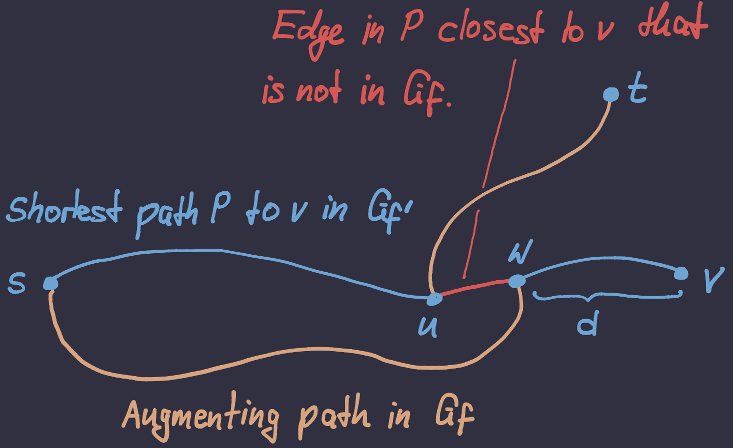




$$\begin{aligned} \text{dist}_{G_{f'}}(s,v) &= \text{dist}_{G_{f'}}(s,u) + d + 1 \\ &\geq \text{dist}_{G_f}(s,u) + d + 1 \end{aligned}$$



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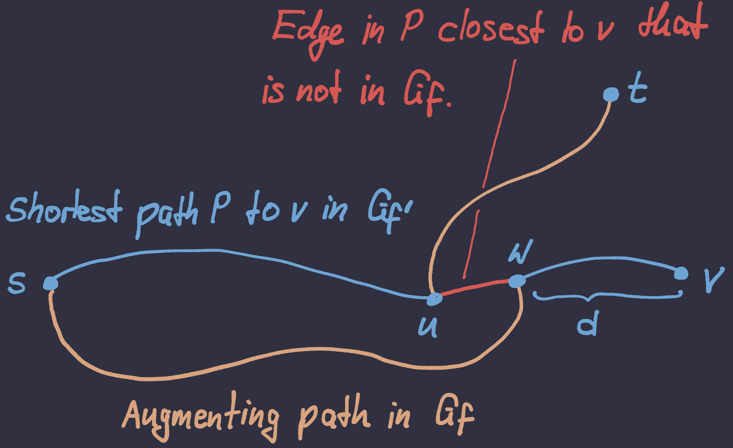
$$\begin{aligned}\mathrm{dist}_{G_f}(s,v) &\leq \mathrm{dist}_{G_f}(s,w) + d\\ &\leq \mathrm{dist}_{G_f}(s,u) + d - 1\end{aligned}$$



$$\mathrm{dist}_{G_{f'}}(s,v) = \mathrm{dist}_{G_{f'}}(s,u) + d + 1 \geq \mathrm{dist}_{G_f}(s,u) + d + 1$$

$$\mathrm{dist}_{G_f}(s,v) \leq \mathrm{dist}_{G_f}(s,w) + d \leq \mathrm{dist}_{G_f}(s,u) + d - 1$$

$$\Rightarrow \mathrm{dist}_{G_f}(s,v) < \mathrm{dist}_{G_{f'}}(s,v)$$