Complementary Slackness

CSCI 4113/6101

LP Duality

Primal LP

Maximize cx

s.t. $Ax \leq b$

$x \geq 0$

Dual LP

Minimize $b^T y$

s.t. $A^T y \geq c^T$

$y \geq 0$

Each variable x_j corresponds to jth constraint in dual.

Each variable y_i corresponds to ith constraint in primal.

Complementary Slackness

Tool to determine optimality of primal and dual solutions without calculating objective function values.

Central to primal-dual schema.

Relaxed version gives approximation algorithms.

Complementary Slackness

$\hat{x}$ = feasible primal solution $\qquad$ $\hat{y}$ = feasible dual solution

Theorem: $\hat{x}$ is an optimal primal solution and $\hat{y}$ is an optimal dual solution if and only if

Primal complementary slackness:

$\forall\, 1 \le j \le n$: either $\hat{x}_j = 0$ or $\sum_{i=1}^{m} a_{ij}\hat{y}_i = c_j$.

Dual complementary slackness:

$\forall\, 1 \le i \le m$: either $\hat{y}_i = 0$ or $\sum_{j=1}^{n} a_{ij}\hat{x}_j = b_i$.

$\hat{x}, \hat{y}$ are feasible $\Rightarrow$

$$A\hat{x} \leq b \text{ and } A^T\hat{y} \geq c^T$$

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$A\hat{x} \leq b$ and $A^T\hat{y} \geq c^T$ $(\hat{y}^T A \geq c)$

$\Rightarrow \quad c\hat{x} \leq \hat{y}^T A \hat{x} \leq \hat{y}^T b = b^T \hat{y}.$

$\hat{x}, \hat{y}$ optimal $\Leftrightarrow$ $c\hat{x} = b^T\hat{y}$

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$\hat{x}, \hat{y}$ optimal $\Leftrightarrow c\hat{x} = b^T\hat{y}$

$$\Leftrightarrow \quad c\hat{x} = \hat{y}^T A\hat{x} \text{ and } \hat{y}^T A\hat{x} = \hat{y}^T b$$

$$\Leftrightarrow \quad (\hat{y}^T A - c)\hat{x} = 0 \text{ and } \hat{y}^T (b - A\hat{x}) = 0.$$

$\hat{x}$ feasible $\Rightarrow$ $\hat{x} \geq 0$

$\hat{x}$ feasible $\Rightarrow$ $\hat{x} \geqslant 0$ $\qquad$ $\hat{y}$ feasible $\Rightarrow$ $\hat{y}^T A - c \geqslant 0$

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$$\Leftrightarrow \forall\, 1 \leq j \leq n: \ \hat{x}_j = 0 \ \text{or} \ \sum_{i=1}^{m} a_{ij} \hat{y}_i = c_j .$$

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$\hat{y}$ feasible $\Rightarrow$ $\hat{y} \geqslant 0$ $\qquad$ $\hat{x}$ feasible $\Rightarrow$ $b - A\hat{x} \geqslant 0$.

$\hat{x}$ feasible $\Rightarrow \hat{x} \geq 0$ $\quad\quad$ $\hat{y}$ feasible $\Rightarrow \hat{y}^T A - c \geq 0$

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$$\hat{y}^T(b - A\hat{x}) = 0$$

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$\Leftrightarrow \forall\, 1 \leq j \leq n$: $\hat{x}_j = 0$ or $\sum_{i=1}^{m} a_{ij} \hat{y}_i = c_j$.

$\hat{y}$ feasible $\Rightarrow$ $\hat{y} \geq 0$ $\hat{x}$ feasible $\Rightarrow$ $b - A\hat{x} \geq 0$.

$\hat{y}^T (b - A\hat{x}) = 0$

$\Leftrightarrow \forall\, 1 \leq i \leq m$: $\hat{y}_i = 0$ or $\sum_{j=1}^{n} a_{ij} \hat{x}_j = b_i$.