The Edmonds-Karp Algorithm

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As mentioned in the previous topic, the Edmonds-Karp algorithm is a variant of the Ford-Fulkerson algorithm that chooses the shortest augmenting path in G^f in each iteration as the path along which to move more flow from s to t. In this topic, we prove that this guarantees that this algorithm terminates and runs in $O\left(nm^2\right)$ time. Since we already proved in the previous topic that, if the Ford-Fulkerson algorithm terminates, it computes a maximum flow, this immediately implies that the Edmonds-Karp algorithm computes a maximum flow in $O\left(nm^2\right)$ time. Further improvements are possible. Dinitz's algorithm is another fairly elementary variant of the Ford-Fulkerson algorithm, with running time $O\left(n^2m\right)$. Further improvements are possible by using efficient data structures to facilitate the search for augmenting paths. We will not discuss these further improvements in this course.

When we say "shortest augmenting path", we mean a path from s to t in G^f with the minimum number of edges. Clearly, we can find such a path, or decide that there exists no path from s to t in G^f, in $O(n+m)$ time by using breadth-first search (BFS). An example of running the Edmonds-Karp algorithm is shown in Fig. 1.

The following lemma shows that, as the algorithm progresses the distances from s to all vertices in G^f do not decrease. As we will see, this is the crucial property that ensures termination of the Edmonds-Karp algorithm.

Lemma 1. *Let f be some st-flow in G, let P be a shortest path from s to t in G^f, and let f' be the st-flow obtained by applying the* Augment *procedure of the Ford-Fulkerson algorithm to f and P. Then $\text{dist}_{G^{f'}}(s,x) \ge \text{dist}_{G^f}(s,x)$, for all $x \in V$.*

Proof. Assume for the sake of contradiction that there exists a vertex x such that $\text{dist}_{G^{f'}}(s,x) < \text{dist}_{G^f}(s,x)$ and choose such a vertex x with minimum distance $\text{dist}_{G^{f'}}(s,x)$. Since distances are integral, $\text{dist}_{G^{f'}}(s,x) < \text{dist}_{G^f}(s,x)$ implies that $\text{dist}_{G^{f'}}(s,x) \le \text{dist}_{G^f}(s,x) - 1$. Let $\Pi_{G^{f'}}(s,x)$ be a shortest path from s to x in $G^{f'}$. Since $\text{dist}_{G^{f'}}(s,s) = 0 = \text{dist}_{G^f}(s,s)$, $x \ne s$, that is, x has a predecessor y in $\Pi_{G^{f'}}(s,x)$. This predecessor satisfies $\text{dist}_{G^{f'}}(s,y) = \text{dist}_{G^{f'}}(s,x) - 1$. Thus, by the choice of x, $\text{dist}_{G^f}(s,y) \le \text{dist}_{G^{f'}}(s,y) = \text{dist}_{G^{f'}}(s,x) - 1 \le \text{dist}_{G^f}(s,x) - 2$. Since every edge $(w,z) \in P$ satisfies $\text{dist}_{G^f}(s,z) = \text{dist}_{G^f}(s,w) + 1$, this implies that $(x,y) \notin P$ and $(y,x) \notin P$. In particular, $f'_{x,y} = f_{x,y}$ and $f'_{y,x} = f_{y,x}$. Thus, since $(y,x) \in G^{f'}$, (y,x) is also an edge in G^f, a contradiction because this would imply that $\text{dist}_{G^f}(s,x) \le \text{dist}_{G^f}(s,y) + 1$, but we observed above that $\text{dist}_{G^f}(s,x) \ge \text{dist}_{G^f}(s,y) + 2$. $\square$

Lem. 1 is all we need to prove that the Edmonds-Karp algorithm terminates quickly:

Theorem 2. *The Edmonds-Karp algorithm computes a maximum flow in $O\left(nm^2\right)$ time.*

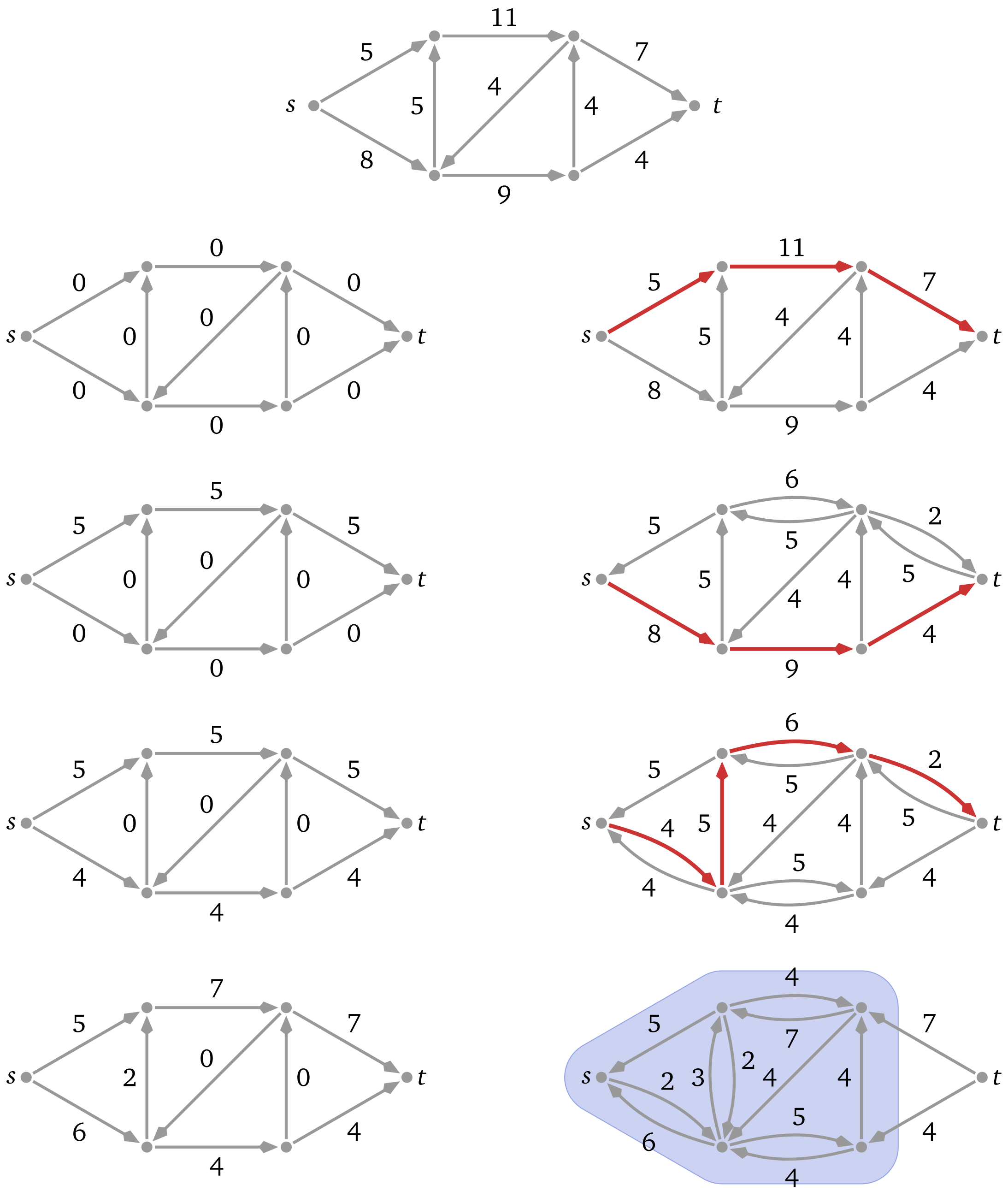

Figure 1: Illustration of an application of the Edmonds-Karp algorithm. The input network is shown at the top, with its edges labelled with their capacities. Each of the following rows show the current flow on the left, and the residual network on the right. Labels in the left figures are the flow values along the edges of the graph. Labels in the right figures are residual capacities. The augmenting path used to augment the flow in each iteration is shown in red. In the last row, there is no augmenting path left, that is, the current flow shown on the left is a maximum flow. This is witnessed by the st-cut consisting of all vertices reachable from s in the residual network, shaded light blue.

Proof. As already observed, the correctness of the Edmonds-Karp algorithm follows because it is a variant of the Ford-Fulkerson algorithm. Next, we analyze its running time.

Constructing G^f, running BFS in G^f, and applying the AUGMENT procedure to the path P found by the BFS take $O(m)$ time. Thus, it suffices to show that the algorithm terminates after at most nm iterations.

Let $f^{(0)}, \ldots, f^{(r)}$ be the sequence of flows constructed by the algorithm. For $0 \le i < r$, let $P^{(i)}$ be the augmenting path used to produce $f^{(i+1)}$ from $f^{(i)}$ and let $\delta^{(i)} = F_s^{(i+1)} - F_s^{(i)}$ be the amount of flow sent along this path. An edge $(x, y) \in G^{f^{(i)}}$ is **critical** for the augmenting path $P^{(i)}$ if $(x, y) \in P^{(i)}$ and $u_{x,y}^{f^{(i)}} = \delta^{(i)}$. In other words, the edge (x, y) is one of the edges that limit the amount of flow sent along $P^{(i)}$ to be no more than $\delta^{(i)}$. Let $E' = \{(x, y), (y, x) \mid (x, y) \in E\}$ be the set of edges in G and their reversals. We have $|E'| \le 2m$ and every residual network $G^{f^{(i)}} = \left(V, E^{f^{(i)}}\right)$ satisfies $E^{f^{(i)}} \subseteq E'$. Thus, it suffices to show that every edge in E' is critical for at most $\frac{n}{2}$ of the paths $P^{(0)}, \ldots, P^{(r-1)}$. Since every augmenting path $P^{(i)}$ has at least one critical edge, this implies that $r \le nm$, that is, the algorithm has at most nm iterations.

Consider an edge $(x, y) \in E'$ and two indices $i < j$ such that (x, y) is critical for $P^{(i)}$ and $P^{(j)}$ but not for any path $P^{(h)}$ with $i < h < j$. Then

$$\delta^{(i)} = u_{x,y}^{f^{(i)}} = u_{x,y} - f_{x,y}^{(i)} + f_{y,x}^{(i)} \ge f_{y,x}^{(i)}.$$

Thus,

$$f_{y,x}^{(i+1)} = f_{y,x}^{(i)} - \min\left(\delta^{(i)}, f_{y,x}^{(i)}\right) = 0$$

and

$$\begin{aligned} f_{x,y}^{(i+1)} &= f_{x,y}^{(i)} + \max\left(0, \delta^{(i)} - f_{y,x}^{(i)}\right) \\ &= f_{x,y}^{(i)} + u_{x,y} - f_{x,y}^{(i)} + f_{y,x}^{(i)} - f_{y,x}^{(i)} \\ &= u_{x,y}. \end{aligned}$$

Therefore, $(x, y) \notin G^{f^{(i+1)}}$. Since $(x, y) \in P^{(j)}$, it does belong to $G^{f^{(j)}}$. Thus, there exists a minimum index $k > i + 1$ such that $(x, y) \in G^{f^{(k)}}$. Since $(x, y) \in G^{f^{(k)}}$ and $(x, y) \notin G^{f^{(k-1)}}$, the augmentation of $f^{(k-1)}$ to produce $f^{(k)}$ must move some flow from y to x, that is, $(y, x) \in P^{(k-1)}$.

By Lem. 1, $\mathrm{dist}_{G^{f^{(k-1)}}}(s, y) \ge \mathrm{dist}_{G^{f^{(i)}}}(s, y)$. Since $(y, x) \in P^{(k-1)}$ and $P^{(k-1)}$ is a shortest path from s to t in $G^{f^{(k-1)}}$, $\mathrm{dist}_{G^{f^{(k-1)}}}(s, x) = \mathrm{dist}_{G^{f^{(k-1)}}}(s, y) + 1$. By Lem. 1 again, $\mathrm{dist}_{G^{f^{(j)}}}(s, x) \ge \mathrm{dist}_{G^{f^{(k-1)}}}(s, x)$. Since $(x, y) \in P^{(j)}$ and $P^{(j)}$ is a shortest path from s to t in $G^{f^{(j)}}$, $\mathrm{dist}_{G^{f^{(j)}}}(s, y) = \mathrm{dist}_{G^{f^{(j)}}}(s, x) + 1$. The combination of these inequalities shows that $\mathrm{dist}_{G^{f^{(j)}}}(s, y) \ge \mathrm{dist}_{G^{f^{(i)}}}(s, y) + 2$. Since every vertex y reachable from s in $G^{f^{(h)}}$ satisfies $\mathrm{dist}_{G^{f^{(h)}}}(s, y) < n$, for all $0 \le h < r$, this implies that the edge (x, y) is critical for at most $\frac{n}{2}$ of the paths $P^{(0)}, \ldots, P^{(r-1)}$. $\square$