Complementary Slackness

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In this topic, we build on the concept of LP duality to derive an alternative optimality criterion for a pair of feasible solutions of a primal-dual pair of LPs, called **complementary slackness**. The complementary slackness conditions have the interesting property that they establish the optimality of two solutions without inspecting their objective function values. Complementary slackness is at the heart of the **primal-dual schema**, a beautiful technique for solving a range of optimtization problems. We will see it in action later in the course in algorithms for finding maximum flows, for finding minimum-weight perfect matchings, and for finding approximate solutions of NP-hard problems.

This chapter is organized as follows: Sec. 1 introduces complementary slackness. When we introduced LP relaxations, we presented an ILP formulation of the MST problem that we claimed had integrality gap 1, but we lacked the tool to prove this. Complementary slackness is this tool. § 2 will apply complementary slackness to prove that this ILP formulation of the MST problem does indeed have integrality gap 1. At the end of the previous topic, we discussed how to construct the dual of an LP not in canonical form. We finish the discussion of complementary slackness by exploring, in **??**, how complementary clackness applies to an LP not in canonical form, and its dual.

1 Complementary Slackness

Once again, consider a primal LP

$$\begin{aligned} \text{Maximize } & cx \\ \text{s.t. } & Ax \le b \\ & x \ge 0 \end{aligned} \tag{1}$$

and its dual

$$\begin{aligned} \text{Minimize } & b^T y \\ \text{s.t. } & A^T y \ge c^T \\ & y \ge 0. \end{aligned} \tag{2}$$

Definition 1 (Complementary Slackness)**.** Let $\hat{x}$ be a feasible solution of (1), and let $\hat{y}$ be a feasible solution of (2). These two solutions are said to satisfy **complementary slackness** if

Primal complementary slackness: For all $1 \le j \le n$,

$$\hat{x}_j = 0 \text{ or } \sum_{i=1}^{m} a_{ij}\hat{y}_i = c_j \text{ and}$$

Dual complementary slackness: For all $1 \le i \le m$,

$$\hat{y}_i = 0 \text{ or } \sum_{j=1}^{n} a_{ij}\hat{x}_j = b_i.$$

We call an inequality constraint $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ **tight** for a given feasible solution $\hat{x}$ if $\sum_{j=1}^{n} a_{ij}\hat{x}_j = b_i$—there is no slack, we cannot increase $\sum_{j=1}^{n} a_{ij}\hat{x}_j$ without violating this constraint. With this terminology, primal complementary slackness states that for every primal variable x_j, either its primal non-negativity constraint is tight or the dual constraint corresponding to x_j is tight. Dual complementary slackness states that for every dual variable y_i, either its dual non-negativity constraint is tight or the primal constraint corresponding to y_i is tight. This explains the name "complementary slackness": the non-negativity constraint of a (primal or dual) variable and the corresponding (dual or primal) constraint in the other LP cannot both have slack; their slackness is complementary.

Theorem 2. *Let $\hat{x}$ be a feasible solution of* (1)*, and let $\hat{y}$ be a feasible solution of* (2)*. Then $\hat{x}$ is an optimal solution of* (1) *and $\hat{y}$ is an optimal solution of* (2) *if and only if $\hat{x}$ and $\hat{y}$ satisfy complementary slackness.*

Proof. Since $\hat{x}$ is a feasible primal solution, we have $A\hat{x} \le b$. Since $\hat{y}$ is a feasible dual solution, we have $A^T\hat{y} \ge c^T$, that is, $\hat{y}^T A \ge c$. Therefore,

$$c\hat{x} \le \hat{y}^T A\hat{x} \le \hat{y}^T b = b^T\hat{y}.$$

By strong LP duality, $\hat{x}$ and $\hat{y}$ are optimal solutions of (1) and (2) if and only if $c\hat{x} = b^T\hat{y}$. Thus, $\hat{x}$ and $\hat{y}$ are optimal solutions if and only if

$$c\hat{x} = \hat{y}^T A\hat{x} \quad \text{and} \quad \hat{y}^T A\hat{x} = \hat{y}^T b,$$

which is equivalent to

$$\left(\hat{y}^T A - c\right)\hat{x} = 0 \quad \text{and} \quad \hat{y}^T(b - A\hat{x}) = 0.$$

Next observe that every entry $\hat{x}_j$ of $\hat{x}$ is non-negative because $\hat{x}$ is a feasible primal solution. Every entry of $\hat{y}^T A - c$ is non-negative because $\hat{y}^T A \ge c$. Thus, $\left(\hat{y}^T A - c^T\right)\hat{x}$ is the inner product of two vectors with non-negative coordinates and is zero if and only if for all $1 \le j \le n$, $\left(\sum_{i=1}^{m}\hat{y}_i a_{ij} - c_j\right)\hat{x}_j = 0$, that is, $\hat{x}_j = 0$ or $\sum_{i=1}^{m}\hat{y}_i a_{ij} = c_j$. In other words, $\left(\hat{y}^T A - c\right)\hat{x} = 0$ if and only if primal complementary slackness holds for $\hat{x}$ and $\hat{y}$.

An analogous argument shows that $\hat{y}^T(b - A\hat{x}) = 0$ if and only if for all $1 \le i \le m$, $\hat{y}_i = 0$ or $b_i - \sum_{j=1}^{n} a_{ij}\hat{x}_j = 0$, that is, if and only if dual complementary slackness holds for $\hat{x}$ and $\hat{y}$.

Since $\hat{x}$ and $\hat{y}$ are optimal if and only if both $\left(\hat{y}^T A - c\right)\hat{x} = 0$ and $\hat{y}^T(b - A\hat{x}) = 0$, we have that $\hat{x}$ and $\hat{y}$ are optimal if and only if both primal and dual complementary slackness hold for $\hat{x}$ and $\hat{y}$. ☐

2 An ILP Formulation of the MST Problem with Integrality Gap 1*

As an application of complementary slackness, let us return to the minimum spanning tree problem. Complementary slackness provides the tool to prove our claim that the LP relaxation of the following ILP formulation of the minimum spanning tree problem has an integral optimal solution:

$$\begin{aligned}
&\text{Minimize} \sum_{e\in E} w_e x_e \\
\text{s.t.} \quad &\sum_{e\in E} x_e \geq n-1 \\
&\sum_{\substack{(u,v)\in E\\ u,v\in S}} x_{u,v} \leq |S|-1 \quad \forall \emptyset \subset S \subseteq V \\
&x_e \in \{0,1\} \qquad \forall e \in E.
\end{aligned} \tag{3}$$

Proposition 3. *The LP relaxation of the ILP* (3) *has an integral optimal solution.*

Proof. This is in fact an alternative (and quite a bit less intuitive) correctness proof of Kruskal's algorithm. Recall how Kruskal's algorithm works: It initializes the MST to have no edges; $T = (V, \emptyset)$. Then it sorts the edges in G by increasing weight and inspects them in order. Every edge (u, v) whose endpoints belong to different connected components of T at the time this edge is inspected is added to T, thereby merging these two connected components.

We can observe how the connected components of T change over the course of the algorithm. Let $\langle e_1, \ldots, e_m\rangle$ be the sorted sequence of edges, let $E_i = \{e_1, \ldots, e_i\}$, let $G_i = (V, E_i)$, and let T_i be the state of T after the ith iteration of the algorithm, that is, after the algorithm has inspected the edges $e_1, \ldots, e_i$. Then the connected components of G_i and T_i have the same vertex sets. For every subset $S \subseteq V$ with $|S| \geq 2$, let $c(S)$ be the minimal index and let $d(S)$ be the maximal index such that $G_{c(S)}, \ldots, G_{d(S)-1}$ have connected components with vertex set S. In other words, $c(S)$ is the "creation time" of the first connected component with vertex set S, graphs $G_{c(S)+1}, \ldots, G_{d(S)-1}$ may add more edges but no vertices to this component, and $d(S)$ is the "destruction time" of the last connected component with vertex set S, by forming a larger connected component from S and some other component of $G_{d(S)-1}$. For $S = V$, $d(V) = \infty$ because once we have obtained a connected componenent of T with vertex set V (T is a spanning tree at this time), we never merge this component with another component to create an even larger component. If $|S| < 2$ or there exists no index $1 \leq i \leq m$ such that S is the vertex set of a connected component of G_i, then $c(S) = d(S) = \infty$. Intuitively, $e_{c(S)}$ is the last edge with both endpoints in S that is added to the MST T, and $e_{d(S)}$ is the first edge with exactly one endpoint in S that is added to T.

Now let us turn to the LP formulation. The primal LP is the LP relaxation of (3):

$$\begin{aligned}
&\text{Minimize} \sum_{e\in E} w_e x_e \\
\text{s.t.} \quad &\sum_{e\in E} x_e \geq n-1 \\
&\sum_{\substack{(u,v)\in E\\ u,v\in S}} x_{u,v} \leq |S|-1 \quad \forall \emptyset \subset S \subseteq V \\
&x_e \geq 0 \qquad \forall e \in E.
\end{aligned} \tag{4}$$

Technically, the constraint $x_e \in \{0,1\}$ from (3) becomes $0 \leq x_e \leq 1$ in the LP relaxation, but the $x_e \leq 1$

part is redundant and can be omitted: if $e = (u, v)$, then the constraint

$$\sum_{\substack{(u,v)\in E\\ u,v\in S}} x_{u,v} \le |S| - 1 \text{ for } S = \{u, v\}$$

already ensures that $x_e \le 1$.

The dual of (4) is

$$\begin{aligned}
\text{Maximize } & (n-1)y_0 - \sum_{\emptyset\subset S\subseteq V} (|S|-1)y_S \\
\text{s.t. } & y_0 - \sum_{\substack{\emptyset\subset S\subseteq V\\ u,v\in S}} y_S \le w_{u,v} \quad \forall (u,v) \in E \\
& y_0 \ge 0 \\
& y_S \ge 0 \qquad \forall \emptyset \subset S \subseteq V.
\end{aligned} \tag{5}$$

Here, y_0 is the dual variable corresponding to the first constraint in (4) and for each $\emptyset \subset S \subseteq V$, y_S is the dual variable corresponding to the constraint $\sum_{(u,v)\in E; u,v\in S} x_{u,v} \le |S| - 1$. Note that the dual is a maximization LP because the primal is a minimization LP, and the coefficients of y_S both in the dual objective function and in all dual constraints that involve y_S are negated because the primal constraint $\sum_{(u,v)\in E; u,v\in S} x_{(u,v)} \le |S| - 1$ is an upper bound constraint but should be a lower bound constraint, given that the primal is a minimization LP.

Now let T be the MST computed by Kruskal's algorithm. By Lemma 4 in our discussion of LP relaxations, the vector $\hat{x}$ defined as

$$\hat{x}_e = \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{otherwise} \end{cases}$$

is a feasible solution of (4) (because it is a feasible solution of (3), and (4) is the LP relaxation of (3)). Next we show that the following is a feasible solution of (5) and that $\hat{x}$ and $\hat{y}$ satisfy complementary slackness:

$$\hat{y}_S = \begin{cases} w_{e_{d(S)}} - w_{e_{c(S)}} & \text{if } d(S) < \infty \\ -w_{e_{c(V)}} & \text{if } S = V \text{ and } w_{e_{c(V)}} \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y}_0 = \begin{cases} w_{e_{c(V)}} & \text{if } w_{e_{c(V)}} \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, by Thm. 2, $\hat{x}$ and $\hat{y}$ are optimal solutions of (4) and (5), respectively. Since $\hat{x}$ is integral, this proves the proposition.

First, we prove that $\hat{y}$ is a feasible solution of (5). The definitions of $\hat{y}_0$ and $\hat{y}_V$ explicitly ensure that $\hat{y}_0 \ge 0$ and $\hat{y}_V \ge 0$. For $\emptyset \subset S \subset V$, if $\hat{y}_S \ne 0$, then $\hat{y}_S = w_{e_{d(S)}} - w_{e_{c(S)}}$ and $d(S) < \infty$. By the definitions of $c(S)$ and $d(S)$, $c(S) < d(S)$, so $w_{e_{c(S)}} \le w_{e_{d(S)}}$, that is, $w_{e_{d(S)}} - w_{e_{c(S)}} \ge 0$. This proves that all entries of $\hat{y}$ are non-negative.

Next, consider any edge $(u, v) \in E$. Let $\mathcal{S}$ be the set of all subsets $S \subseteq V$ such that $|S| \ge 2$ and S is the vertex set of a connected component of some graph G_i, and let $\mathcal{S}_{u,v} = \{S \in \mathcal{S} \mid u, v \in S\}$. We have $\hat{y}_S = 0$

for all $S \notin \mathcal{S}$. Thus,

$$\hat{y}_0 - \sum_{\substack{S \subseteq V \\ u,v \in S}} \hat{y}_S = \hat{y}_0 - \sum_{S \in \mathcal{S}_{u,v}} \hat{y}_S.$$

Let $\mathcal{S}_{u,v} = \{S_1, \dots, S_k\}$ such that $S_1 \subset \cdots \subset S_k = V$, and let $f_i = e_{c(S_i)}$, for all $1 \le i \le k$. Then $f_k = e_{c(V)}$ and $f_i = e_{d(S_{i-1})}$ for all $1 < i < k$. Thus,

$$\begin{aligned}
\sum_{i=1}^{k} \hat{y}_{S_i} &= \sum_{i=1}^{k-1} \hat{y}_{S_i} + \hat{y}_V \\
&= \sum_{i=1}^{k-1} \left(w_{e_{d(S_i)}} - w_{e_{c(S_i)}} \right) + \hat{y}_V \\
&= \sum_{i=1}^{k-1} \left(w_{f_{i+1}} - w_{f_i} \right) + \hat{y}_V \\
&= w_{f_k} - w_{f_1} + \hat{y}_V \\
\hat{y}_0 - \sum_{i=1}^{k} \hat{y}_{S_i} &= (\hat{y}_0 - \hat{y}_V) + w_{f_1} - w_{f_k} \\
&= w_{f_k} + w_{f_1} - w_{f_k} \\
&= w_{f_1}.
\end{aligned}$$

Now let i be the index such that $e_i = (u, v)$. If $(u, v) \in T$, then $c(S_1) = i$ and $f_1 = (u, v)$, so $\hat{y}_0 - \sum_{i=1}^{k} \hat{y}_{S_i} = w_{u,v}$. If $(u, v) \notin T$, then $c(S_1) < i$ because G_i has a connected component that includes both u and v. Thus, $w_{f_1} \le w_{e_i} = w_{u,v}$, that is, $\hat{y}_0 - \sum_{i=1}^{k} \hat{y}_{S_i} \le w_{u,v}$. In both cases, $\hat{y}$ satisfies the constraint in (5) corresponding to the edge (u, v). Since this argument holds for every edge $(u, v) \in E$, this shows that $\hat{y}$ is a feasible solution of (5).

It remains to prove that $\hat{x}$ and $\hat{y}$ are *optimal* solutions of (4) and (5), respectively. To this end, it suffices to prove that they satisfy complementary slackness. As just observed, if $(u, v) \in T$, that is, if $\hat{x}_{u,v} \ne 0$, then $\hat{y}_0 - \sum_{i=1}^{k} \hat{y}_{S_i} = w_{u,v}$. Thus, $\hat{x}$ and $\hat{y}$ satisfy primal complementary slackness.

Next consider dual complementary slackness. Since $\hat{x}$ is a feasible solution of (4), we have $\sum_{e \in E} \hat{x}_e \ge n - 1$ and $\sum_{e \in E} \hat{x}_e \le n - 1$, that is, $\sum_{e \in E} \hat{x}_e = n - 1$ and the primal constraints corresponding to both y_0 and y_V are tight. If $\hat{y}_S \ne 0$ for some $\emptyset \subset S \subset V$, then S is a connected component of some graph G_i and thus of T_i. Since every connected graph on $|S|$ vertices has at least $|S| - 1$ edges, this shows that

$$\sum_{\substack{(u,v) \in E \\ u,v \in S}} \hat{x}_{u,v} \ge |S| - 1.$$

Since $\hat{x}$ is a feasible solution of (4), we also have

$$\sum_{\substack{(u,v) \in E \\ u,v \in S}} \hat{x}_{u,v} \le |S| - 1.$$

Thus,

$$\sum_{\substack{(u,v)\in E\\ u,v\in S}} \hat{x}_{u,v} = |S| - 1,$$

and the primal constraint corresponding to y_S is tight. Thus, $\hat{x}$ and $\hat{y}$ satisfy dual complementary slackness.

Since $\hat{x}$ and $\hat{y}$ satisfy both primal and dual complementary slackness, Thm. 2 shows that they are optimal solutions of (4) and (5), respectively. □