Assignment 8

CSCI 4113/6101
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Solutions

Question 1

Since G is connected, so is H. Thus, there exists a path $P = \langle v_1 = v_{j_1}, \ldots, v_{j_t} = v_i \rangle$ from v_1 to v_i in H, for all $i \in [2, n]$. Assume that P is a shortest path from v_1 to v_i. Let $h \in [t]$ such that $j_h = \max\{j_1, \ldots, j_t\}$.

If $h = t$, then $j_{t-1} < j_t = i$. Since $v_{j_{t-1}}$ is a neighbour of v_i in H, this shows that $v_{j_{t-1}} \in N_H^\sigma(v_i)$, that is, $H_H^\sigma(v_i) \neq \emptyset$.

If $h \neq t$, then $1 < h < t$, because $j_1 = 1 < j_{h'}$, for all $h' \in [2, t]$. Thus, v_{j_h} has a predecessor $v_{j_{h-1}}$ and a successor $v_{j_{h+1}}$ in P, and they satisfy $j_{h-1}, j_{n+1} < j_h$. This implies that $v_{j_{h-1}}, v_{j_{h+1}} \in N_H^\sigma\left(v_{j_h}\right)$. Therefore, since $H\left[N_H^\sigma\left(v_{j_h}\right)\right]$ is a clique, the edge $\left\{v_{j_{h-1}}, v_{j_{h+1}}\right\}$ is an edge of H. This makes $\left\langle v_{j_1}, \ldots, v_{j_{h-1}}, v_{j_{h+1}}, \ldots, v_{j_t}\right\rangle$ a path from v_1 to v_i in H, one with fewer edges than P. Since P is a shortest path from v_1 to v_i in H, no such path can exist. Thus, the case when $h \neq t$ cannot arise.

Question 2

Since v_1 is the root of T, all vertices of G are descendants of v_1 in T. Thus, $G_1 = G$ and $X_1 = V(G)$. In particular $w_1(I) = w(I)$, for every independent set I of G. Since v_1 is the first vertex in σ, $N_H^\sigma(v_1) = \emptyset$ and $N_H^\sigma[v_1] = \{v_1\}$. Thus, there exist exactly two subsets of $N_H^\sigma[v_1]$, $\{v_1\}$ and $\emptyset$. Since $w_1(I) = w(I)$, for every independent set I of G, $I[1, \{v_1\}]$ is the maximum weight of all independent sets of $G_1 = G$ that contain v_1, and $I[1, \emptyset]$ is the maximum weight of all independent sets of $G_1 = G$ that do not contain v_1. Since every independent set of G either contains v_1 or not, the weight of a maximum-weight independent set of G is $\max(I[1, \emptyset], I[1, \{v_1\}])$, which can clearly be computed in constant time.

Question 3

If v_i is a leaf of T, then it has no proper descendants. Thus, $X_i = \{v_i\}$ and $G_i = G[N_H^\sigma[v_i]]$. Moreover, every neighbour v_j of v_i in H satisfies $j < i$. Assume the contrary, that is, there exists a neighbour v_j of v_i with $j > i$, and choose this neighbour so that j is minimized. Since v_i is not the parent of v_j in T (v_i is a leaf), there exists a neighbour v_h of v_j in H with $i < h < j$. This implies that $v_h, v_i \in N_H^\sigma(v_j)$ though. Since $N_H^\sigma(v_j)$ is a clique, v_h is a neighbour of v_i in H. Since $i < h < j$, this contradicts the choice of v_j.

Since every neighbour v_j of v_i in H satisfies $j < i$, we have $N_H^\sigma[v_i] = N_H[v_i]$. Thus, since $|N_H^\sigma[v_i]| \leq k$, we can identify $N_H^\sigma[v_i]$ in $O(k)$ time simply by scanning the adjacency list of v_i in H and collecting the vertices it contains plus v_i itself. As already observed, this is the vertex set V_i of G_i. To identify the edge set E_i of G_i, we scan the list of all edges in G and add all those edges to E_i that have both endpoints in V_i. This takes $O(m)$ time. Thus, G_i can be constructed in $O(k+m)$ time.

Now, if I is an independent set of G_i with $I \cap N_H^\sigma[v_i] = U$, then $I = U$ because $N_H^\sigma[v_i] = V(G_i)$. Thus, there exists such an independent set if and only if U itself is an independent set of G_i, a condition we can test in $O(m)$ time by scanning all edges of G_i and testing for each whether it has both endpoints in U. If U is independent, then $I[i,U] = w(U \cap X_i) = w(U \cap \{v_i\})$. In particular, $I[i,U] = w(v_i)$ if $v_i \in U$, and $I[i,U] = 0$ otherwise. If U is not independent, then there is no independent set of G_i that contains U, so $I[i,U] = -\infty$. Whichever case applies, $I[i,U]$ can be computed in constant time after testing whether U is idependent.

The total cost of all steps just described adds to $O(k+m)$, so $I[i,U]$ can be computed in $O(k+m)$ time, as claimed.

QUESTION 4

Now consider a vertex v_i that is not a leaf of T, and let $v_{j_1}, \dots, v_{j_d}$ be its children in T. Then $X_i = \{v_i\} \cup X_{j_1} \cup \dots \cup X_{j_d}$, and all these subsets of X_i are disjoint. We start by proving a number of claims that we will use to relate independent sets of G_i and $G_{j_1}, \dots, G_{j_d}$ to each other, which is the key to computing $I[i,U]$ from appropriate table entries associated with the children of v_i in T.

CLAIM 1. *For all $j \in [d]$, $N_H^\sigma\left(v_{j_h}\right) \subseteq N_H^\sigma[v_i]$.*

Proof. Assume the contrary. Then there exists a vertex $v_a \in N_H^\sigma\left(v_{j_h}\right) \setminus N_H^\sigma[v_i]$. We have $a \neq i$ because $v_i \in N_H^\sigma[v_i]$. We also have $a \leq i$ because, by the definition of T, v_i is the vertex with maximum index i in $N_H^\sigma\left(v_{j_h}\right)$. Thus, $a < i$.

Since $v_i, v_a \in N_H^\sigma\left(v_{j_h}\right)$ and $H\left[N_H^\sigma\left(v_{j_h}\right)\right]$ is a clique, H contains the edge $\{v_a, v_i\}$, that is, $v_a \in N_h(v_i)$. Therefore, since $a < i$, $v_a \in N_H^\sigma(v_i)$, a contradiction. This shows that $N_H^\sigma\left(v_{j_h}\right) \subseteq N_H^\sigma[v_i]$. □

CLAIM 2. *For all $j \in [d]$, $G_{j_h} \subseteq G_i$.*

Proof. By Clm. 1, we have $N_H^\sigma\left(v_{j_h}\right) \subseteq N_H^\sigma[v_i]$. Since v_{j_h} is a child of v_i in T, we also have $X_{j_h} \subset X_i$. Thus, $G_{j_h} = G\left[N_H^\sigma\left[v_{j_h}\right] \cup X_{j_h}\right] \subseteq G\left[N_H^\sigma[v_i] \cup X_i\right] = G_i$. □

Next, we prove the claim hinted at in the assignment, that each set $N_H^\sigma[v_i]$ is a separator.

CLAIM 3. *For all $i \in [n]$, every edge $\{v_a, v_b\} \in G$ with $v_a \notin G_i$ and $v_b \in G_i$ satisfies $v_b \in N_H^\sigma[v_i]$.*

Proof. Assume the contrary. Then there exists an edge $\{v_a, v_b\}$ with $v_a \notin G_i$ and $v_b \in X_i \setminus \{v_i\}$. This implies in particular that v_b is a proper descendant of v_i in T. Now we distinguish two cases.

Case 1: v_a is an ancestor of v_i in T. Then $a \leq i$. Since $v_i \in G_i$ but $v_a \notin G_i$, we have in fact that $a < i$. Since v_b is a proper descendant of v_i, the path $P = \langle v_i = v_{j_1}, \dots, v_{j_t} = v_b \rangle$ from v_i to v_b in T satisfies $i = j_1 < \dots < j_t = b$. Thus, since $a < i$, we also have that $a < j_1 < \dots < j_t = b$. H contains the edge $\left\{v_a, v_{j_t}\right\}$.

We use induction on t to prove that $\{v_a, v_i\} \in H$. Thus, since $a < i$, this implies that $v_a \in N_H^\sigma(v_i)$ and, therefore, that $v_a \in G_i$. This is the desired contradiction.

If $t = 1$, then the claim follows immediately because, in this case, $i = j_t$, and H contains the edge $\left\{v_a, v_{j_t}\right\}$.

If $t > 1$, then v_a and $v_{j_{t-1}}$ are neighbours of v_{j_t}. Since $a < j_{t-1} < j_t$, this implies that $v_a, v_{j_{t-1}} \in N_H^\sigma\left(v_{j_t}\right)$. Since $H\left[N_H^\sigma\left(v_{j_t}\right)\right]$ is a clique, this implies that $\left\{v_a, v_{j_{t-1}}\right\}$ is an edge of H. By the induction hypothesis, this implies that $\{v_a, v_i\} \in H$.

Case 2: v_a is not an ancestor of v_i in T. Let v_q be the lowest common ancestor of v_a and v_i in T. Since v_a is not an ancestor of v_i in T, v_q is a proper ancestor of v_a. Since v_b is a proper descendant of v_i, v_q is also a proper ancestor of v_b. Let $P = \left\langle v_q = v_{j_0}, v_{j_1}, \ldots, v_{j_t} = v_b \right\rangle$ and $Q = \left\langle v_q = v_{i_0}, v_{i_1}, \ldots, v_{i_s} = v_a \right\rangle$ be the paths from v_q to v_b and v_a in T, respectively. H contains the edge $\{v_a, v_b\} = \left\{v_{i_s}, v_{j_t}\right\}$. We use induction on $s + t$ to prove that this leads to a contradiction. We assume w.l.o.g. that $i_s < j_t$. The case when $i_s > j_t$ is analogous, with the roles of v_{i_s} and v_{j_t} swapped. Since $\left\{v_{i_s}, v_{j_t}\right\}$ is an edge of H, the assumption that $i_s < j_t$ implies that $v_{i_s} \in N_H^\sigma\left(v_{j_t}\right)$.

If $t = 1$, then v_q is the parent of v_{j_t} in T. Since v_q is a proper ancestor of v_{i_s}, $q < i_s$. Since $v_{i_s} \in N_H^\sigma\left(v_{j_t}\right)$, this shows that v_q is not the vertex with maximum index q in $N_H^\sigma\left(v_{j_t}\right)$, that is, v_q is not the parent of v_{j_t} in T, which is the desired contradiction.

If $t > 1$, then $i_s, j_{t-1} < j_t$ and $j_{t-1} \neq q$. Since H contains both $\left\{v_{i_s}, v_{j_t}\right\}$ and $\left\{v_{j_{t-1}}, v_{j_t}\right\}$, this shows that $v_{i_s}, v_{j_{t-1}} \in N_H^\sigma\left(v_{j_t}\right)$. Since $H\left[N_H^\sigma\left(v_{j_t}\right)\right]$ is a clique, this shows that $\left\{v_{i_s}, v_{j_{t-1}}\right\}$ is an edge of H. By the induction hypothesis, this leads to a contradiction. □

Now consider any subset $I \subseteq V(G_i)$, let $U = I \cap N_H^\sigma[v_i]$, and, for all $h \in [d]$, let $I_h = I \cap V\left(G_{j_h}\right)$ and $U_h = I_h \cap N_H^\sigma\left[v_{j_h}\right]$.

Claim 4. *For all $h \in [d]$, $U_h = \left(U \cap N_H^\sigma\left(v_{j_h}\right)\right) \cup \left(I \cap \left\{v_{j_h}\right\}\right)$.*

Proof. We have

$$U_h = I \cap V\left(G_{j_h}\right) \cap N_H^\sigma\left[v_{j_h}\right] = I \cap N_H^\sigma\left[v_{j_h}\right] = I \cap \left(N_H^\sigma\left(v_{j_h}\right) \cup \left\{v_{j_h}\right\}\right) = \left(I \cap N_H^\sigma\left(v_{j_h}\right)\right) \cup \left(I \cap \left\{v_{j_h}\right\}\right). \quad (1)$$

By Clm. 1, $N_H^\sigma\left(v_{j_h}\right) \subseteq N_H^\sigma[v_i]$. Thus,

$$I \cap N_H^\sigma\left(v_{j_h}\right) = I \cap N_H^\sigma[v_i] \cap N_H^\sigma\left(v_{j_h}\right) = U \cap N_H^\sigma\left(v_{j_h}\right). \quad (2)$$

The claim follows by substituting (2) into (1). □

The following claim is the heart of the dynamic programming algorithm.

Claim 5. *I is an independent set of G_i if and only if U is an independent set of G_i and I_h is an independent set of G_{j_h}, for all $h \in [d]$.*

Proof. First, observe that $I = U \cup I_1 \cup \cdots \cup I_d$. Indeed, every vertex of G_i belongs either to $N_H^\sigma[v_i]$ or to $X_i \setminus \{v_i\}$. Any vertex in $I \cap N_H^\sigma[v_i]$ belongs to U. Any vertex in $I \cap (X_i \setminus \{v_i\})$ belongs to X_{j_h}, for some $h \in [d]$, and, thus, to I_h. This proves that $I \subseteq U \cup I_1 \cup \cdots \cup I_d$. Since $U, I_1, \ldots, I_d$ are all subsets of I, the converse inclusion is trivial.

The "only if" direction of the claim is trivial now. Since $U \subseteq I$, U is an independent set of G_i if I is an independent set of G_i. Since $G_{j_h} \subseteq G_i$ and $I_h \subseteq I$, I_h is independent in G_{j_h} if I is independent in G_i, for all $h \in [d]$.

For the "if" direction, assume I is not an independent set. Then there exists an edge $\{v_a, v_b\} \in G \subseteq H$ with $v_a, v_b \in I$. If $v_a, v_b \in U$, then U is not an independent set. So assume that $v_a \notin U$. Then $v_a \in I_h \setminus U$, for some $h \in [d]$, and, therefore, $v_a \in G_{j_h}$. If $v_b \in I_h$, then I_h is not an independent set. Thus, we can also assume that $v_b \notin I_h$ and, therefore, $v_b \notin G_{j_h}$. By Clm. 3, this implies that $v_a \in U_h$. By Clm. 4, this implies that $v_a \in U$. But we just assumed that $v_a \notin U$, so the case when $v_a \in I_h \setminus U$ and $v_b \notin I_h$ cannot arise. □

Now let $U \subseteq N_H^\sigma[v_i]$. If U is not an independent set, then there is no independent set $I \supseteq U$ with $I \cap N_H^\sigma[v_i] = U$. Thus, $I[i, U] = -\infty$ in this case. Otherwise, let $U'_h = U \cap N_H^\sigma\left(v_{j_h}\right)$, for all $h \in [d]$. Then

Claim 6. $I[i,U] = w(U \cap \{v_i\}) + \sum_{h=1}^{d} \max\left(I\left[j_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right)$.

Proof. Since U is independent, there exists an independent set $I \supseteq U$ of G_i with $I \cap N_H^\sigma[v_i] = U$: $I = U$ does the trick. Assume that I has maximum weight $w_i(I)$ among all such independent sets. Then $I[i,U] = w_i(I)$.

By Clm. 5, $I_1, \ldots, I_d$ as defined before Clm. 4 are independent sets of $G_{j_1}, \ldots, G_{j_d}$, respectively. Since $\{v_i\}, X_{j_1}, \ldots, X_{j_d}$ form a partition of X_i, the sets $I \cap \{v_i\}, I \cap X_{j_1}, \ldots, I \cap X_{j_d}$ form a partition of $I \cap X_i$. However, since the vertex set of G_{j_h} includes X_{j_h}, we have $I \cap X_{j_h} = I_h \cap X_{j_h}$, for all $h \in [d]$. Therefore,

$$w_i(I) = w(I \cap X_i) = w(I \cap \{v_i\}) + \sum_{h=1}^{d} w\left(I_h \cap X_{j_h}\right) = w(U \cap \{v_i\}) + \sum_{h=1}^{d} w_{j_h}(I_h).$$

Since I_h is an independent set of G_{j_h}, for all $h \in [d]$, we have $w_{j_h}(I_h) \le I[j_h, U_h]$, for U_h as defined before Clm. 4. Thus, since $I[i,U] = w_i(I)$, we have

$$I[i,U] \le w(U \cap \{v_i\}) + \sum_{h=1}^{d} I[j_h, U_h].$$

Now, by Clm. 4,

$$U_h = \left(U \cap N_H^\sigma\left(v_{j_h}\right)\right) \cup \left(I \cap \left\{v_{j_h}\right\}\right) = U'_h \cup \left(I \cap \left\{v_{j_h}\right\}\right) \in \left\{U'_h, U'_h \cup \left\{v_{j_h}\right\}\right\}, \tag{3}$$

for all $h \in [d]$. Thus, $I[j_h, U_h] \le \max\left(I\left[j_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right)$, for all $h \in [d]$. This implies that

$$I[i,U] \le w(U \cap \{v_i\}) + \sum_{h=1}^{d} \max\left(I\left[j_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right). \tag{4}$$

Now, since I_h is an independent set of G_h with $I_h \cap N_H^\sigma\left[v_{j_h}\right] = U_h$, for all $h \in [d]$, (3) shows that there exists an independent set I''_h of G_h with $I''_h \cap N_H^\sigma\left[v_{j_h}\right] \in \left\{U'_h, U'_h \cup \left\{v_{j_h}\right\}\right\}$, for all $h \in [d]$. Choose I'_h from among all independent sets I''_h of G_h that satisfy this condition so that $w_{j_h}\left(I'_h\right)$ is maximized, for all $h \in [d]$. Then $w_{j_h}\left(I'_h\right) = \max\left(I\left[j_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right)$, for all $h \in [d]$.

Now let $I' = U \cup \bigcup_{h=1}^{d} I'_h$. Then, similar to the partition of $I \cap X_i$ into $U \cap \{v_i\}, I_1 \cap X_{j_1}, \ldots, I_d \cap X_{j_d}$, the sets $U \cap \{v_i\}, I'_1 \cap X_{j_1}, \ldots, I'_d \cap X_{j_d}$ form a partition of $I' \cap X_i$. Thus,

$$w_i\left(I'\right) = w(U \cap \{v_i\}) + \sum_{h=1}^{d} w_{j_h}\left(I'_h\right) = w(U \cap \{v_i\}) + \sum_{h=1}^{d} \max\left(I\left[j_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right).$$

Next, we prove that I' is an independent set of G_i and that $I' \cap N_H^\sigma[v_i] = U$. Thus, $S[i,U] \geq w_i\left(I'\right)$ and

$$I[i,U] \geq w(U \cap \{v_i\}) + \sum_{h=1}^{d} \max\left(I\left[j_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right). \tag{5}$$

Together, (4) and (5) prove the claim.

Since $U \subseteq I'$ and $U \subseteq N_H^\sigma[v_i]$, we immediately conclude that $I' \cap N_H^\sigma[v_i] \supseteq U$. For each $h \in [d]$, we have $I'_h \cap N_H^\sigma\left[v_{j_h}\right] \in \left\{U'_h, U'_h \cup \left\{v_{j_h}\right\}\right\}$. Thus, $I'_h \cap N_H^\sigma\left(v_{j_h}\right) = U'_h \subseteq U$. Since $I'_h = \left(I'_h \cap N_H^\sigma\left(v_{j_h}\right)\right) \cup \left(I'_h \cap X_{j_h}\right)$ and $X_{j_h} \cap N_H^\sigma[v_i] = \emptyset$, this shows that $I'_h \cap N_H^\sigma[v_i] \subseteq U$, for all $n \in [d]$. Thus, we also have $I' \cap N_H^\sigma[v_i] \subseteq U$, that is, $I' \cap N_N^\sigma[v_i] \subseteq U$.

Since $I'_h \subseteq I'$, for all $h \in [d]$, we have $I' \cap V\left(G_{j_h}\right) \supseteq I'_h$. If this is not an equality, then there exists a vertex $x \in \left(I' \cap V\left(G_{j_h}\right)\right) \setminus I'_h$. Since $X_{j_h} \cap \left(U \cup V\left(V_{j_{h'}}\right)\right) = \emptyset$, for all $h' \neq h$, this vertex x must belong to $N_H^\sigma\left(v_{j_h}\right)$ and, therefore, to $N_H^\sigma[v_i]$. Since $I' \cap N_N^\sigma[v_i] = U$, this shows that $x \in U$. However, $U'_h = U \cap N_H^\sigma\left(v_{j_h}\right)$ and $U'_h = I'_h \cap N_H^\sigma\left(v_{j_h}\right) \subseteq I'_h$. Thus, $x \in I'_h$, a contradiction. This shows that $I' \cap V\left(G_{j_h}\right) = I'_h$, for all $h \in [d]$.

Since $U, I'_1, \ldots, I'_d$ are all independent sets, Clm. 5 shows that I' is an independent set of G_i. This finishes the proof. □

To compute $I[i,U]$, we need to test whether U is an independent set of G. If not, then $S[i,U] = -\infty$. Otherwise, we need to apply the formula in Clm. 6 to compute $I[i,U]$. To do the latter, we need to identify $U'_1, \ldots, U'_d$. Testing whether U is an independent set is a matter of marking all vertices in G that belong to U, which can be done in $O(|U|) = O(k)$ time. Then we scan the edges of G and test for each whether both its endpoints belong to U or not. This takes $O(m)$ time. If we find such an edge, then U is not independent in G and, therefore, not in G_i either. We set $I[i,U] = -\infty$ in this case. Otherwise, U *is* independent in G and, therefore, also in G_i. Since $U'_h = U \cap N_H^\sigma\left(v_{j_h}\right)$, for all $h \in [d]$, we can scan the adjacency list of v_{j_h} in H and collect all those neighbours of v_{j_h} that are marked as belonging to U. The resulting set is U'_h. This construction of U'_h takes $O\left(\deg_H\left(v_{j_h}\right)\right)$ time. Summed over all $h \in [d]$, this is $O(m)$. Given $U'_1, \ldots, U'_d$, the expression in Clm. 6 can be evaluated in $O(d)$ time. Thus, we can compute $I[i,U]$ in $O(d+m)$ time.[1]

Question 5

The equation for $I[i,U]$ in Clm. 6 depends only on table entries associated with the children of v_i in T. Thus, if we visit the nodes of T in postorder, from the leaves towards the root, then the table entries needed to compute $I[i,U]$, for each $i \in [n]$, are available when we compute $I[i,U]$. Since $|N_H^\sigma[v_i]| \leq k+1$, for all $i \in [n]$, there are at most 2^{k+1} subsets $U \subseteq N_H^\sigma[v_i]$ to be considered for every

[1] When I wrote the assignment, I thought there more than two table entries to consider for each j_h, up to 2^k in fact. Hence the less strict expectation in the assignment.

vertex v_i. Therefore, the cost of computing all table entries for all pairs (i, U) is bounded by

$$\sum_{i \in [n]} O\left(2^k(k + d_i + m)\right),$$

where d_i is the number of children of v_i in T. This is bounded by

$$\sum_{i \in [n]} O\left(2^k m\right) = O\left(2^k nm\right)$$

because $k, d_i < n$ and, since G is connected, $n \in O(m)$. As shown in the answer to Question 2, the weight of a maximum-weight independent set can be computed from the entries in I in constant time. What we still have to figure out is how to find an actual independent set of this weight. For this, we retrace the steps the algorithm took to compute this weight.

We find an independent set of G of maximum weight by making recursive calls on the nodes of T. Each recursive call on a node v_i is given as input a set $U \subseteq N_H^\sigma[v_i]$ and returns the set $I_i = I \cap X_i$, where I is an independent set of G_i that includes U and satisfies $w_i(I) = S[i, U]$.

For the root v_1 of T, we observed in the answer to Question 2 that $N_H^\sigma[v_1] = \{v_1\}$ and $X_1 = V(G)$, and that the maximum-weight independent set of G has weight $\max(I[1, \emptyset], I[1, \{v_1\}])$. Let $U \subseteq N_H^\sigma[v_1]$ such that $I[1, U]$ is maximized. The recursive call on v with argument U then returns an independent set $I \supseteq U$ with $w(I) = w_1(I) = I[1, U]$, that is, I as a maximum-weight independent set.

We need to figure out how to implement each recursive call.

If v_i is a leaf, then $X_i = \{v_i\} \subseteq U$. Thus, for any independent set $I \supseteq U$ of G_i, we have $I \cap \{v_i\} = U \cap \{v_i\}$. Thus, if $v_i \in U$, we have $I \cap X_i = \{v_i\}$. Otherwise, we have $I \cap X_i = \emptyset$. Thus, the invocation tests whether $v_i \in U$ and accordingly returns $\{v_i\}$ or $\emptyset$.

If v_i as an internal node with children $v_{j_1}, \ldots, v_{j_d}$, then, by Clm. 6,

$$I[i, U] = w(U \cap \{v_i\}) + \sum_{h=1}^{d} \max\left(I\left[y_h, U'_h\right], I\left[j_h, U'_h \cup \left\{v_{j_h}\right\}\right]\right),$$

where

$$U'_h = U \cap N_H^\sigma\left[v_{j_h}\right] \quad \forall h \in [d].$$

The proof of Clm. 6 arrived at this equation by observing that, for any independent set I of G_i with $I \cap N_H^\sigma[v_i] = U$, $I \cap X_i$ decomposes into the sets $U \cap \{v_i\}, I \cap X_{j_1}, \ldots, I \cap X_{j_d}$, and that $U_h = I \cap N_H^\sigma\left[v_{j_h}\right] \in \left\{U'_h, U'_h \cup \left\{v_{j_h}\right\}\right\}$. Thus, if we choose $U_h \in \left\{U'_h, U'_h \cup \left\{v_{j_h}\right\}\right\}$ such that $I[j_h, U_h]$ is maximized, for all $n \in [d]$, then we can find the sets $I \cap X_{j_1}, \ldots, I \cap X_{j_d}$ by making recursive calls on $v_{j_1}, \ldots, v_{j_d}$ with arguments $U_1, \ldots, U_d$, respectively. The set $U \cap \{v_i\}$ is trivial to compute.

Overall, we spend constant time to compute the set U passed to the initial invocation on the root. Then we spend $O(k)$ time per node v_i to compute its input set U from the input set of its parent and $N_H^\sigma[v_i]$. This takes $O(kn)$ time in total. Given each node's input set, we decide in constant time whether to add v_i to the independent set I based on whether $v_i \in U$ or not.

This shows that, given the table I, wa can compute an independent set of maximum weight in $O(kn)$ time. Since we argued that we can fill in this table in $O\left(2^k nm\right)$ time, wa can find an independent set of maximum weight in $O\left(2^k nm\right)$ time.