Assignment 7

CSCI 4113/6101
Instructor: Norbert Zeh
Due: Dec 2, 2025, 11:59pm

This assignment is about kernelization.

Question 1

The 15-puzzle is a classical puzzle game. It can be described as a 4×4 grid. 15 of the 16 grid cells are filled with the numbers 1 through 15. One grid cell is left empty. Your goal is to rearrange the numbers so that the numbers 1 through 4 appear left to right in the top row; the numbers 5 through 8, in the next row; and so on with the numbers 13, 14, 15 in the last row and the empty cell in the bottom-right corner. This is illustrated in Fig. 1. We will call this "sorting" the puzzle.

In one move, you can shift one of the numbers adjacent to the empty cell into the empty cell. This is illustrated in Fig. 2. Your goal is to sort the puzzle in as few moves as possible.

We can generalize the 15-puzzle to an (n^2-1)-puzzle, for any integer $n \geq 2$, in which we have an $n \times n$ grid filled with the numbers 1 through n^2-1 and, once again, one of the grid cells is empty. In the parameterized version of the problem, we are given the $n \times n$ grid cells filled in this way, and an integer k, and we want to decide whether this puzzle can be sorted in at most k moves. Prove that this problem has a kernel of size $O\left(k^2\right)$.

Hint: *This question is really a warm-up. Think about how far away from the empty cell any tile can be that can be moved as past of a solution with at most k moves.*

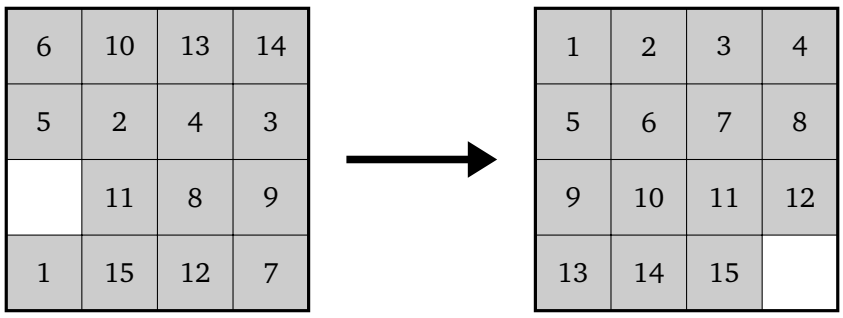

Figure 1: A random input of the 15-puzzle and the sorted puzzle into which it is to be transformed.

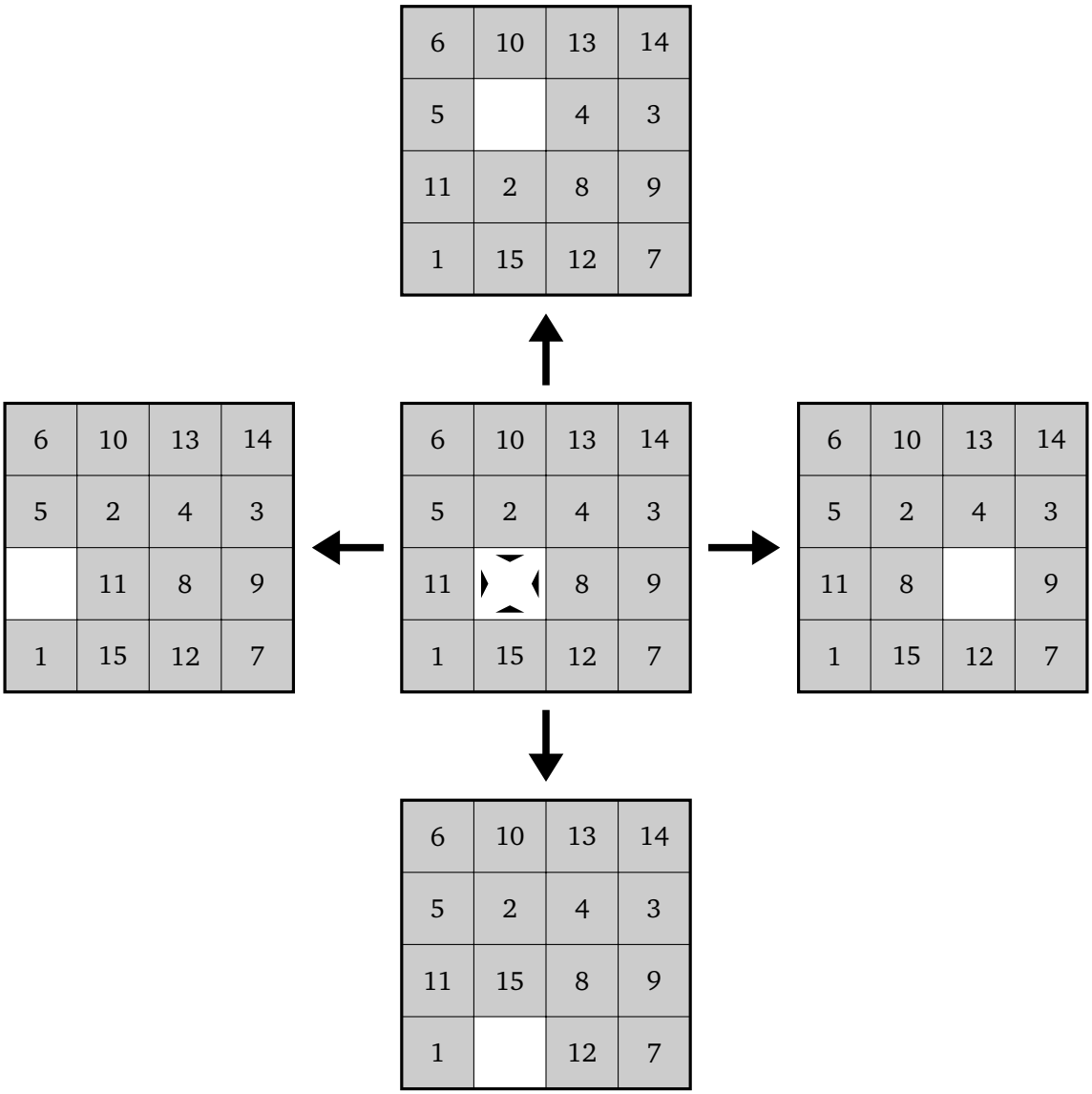

Figure 2: The four configurations reachable from the middle configuration in a single move.

QUESTION 2

For most problems in this course, we use the solution size as the parameter. *Structural* parameterizations use some information about the structure of the input to help the search for a solution. The size of a vertex cover turns out to be a rather useful parameter. In this question, you are asked to show that a small vertex cover helps with finding long cycles.

You may have heard of the HAMILTONIAN CYCLE problem. A Hamiltonian cycle in a graph G is a cycle in G that visits every vertex of G exactly once. Deciding whether such a cycle exists in a graph is a classical NP-hard problem. The optimization version looks for a longest cycle. In the parameterized version, we are given a pair (G, ℓ), where ℓ is an integer, and we are asked to decide whether there exists a cycle of length at least ℓ in G. We call this the ℓ-CYCLE PROBLEM.

Now assume that C is a vertex cover of G, and let $k = |C|$. Prove that C can be used to construct a kernel for the ℓ-CYCLE PROBLEM on G of polynomial size.

Hint: *Consider an auxiliary bipartite graph H. One side of the bipartition is $V \setminus C$. The other side of the bipartition contains two vertices $v^1_{a,b}$ and $v^2_{a,b}$, for every pair $\{a, b\} \in \binom{C}{2}$. There exist edges $\left\{v^1_{a,b}, c\right\}$ and $\left\{v^2_{a,b}, c\right\}$, for $\{a, b\} \in \binom{C}{2}$ and $c \in V \setminus C$ if c is adjacent to both a and b. Let M be a maximum matching of H, let D be the set of vertices in $V \setminus C$ matched by M, and let $G[C \cup D]$ be the subgraph of G induced by the vertices in $C \cup D$. Prove that this graph has $O\left(k^2\right)$ vertices and that it contains a cycle of length ℓ if and only if G contains such a cycle.*

Marking Scheme

No more marking scheme. Submit this for formative assessment only, but do give it your best shot and try to have fun.

Submission Instructions

Follow the submission link for this assignment on the course webpage in the email you should have received from Crowdmark. Upload the assignment as a single PDF file.