Assignment 7

CSCI 4113/6101
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Solutions

Question 1

We refer to the given $n \times n$ grid as G. For row and column indices r_ℓ, r_u, c_ℓ, c_u, let $G[r_\ell, r_u, c_\ell, c_u]$ be the subgrid of G containing all those cells (r, c) with $r \in [r_\ell, r_u]$ and $c \in [c_\ell, c_u]$. Let (r, c) be the position of the empty cell, and let

$$\begin{aligned} r_\ell &= \max(1, r - k) \\ r_u &= \min(n, r + k) \\ c_\ell &= \max(1, c - k) \\ c_u &= \min(n, c + k). \end{aligned}$$

If all tiles in G that are in incorrect positions are part of $G' = G[r_\ell, r_u, c_\ell, c_u]$, then the kernel is (G', k). Otherwise, (G, k) is a no-instance. Indeed, to move any tile in a cell outside G' to an adjacent cell, we first need to move the "hole" to that adjacent cell. However, the hole is intially in cell (r, c), and moving the hole one row up or down or one column left or right takes one move. Thus, moving the hole to a cell adjacent to a a cell not in G' takes et least k moves and, therefore, moving any tile in a cell not in G' to a different cell takes at least $k + 1$ moves. Thus, if sorting G requires moving such a tile, then (G, k) is a no-instance.

So assume that all misplaced tiles are part of G'. Then, clearly, the kernel (G', k) can be constructed in linear time. We need to prove that (G, k) is a yes-instance if and only if $(G[r_\ell, r_u, c_\ell, c_u], k)$ is a yes-instance.

The "if" direction is trivial: A sequence M of at most k moves that sorts G' also sorts G because all the tiles in G outside of G' are already in the correct posititons, and the moves in M sort G' and do not touch any cells not in G'.

For the "if" direction, let M be a sequence of at most k moves that sorts G. As observed above, it cannot touch any cells outside of G', as doing so would require more than k moves. Thus, all the moves in M stay completely inside G'. Since M sorts G, and G' is part of G, M also sorts G'.

Question 2

Consider the graph H and the matching M in G defined in the hint. The graph H has all the vertices in $V \setminus C$ as one partition. The other partition contains $2 \cdot \binom{k}{2} = k(k-1)$ vertices, two for every pair of vertices in C. Thus, $|D| = |M| \le k(k-1)$ and $|C \cup D| \le k + k(k-1) = k^2$. This shows that $G[C \cup D]$ has k^2 vertices. We need to show that G contains a cycle $Q = \langle x_0, \ldots, x_t \rangle$ of length $t \ge \ell$ if and only if

$G[C \cup D]$ contains such a cycle.

The "if" direction is trivial because $G[C \cup D] \subseteq G$, that is, any cycle in $G[C \cup D]$ is also a cycle in G.

For the "only if" direction, assume that G contains a cycle $Q = \langle x_0, \dots, x_t \rangle$ of length $t \geq \ell$. Let I be the set of indices $i \in [t]_0$ such that $x_i \in C$, and let J be the set of indices $i \in [t]_0$ such that $x_i \in V \setminus C$. Then we distinguish two cases.

First, assume that $t = 4$ and $|J| = 2$. Then, w.l.o.g., $x_0, x_2 \in C$ and $x_1, x_3 \notin C$ because no two vertices in $V \setminus C$ are adjacent in G. In particular, both x_1 and x_3 are adjacent to x_0 and x_2. Therefore, in H, both x_1 and x_3 are adjacent to $v^1_{x_0,x_2}$ and $v^2_{x_0,x_2}$. If D contains two vertices a and b adjacent to x_0 and x_2, then $Q' = \langle x_0, a, x_2, b, x_0 \rangle$ is a cycle of length $|Q'| = 4$ in $G[C \cup D]$. If D contains at most one vertex adjacent to x_0 and x_2, then w.l.o.g., $x_1 \notin D$ and $v^1_{x_0,x_2}$ is unmatched by M. This is a contradiction because M is a maximum matching in H but adding the edge $\left\{x_1, v^1_{x_0,x_2}\right\}$ to M would produce a bigger matching. Thus, D must contain at least two common neighbours of x_0 and x_2, and $G[C \cup D]$ contains a cycle of length 4.

If $t > 4$ or $|J| \neq 2$, then $\{x_{h-1}, x_{h+1}\} \neq \{x_{i-1}, x_{i+1}\}$, for all $h, i \in J$, $h \neq i$. We partition J into two (possibly empty) subsets J_1 and J_2 such that

(i) For all $i \in J_1$, $x_i \in D$,
(ii) For all $i \in J_2$, $v^1_{x_{i-1},x_{i+1}}$ is matched to some vertex $y_i \in D$, and
(iii) For all $i \in J_2$, $y_i \notin \{x_h \mid h \in J_1\}$.

We prove below that such a partition exists. Given such a partition, we define a sequence $Q' = \left\langle x'_0, \dots x'_t \right\rangle$ by setting

$$x'_i = \begin{cases} x_i & \text{if } i \in I \cup J_1 \\ y_i & \text{if } i \in J_2. \end{cases}$$

Then Q' is a cycle in $G[C \cup D]$:

- By definition, all vertices in Q' belong to $C \cup D$.
- Every pair of consecutive vertices in Q' are adjacent in G and, thus, also in $G[C \cup D]$:
  - If $i, i+1 \in I \cup J_1$, then $\left\{x'_i, x'_{i+1}\right\} = \{x_i, x_{i+1}\}$ is an edge of Q, so this edge exists in G.
  - If $i \in J_2$, then $x'_i = y_i$ is the mate of $v^1_{x_{i-1},x_{i+1}}$ in M. Thus, the edge $\left\{v^1_{x_{i-1},x_{i+1}}, y_i\right\}$ exists in H. By the definition of H, this implies that y_i is adjacent to x_{i-1} and x_i in G. However since $i \in J_2 \subseteq J$, $x_i \notin C$, so $x_{i-1}, x_{i+1} \in C$, because C is a vertex cover. Therefore, $x'_{i-1} = x_{i-1}$ and $x'_{i+1} = x_{i+1}$, that is, G contains the edges $\left\{x'_{i-1}, x'_i\right\}$ and $\left\{x'_{i+1}, x'_i\right\}$.
- For all $h \neq i$, $x'_h \neq x'_i$:
  - If $h, i \in I \cup J_1$, then $x'_h = x_h$ and $x'_i = x_i$. Since Q is a cycle, we have $x_h \neq x_i$, so $x'_h \neq x'_i$.
  - If $h, i \in J_2$, then $\left\{v^1_{x_{h-1},x_{h+1}}, y_h\right\}$ and $\left\{v^1_{x_{i-1},x_{i+1}}, y_i\right\}$ are edges in M. Since M is a matching, this implies that $y_h \neq y_i$. Since $x'_h = y_h$ and $x'_i = y_i$, this shows that $x'_h \neq x'_i$.
  - If, w.l.o.g., $h \in I \cup J_1$ and $i \in J_2$, then $x'_h = x_h$ and $x'_i = y_i$. If $h \in I$, then $x_h \neq y_i$ because $x_h \in C$ but the mate of $v^1_{x_{i-1},x_{i+1}}$ in M belongs to $V \setminus C$. If $h \in J_1$, then $x_h \neq y_i$, by (iii).

It remains to show how to find the partition (J_1, J_2) of J. We use induction on $|J|$.

If $v^1_{x_{i-1},x_{i+1}}$ is matched for all $i \in J$ — this is vacuously true if $J = \emptyset$ — then we choose the partition $(J_1 = \emptyset, J_2 = J)$. This satisfies (ii), and (i) and (iii) hold vacuously because $J_1 = \emptyset$.

So assume that there exists an index $i \in J$ such that $v^1_{x_{i-1},x_{i+1}}$ is unmatched. In this case, consider the subgraph H' of H with vertex set $V' = \left\{x_i, v^1_{x_{i-1},x_{i+1}} \,\middle|\, i \in J\right\}$ and with edge set $E' = M' \cup M''$, where $M' \subset M$ is the subset of edges with both endpoints in V', and M'' is the perfect matching $\left\{\left\{x_i, v^1_{x_{i-1},x_{i+1}}\right\} \,\middle|\, i \in J\right\}$. This is indeed a subgraph of H, as all edges in $M \supset M'$ are edges of H, and $\left\{x_i, v^1_{x_{i-1},x_{i+1}}\right\}$ is an edge of H, for all $i \in J$, because x_i is adjacent to both x_{i-1} and x_{i+1} in G (because Q is a cycle in G). Let $\Delta = M' \oplus M''$. As we discussed in class, every path in Δ is alternating for M'. Since $v^1_{x_{i-1},x_{i+1}}$ is unmatched by $M \supseteq M'$, there exists such a path P with $v^1_{x_{i-1},x_{i+1}}$ as an endpoint. This path must have odd length, that is, it is of the form $P = \left\langle v^1_{x_{i_1-1},x_{i_1+1}}, x_{i_1}, \dots, v^1_{x_{i_q-1},x_{i_q+1}}, x_{i_q}\right\rangle$, where $i_1 = i$. Indeed, if it is of even length, then we have $P = \left\langle v^1_{x_{i_1-1},x_{i_1+1}}, x_{i_1}, \dots, v^1_{x_{i_q-1},x_{i_q+1}}\right\rangle$. Since $v^1_{x_{i-1},x_{i+1}}$ is unmatched by M', the first edge in P is not in M'. Thus, the last edge is in M', and the edge $\left\{v^1_{x_{i_q-1},x_{i_q+1}}, x_{i_q}\right\}$ is in $M'' \setminus M'$ (because $v^1_{x_{i_q-1},x_{i_q+1}}$ has at most one incident edge in M'). Thus, $v^1_{x_{i_q-1},x_{i_q+1}}$ has two incident edges in Δ, that is, it is not an endpoint of a path in Δ, a contradiction.

Now we define $J' = \{i_1, \dots, i_q\}$ and $J'' = J \setminus J'$. Since $J' \neq \emptyset$, $|J''| < |J|$. Thus, by the induction hypothesis, there exists a partition (J'_1, J'_2) of J'' that satisfies (i)–(iii). We obtain a partition (J_1, J_2) of J by defining $J_1 = J'_1 \cup J'$ and $J_2 = J'_2$. This partition satisfies (i)–(iii). Indeed, since J'_2 satisfies (ii) and $J_2 = J'_2$, J_2 satisfies (ii).

For $i \in J_1$, if $i \in J'_1$, then $x_i \in D$ because J'_1 satisfies (i). For $i \in J'$, we show that x_i is matched by M. By the definition of D, this implies that $x_i \in D$. Thus, J_1 satisfies (i).

If $i \in \{i_1, \dots, i_{q-1}\}$, then x_i is matched by M because $x_{i_1}, \dots, x_{i_{q-1}}$ are internal vertices of P, and P, being an alternating path for M contains an edge in M incident to each of its internal vertices. So assume that $i = i_q$. The vertex x_{i_q} is matched by M because otherwise, P would be an augmenting path for M, but M is a maximum matching, so no augmenting path exists for M. Indeed, P is an alternating path for $M' \subseteq M$ and, thus, also for M. By the choice of P, $v^1_{x_{i_1-1},x_{i_1+1}}$ is unmatched. Thus, if x_{i_q} is unmatched, P is an alternating path with two unmatched endpoints, that is, P is an augmenting path for M.

It remains to prove (iii). Since (J'_1, J'_2) satisfy (iii) and $J_2 = J'_2$, we have $y_i \notin \left\{x_h \mid h \in J'_1\right\}$, for all $i \in J_2$. For $h \in J'$, we have $h = i_j$, for some $j \in [q]$. If $j \neq q$, then the mate of x_{i_j} is $v^1_{x_{i_{j+1}-1},x_{i_{j+1}+1}}$. Since $i_{j+1} \in J' \subseteq J_1$, this shows that x_{i_j} is not the mate of any vertex in $\left\{v^1_{x_{i-1},x_{i+1}} \,\middle|\, i \in J_2\right\}$. Since x_{i_q} is an endpoint of P and we observed that the last edge in P is in M'', x_{i_q} has no incident edge in M'. Since M' contains all edges in M incident to vertices in J, the mate of x_{i_q} is not in $\left\{v^1_{x_{i-1},x_{i+1}} \,\middle|\, i \in J\right\}$ and, thus, not in $\left\{v^1_{x_{i-1},x_{i+1}} \,\middle|\, i \in J_2\right\}$. Thus, (J_1, J_2) satisfies (iii). This finishes the proof that $G[C \cup D]$ is a quadratic kernel for the ℓ-Path Problem parameterized by the size of a vertex cover of G.