# Assignment 6

## CSCI 4113/6101
## Instructor: Norbert Zeh
## Due: Nov 25, 2025, 11:59pm

Maximal matchings are easy to find, but they may not always be as large as they could be; they may not be maximum. Sometimes, this can be a good thing. For example, we showed that if $M$ is a maximal matching, then the set of endpoints of $M$ is a vertex cover. Thus, if we find a small maximal matching, then we also find a small vertex cover. This motivates the problem of finding a minimum maximal matching. This is a maximal matching of minimum size among all maximal matchings. (Since we want the matching to be maximal, we cannot simply choose the empty matching, so finding a maximal matching of minimum size is certainly not entirely trivial. It's NP-hard in fact.)

The parameterized version of this problem is given a pair $(G, k)$ as input and asks whether $G$ has a maximal matching of size at most $k$. Your goal in this assignment is to design an algorithm that can decide whether $G$ has such a matching in time $O^*\left(4^k\right)$. As always, you'll build up the solution slowly.

## Question 1

A minimal vertex cover of a graph $G$ is a vertex cover $C$ with the property that no proper subset of $C$ is a vertex cover of $G$. Prove that you can test in linear time whether $C$ is a minimal vertex cover.

**Hint:** *Prove that a vertex cover $C$ is minimal if and only if every vertex $v \in C$ has an incident edge $\{u, v\}$ such that $u \notin C$. Explain how to test this condition in linear time.*

## Question 2

Consider the following algorithm for enumerating all minimal vertex covers of size at most $k$ of $G$: The algorithm proceeds in two phases. In the first phase, it constructs a set $\mathcal{C}$ of vertex covers of size at most $k$. The second phase inspects the vertex covers in $\mathcal{C}$ and discards all those that are not minimal, by using your answer to Question 1.

To implement the first phase, we use a recursive procedure. Given input $(G, k)$ with $k \geq 0$, we consider the following cases:

- If $G$ has no edges, then return $\mathcal{C} = \{\emptyset\}$. Clearly, $\emptyset$ is a vertex cover of $G$ in this case, and its size is $0 \leq k$.
- If $G$ has at least one edge but $k = 0$, then return $\mathcal{C} = \emptyset$. Clearly, $G$ has no vertex cover of size at most $k$ in this case.
- IF $G$ has at least one edge and $k > 0$, then choose an arbitrary edge $\{u, v\}$. Recursively call the algorithm on $(G - u, k - 1)$ and $(G - v, k - 1)$. Let $\mathcal{C}_u$ and $\mathcal{C}_v$ be the sets returned by these recursive calls. Then return the set $\mathcal{C} = \{C \cup \{u\} \mid C \in \mathcal{C}_u\} \cup \{C \cup \{v\} \mid C \in \mathcal{C}_v\}$.

(a) Prove that every set $C \in \mathcal{C}$ is a vertex cover of $G$ of size at most $k$.

(b) Prove that $\mathcal{C}$ contains all minimal vertex covers of $G$ of size at most $k$.

(c) Show that this implies that we can produce the set of all minimal vertex covers of $G$ of size $k$ by constructing $\mathcal{C}$ as just described and then discarding all those vertex covers from $\mathcal{C}$ that are not minimal, and argue that this takes $O^*\left(2^k\right)$ time. (**Hint:** *Prove that constructing* $\mathcal{C}$ *takes* $O^*\left(2^k\right)$ *time and that* $|\mathcal{C}| = O\left(2^k\right)$*. The rest should follow fairly easily.*)

## Question 3

Let $M$ be an arbitrary maximal matching of $G$. Let $V(M)$ be the subset of vertices of $G$ matched by $M$. We proved in class that $V(M)$ is a vertex cover of $G$. Thus, there also exists a minimal vertex cover $C$ of $G$ such that $C \subseteq V(M)$. If $|M| \le k$, then $|C| \le |V(M)| \le 2k$, that is, $G$ has a minimal vertex cover of size at most $2k$. Let $M_1$ be a *maximum* matching of $G[C]$, and let $M_2$ be a maximal matching of $G[V(G) \setminus V(M_1)]$. Prove that $M_1 \cup M_2$ is a maximal matching of $G$ of size $|M_1 \cup M_2| \le |M|$. In particular, if $M$ is a minimum maximal matching, then so is $M_1 \cup M_2$!

**Hint:** *Proving that* $M_1 \cup M_2$ *is a maximal matching should be really easy. To bound the size of* $M_1 \cup M_2$*, prove that* $|M_2| \le |X| - 2|M_2|$*. Then split* $M$ *into two subsets* $M_X$ *and* $\bar{M}_X$*, where all edges in* $M_X$ *have both their endpoints in* $X$*, and* $\bar{M}_X = M \setminus M_X$*. Then prove that* $|\bar{M}_X| = |X| - 2|M_X|$*. Finally, use that* $M_2$ *is a* maximum *matching to conclude that* $|M_X| \le |M_2|$*.*

## Question 4

Combine your answers to Questions 1–3 to show that a minimum maximal matching can be found in $O^*\left(4^k\right)$ time.

## Marking Scheme

### Question 1 (5 marks)

| | Yes | Minor mistakes | Major mistakes | No |
|---|---|---|---|---|
| Algorithm correctly tests whether the vertex cover is minimal | 1 mark | | | 0 marks |
| Algorithm runs in linear time | 1 mark | | | 0 marks |
| Correct proof that the algorithms decides correctly whether a vertex cover is minimal | 3 marks | 2 marks | 1 mark | 0 marks |

### Question 2 (10 marks)

| | Yes | Minor mistakes | No |
|---|---|---|---|
| Correct proof that every set in $\mathcal{C}$ is a vertex cover of $G$ (a) | 2 marks | 1 mark | 0 marks |
| Correct proof that $\mathcal{C}$ contains all minimal vertex covers of $G$ of size at most $k$ (b) | 2 marks | 1 mark | 0 marks |
| Correct proof that $\lvert\mathcal{C}\rvert = O\left(2^k\right)$ (c) | 2 marks | 1 mark | 0 marks |
| Correct proof that constructing $\mathcal{C}$ takes $O^*\left(2^k\right)$ time (c) | 2 marks | 1 mark | 0 marks |
| Correct proof that filtering out the non-minimal vertex covers in $\mathcal{C}$ takes $O^*\left(2^k\right)$ time (c) | 2 marks | 1 mark | 0 marks |

### Question 3 (10 marks)

| | Yes | Minor mistakes | Major mistakes | No |
|---|---|---|---|---|
| Correct proof that $M_1 \cup M_2$ is a matching | 2 marks | 1 mark | 0 marks | 0 marks |
| Correct proof that $M_1 \cup M_2$ is maximal | 2 marks | 1 mark | 0 marks | 0 marks |
| Correct proof that $\lvert M_1 \cup M_2\rvert \le \lvert M\rvert$ | 6 marks | 4 marks | 2 marks | 0 marks |

### Question 4 (6 marks)

| | Yes | Minor mistakes | Major mistakes | No |
|---|---|---|---|---|
| Correct proof that a minimum maximal matching can be found in $O^*\left(4^k\right)$ time | 6 marks | 4 marks | 2 marks | 0 marks |

## Submission Instructions

Follow the submission link for this assignment on the course webpage in the email you should have received from Crowdmark. Upload the assignment as a single PDF file.