# Assignment 5

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Solutions

## Question 1

We construct a bipartite multigraph $G = (I, O, E)$, where $I = \{i_1, \ldots, i_n\}$ is the set of input wires, $O = \{o_1, \ldots, o_n\}$ is the set of output wires, and $E = P$ (each packet $(i_j, o_j) \in P$ is interpreted as an edge between $i_j$ and $o_j$).

As observed in Question 2, an edge colouring of $G$ with $k$ colours is simply a partition of $E$ into $k$ matchings $M_1, \ldots, M_k$. This makes $\mathcal{P} = \{M_1, \ldots, M_k\}$ a valid packet routing because no two edges in the same matching $M_h$ share an endpoint, that is, all packets in $M_h$ originate at different input wires and are sent to different output wires. Thus, if $G$ has a valid edge colouring with $k$ colours, then all packets in $P$ can be sent in $k$ time steps.

Conversely, given a valid packet routing $\mathcal{P} = \{P_1, \ldots, P_k\}$, each set $P_h \in \mathcal{P}$, interpreted as a set of edges in $G$, is a matching of $G$ because no two packets in $P_h$ share the same input wire or output wire. Thus, by assigning colour $h$ to the edges in each set $P_h$, we obtain a valid edge colouring of $G$ with $k$ colours.

This implies that $\mathcal{P} = \{P_1, \ldots, P_k\}$ is an optimal packet routing if and only if the corresponding edge colouring of $G$ is an optimal edge colouring of $G$. Indeed, we can represent this colouring as the collection $\mathcal{C} = \{P_1, \ldots, P_k\}$ of matchings. Let $\mathcal{P}^* = \{P^*_1, \ldots, P^*_\ell\}$ be an optimal packet routing, and let $\mathcal{C}^* = \{M^*_1, \ldots, M^*_t\}$ be an optimal edge colouring of $G$ represented as a set of matchings. If $\mathcal{C}$ is not an optimal edge colouring of $G$, then $k > t$. Since $\{M^*_1, \ldots, M^*_t\}$ is also valid packet routing, this shows that $\mathcal{P}$ is not an optimal packet routing. Conversely if $\mathcal{P}$ is not an optimal packet routing, then $k > \ell$. Since $\{P^*_1, \ldots, P^*_\ell\}$ is a valid edge colouring of $G$, this shows that $\mathcal{C}$ is not an optimal edge colouring of $G$.

## Question 2

We describe a recursive algorithm to find an edge colouring of $G$ with $\Delta$ colours.[1]

If $G$ has no edges, which can be checked in constant time, then $\Delta = 0$, and $\mathcal{C} = \emptyset$ is a valid edge colouring of $G$ with 0 colours. Thus, a valid edge colouring of $G$ can be found in at most $c(1 + \Delta nm)$ time in this case, where $c$ is an appropriate constant.

If $\Delta > 0$, then let $M_\Delta$ be a matching of $G$ that matches every degree-$\Delta$ vertex. Then the graph $G' = (V, E \setminus M_\Delta)$ has maximum vertex degree $\Delta - 1$. Indeed, every vertex $v$ of degree $\deg_G(v) < \Delta$ satisfies $\deg_{G'}(v) \le \deg_G(v) < \Delta$. Every vertex $v$ of degree $\deg_G(v) = \Delta$ has an incident edge in $M_\Delta$

[1] This algorithm is easy to convert into an iterative one, but its correctness is easier to establish when describing the algorithm recursively.

and, therefore, satisfies $\deg_{G'}(v) = \deg_G(v) - 1 = \Delta - 1$. Thus, by the induction hypothesis, a valid edge colouring $\mathcal{C}' = \{M_1, \dots, M_{\Delta-1}\}$ of $G'$ can be found in at most $c(1 + (\Delta - 1)nm)$ time. Since $M_\Delta$ is a matching, this makes $\mathcal{C} = \mathcal{C}' \cup \{M_\Delta\}$ a valid edge colouring of $G$ with $\Delta$ colours. As shown in Question 3, the matching $M_\Delta$ can be found in $O(nm)$ time, and the construction of $G'$ from $G$ and $M_\Delta$ takes $O(n + m) = O(nm)$ time. Thus, if we choose $c$ large enough, the construction of $M_\Delta$, of $G'$ from $G$ and $M_\Delta$, and of $\mathcal{C}$ from $\mathcal{C}'$ and $M_\Delta$ takes at most $cnm$ time. Thus, computing $\mathcal{C}$ takes at most $c(1 + \Delta nm) = O(\Delta nm) = O\left(n^2 m\right)$ time.

Since $\Delta$ is the maximum vertex degree of $G$, there exists a vertex of $G$ with $\Delta$ incident edges. Each of these edges must be given a different colour, so there does not exist a valid edge colouring of $G$ with fewer than $\Delta$ colours.

## Question 3

If $G = (U, W, E)$ has no edges, then the desired matching is $M = \emptyset$, which can be constructed in constant time. So assume that $G$ has at least one edge, so the maximum vertex degree $\Delta$ is non-zero. We use the same strategy as in the maximum matching algorithm: We start with the empty matching $M = \emptyset$ and iteratively update this matching until it matches all vertices of degree $\Delta$. In particular, each iteration checks whether $M$ matches all vertices of degree $\Delta$. If so, we return $M$. Otherwise, we construct from $M$ a new matching $M'$ that matches at least one more vertex of degree $\Delta$ than $M$ does. Thus, after at most $n$ iterations, we obtain a matching that matches all vertices of degree $\Delta$. Therefore, it suffices to show how to implement each iteration in $O(m)$ time. We can assume that $G$ does not contain any isolated vertices, as they can be removed in $O(n)$ time before starting the algorithm. Including this preprocessing step, the cost of the algorithm becomes $O(n + nm) = O(nm)$.

So consider the current matching $M$. We can test whether $M$ matches all vertices of degree $\Delta$ in $O(n) = O(m)$ time. If this is the case, then we return $M$. Otherwise, assume that $U$ contains an unmatched vertex of degree $\Delta$. If this is not the case, then $W$ must contain such a vertex, so we can simply exchange the roles of $U$ and $W$. We run alternating BFS from the unmatched vertices of degree $\Delta$ in $U$. This takes $O(n + m) = O(m)$ time. If $F$ contains an unmatched vertex $w \in W$, then the path $P$ in $F$ from $w$ to the root $u$ of the tree in $F$ that contains $w$ is an augmenting path for $M$. Thus, $M \oplus P$ is a matching, and this matching matches $u$ and all vertices matched by $M$. Since $u$ is unmatched by $M$ and has degree $\Delta$, this shows that $M' = M \oplus P$ matches more vertices of degree $\Delta$ than $M$ does. If $F$ contains a matched vertex $u' \in U$ of degree less than $\Delta$, then let $P$ be the path in $F$ from $u'$ to the root $u$ of the tree in $F$ that contains $u'$. Since $G$ is bipartite, this path must have even length. Since $u$ is unmatched, $P$ starts with an edge not in $M$ and, therefore, ends with an edge in $M$. Since $u$ is unmatched by $M$, this implies that $M' = M \oplus P$ is a matching (by the same argument we used in class to show that $M \oplus P$ is a matching if $P$ is an augmenting path). $M'$ matches $u$, which was unmatched by $M$. The only vertex matched by $M$ that is unmatched by $M'$ is $u'$. Since $u$ has degree $\Delta$ and $u'$ has degree less than $\Delta$, $M'$ once again matches one more vertex of degree $\Delta$ than $M$ does. It remains to prove that $F$ must contain one of these two types of vertices: an unmatched vertex in $W$ or a matched vertex of degree less than $\Delta$ in $U$.

Assume the contrary. Then observe that all vertices in $U$ that belong to $F$ have degree $\Delta$. Indeed, the unmatched vertices in $U$ of degree $\Delta$ are chosen as the roots of $F$. All matched vertices in $U$ that belong to $F$ have degree $\Delta$ because we assume that $F$ does not contain a matched vertex of degree less than $\Delta$

in $U$. Finally observe that, apart from the roots of $F$, $F$ contains no unmatched vertices in $U$ because the only non-root vertics in $U$ that alternating BFS adds to $F$ are mates of vertices in $W$ added to $F$.

Since we also assume that $F$ does not contain any unmatched vertex in $W$, all vertices in $W$ that belong to $F$ are matched and, therefore, $F$ also contains their mates. Therefore, $F$ contains the same number of vertices from $W$ as matched vertices from $U$. Since there is at least one unmatched vertex of degree $\Delta$ in $U$, and $F$ contains these vertices as roots, this implies that $F$ contains more vertices from $U$ than from $W$. Let $n_U$ be the number of vertices from $U$ that belong to $F$, let $n_W$ be the number of vertices from $W$ that belong to $F$, and let $t$ be the number of edges in $G$ with both endpoints in $F$. As just observed, we have $n_U > n_W$. Since every vertex in $W$ has degree at most $\Delta$, we have $t \leq \Delta n_W < \Delta n_U$.

A vertex $u \in U$ that belongs to $F$ does not have any neighbours not in $F$. Indeed, if $u$ is unmatched, then none of its incident edges is in $M$. Alternating BFS adds all neighbours of $u$ to $F$ that are connected to $u$ by edges not in $M$. Thus, if $u$ is unmatched, then all its neighbours are in $F$. If $u$ is matched, then its mate in $W$ is in $F$ because mates are added to $F$ together. Once again, neighbours of $u$ connected to $u$ by edges not in $M$ are added to $F$ by alternating BFS. Thus, $F$ contains all neighbours of $u$ also if $u$ is matched. Now recall that we showed above that, under the assumption that $F$ contains no matched vertex of degree less than $\Delta$ in $U$, all vertices in $U$ that belong to $F$ have degree $\Delta$. Thus, $t = \Delta n_U$. Since we proved in the previous paragraph that $t < \Delta n_u$, this is the desired contradiction, that is, $F$ must contain an unmatched vertex in $W$ or a matched vertex of degree less than $\Delta$ in $U$.