Assignment 4

CSCI 4113/6101
Instructor: Norbert Zeh
Due: Nov 4, 2025, 11:59pm

Consider the following **table rounding problem** first encountered when reporting US census data. You have a number of statistics to be reported in a table, represented as an $m \times n$ matrix. In addition, you also want to report the totals for all rows and the totals for all columns. The numbers in the raw data you want to report are arbitrary real numbers. Here's an example:

3.4	4.8	9.1	17.3
2.7	9.9	0.9	13.5
6.1	14.7	10.0	Σ

For the consumer of your information, the data looks less messy and is easier to digest if all the numbers in the table are integers. Thus, you want to round the table entries to be integers. You do not need to round to the closest integer: the only requirement is that each table entry x gets rounded to $\lfloor x \rfloor$ or $\lceil x \rceil$.

Rounding these entries up or down is easy enough, but we want to ensure that the rounded row and column totals are the totals of the rounded entries in each row or column:

3	5	9	17
3	10	1	14
6	15	10	Σ

In this example, rounding each number to its closest integer worked well. In general, it doesn't. For example, if we have a row with entries 1.6, 2.6, 3.6, 4.6, whose total is 12.4, then it would be natural to round the total down to 12 and the row entries up to 2, 3, 4, 5. The total of the row entries is thus $14 \neq 12$. Even worse, even if we rounded the row total up, we would obtain a total of 13, which is still not the same as the total of the rounded row entries. Thus, we need to round one entry in the row down, but choosing this entry arbitrarily may create similar problems with the column totals. We need to take all table entries into account in order to decide which entries should be rounded up or down.

Formally, let us refer to the input matrix as A, to the vector of row totals as R, and to the vector of column totals as C. We call a triple $\left(\tilde{A}, \tilde{R}, \tilde{C}\right)$ composed of an integer matrix $\tilde{A}$, and two integer vectors $\tilde{R}$ and $\tilde{C}$ a **feasible rounding** of (A, R, C) if

- $\tilde{A}$, $\tilde{R}$, and $\tilde{C}$ have the same dimensions as A, R, and C, respectively,
- $\tilde{A}_{ij} \in \{\lfloor A_{ij} \rfloor, \lceil A_{ij} \rceil\}$, for all $1 \le i \le m, 1 \le j \le n$,
- $\tilde{R}_i \in \{\lfloor R_i \rfloor, \lceil R_i \rceil\}$, for all $1 \le i \le m$,
- $\tilde{C}_j \in \{\lfloor C_j \rfloor, \lceil C_j \rceil\}$, for all $1 \le j \le n$,
- $\sum_{j=1}^{n} \tilde{A}_{ij} = \tilde{R}_i$, for all $1 \le i \le m$, and
- $\sum_{i=1}^{m} \tilde{A}_{ij} = \tilde{C}_j$, for all $1 \le j \le n$.

Our goal is to find a feasible rounding of (A, R, C). Since the Edmonds-Karp Algorithm can find a maximum flow in a network in $O\left(nm^2\right)$ time, the answers to the following four questions produce an algorithm that can find a feasible rounding of any triple (A, R, C) in $O\left((m+n)(nm)^2\right)$ time.[1]

Question 1

Given a directed graph $G = (V, E)$ and two functions $\ell, u : E \to \mathbb{R}$ with $\ell_e \leq u_e$, for all $e \in E$, a feasible circulation is a function $f : E \to \mathbb{R}$ such that

Capacity constraints: $\ell_e \leq f_e \leq u_e$, for all $e \in E$, and
Flow conservation: $\sum_{v \in V}(f_{u,v} - f_{v,u}) = 0$, for all $v \in V$.

Given an input (A, R, C) to the table rounding problem, construct from it a triple (G, ℓ, u) such that every integral feasible circulation corresponds to a feasible rounding of (A, R, C) and vice versa. Specifically, G should (possibly in addition to other edges) contain one edge per entry in A, R, and C, and the integral flow along each such edge should represent the corresponding entry in $\tilde{A}$, $\tilde{R}$ or $\tilde{C}$. The graph G should have $O(m+n)$ vertices and $O(mn)$ edges. Prove that this one-to-one correspondence between feasible roundings and integral feasible circulations in (G, ℓ, u) holds.

Question 2

Given an input (G, ℓ, u) of the feasible circulation problem, show how to construct from it a network $G' = (V', E')$ along with non-negative edge capacities $c : E' \to \mathbb{R}_{\geq 0}$ and two vertices $s, t \in V'$ such that there exists a feasible circulation in G if and only if the maximum st-flow in G' has at least some value F. The graph G' should have $O(n)$ vertices and $O(m)$ edges, where, in this question, n and m denote the numbers of vertices and edges in G, respectively. As part of your proof, show how to construct a feasible circulation in G from a maximum flow of value F in G'. Prove that if this maximum flow is integral, then the corresponding feasible circulation is also integral.

Question 3

Prove that, if all edge capacities are integers, then the Ford-Fulkerson Algorithm (and, thus, the Edmonds-Karp Algorithm) finds an integral maximum flow.

Question 4

Use your answers to Question 4 to prove that there always exists a feasible rounding of any triple (A, R, C) such that R is the vector of row totals in A, and C is the vector of column totals in A.

[1]Using a faster maximum flow algorithm also produces a faster algorithm for the table rounding problem.

Marking Scheme

Question 1 (9 marks)

	Yes	Minor mistakes	Major mistakes	No
Integral feasible circulation in (G, ℓ, u) corresponds to feasible rounding	1 mark			0 marks
Feasible rounding corresponds to integral feasible circulation in (G, ℓ, u)	1 mark			0 marks
G has the required number of vertices and edges	1 mark			0 marks
Correct proof that a feasible rounding corresponds to an integral feasible circulation	3 marks	2 marks	1 mark	0 marks
Correct proof that an integral feasible circulation corresponds to a feasible rounding	3 marks	2 marks	1 mark	0 marks

Question 2 (11 marks marks)

	Yes	Minor mistakes	Major mistakes	No
Feasible circulation in (G, ℓ, u) corresponds to *st*-flow of value F in (G', c)	1 mark			0 marks
st-flow of value F in (G', c) corresponds to feasible circulation in (G, ℓ, u)	1 mark			0 marks
G' has the required number of vertices and edges	1 mark			0 marks
Correct proof that feasible circulation in (G, ℓ, u) corresponds to *st*-flow of value F in (G', c)	3 marks	2 marks	1 mark	0 marks
Correct proof that *st*-flow of value F in (G', c) corresponds to feasible circulation in (G, ℓ, u)	3 marks	2 marks	1 mark	0 marks
Proof gives construction of feasible circulation from maximum flow of value F	1 mark			0 marks
Proof that the circulation is integral if the flow is integral	1 mark			0 marks

Question 3 (3 marks)

	Yes	Minor mistakes	Major mistakes	No
Correct proof	3 marks	2 marks	1 mark	0 marks

Question 4 (3 marks)

	Yes	Minor mistakes	Major mistakes	No
Correct proof	3 marks	2 marks	1 mark	0 marks

Submission Instructions

Follow the submission link for this assignment on the course webpage in the email you should have received from Crowdmark. Upload the assignment as a single PDF file.