Assignment 4

CSCI 4113/6101
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Solutions

Question 1

Given (A,R,C), we construct the following triple (G,ℓ,u):

- G has m vertices $r_1,\ldots,r_m$, n vertices $c_1,\ldots,c_n$, and one additional vertex s.
- G has edges (s,r_i), for all $i \in [m]$, edges (c_j,s), for all $j \in [n]$, and edges (r_i,c_j), for all $i \in [m]$, $j \in [n]$.
- For each edge (s,r_i), we set $\ell_{s,r_i} = \lfloor R_i \rfloor$ and $u_{s,r_i} = \lceil R_i \rceil$.
- For each edge (c_j,s), we set $\ell_{c_j,s} = \lfloor C_j \rfloor$ and $u_{c_j,t} = \lceil C_j \rceil$.
- For each edge (r_i,c_j), we set $\ell_{r_i,c_j} = \lfloor A_{i,j} \rfloor$ and $u_{r_i,c_j} = \lceil A_{i,j} \rceil$.

The graph for the example in the assignment is shown in Fig. 1. This graph as $m+n+1 = O(m+n)$ vertices and $mn+m+n = O(mn)$ edges. We need to prove that there exists a one-to-one correspondence between feasible roundings of (A,R,C) and integral feasible circulations in (G,ℓ,u).

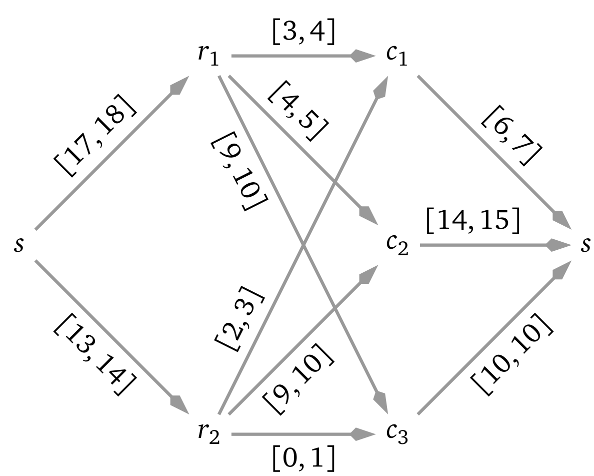

Figure 1: The graph (G,ℓ,u) constructed from the rounding instance in the assignment. Edge labels are of the form $[\ell_e,u_e]$. For ease of drawing the edges incident to s, this vertex is shown twice, once on the left, once on the right.

First, assume that $(\tilde{A}, \tilde{R}, \tilde{C})$ is a feasible rounding of (A, R, C). Then define a circulation f as

$$\begin{aligned} f_{s,r_i} &= \tilde{R}_i && \forall i \in [m] \\ f_{c_j,s} &= \tilde{C}_j && \forall j \in [n] \\ f_{r_i,c_j} &= \tilde{A}_{i,j} && \forall i \in [m], j \in [n] \end{aligned}$$

This circulation f satisfies the capacity constraints because

$$\begin{aligned} \ell_{s,r_i} &= \lfloor R_i \rfloor \leq \tilde{R}_i = f_{s,r_i} \leq \lceil R_i \rceil = u_{s,r_i} && \forall i \in [m] \\ \ell_{c_j,s} &= \lfloor C_j \rfloor \leq \tilde{C}_j = f_{c_j,s} \leq \lceil C_j \rceil = u_{c_j,s} && \forall j \in [n] \\ \ell_{r_i,c_j} &= \lfloor A_{i,j} \rfloor \leq \tilde{A}_{i,j} = f_{r_i,c_j} \leq \lceil A_{i,j} \rceil = u_{r_i,c_j} && \forall i \in [m], j \in [n] \end{aligned}$$

Flow conservation is satisfied because $(\tilde{A}, \tilde{R}, \tilde{C})$ is a feasible rounding, that is,

$$\begin{aligned} f_{s,r_i} = \quad & \tilde{R}_i = \sum_{j \in [n]} \tilde{A}_{i,j} = \sum_{j \in [n]} f_{r_i,c_j} && \forall i \in [m] \\ f_{c_j,t} = \quad & \tilde{C}_j = \sum_{i \in [m]} \tilde{A}_{i,j} = \sum_{i \in [m]} f_{r_i,c_j} && \forall j \in [n] \\ \sum_{j \in [n]} f_{c_j,s} = \sum_{j \in [n]} \tilde{C}_j &= \sum_{i \in [m]} \sum_{j \in [n]} \tilde{A}_{i,j} = \sum_{i \in [m]} \tilde{R}_i = \sum_{i \in [m]} f_{s,r_i}. \end{aligned}$$

Since we have shown that f is a feasible circulation, and it is obviously integral, every feasible rounding gives rise to an integral feasible circulation.

Now assume we have a feasible integral circulation f in (G, ℓ, u). Then we define a feasible rounding $(\tilde{A}, \tilde{R}, \tilde{C})$ as

$$\begin{aligned} \tilde{R}_i &= f_{s,r_i} && \forall i \in [m] \\ \tilde{C}_j &= f_{c_j,s} && \forall j \in [n] \\ \tilde{A}_{i,j} &= f_{r_i,c_j} && \forall i \in [m], j \in [n]. \end{aligned}$$

Since f is integral, all entries in $\tilde{A}$, $\tilde{R}$, and $\tilde{C}$ are integral.

Since f_{s,r_i} is integral in $[\lfloor R_i \rfloor, \lceil R_i \rceil]$, we have $\tilde{R}_i = f_{s,r_i} \in \{\lfloor R_i \rfloor, \lceil R_i \rceil\}$, for all $i \in [m]$, as required. By an analogous argument, we have $\tilde{C}_j \in \{\lfloor C_j \rfloor, \lceil C_j \rceil\}$, for all $j \in [n]$, and $\tilde{A}_{i,j} \in \{\lfloor A_{i,j} \rfloor, \lceil A_{i,j} \rceil\}$, for all $i \in [m], j \in [n]$.

By flow conservation, we have

$$\begin{aligned} \tilde{R}_i &= f_{s,r_i} = \sum_{j \in [n]} f_{r_i,c_j} = \sum_{j \in [n]} \tilde{A}_{i,j} && \forall i \in [m] \\ \tilde{C}_j &= f_{c_j,s} = \sum_{i \in [m]} f_{r_i,c_j} = \sum_{i \in [m]} \tilde{A}_{i,j} && \forall j \in [n]. \end{aligned}$$

Thus, $(\tilde{A}, \tilde{R}, \tilde{C})$ is a feasible rounding of (A, R, C).

QUESTION 2

We construct (G', c') from G so that there exists a feasible circulation in (G, ℓ, u) if and only if there exists a flow in (G', c') that saturates all out-edges of s. Clearly, such a flow is a maximum flow.

Given (G, ℓ, u), we define (G', c') as follows: We start with G and add two vertices s and t. For each vertex $x \in V$, we add two edges (s, x) and (x, t) to G'. For each edge (x, y) of G, we define

$$c'_{x,y} = u_{x,y} - \ell_{x,y}.$$

For each vertex x of G, let

$$L_x = \sum_{y \in V} (\ell_{x,y} - \ell_{y,x}).$$

Then we define

$$c'_{s,x} = \max(0, -L_x) \quad \text{and} \quad c'_{x,t} = \max(0, L_x).$$

We need to prove that there exists a feasible circulation in G if and only if there exists a flow in G' that saturates all out-edges of s.

First, assume that there exists a feasible circulation f in (G, ℓ, u). Then we define a flow f' in G' as follows: For each edge $(x, y) \in G$, we set $f'_{x,y} = f_{x,y} - \ell_{x,y}$. For each vertex x of G, we set $f'_{s,x} = c'_{s,x}$ and $f'_{x,t} = c'_{x,t}$. This flow clearly saturates all out-edges of s. We need to show that it is feasible.

The capacity constraints are satisfied:

$$\begin{aligned}
0 = \ell_{x,y} - \ell_{x,y} \le f'_{x,y} &= f_{x,y} - \ell_{x,y} \le u_{x,y} - \ell_{x,y} = c'_{x,y} && \forall (x, y) \in G \\
0 \le c'_{s,x} &= f'_{s,x} = c'_{s,x} && \forall x \in V \\
0 \le c'_{x,t} &= f'_{x,t} = c'_{x,t} && \forall x \in V,
\end{aligned}$$

where the first row holds because f is a feasible circulation, that is, $\ell_{x,y} \le f_{x,y} \le u_{x,y}$, for all $(x, y) \in G$.

Flow conservation is also satisfied: Since f is a feasible circulation, we have

$$\sum_{y \in V} (f_{x,y} - f_{y,x}) = 0 \quad \forall x \in V.$$

We also have

$$f'_{x,t} - f'_{s,x} = \max(0, L_x) - \max(0, -L_x) = L_x = \sum_{y \in V} (\ell_{x,y} - \ell_{y,x}) \quad \forall x \in V.$$

Thus,

$$\begin{aligned}
\sum_{y \in V} (f'_{x,y} - f'_{y,x}) + f'_{x,t} - f'_{s,x} &= \sum_{y \in V} (f_{x,y} - \ell_{x,y} - f_{y,x} + \ell_{y,x}) + \sum_{y \in V} (\ell_{x,y} - \ell_{y,x}) \\
&= \sum_{y \in V} (f_{x,y} - f_{y,x}) = 0.
\end{aligned}$$

Conversely, assume that there exists a feasible flow f' in G' that saturates all out-edges of s. The total out-flow of s equals the total in-flow of t, due to flow-conservation. Next observe that the total capacity of the out-edges of s also equals the total capacity of the in-edges of t:

$$\sum_{x \in V} c'_{x,t} - \sum_{x \in V} c'_{s,x} = \sum_{x \in V} \max(0, L_x) - \sum_{x \in V} \max(0, -L_x) = \sum_{x \in V} L_x = \sum_{x \in V} \sum_{y \in V} (\ell_{x,y} - \ell_{y,x}) = 0,$$

because the final sum counts every edge in G twice, once with positive sign and once with negative sign. Since the total out-flow of s equals the total in-flow of t, and the total capacity of the out-edges of s equals the total capacity of the in-edges of G, a feasible flow that saturates the out-edges of s also saturates the in-edges of t.

Now define $f_{x,y} = f'_{x,y} + \ell_{x,y}$, for all $(x, y) \in G$. Then this is a feasible circulation in G. Indeed, the capacity constraints are satisfied:

$$\ell_{x,y} = 0 + \ell_{x,y} \leq f_{x,y} = f'_{x,y} + \ell_{x,y} \leq c'_{x,y} + \ell_{x,y} = u_{x,y} \quad \forall (x, y) \in G$$

Flow conservation is also satisfied: Since f' is a feasible flow, we have

$$\sum_{y \in V} (f'_{x,y} - f'_{y,x}) + f'_{x,t} - f'_{s,x} = 0 \quad \forall x \in V,$$

so

$$\begin{aligned}
\sum_{y \in V} (f_{x,y} - f_{y,x}) &= \sum_{y \in V} (f'_{x,y} + \ell_{x,y} - f'_{y,x} - \ell_{y,x}) \\
&= \sum_{y \in V} (f'_{x,y} - f'_{y,x}) + \sum_{y \in V} (\ell_{x,y} - \ell_{y,x}) \\
&= \sum_{y \in V} (f'_{x,y} - f'_{y,x}) + L_x \\
&= \sum_{y \in V} (f'_{x,y} - f'_{y,x}) + \max(0, L_x) - \max(0, -L_x) \\
&= \sum_{y \in V} (f'_{x,y} - f'_{y,x}) + f'_{x,t} - f'_{s,x} = 0.
\end{aligned}$$

From an (integral) flow to a feasible (integral) circulation. The above proofs explicitly demonstrate how to construct a feasible circulation f in G from a feasible flow f' in G' that saturates all out-edges of s: We define

$$f_{x,y} = f'_{x,y} + \ell_{x,y}.$$

If f' is an integral flow and all flow lower bounds ℓ are integral, this ensures that f is an integral circulation.

Question 3

Ford-Fulkerson starts with the all-zero flow, which is clearly integral. If all edge capacities are integers and f is an integral flow, then all residual capacities in G^f are integral. Therefore, the minimum residual capacity of the edges on an augmenting path is an integer. Thus, the integral flow along each edge on this path gets adjusted by an integral amount and, therefore, remains integral.

We can now use induction on the number of augmentations applied to the flow to show that it is always integral. As just observed, this is true before the first augmentation because the flow along all edges in 0. If the flow before the ith augmentation is integral, then, as just shown, the flow after this augmentation is also integral. Thus, the flow returned by the algorithm is integral.

(A similar argument shows that the Push-Relabel Algorithm also computes an integral flow if all edge capacities are integers, so we can also use this faster algorithm to find a feasible rounding.)

QUESTION 4

Consider an input (A, R, C) to the table rounding problem, the corresponding circulation instance (G, ℓ, u), and the corresponding flow network $\left(G', c'\right)$. The construction in Question 1 ensures that all values in ℓ and u are integers. Thus, all capacities in c' are integers. Thus, as shown in Question 3, Ford-Fulkerson finds an integral maximum flow f' in $\left(G', c'\right)$. If this flow saturates all out-edges of s in G', then Question 2 shows that this flow f' corresponds to an integral feasible circulation f in G, which in turn corresponds to a feasible rounding of (A, R, C), as shown in Question 1. What we need to show is that a maximum flow in G' saturates all out-edges of G'.

Question 2 shows that there exists a flow in G' that saturates all out-edges of s in G' (which is trivially a maximum flow) if there exists a feasible circulation in G. Thus, we need to show that such a circulation exists. Such a circulation f is easy to obtain from (A, R, C) though, using a similar construction as in Question 1. We define

$$\begin{aligned} f_{s,r_i} &= R_i && \forall i \in [m] \\ f_{c_j,s} &= C_j && \forall j \in [n] \\ f_{r_i,c_j} &= A_{i,j} && \forall i \in [m], j \in [n]. \end{aligned}$$

This circulation f satisfies the capacity constraints because

$$\begin{aligned} \ell_{s,r_i} &= \lfloor R_i \rfloor \le R_i = f_{s,r_i} \le \lceil R_i \rceil = u_{s,r_i} && \forall i \in [m] \\ \ell_{c_j,s} &= \lfloor C_j \rfloor \le C_j = f_{c_j,s} \le \lceil C_j \rceil = u_{c_j,s} && \forall j \in [n] \\ \ell_{r_i,c_j} &= \lfloor A_{i,j} \rfloor \le A_{i,j} = f_{r_i,c_j} \le \lceil A_{i,j} \rceil = u_{r_i,c_j} && \forall i \in [m], j \in [n]. \end{aligned}$$

Flow conservation is satisfied because R and C are the row and column totals of A, so

$$\begin{aligned} f_{s,r_i} &= R_i = \sum_{j\in[n]} A_{i,j} = \sum_{j\in[n]} f_{r_i,c_j} && \forall i \in [m] \\ f_{c_j,s} &= C_j = \sum_{i\in[m]} A_{i,j} = \sum_{i\in[m]} f_{r_i,c_j} && \forall j \in [n] \\ \sum_{j\in[n]} f_{c_j,s} &= \sum_{j\in[n]} C_j = \sum_{i\in[m]} \sum_{j\in[n]} A_{i,j} = \sum_{i\in[m]} R_i = \sum_{i\in[m]} f_{s,r_i}. \end{aligned}$$