

ASSIGNMENT 3

CSCI 4113/6101

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Consider an LP

$$\begin{aligned} &\text{Maximize } c^T x \\ &\text{s.t. } Ax \leq b \\ &\quad x \geq 0 \end{aligned} \tag{1}$$

and its dual

$$\begin{aligned} &\text{Minimize } b^T y \\ &\text{s.t. } A^T y \geq c \\ &\quad y \geq 0. \end{aligned} \tag{2}$$

In class, we proved that two feasible solutions $\hat{x}$ and $\hat{y}$ of (1) and (2), respectively, are optimal solutions if and only if they satisfy complementary slackness:

$$\begin{aligned} \text{Primal complementary slackness: } \hat{x}_j = 0 \quad &\text{or} \quad \sum_{i=1}^m a_{ij} \hat{y}_i = c_j \quad \forall j \in [n] \\ \text{Dual complementary slackness: } \hat{y}_i = 0 \quad &\text{or} \quad \sum_{j=1}^n a_{ij} \hat{x}_j = b_i \quad \forall i \in [m] \end{aligned}$$

If we relax these conditions, then we obtain what is called (α, β) -relaxed complementary slackness:

$$\begin{aligned} \alpha\text{-Relaxed primal complementary slackness: } \hat{x}_j = 0 \quad &\text{or} \quad \sum_{i=1}^m a_{ij} \hat{y}_i \leq \alpha c_j \quad \forall j \in [n] \\ \beta\text{-Relaxed dual complementary slackness: } \hat{y}_i = 0 \quad &\text{or} \quad \sum_{j=1}^n a_{ij} \hat{x}_j \geq \frac{b_i}{\beta} \quad \forall i \in [m] \end{aligned}$$

QUESTION 1: A GENERAL APPROXIMATION GUARANTEE

While $\hat{x}$ and $\hat{y}$ are optimal solutions if they satisfy complementary slackness, we should not hope that this holds if they satisfy only (α, β) -relaxed complementary slackness, for some $\alpha > 1$ and $\beta > 1$. In fact, they can't be because strict complementary slackness is a sufficient and *necessary* condition for $\hat{x}$ and $\hat{y}$ to both be optimal solutions. We can still prove that they are not too far away from optimality though:

Let $\hat{x}$ and $\hat{y}$ be feasible solutions of (1) and (2), respectively, let $\tilde{x}$ and $\tilde{y}$ be optimal solutions of (1) and (2), respectively, and assume that $\hat{x}$ and $\hat{y}$ satisfy (α, β) -relaxed complementary slackness. Prove that

$$c^T \hat{x} \geq \frac{c^T \tilde{x}}{\alpha \beta}$$

and

$$b^T \hat{y} \leq \alpha \beta \cdot b^T \tilde{y}.$$

QUESTION 2: A 2-APPROXIMATION FOR WEIGHTED VERTEX COVER

A vertex cover of an undirected graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that every edge has at least one endpoint in C . In the *unweighted vertex cover problem*, we are given an undirected graph G , and our goal is to find a vertex cover C^* of G of minimum size: for every vertex cover C of G , we have $|C| \geq |C^*|$. This is an NP-hard problem. A simple, greedy, linear-time algorithm is able to find a vertex cover $\tilde{C}$ such that $|\tilde{C}| \leq 2|C^*|$, for every vertex cover C of G . In particular, $\tilde{C}$ is at most twice as big as the smallest vertex cover we could possibly find.

Sometimes, not all vertices are created equal. In the *weighted vertex cover problem*, we are given an undirected graph $G = (V, E)$ and a weight function $w : V \rightarrow \mathbb{R}^+$ assigning a positive weight to every vertex. Our goal is to find a vertex cover C^* of minimum weight: for every vertex cover C of G , we have $\sum_{v \in C^*} w_v \leq \sum_{v \in C} w_v$. We refer to the quantity $\sum_{v \in C} w_v$ as the weight $w(C)$ of C . Clearly, this problem is also NP-hard. Moreover, the simple greedy algorithm no longer finds a good approximation of an optimal solution. However, we can still find a good vertex cover in polynomial time. Here is the algorithm to do this:

We start by setting $C = \emptyset$ and assigning a potential $\pi_{u,v} = 0$ to every pair $\{u, v\}$ of vertices in V . Note that this guarantees that

$$\sum_{v \in V} \pi_{u,v} \leq w_u,$$

for every vertex $u \in V$. This is an invariant we maintain.

Now, as long as there exists an edge $\{u, v\} \in E$ with $u, v \notin C$, let

$$\delta_u = w_u - \sum_{z \in V} \pi_{u,z}$$

and

$$\delta_v = w_v - \sum_{z \in V} \pi_{v,z}.$$

If $\delta_u \leq \delta_v$, then we increase $\pi_{u,v}$ by δ_u and add u to C . If $\delta_v < \delta_u$, then we increase $\pi_{u,v}$ by δ_v and add v to C .

Once every edge has an endpoint in C , the algorithm terminates and returns the current set C , which is clearly a vertex cover of G .

It is not difficult to prove that this algorithm runs in $O(nm)$ time: we simply iterate over all edges. For each edge $\{u, v\}$, we test in constant time whether one of u and v already belongs to C . If so, we proceed to the next edge. Otherwise, we calculate δ_u and δ_v in $O(n)$ time, by iterating over all edges incident to u and v , and then we update $\pi_{u,v}$ as above and add u or v to C in constant time. Overall, the algorithm runs for m iterations, each of which takes $O(n)$ time.

Your goal in this question is to prove that if C is the vertex cover returned by the algorithm and C^* is a vertex cover of G of minimum weight, then $w(C) \leq 2w(C^*)$.

QUESTION 2A

Formulate the weighted vertex cover problem as an ILP. In this formulation, you should have a variable $x_v \in \{0, 1\}$, for every vertex $v \in V$, such that $x_v = 1$ means that the vertex is in the vertex cover, and $x_v = 0$ means that v is not in the vertex cover.

QUESTION 2B

Construct the dual of the LP relaxation of the ILP in Question 2a. In this dual, you should have a variable $y_{u,v}$ associated with every edge $\{u, v\} \in E$.

QUESTION 2C

Consider the vertex cover C returned by the algorithm, consider the edge potentials $\pi_{u,v}$ at the time the algorithm terminates, let

$$\hat{x}_v = \begin{cases} 1 & \text{if } v \in C \\ 0 & \text{otherwise,} \end{cases}$$

and let $\hat{y}_{u,v} = \pi_{u,v}$, for every edge $\{u, v\} \in E$.

Prove that these two solutions $\hat{x}$ and $\hat{y}$ are feasible solutions of the (I)LPs in Questions 2a and 2b, and that they satisfy (1, 2)-relaxed complementary slackness. Argue that this implies that $w(C) \leq 2w(C^*)$.

MARKING SCHEME

QUESTION 1 (5 MARKS)

	Yes	Minor mistakes	Major mistakes	No
Correct proof	5 marks	3 marks	1 mark	0 marks

QUESTION 2A (3 MARKS)

	Yes	Some mistakes	No
Objective function expresses that we need to minimize $w(C)$	1 mark		0 marks
Constraints express that C is a vertex cover	2 marks	1 mark	0 marks

QUESTION 2B (4 MARKS)

	Yes	Some mistakes	No
Correct dual objective function	1 mark		0 marks
Correct dual constraints corresponding to primal variables	2 mark	1 mark	0 marks
Non-negativity constraints for the dual variables	1 mark		0 marks

QUESTION 2C (11 MARKS)

	Yes	Minor mistakes	Major mistakes	No
Correct proof that $\hat{x}$ is a feasible primal solution	1 mark			0 marks
Correct proof that $\hat{y}$ is a feasible dual solution	4 marks	2 marks	1 mark	0 marks
Correct proof that $\hat{x}$ and $\hat{y}$ satisfy 2-relaxed dual complementary slackness	1 mark			0 marks
Correct proof that $\hat{x}$ and $\hat{y}$ satisfy strict primal complementary slackness	4 marks	2 marks	1 mark	0 marks
Correct argument why this implies that $w(C) \leq w(C^*)$, for an optimal vertex cover C^*	1 mark			0 marks

SUBMISSION INSTRUCTIONS

Follow the submission link for this assignment on the course webpage in the email you should have received from Crowdmark. Upload the assignment as a single PDF file.