Assignment 3

CSCI 4113/6101
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Solutions

Question 1

Since $\hat{x}$ and $\hat{y}$ are feasible solutions, and $\tilde{x}$ and $\tilde{y}$ are optimal solutions, we have

$$c^T\hat{x} \le c^T\tilde{x} \text{ and } b^T\tilde{y} \le b^T\hat{y}.$$

By weak LP duality, we also have

$$c^T\tilde{x} \le b^T\tilde{y}.$$

Thus,

$$c^T\hat{x} \le c^T\tilde{x} \le b^T\tilde{y} \le b^T\hat{y}.$$

Therefore, it suffices to prove that

$$b^T\hat{y} \le \alpha\beta \cdot c^T\hat{x}. \tag{1}$$

Indeed, this implies that

$$b^T\hat{y} \le \alpha\beta \cdot c^T\hat{x} \le \alpha\beta\, b^T\tilde{y}$$

and

$$c^T\hat{x} \ge \frac{b^T\hat{y}}{\alpha\beta} \ge \frac{c^T\tilde{x}}{\alpha\beta}.$$

To prove (1), note that, for all $i \in [m]$,

$$b_i\hat{y}_i \le \beta\hat{y}_i\sum_{j=1}^{n} a_{ij}\hat{x}_j = \beta\sum_{j=1}^{n} a_{ij}\hat{y}_i\hat{x}_j, \tag{2}$$

no matter whether $\hat{y}_i = 0$ or not, due to β-relaxed dual complementary slackness.

Similarly, for all $i \in [n]$,

$$\sum_{i=1}^{m} a_{ij}\hat{y}_i\hat{x}_j = \hat{x}_j\sum_{i=1}^{m} a_{ij}\hat{y}_i \le \alpha c_j\hat{x}_j, \tag{3}$$

no matter whether $\hat{x}_j = 0$ or not, due to α-relaxed primal complementary slackness.

Thus,

$$\begin{aligned}\sum_{i=1}^{m} b_i\hat{y}_i &\overset{(2)}{\le} \beta\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\hat{y}_i\hat{x}_j\\ &\overset{(3)}{\le} \alpha\beta\sum_{j=1}^{0} c_j\hat{x}_j,\end{aligned}$$

which is exactly what (1) states.

Question 2a

$$\begin{aligned} \text{Minimize} \sum_{v \in V} w_v x_v & \\ \text{s.t. } x_u + x_v \geq 1 \qquad & \forall \{u, v\} \in E \\ x_v \in \{0, 1\} \quad & \forall v \in V. \end{aligned} \tag{4}$$

Question 2b

$$\begin{aligned} \text{Maximize} \sum_{\{u,v\} \in E} y_{u,v} & \\ \text{s.t.} \sum_{\substack{v \in V \\ \{u,v\} \in E}} y_{u,v} \leq w_u \quad & \forall u \in V \\ y_{u,v} \geq 0 \qquad & \forall \{u, v\} \in E. \end{aligned} \tag{5}$$

Question 2c

Feasibility of the primal solution. By construction, every edge $\{u, v\} \in E$ has at least one endpoint in C, so

$$\hat{x}_u + \hat{x}_v \geq 1,$$

for every edge $\{u, v\} \in E$. Since we also have $\hat{x}_v \in \{0, 1\}$, for all $v \in V$, this makes $\hat{x}$ a feasible solution of (4).

Feasibility of the dual solution. To prove that $\hat{y}$ is a feasible solution of (5), we do not set $\hat{y}_{u,v} = \pi_{u,v}$, for all $\{u, v\} \in V$, only at the end of the algorithm but maintain this equality at all times while the algorithm runs. This allows us to prove that $\hat{y}$ is a feasible solution initially and that every update of π and the corresponding update of $\hat{y}$ keeps $\hat{y}$ feasible.

Initially, we have $\hat{y}_{u,v} = \pi_{u,v} = 0$, for all $\{u, v\} \in E$. Thus, $\hat{y}$ satisfies all non-negativity constraints in (5). Since $w_u > 0$, for all $u \in V$, this also ensures that

$$\sum_{\substack{v \in V \\ \{u,v\} \in E}} \hat{y}_{u,v} \leq w_u,$$

for all $u \in V$. Thus, $\hat{y}$ is a feasible solution of (5) initially.

Each iteration of the algorithm that updates π inspects an edge $\{u, v\} \in E$ and calculates

$$\delta_u = w_u - \sum_{z \in V} \pi_{u,z} = w_u - \sum_{\substack{z \in V \\ \{u,z\} \in E}} \hat{y}_{u,z}$$

and

$$\delta_v = w_v - \sum_{z \in V} \pi_{v,z} = w_v - \sum_{\substack{z \in V \\ \{v,z\} \in E}} \hat{y}_{v,z},$$

where the second equality in each of these equations follows because the algorithm never changes $\pi_{x,y}$ from its initial value of 0 unless $\{x, y\} \in E$, and we have $\hat{y}_{x,y} = \pi_{x,y}$, for every edge $\{x, y\} \in E$. Then it increases $\hat{y}_{u,v} = \pi_{u,v}$ by $\delta = \min(\delta_u, \delta_v)$. Let us refer to the value of $\hat{y}$ before the current iteration as $\hat{y}$, and to the value of $\hat{y}$ after the current iteration as $\bar{y}$.

Since we have

$$\sum_{\substack{z \in V \\ \{u,z\} \in E}} \hat{y}_{u,z} \le w_u$$

and

$$\sum_{\substack{z \in V \\ \{v,z\} \in E}} \hat{y}_{v,z} \le w_v$$

we have $\delta_u, \delta_v \ge 0$ and, therefore, $\delta = \min(\delta_u, \delta_v) \ge 0$. Thus, $\bar{y}_{u,v} \ge \hat{y}_{u,v} \ge 0$. Since $\bar{y}_{x,z} = \hat{y}_{x,z} \ge 0$, for all $\{x, z\} \ne \{u, v\}$, this shows that $\bar{y} \ge 0$.

For all $x \notin \{u, v\}$, we have

$$\sum_{\substack{z \in V \\ \{x,z\} \in E}} \bar{y}_{x,z} = \sum_{\substack{z \in V \\ \{x,z\} \in E}} \hat{y}_{x,z} \le w_x.$$

For u and v, we have

$$\sum_{\substack{z \in V \\ \{u,z\} \in E}} \bar{y}_{u,z} = \sum_{\substack{z \in V \\ \{u,z\} \in E}} \hat{y}_{u,z} + \delta \le \sum_{\substack{z \in V \\ \{u,z\} \in E}} \hat{y}_{u,z} + \delta_u = w_u$$

and

$$\sum_{\substack{z \in V \\ \{v,z\} \in E}} \bar{y}_{v,z} = \sum_{\substack{z \in V \\ \{v,z\} \in E}} \hat{y}_{v,z} + \delta \le \sum_{\substack{z \in V \\ \{v,z\} \in E}} \hat{y}_{v,z} + \delta_v = w_u.$$

Thus, $\bar{y}$ also satisfies all the constraints of (5) corresponding to the vertices of G, that is, $\bar{y}$ is a feasible solution of (5).

Since we have shown that $\hat{y}$ is feasible at the beginning of the algorithm and that every update of $\hat{y}$ maintains the feasibility of $\hat{y}$, $\hat{y}$ is also a feasible solution of (5) at the end of the algorithm.

Complementary slackness. Next, we verify the complementary slackness conditions. Again, we prove that the two solutions satisfy these conditions at all times during the execution of the algorithm.

2-relaxed dual complementary slackness is easy to verify: Since $\hat{x}_v \in \{0, 1\}$, for all $v \in V$, we have $\hat{x}_u + \hat{x}_v \le 2$, for all $\{u, v\} \in E$, no matter whether $\hat{y}_{u,v} = 0$ or not.

Proving 1-relaxed primal complementary slackness means that we need to prove that $\hat{x}$ and $\hat{y}$ satisfy strict primal complementary slackness. Initially, strict primal complementary slackness holds because $\hat{x}_v = 0$, for all $v \in V$. Whenever we update $\hat{x}$, this is done be inspecting an edge $\{u, v\} \in E$ and adding u or v to C. We add u to C (and thereby set $\hat{x}_u = 1$) only after increasing $\hat{y}_{u,v}$ by δ_u, which ensures that

$$\sum_{\substack{z \in V \\ \{u,z\} \in E}} \hat{y}_{u,z} = w_u.$$

Similarly, we add v to C only after increasing $\hat{y}_{u,v}$ by δ_v, which ensures that

$$\sum_{\substack{z \in V \\ \{v,z\} \in E}} \hat{y}_{v,z} = w_v.$$

Since we already proved that no $\hat{y}_{x,z}$ value ever decreases, any constraint corresponding to a vertex v that is made tight before adding v to C remains tight until the algorithm terminates. Thus, we have

$$\sum_{\substack{z \in V \\ \{v,z\} \in E}} \hat{y}_{v,z} = w_v$$

for all $v \in C$, that is, primal complementary slackness holds.

The approximation guarantee. Consider an optimal vertex cover C^* and its corresponding solution x^* of (4). Then, as shown in Question 1, we have $w(C) = w^T \hat{x} \le 2w^T x^* = w(C^*)$ because, $\hat{x}$ and $\hat{y}$ are feasible solutions of (4) and (5) that satisfy $(1,2)$-relaxed complemantary slackness.

A note. This algorithm is in fact an application of the primal-dual schema discussed later in class. Algorithms based on the primal-dual schema maintain an infeasible primal solution and a feasible dual solution and ensure that both solutions satisfy (relaxed) complementary slackness at all times. They then update the primal and dual solutions to (1) move the primal solution closer to feasibility, (2) maintain feasibility of the dual solution, and (3) maintain complementary slackness. Thus, once the primal solution becomes feasible, we have feasible primal and dual solutions, and these two solutions satisfy (relaxed) complementary slackness. Thus, the primal solution is an optimal solution if we ensure strict complementary slackness, or a good approximate solution if we ensure only relaxed complementary slackness.

Here, the primal solution is the vertex cover, which is initially infeasible because it does not cover any of the edges yet. The vertex potentials are the dual solution, and we proved that this solution is feasible at all times. We also proved that these two solutions satisfy $(1,2)$-relaxed complementary slackness at all times. As we proved in the answer to Question 2c, this ensures that once C becomes a vertex cover, it satisfies $w(C) \le 2w(C^*)$.