Assignment 2

CSCI 4113/6101
Instructor: Norbert Zeh
Due: Oct 21, 2025, 11:59pm

$$
\begin{aligned}
\text{Minimize } & -4x - y + 3z \\
\text{s.t. } & -8x - 5y + 3z \ge -10 \\
& -7x - 4y + 2z \le 20 \\
& 3x + y - z = 4 \\
& x \ge 0 \\
& y \le 0
\end{aligned}
\tag{1}
$$

Question 1: Canonical Form

First, we turn (1) into a maximization LP by negating the objective function:

$$
\begin{aligned}
\text{Maximize } & 4x + y - 3z \\
\text{s.t. } & -8x - 5y + 3z \ge -10 \\
& -7x - 4y + 2z \le 20 \\
& 3x + y - z = 4 \\
& x \ge 0 \\
& y \le 0
\end{aligned}
$$

Next, we split the equality constraint into an upper bound constraint and a lower bound constraint:

$$
\begin{aligned}
\text{Maximize } & 4x + y - 3z \\
\text{s.t. } & -8x - 5y + 3z \ge -10 \\
& -7x - 4y + 2z \le 20 \\
& 3x + y - z \le 4 \\
& 3x + y - z \ge 4 \\
& x \ge 0 \\
& y \le 0
\end{aligned}
$$

We negate all the lower bound constraints to turn them into upper bound constraints:

$$\begin{aligned}
&\text{Maximize } 4x + y - 3z \\
\text{s.t.}\quad & 8x + 5y - 3z \le 10 \\
& -7x - 4y + 2z \le 20 \\
& 3x + y - z \le 4 \\
& -3x - y + z \le -4 \\
& x \ge 0 \\
& y \le 0
\end{aligned}$$

Finally, we define $y' = -y$ and replace every occurrence of y with $-y'$. This ensures that y' is non-negative because y is non-positive. We also replace z, which unconstrained, with the difference $z = z' - z''$ of two non-negative variables. This gives the following LP in canonical form:

$$\begin{aligned}
&\text{Maximize } 4x - y' - 3z' + 3z'' \\
\text{s.t.}\quad & 8x - 5y' - 3z' + 3z'' \le 10 \\
& -7x + 4y' + 2z' - 2z'' \le 20 \\
& 3x - y' - z' + z'' \le 4 \\
& -3x + y' + z' - z'' \le -4 \\
& x, y', z', z'' \ge 0
\end{aligned} \tag{2}$$

Question 2: Standard Form

To convert the LP into standard form, we introduce a non-negative slack variable for each constraint. Adding this variable to its corresponding constraint turns all upper bound constraints into equality constraints:

$$\begin{aligned}
&\text{Maximize } 4x - y' - 3z' + 3z'' \\
\text{s.t. } & y_1 + 8x - 5y' - 3z' + 3z'' = 10 \\
& y2 - 7x + 4y' + 2z' - 2z'' = 20 \\
& y3 + 3x - y' - z' + z'' = 4 \\
& y4 - 3x + y' + z' - z'' = -4 \\
& y_1, y_2, y_3, y_4, x, y', z', z'' \ge 0
\end{aligned}$$

Question 3: Tableau

$$\begin{array}{|r|cccc|rrrr|}
\hline
 & y_1 & y_2 & y_3 & y_4 & x & y' & z' & z'' \\
\hline
10 & 1 & & & & 8 & -5 & -3 & 3 \\
20 & & 1 & & & -7 & 4 & 2 & -2 \\
4 & & & 1 & & 3 & -1 & -1 & 1 \\
-4 & & & & 1 & -3 & 1 & 1 & -1 \\
\hline
0 & & & & & 4 & -1 & -3 & 3 \\
\hline
\end{array} \tag{3}$$

The basic solution of this tableau is

$$y_1 = 10 \qquad y_2 = 20 \qquad y_3 = 4 \qquad y_4 = -4 \qquad x = y' = z' = z'' = 0.$$

This is not a feasible solution because $y_4 < 0$.

Question 4: Simplex Initialization

To test whether (3) is feasible and, if so, transform it into an equivalent tableau with a BFS, we construct an auxiliary tableau with an additional variable s and with objective function $-s$. This variable s is subtracted from each constraint:

	y_1	y_2	y_3	y_4	x	y'	z'	z''	s
10	1				8	−5	−3	3	−1
20		1			−7	4	2	−2	−1
4			1		3	−1	−1	1	−1
−4				1	−3	1	1	−1	−1
0									−1

(4)

Our goal is to verify that this LP has a feasible solution with objective function value 0, as this is the condition that determines whether (3) has a feasible solution.

To test this, we use the Simplex Algorithm to find an optimal of (4). We start by transforming (4) into an equivalent tableau with a BFS. To do this, we perform a special pivoting step to move y_4 out of the basis and s into basis because $b_4 = -4$ is the smallest of the constants in the equality constraints:

	y_1	y_2	y_3	s	x	y'	z'	z''	y_4
14	1				11	−6	−4	4	−1
24		1			−4	3	1	−1	−1
8			1		6	−2	−2	2	−1
4				1	3	−1	−1	1	−1
4					3	−1	−1	1	−1

The BFS of this tableau has objective function value −4. It is not guaranteed to be an optimal solution yet because both x and z'' have positive objective function coefficients. So we pivot to try to find a better solution. We choose to move x into the basis (we could also have chosen z''). To decide which variable should leave the basis, we need to inspect all constraints in which x has a positive coefficient. Of the three values

$$^{14}/_{11} \qquad ^{8}/_{6} \qquad ^{4}/_{3},$$

$^{14}/_{11}$ is the smallest, so we need to move the variable corresponding to the first constraint, y_1, out of the basis:

	x	y_2	y_3	s	y_1	y'	z'	z''	y_4
$14/11$	1				$1/11$	$-6/11$	$-4/11$	$4/11$	$-1/11$
$320/11$		1			$4/11$	$9/11$	$-5/11$	$5/11$	$-15/11$
$4/11$			1		$-6/11$	$14/11$	$2/11$	$-2/11$	$-5/11$
$2/11$				1	$-3/11$	$7/11$	$1/11$	$-1/11$	$-8/11$
$2/11$					$-3/11$	$7/11$	$1/11$	$-1/11$	$-8/11$

We still cannot guarantee that the BFS is optimal because we still have variables with positive objective function coefficients. Let's move z' into the basis this time. We have

$$4/2 = 2/1,$$

so we can move either y_3 or s out of the basis. We can make our life a bit easier by moving s out of the basis because then we don't need to pivot again to make s non-basic before dropping it:

	x	y_2	y_3	z'	y_1	y'	s	z''	y_4
2	1				-1	2	4		-3
30		1			-1	4	5		-5
0			1				-2		1
2				1	-3	7	11	-1	-8
0							-1		

(5)

Now all variables have non-positive objective function coefficients, so the BFS of this tableau is an optimal solution. Since its objective function value (negation of the bottom-left corner) is 0, this shows that (3) is feasible. To transform (3) into an equivalent tableau with a BFS, we drop s from (5) and restore the original objective function:

	x	y_2	y_3	z'	y_1	y'	z''	y_4
2	1				-1	2		-3
30		1			-1	4		-5
0			1					1
2				1	-3	7	-1	-8
0	4			-3		-1	3	

By adding -4 times the first constraint row to the and 3 times the last constraint row to the objective function row, we restore this tableaux to standard form:

	x	y_2	y_3	z'	y_1	y'	z''	y_4
2	1				-1	2		-3
30		1			-1	4		-5
0			1					1
2				1	-3	7	-1	-8
-2					-5	12		-12

(6)

Since this tableau can be obtained from (3) by permuting the columns and applying the same basic row operations that transformed (4) into (5), this tableau is equivalent to (3). Its basic solution feasible because it is in fact the BFS of (5) minus the value assigned to s.

Question 5: Simplex Optimization

With (6) in hand, we can now continue to pivot to transform this into an equivalent tableau whose BFS is an optimal solution. Currently, we cannot guarantee this yet because y' has a positive objective function coefficient, so we should pivot to make y' basic. Of the values

$$2/2 \qquad 30/4 \qquad 2/7,$$

$2/7$ is the smallest, so z' must leave the basis:

	x	y_2	y_3	y'	y_1	z'	z''	y_4
$10/7$	1				$-1/7$	$-2/7$	$2/7$	$-5/7$
$202/7$		1			$5/7$	$-4/7$	$4/7$	$-3/7$
0			1					1
$2/7$				1	$-3/7$	$1/7$	$-1/7$	$-8/7$
$-38/7$					$1/7$	$-12/7$	$12/7$	$12/7$

Next, let's move y_1 into the basis. The only constraint where y_1 has a positive coefficient is the second constraint, so y_2 has to leave the basis:

	x	y_1	y_3	y'	y_2	z'	z''	y_4
$36/5$	1				$1/5$	$-2/5$	$2/5$	$-4/5$
$202/5$		1			$7/5$	$-4/5$	$4/5$	$-3/5$
0			1					1
$88/5$				1	$3/5$	$-1/5$	$1/5$	$-7/5$
$-56/5$					$-1/5$	$-8/5$	$8/5$	$9/5$

Next, let's move z'' into the basis. Of the three values

$$\frac{36/5}{2/5} = \frac{36}{5} \qquad \frac{202/5}{4/5} = \frac{202}{4} \qquad \frac{88/5}{1/5} = 88,$$

$36/5$ is the smallest, so x has to leave the basis:

	z''	y_1	y_3	y'	y_2	z'	x	y_4
18	1				$1/2$	-1	$5/2$	-2
26		1			1		-2	1
0			1					1
14				1	$1/2$		$-1/2$	-1
-40					-1		-4	5

The BFS of this tableau is in fact optimal, but we cannot confirm this yet becaus y_4 still has a positive objective function coefficient. Of the two values

$$0/1 = 0 \qquad 26/1 = 26,$$

0 is the smaller, so y_3 has to leave the basis:

	z''	y_1	y_4	y'	y_2	z'	x	y_3
18	1				$1/2$	-1	$5/2$	2
26		1			1		-2	-1
0			1					1
14				1	$1/2$		$-1/2$	1
-40					-1		-4	-5

Note that this pivot operation did not change the BFS at all (which is always the case then the basic variable that leaves the basis is already 0), but now we have a tableau in which all objective coefficients are non-positive, so the current BFS is an optimal solution.[1]

The BFS of this tableau is

$$z'' = 18 \qquad y_1 = 16 \qquad y_4 = 0 \qquad y' = 14 \qquad y_2 = z' = x = y_3 = 0.$$

By dropping the slack variables $y_1, \ldots, y_4$ from this solution, we obtain an optimal solution of (2):

$$x = 0 \qquad y' = 14 \qquad z' = 0 \qquad z'' = 18$$

During the transformation of (1) into (2), we defined that $y = -y'$ and $z = z' - z''$. Thus, the solution

$$x = 0 \qquad y = -y' = -14 \qquad z = z' - z'' = -18$$

is an optimal solution of (1) with objective function value -40.

[1] It is not always the case that after performing such a "degenerate" pivot operation that does not change the BFS, we cannot make further improvements. This only happened in this example.