Assignment 1

CSCI 4113/6101
Instructor: Norbert Zeh
Due: Oct 7, 2025, 11:59pm

Question 1: Maximum Matching as an LP

Later in this course, we will study matching problems at some length. A **matching** in a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ such that no two edges in M share an endpoint. Such a matching M is a **maximum matching** if there exists no matching M' of greater cardinality. The problem of finding a maximum matching in a graph $G = (V, E)$ is easily expressed as an *integer* linear program: We assign a variable $x_e \in \{0, 1\}$ to every edge $e \in E$. If $e = \{u, v\}$, then we also refer to this variable as $x_{u,v}$. This variable x_e is 1 if e is included in the matching, and 0 otherwise. Thus, $M = \{e \in E \mid x_e = 1\}$. Our goal then is to maximize

$$|M| = \sum_{e \in E} x_e.$$

The condition that no vertex has more than one incident edge in M can be expressed as the set of constraints

$$\sum_{\{u,v\} \in E} x_{u,v} \le 1 \quad \forall u \in V.$$

Thus, we obtain the following ILP formulation of the maximum matching problem:

$$\begin{aligned} \text{Maximize } & \sum_{e \in E} x_e \\ \text{s.t. } & \sum_{\{u,v\} \in E} x_{u,v} \le 1 && \forall u \in V \\ & x_e \in \{0, 1\} && \forall e \in E \end{aligned} \tag{1}$$

As already mentioned in class, we will prove that integer linear programming is NP-hard. Given that we will prove later in this course that a maximum matching in any graph can be found in polynomial time, this ILP formulation does not seem to be particularly useful. Let's turn it into an LP (without the constraint that variables should have integer values):

$$\begin{aligned} \text{Maximize } & \sum_{e \in E} x_e \\ \text{s.t. } & \sum_{\{u,v\} \in E} x_{u,v} \le 1 && \forall u \in V \\ & x_e \le 1 && \forall e \in E \\ & x_e \ge 0 && \forall e \in E \end{aligned} \tag{2}$$

As we will discuss later in class, this is called the **LP relaxation** of the ILP (1), but you can ignore this for now. Our goal is to prove that, at least for bipartite graphs (see below for a definition), (2) is also a correct formulation of the maximum matching problem in the following sense:

(a) Since we dropped the requirement that the variables in (2) should be given integer values, there may exist optimal solutions of (2) that assign fractional values to the variables in (2). These solutions do not represent matchings. This is not a coincidence. Provide an example of a graph for which the optimal solution of (2) has a strictly greater objective function value than the optimal solution of (1), along with such optimal solutions. Argue briefly that both solutions are optimal.

Hint: *Look at (b) to get an idea of what such a graph should look like.*

(b) A graph $G = (V, E)$ is **bipartite** if we can partition the vertex set V into two subsets U and W—that is, $U \cup W = V$ and $U \cap W = \emptyset$—such that every edge $\{u, v\} \in E$ has one endpoint in U and one endpoint in W. A useful property of bipartite graphs, which you do not need to prove is that every cycle in such a graph has an even number of edges. Exploit this property to prove that, if G is bipartite, then optimal solutions of (1) and (2) have the same objective function value, that is, there exists an integral optimal solution of (2).

Hint: *Consider an optimal solution $\tilde{x}$ of (2). If $\tilde{x}_e$ is an integer, for all $e \in E$, then we already have the desired integral solution. If not, then prove that you can construct from $\tilde{x}$ another optimal solution $\hat{x}$ in which fewer variables have fractional values than in $\tilde{x}$. By applying this construction inductively, you will eventually obtain an integral optimal solution. How can you obtain such a solution $\hat{x}$ from $\tilde{x}$? Consider the subgraph $H = (V, E')$ of G such that E' is the set of all edges $e \in E$ such that $0 < \tilde{x}_e < 1$, that is, such that $\tilde{x}_e$ is fractional. If H contains a cycle C, argue that you can increase the values $\tilde{x}_e$ for some edges $e \in C$, and decrease the values $\tilde{x}_e$ of other edges $e \in C$ so that the solution remains feasible, the objective function value does not change, and at least one of the values assigned to variables in C becomes an integer. If H does not contain any cycle, then H is a forest. Pick a path P between two leaves in H. Argue that you can increase $\tilde{x}_e$ for some edges in P and decrease $\tilde{x}_e$ for other edges in P so that the solution remains feasible and the objective function value strictly increases. Since $\tilde{x}$ was an optimal solution, this is impossible, so H must in fact contain a cycle.*

(c) Prove that (b) implies that, if G is bipartite, then an integral solution $\hat{x}$ of (2) is an optimal solution of (2) if and only if the set $M = \{e \in E \mid \hat{x}_e = 1\}$ is a maximum matching.

(d) To make the statement in (c) algorithmically useful, we need a method to find an optimal integral solution of (2) efficiently. We will soon discuss efficient algorithms for solving linear programs. Thus, we can efficiently compute an optimal solution of (2), but the produced solution $\tilde{x}$ may not be integral. Describe a polynomial-time algorithm for constructing from such an optimal solution $\tilde{x}$ an integral solution $\hat{x}$ such that

$$\sum_{e \in E} \hat{x}_e = \sum_{e \in E} \tilde{x}_e, \tag{3}$$

that is, $\hat{x}_e$ is an integral optimal solution. Prove that your algorithm runs in polynomial time, that the solution $\hat{x}$ it outputs is feasible, and that $\hat{x}$ satisfies (3).

Hint: *This should be really easy once you proved (b) because the proof in (b) should be constructive, and it should be easy to describe an algorithm that implements the construction in (b).*

QUESTION 2: VERTEX COLOURING AS AN ILP

Graph colouring is a classical NP-hard problem. Given a graph $G = (V, E)$, the goal is to assign a color c_v to each vertex $v \in V$ such that

- Adjacent vertices have different colours: For every edge $\{u, v\} \in E$, $c_u \neq c_v$, and
- The number of unique colours used is minimized: $|\{c_v \mid v \in V\}|$ is minimized.

The exact values of the colours clearly don't matter. If the graph can be coloured with three colours, then it doesn't matter whether we call them red, green, blue or 1, 2, 3 or monkey, donkey, lion. We may therefore assume that every vertex is given a colour in the set $[n-1]_0 = \{0, \ldots, n-1\}$ because there is always a valid colouring of a graph with at most n colours, namely the one that assigns a different colour to each vertex.

(a) Express this problem as an ILP.
(b) Prove that an assignment $c : V \to [n-1]_0$ is a valid colouring if and only if it is a feasible solution of your ILP.
(c) Argue why this implies that an assignment $c : V \to [n]_0$ is a valid colouring that uses the minimum number of colours if and only if it is an optimal solution of your ILP.

Hint: *Given that the vertices are given colours from the set $[n-1]_0$ and the exact colours do not matter, we can always assume that a colouring with k distinct colours uses the colours 0 through $k-1$. Thus, our goal is to minimize $1 + \max\{c_v \mid v \in V\}$. As just expressed, this is not a linear combination of the different vertex colours, but you can introduce a helper variable k, representing the number of colours used, and you can use linear constraints to enforce that k is greater than the maximum colour assigned to any vertex. The objective function to be minimized then simply becomes k.*

To enforce that any two adjacent vertices have different colours, you should try to enforce that at least one of the two inequalities

$$c_u \geq c_v + 1$$
$$c_v \geq c_u + 1$$

needs to be satisfied for every edge $\{u, v\} \in E$. This may seem hard at first, because you cannot choose a priori which of these two constraints should be satisfied by a solution. However, you can introduce a "switching variable" $s_{u,v} \in \{0, 1\}$, for every edge $\{u, v\} \in E$. By incorporating a multiple of this variable and an additive term into each of these two constraints, you can enforce that the first constraint must be satisfied if $s_{u,v} = 0$ but can be violated if $s_{u,v} = 1$, and the second constraint must be satisfied if $s_{u,v} = 1$ but can be violated if $s_{u,v} = 0$.

Marking Scheme

Question 1 (20 marks)

Question 1a	Yes	Minor mistakes	No/major mistakes
For the given graph, an optimal solution of (2) has a greater objective function value than an optimal solution of (1).	1 mark	0.5 marks	0 marks
The provided solution of (2) is optimal.	1 mark	0.5 marks	0 marks
The provided solution of (1) is optimal.	1 mark	0.5 marks	0 marks
The argument that the provided solution of (2) is optimal is correct.	1 mark	0.5 marks	0 marks
The argument that the provided solution of (1) is optimal is correct.	1 mark	0.5 marks	0 marks

Question 1b	Yes	Minor mistakes	Major mistakes	No
Solution $\hat{x}$ constructed from $\tilde{x}$ is integral.	3 marks	2 marks	1 mark	0 marks
Solution $\hat{x}$ constructed from $\tilde{x}$ is feasible.	3 marks	2 marks	1 mark	0 marks
Solution $\hat{x}$ constructed from $\tilde{x}$ has the same objective function value.	3 marks	2 marks	1 mark	0 marks

Question 1c	Yes	Minor mistakes	No/major mistakes
Correct proof that an optimal integral solution corresponds to a maximum matching.	1 mark	0.5 marks	0 marks
Correct proof that a maximum matching corresponds to an optimal integral solution.	1 mark	0.5 marks	0 marks

Question 1d	Yes	Minor mistakes	No/major mistakes
Algorithm is correct.	1 mark	0.5 marks	0 marks
Algorithm runs in polynomial time.	1 mark	0.5 marks	0 marks
Correct proof that the algorithm is correct.	1 mark	0.5 marks	0 marks
Correct proof that the algorithm runs in polynomial time.	1 mark	0.5 marks	0 marks

Question 2 (12 marks)

	Yes	Minor mistakes	Major mistakes	No
Objective function expresses that the number of colours should be minimized.	2 marks	1 mark	0.5 marks	0 marks
Constraints express that a feasible solution represents a valid colouring.	3 marks	2 marks	1 mark	0 marks
Proof that a valid colouring corresponds to a feasible solution of the ILP.	2 marks	1 mark	0.5 marks	0 marks
Proof that a feasible solution of the ILP corresponds to a valid colouring.	2 marks	1 mark	0.5 marks	0 marks
Proof that an optimal colouring corresponds to an optimal solution of the ILP.	1 mark	0.5 marks		0 marks
Proof that an optimal solution of the ILP corresponds to an optimal colouring.	1 mark	0.5 marks		0 marks

Submission Instructions

I'll provide them by the end of the week, as soon as I figure out how to use CrowdMark.