Sample Solutions

Assignment 6

CSCI 3110 — Summer 2018

Question 1

(a) **Claim:** $T(n) \in O(n^{\log_2 3})$, that is, $T(n) \le cn^{\log_2 3}$, for some $c > 0$ and $n_0 \ge 0$ and all $n \ge n_0$. In fact, we claim that $T(n) \le cn^{\log_2 3} - dn$, for some constants $c > 0$ and $d > 0$. This is necessary to make the inductive proof work.

Proof: For $n = 1$, we have $T(n) \le c - d \le cn^{\log_2 3} - dn$, for some constant $c > 0$ and $d = c/2$ because $T(1) \in \Theta(1)$.

For $n > 1$, we have $n/2 \ge 1$, so we can apply the inductive hypothesis to $T(n/2)$. This gives

$$\begin{aligned}
T(n) &= 3T\left(\frac{n}{2}\right) + n \\
&\le 3\left[c\left(\frac{n}{2}\right)^{\log_2 3} - \frac{dn}{2}\right] + n \\
&= cn^{\log_2 3} - \frac{3dn}{2} + n \\
&= cn^{\log_2 3} - \left(\frac{3d}{2} - 1\right)n \\
&\le cn^{\log_2 3} - dn, \text{ for all } 3d/2 - 1 \ge d, \text{ that is, } d \ge 2.
\end{aligned}$$

Thus, the claims holds for $n_0 = 1$, $c \ge 4$, $d = c/2$, and c large enough to ensure that $T(1) \le c/2$.

Claim: $T(n) \in \Omega(n^{\log_2 3})$, that is, $T(n) \ge cn^{\log_2 3}$, for some $c > 0$ and $n_0 \ge 0$ and all $n \ge n_0$.

Proof: For $n = 1$, we have $T(n) \ge c = cn^{\log_2 3}$, for some constant $c > 0$ because $T(1) \in \Theta(1)$ and $n^{\log_2 3} = 1$.

For $n > 1$, we have $n/2 \ge 1$, so we can apply the inductive hypothesis to $T(n/2)$. This gives

$$\begin{aligned}T(n) &= 3T\left(\frac{n}{2}\right) + n\\ &\geq 3c\left(\frac{n}{2}\right)^{\log_2 3} + n\\ &= cn^{\log_2 3} + n\\ &> cn^{\log_2 3}\end{aligned}$$

Thus, the claim holds for $c > 0$ small enough that $T(1) \geq c$ and for $n_0 = 1$.

(b) **Claim:** $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$, $n_0 \geq 0$, and all $n \geq n_0$.

Proof: For $1 \leq n < 5$, we have $T(n) \in \Theta(1)$, that is, $T(n) \leq c \leq cn$, for c sufficiently large.

For $n \geq 5$, we have $n/4 > n/5 \geq 1$, that is, we can apply the inductive hypothesis to $T(n/4)$ and $T(n/5)$. This gives

$$\begin{aligned}T(n) &= 3T\left(\frac{n}{4}\right) + T\left(\frac{n}{5}\right) + n\\ &\leq \frac{3cn}{4} + \frac{cn}{5} + n\\ &= \left(\frac{19c}{20} + 1\right)n\\ &\leq cn, \text{ for all } c \geq 20.\end{aligned}$$

Thus, the claim holds for $c \geq 20$ and sufficiently large to ensure that $T(1) \leq c$ and for $n_0 = 1$.

Claim: $T(n) \in \Omega(n)$, that is, $T(n) \geq cn$, for some $c > 0$, $n_0 \geq 0$, and all $n \geq n_0$.

Proof: This is trivial for $c = 1$ and all n because $T(n) \geq n$ by definition.

(c) **Claim:** $T(n) \in O(n \lg n)$, that is, $T(n) \leq cn \lg n$, for some $c > 0$, $n_0 \geq 0$, and all $n \geq n_0$. For the inductive proof to work, we do in fact prove the stronger claim that $T(n) \leq cn \lg n - dn$, for some $c > 0$ and $d > 0$.

Proof: For $2 \leq n < 4$, we have $T(n) \in \Theta(1)$, that is, $T(n) \leq c - d \leq cn \lg n - dn$, for c large enough and $d = c/2$.

For $n \geq 4$, we have $\sqrt{n} \geq 2$, that is, we can apply the inductive hypothesis to $T(\sqrt{n})$.

This gives

$$
\begin{aligned}
T(n) &= 2\sqrt{n}T(\sqrt{n}) + n \\
&\le 2\sqrt{n}\left[c\sqrt{n}\lg\sqrt{n} - d\sqrt{n}\right] + n \\
&= 2cn\lg\sqrt{n} - 2dn + n \\
&= cn\lg n - (2d-1)n \quad \text{because } \lg\sqrt{n} = \frac{1}{2}\lg n \\
&\le cn\lg n - dn, \text{ for all } d \ge 1.
\end{aligned}
$$

Thus, the claim holds for $n_0 = 2$, $c \ge 2$, $d = c/2$, and c large enough that $T(n) \le c/2$ for all $2 \le n < 4$.

Claim: $T(n) \in \Omega(n\lg n)$, that is, $T(n) \ge cn\lg n$, for some $c > 0$, $n_0 \ge 0$, and all $n \ge n_0$.

Proof: For $2 \le n < 4$, we have $T(n) \in \Theta(1)$, so $T(n) \ge 8c > cn\lg n$, for some $c > 0$.

For $n \ge 4$, we have $\sqrt{n} \ge 2$, so we can apply the inductive hypothesis to $T(\sqrt{n})$. This gives

$$
\begin{aligned}
T(n) &= 2\sqrt{n}T(\sqrt{n}) + n \\
&\ge 2\sqrt{n}c\sqrt{n}\lg\sqrt{n} + n \\
&= cn\lg n + n \quad \text{because } \lg\sqrt{n} = \frac{1}{2}\lg n \\
&> cn\lg n.
\end{aligned}
$$

Thus, the claim holds for $n_0 = 2$ and c small enough that $T(n) \ge 8c$ for $2 \le n < 4$.

Question 2

(a) Here we have $n\lg n \in o(n^{1.1}) \subset O(n^{1.1}) = O(n^{\log_3 4 - \varepsilon})$, where $\varepsilon = \log_3 4 - 1.1 > 0$. This holds because $\lg n \in o(n^\delta)$ for all $\delta > 0$. Thus, the second case of the Master Theorem applies and $T(n) \in \Theta(n^{\log_3 4})$.

(b) Here we have $n^2/\lg n = n^{\log_2 4}/\lg n$. Since $\lg n \in \omega(1)$, we thus don't have $n^2/\lg n \in \Theta(n^{\log_2 4})$ or $n^2/\lg n \in \Omega(n^{\log_2 4+\epsilon})$. We also do not have $n^2/\lg n \in O(n^{\log_2 4-\epsilon})$ for any $\epsilon > 0$ because $\lg n \in o(n^\epsilon)$ for all $\epsilon > 0$. Thus, the restrictive version of the Master theorem discussed in class cannot be used to solve this recurrence.

(c) Here we have $n^2 = n^{\log_3 9}$. Thus, the third case of the Master Theorem applies and $T(n) \in \Theta(n^2\lg n)$.

(d) Here we have $n = n^{\log_4 3+\varepsilon}$, where $\varepsilon = 1 - \log_4 3 > 0$. Since we also have $3n/4 \le cn$ for $c = 3/4$, the first case of the Master Theorem applies and $T(n) \in \Theta(n)$.

(e) Here we have $n \lg n = n^{\log_2 2} \lg n$. Thus, similar to (b), we neither have $n \lg n \in \Theta(n^{\log_2 2})$ nor is the difference polynomial in n. Thus, once again, the Master Theorem cannot be used to solve this recurrence.