Sample Solution

Assignment 1 — CSCI 3110

Summer 2018

Question 1

(a) First observe that the function $f(n) = 8n^2 - 5n + 12\sqrt{n} - 10n \lg n$ is well-defined only for $n > 0$, so we impose $n \geq 1$ as an initial constraint. Under this assumption, we have:

$$
\begin{array}{rlr r}
f(n) = & 8n^3 + 5n^2 + 12\sqrt{n} - 10n \lg n & \forall n \geq 1 & (1) \\
0 \leq & \phantom{8n^3 + 5n^2 + 12\sqrt{n} -} 10n \lg n & \forall n \geq 1 & (2) \\
\hline
f(n) \leq & 8n^3 + 5n^2 + 12\sqrt{n} & \forall n \geq 1 & (3) \\
0 \leq & n^3 - 12\sqrt{n} & \forall n \geq 4 & (4) \\
\hline
f(n) \leq & 9n^3 + 5n^2 & \forall n \geq 4 & (5) \\
0 \leq & n^3 - 5n^2 & \forall n \geq 5 & (6) \\
\hline
f(n) \leq & 10n^3 & \forall n \geq 5 & (7)
\end{array}
$$

- Inequality (2) holds because, for $n \geq 1$, $\lg n \geq 0$, so $10n \lg n \geq 0$.
- Inequality (4) holds because, for $n \geq 4$, $n \geq \sqrt{n}$ and $n^2 \geq 16$, so $n^3 = n^2 \cdot n \geq 16 \cdot \sqrt{n}$.
- Inequality (6) holds because, for $n \geq 5$, we have $n^3 = n \cdot n^2 \geq 5 \cdot n^2$.

For the lower bound, we obtain:

$$
\begin{array}{rlr r}
f(n) = & 8n^3 + 5n^2 + 12\sqrt{n} - 10n \lg n & \forall n \geq 1 & (8) \\
0 \geq & - 5n^2 - 12\sqrt{n} & \forall n \geq 0 & (9) \\
\hline
f(n) \geq & 8n^3 \phantom{+ 5n^2 + 12\sqrt{n}} - 10n \lg n & \forall n \geq 1 & (10) \\
0 \geq & -5n^3 \phantom{+ 5n^2 + 12\sqrt{n}} + 10n \lg n & \forall n \geq 2 & (11) \\
\hline
f(n) \geq & 3n^3 & \forall n \geq 2 & (12)
\end{array}
$$

- Inequality (9) holds because, for $n \geq 0$, $5n^2 \geq 0$ and $12\sqrt{n} \geq 0$.
- Inequality (11) holds because, for $n = 2$, we have $5n^3 = 40$ and $10n \lg n = 20$, so $5n^3 \geq 10n \lg n$. Now observe that $\frac{d(5n^2)}{dn} = 10n$ and $\frac{d(10 \lg n)}{dn} = \frac{10}{n \ln 2} \leq \frac{20}{n}$. For $n \geq 2$, $10n \geq 20$ and $20/n \leq 10$, that is, $5n^2$ grows faster than $10 \lg n$ and thus $5n^3$ grows faster than $10n \lg n$. Since $5n^3 \geq 10n \lg n$ for $n = 2$, this implies that $5n^3 \geq 10n \lg n$ for all $n \geq 2$.

By combining (7) and (12), we obtain

$$3n^3 \le f(n) \le 10n^3 \quad \forall n \ge 5,$$

that is, $f(n) \in \Theta(n^3)$.

(b) Again, we constrain n to be at least 1 because this ensures that $\lg n$ is well defined and non-negative. This gives:

$$\begin{array}{rlr}
f(n) = & n\lg n + 13n - 40\lg n \quad \forall n \ge 1 & (1)\\
0 \le & 40\lg n \quad \forall n \ge 1 & (2)\\
\hline
f(n) \le & n\lg n + 13n \quad \forall n \ge 1 & (3)\\
0 \le & n\lg n - 13n \quad \forall n \ge 2^{13} & (4)\\
\hline
f(n) \le & 2n\lg n \quad \forall n \ge 2^{13} & (5)
\end{array}$$

- Inequality (2) is obvious.
- Inequality (4) holds because, for $n \ge 2^{13}$, $\lg n \ge 13$ and, thus, $n\lg n \ge 13n$.

For the lower bound, we have:

$$\begin{array}{rlr}
f(n) = & n\lg n + 13n - 40\lg n \quad \forall n \ge 1 & (6)\\
0 \ge & -13n \quad \forall n \ge 0 & (7)\\
\hline
f(n) \ge & n\lg n - 40\lg n \quad \forall n \ge 1 & (8)\\
0 \ge & -\tfrac{1}{2}n\lg n + 40\lg n \quad \forall n \ge 80 & (9)\\
\hline
f(n) \ge & \tfrac{1}{2}n\lg n \quad \forall n \ge 80 & (10)
\end{array}$$

- Inequality (7) is once again obvious.
- Inequality (9) holds because, for $n \ge 80$, $n/2 \ge 40$ and thus $(n/2)\lg n \ge 40\lg n$.

By combining (5) and (10), we obtain

$$\frac{1}{2}\lg n \le f(n) \le 2\lg n \quad \forall n \ge 2^{13},$$

that is, $f(n) \in \Theta(n\lg n)$.

Question 2

The correct order is

$$4^{\lg\lg n} \quad (4/3)^{\lg n} \quad \sqrt{n} \quad n \quad n\lg n \quad 3^n.$$

$4^{\lg\lg n} \in o((4/3)^{\lg n})$: First observe that

$$4^{\lg\lg n} = (2^2)^{\lg\lg n} = (2^{\lg\lg n})^2 = (\lg n)^2.$$

Similarly,

$$(4/3)^{\lg n} = (2^{\lg(4/3)})^{\lg n} = (2^{\lg n})^{\lg(4/3)} = n^{\lg(4/3)}.$$

Now we have

$$\lim_{n\to\infty} \frac{(\lg n)^2}{n^{\lg(4/3)}} = \lim_{n\to\infty} \left(\frac{\lg n}{n^{\frac{\lg(4/3)}{2}}}\right)^2 = 0 \Longleftrightarrow \lim_{n\to\infty} \frac{\lg n}{n^{\frac{\lg(4/3)}{2}}} = 0$$

By l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\lg n}{n^{\frac{\lg(4/3)}{2}}} = \lim_{n\to\infty} \frac{\frac{1}{\ln 2\cdot n}}{\frac{\lg(4/3)}{2}\cdot n^{\frac{\lg(4/3)}{2}-1}} = \lim_{n\to\infty} \frac{1}{\frac{\lg(4/3)\cdot\ln 2}{2}\cdot n^{\frac{\lg(4/3)}{2}}} = 0.$$

Thus,

$$\lim_{n\to\infty} \frac{(\lg n)^2}{n^{\lg(4/3)}} = 0,$$

that is, $(\lg n)^2 \in o(n^{\lg(4/3)})$.

$(4/3)^{\lg n} \in o(\sqrt{n})$: As observed above, $(4/3)^{\lg n} = n^{\lg(4/3)}$.

$$\lim_{n\to\infty} \frac{n^{\lg(4/3)}}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{n^{1/2-\lg(4/3)}} = 0$$

because $4/3 < \sqrt{2}$ and, thus, $\lg(4/3) < 1/2$, that is, $1/2-\lg(4/3) > 0$. This proves that $n^{\lg(4/3)} \in o(\sqrt{n})$.

$\sqrt{n} \in o(n)$:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Thus, $\sqrt{n} \in o(n)$.

$n \in o(n \lg n)$:

$$\lim_{n\to\infty} \frac{n}{n\lg n} = \lim_{n\to\infty} \frac{1}{\lg n} = 0.$$

Thus, $n \in o(n\lg n)$.

$n \lg n \in o(3^n)$: By l'Hôpital's rule (applied twice),

$$\lim_{n\to\infty} \frac{n\lg n}{3^n} = \lim_{n\to\infty} \frac{\lg n + n\cdot\frac{1}{\ln 2\cdot n}}{3^n \ln 3} = \lim_{n\to\infty} \frac{\lg n + \frac{1}{\ln 2}}{3^n \ln 3} = \lim_{n\to\infty} \frac{\frac{1}{\ln 2\cdot n}}{3^n \ln^2 3} = \lim_{n\to\infty} \frac{1}{3^n\cdot n\cdot \ln^2 3\cdot \ln 2} = 0.$$

Thus, $n\lg n \in o(3^n)$.