On Reverse Nearest Neighbor Queries

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Abstract

In this paper, we discuss static and dynamic-data structures for answering *reverse nearest neighbor* queries in two dimensions.

1 Introduction

A well-know proximity concept used, e.g., in clustering applications is the notion of a point's nearest neighbor according to some given metric. The problem of finding nearest neighbors, formally defined below, has received considerable attention in the past, and a variety of algorithms have been devised for this problem and its variants—see the recent survey by Smid [7].

Problem 1 (Nearest Neighbor) *Given a set $\mathcal{S}$ of n points in $\mathbb{R}^d$, a distance metric $\mathbf{d}$, and a query point p in $\mathbb{R}^d$, report a point $q \in \mathcal{S}$, for which $\mathbf{d}(p,q) = \min\{\mathbf{d}(p,r) \mid r \in \mathcal{S}\}$.*

The property of being a point's nearest neighbor, however, is not symmetric: while a point p can be the nearest neighbor of a point q, the reverse is not necessarily true. This asymmetry has led to defining the so-called *reverse nearest neighbor* problem [4], which, given a point p, asks for all points for which p is the nearest neighbor.

Problem 2 (Reverse Nearest Neighbors) *Given a set $\mathcal{S}$ of n points in $\mathbb{R}^d$, a distance metric $\mathbf{d}$, and a query point p in $\mathbb{R}^d$, report all points $q \in \mathcal{S}$, for which $\mathbf{d}(q,p) = \min\{\mathbf{d}(q,r) \mid r \in \mathcal{S} \cup \{p\},\ r \neq q\}$. (see Figure 1).*

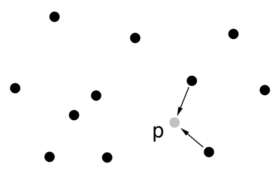

Figure 1: Reverse nearest neighbors for p.

Motivation and Related Work: The *Reverse Nearest Neighbor* problem (Problem 2) has been introduced in the database setting by Korn and Muthukrishnan [4] along with several applications, e.g., in decision support systems, continuous referral systems, or profile based marketing. Korn and Muthukrishnan presented static and dynamic solutions and also discussed a bichromatic setting: each point has one out of two possible colors, and reverse nearest neighbors are only sought among points of the other color. In this paper, we only consider the monochromatic problem. The algorithm of [4] performs a preprocessing step to solve the *All Nearest Neighbor* problem for $\mathcal{S}$, that is, to determine for each point in $\mathcal{S}$ its nearest neighbor. Each point q and its nearest neighbor r define a so-called $\mathcal{NN}$-ball centered at q with radius $\mathbf{d}(q,r)$. Then, all $\mathcal{NN}$-balls are stored in a data structure that can be used to report, given a query point p, all $\mathcal{NN}$-balls containing p. It is easy to verify that the points corresponding to the $\mathcal{NN}$-balls that contain p are exactly the points having p as their nearest neighbor in $\mathcal{S} \cup \{q\}$.

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Korn and Muthukrishnan approximate each $\mathcal{NN}$-ball by its axis-aligned bounding box and then use an R-tree [3] to maintain these approximations. Insertions and deletions can be handled in logarithmic time, but the worst-case complexity for point-containment queries is linear—even if no containing boxes are reported. Moreover, approximating the $\mathcal{NN}$-balls by their bounding boxes can lead to wrong answers (unless appropriate postprocessing is done).

Iterating the situation sketched in Figure 2, we can construct a data set where all bounding boxes but no $\mathcal{NN}$-balls contain a specific query point p.

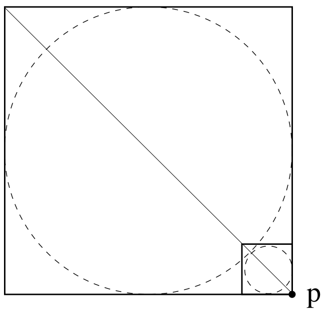


Figure 2: Worst-case situation for an approach based on bounding boxes.

Our Results: We present efficient data structures for the static and dynamic version of the reverse nearest neighbor problem in two dimensions that avoid using bounding boxes. The static data structure for a set of n points uses linear space, can be constructed in $\mathcal{O}(n \log n)$ time, and can be used to answer a reverse nearest neighbor query in $\mathcal{O}(\log n)$ time. The dynamic data structure requires $\mathcal{O}(n \log n)$ space, it can be updated $\mathcal{O}(Q(n) + \log^2 n \log\log n)$ time, and a query can be answered in $\mathcal{O}(\log^2 n \log\log n)$ time. Here, the quantity $Q(n)$ denotes the complexity of adding or removing a point to resp. from $\mathcal{S}$ while at the same time maintaining for each point in the $\mathcal{S}$ the *distance* to its nearest neighbor using only $\mathcal{O}(n \log n)$ space.

The static data structure is a persistent one-dimensional search tree created during a plane-sweep the set of (two-dimensional) $\mathcal{NN}$-discs defined by the given set of points. The dynamic data structure is a fractional cascaded interval tree over the same set of $\mathcal{NN}$-discs (or, rather, quarter-$\mathcal{NN}$-discs). Both data structures exploit the (limited) intersection properties of these discs.

2 The static case

Definition 1 *For a set $\mathcal{S}$ of points in the plane and any point $p \in \mathcal{S}$, let its $\mathcal{NN}$-disc $D(p, N(p))$ be the disc centered at p with radius $N(p) := \mathbf{d}(p, p')$, where p' is the nearest neighbor of p with respect to $\mathcal{S} \setminus \{p\}$.*

The main observation in [4] is that the points in $\mathcal{RNN}(q)$ are exactly those points $p \in \mathcal{S}$ for which $q \in D(p, N(p))$. Hence, to answer a reverse nearest neighbor query, it is sufficient to find all $\mathcal{NN}$-discs containing q.

Observation 1 *Empty disc property: For any point $p \in \mathcal{S}$, the $\mathcal{NN}$-disc $D(p, N(p))$ does not contain any other point of $\mathcal{S}$.*

In the following, we will discuss the arrangement $\mathcal{A}$ of the set of n $\mathcal{NN}$-discs. The complexity of an arrangement of discs in two dimensions is the total number of its 0-dimensional faces (vertices) and 1-dimensional faces (circular arcs). Although, in general, the complexity of an arrangement of n discs can be $\Theta(n^2)$ [2], we show in the following lemma that the complexity of the arrangement $\mathcal{A}$ of the $\mathcal{NN}$-discs in our setting is only $\mathcal{O}(n)$.

Lemma 1 *The complexity of the arrangement $\mathcal{A}$ of all $\mathcal{NN}$-discs for a set $\mathcal{S}$ of n points is $\mathcal{O}(n)$ (see the appendix for the proof).*

Proof. From the empty disc property we know that for any given point $p \in \mathcal{S}$, the $\mathcal{NN}$-disc $D(p, N(p))$ does not contain any other point of $\mathcal{S}$. Observe that the boundary of any two discs can intersect in at most two points. We show that in all only $\mathcal{O}(n)$ pairs of $\mathcal{NN}$-discs can intersect, and hence we can conclude that the complexity of this arrangement is $\mathcal{O}(n)$.

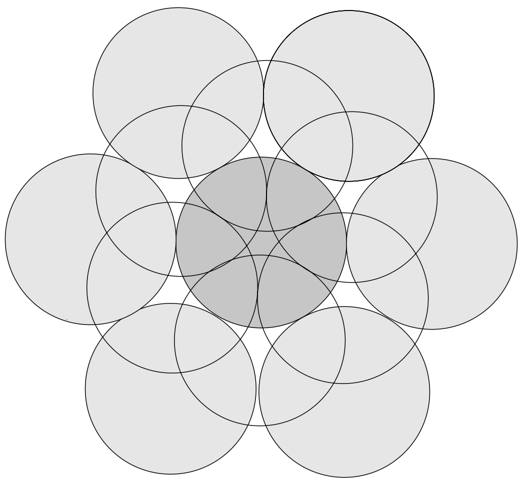


Figure 3: Central disc overlaps with 12 other large discs.

Construct a directed graph $G = (V, E)$ over the set of $\mathcal{NN}$-discs as follows. Each $\mathcal{NN}$-disc corresponds to a vertex in V. Two vertices u and v are joined by a directed edge from u to v if and only if the corresponding $\mathcal{NN}$-discs intersect, and the radius of the $\mathcal{NN}$-disc corresponding to u is at least as big as the radius of the $\mathcal{NN}$-disc corresponding to v. We show that the indegree of any vertex is bounded. More specifically, we show that the indegree of any vertex is no larger than 12.

Consider any disc D, say the one corresponding to a given node v (the center disc in Figure 3). By a standard packing argument, we know that there are at most six discs touching D and having radius at least equal to that of the radius of D (the light gray discs in Figure 3). These discs contribute at most six to the indegree of v. In each of the cavities between the disc D and these six discs, we can place the center of one of at most six other discs satisfying the empty disc property and having radii at least equal to that of D (the hollow discs in Figure 3). These discs account for an increase in the indegree of v of at most six. $\square$

If we store for each 2-dimensional face of the arrangement $\mathcal{A}$ the $\mathcal{NN}$-disc(s) overlapping it, we can answer a reverse nearest neighbor query for a point q by simply performing a point location query on $\mathcal{A}$. The answer to the reverse nearest neighbor query then consists of the points whose $\mathcal{NN}$-discs overlap (the face containing) q.

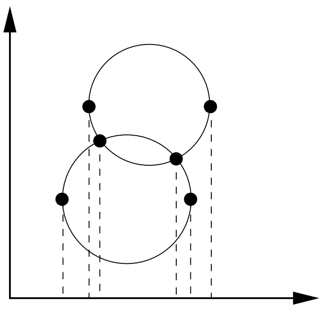


Figure 4: Events used for the plane-sweep.

Next we show how the arrangement $\mathcal{A}$ can be constructed and at the same time preprocessed to efficiently support point location queries. The main ingredient of our algorithm is a modification of the well-known line segment intersection algorithm by Bentley and Ottmann [1]. The original plane-sweep algorithm maintains all segments intersecting the sweep-line ordered by the y-coordinate of their intersection point. Intersections are detected by comparing neighboring segments in this y-structure, and the overall complexity of this algorithm is $\mathcal{O}((n + k)\log n)$ where k is the number of intersections.

Our algorithm implicitly computes the arrangement $\mathcal{A}$ by also sweeping the plane by a vertical sweep line from left to right, by increasing x-coordinate, but additionally maintaining the circles intersecting the sweep-line in a persistent y-structure [6]. The event points are the extremal left and right end points of each disc as well as points corresponding to intersection points of pair of discs (see Figure 4). More specifically, we split each circle into an upper and a lower half-circle such that the resulting circular arcs are monotone with respect to the sweeping direction. At each extremal left or right point of a circle we insert resp. delete the two half-circles into resp. from the y-structure, and at each intersection point of two half-circles, we exchange the intersecting circular arcs in the y-structure.

From Lemma 1 we know that in all there are $\mathcal{O}(n)$ event points. Since the circular arcs defining the boundary of cells can be expressed by a simple second order function, it is easy to maintain the order of circular arcs along the sweep line as well as to insert an arc, delete an arc, or compute the intersection point of a pair of neighboring arcs. All of these operations can be performed in $\mathcal{O}(\log n)$ time. Moreover, for each face f in the arrangement $\mathcal{A}$, we maintain a list of all the discs overlapping face f. Exploiting a standard packing argument, we know that there are only a constant number of discs overlapping any given face, and hence this additional information can be stored without asymptotically increasing the space complexity bounds. We conclude that the arrangement of n discs satisfying the empty disc property can be computed in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space.

To answer a reverse nearest neighbor query for a query point $q = (x_q, y_q)$, we use the persistent data structure to recover the status of the sweep-line as of passing over $x = x_q$. By searching along the y-coordinate in the recovered y-structure, we locate the face f containing the query point q, and along with f, we also find all $\mathcal{NN}$-discs overlapping f. As discussed above, the center points defining these $\mathcal{NN}$-discs are the solution to the the reverse nearest neighbor query. It takes $\mathcal{O}(\log n)$ time to recover the y-structure corresponding to a specific x-coordinate using the persistent data structure and then it takes an additional $\mathcal{O}(\log n)$ time to locate the face f containing the query point q. Since there are only a constant number of $\mathcal{NN}$-discs overlapping a face, and they are also stored with the face, these $\mathcal{NN}$-discs can be retrieved in additional constant time. We summarize our discussion in the following theorem.

Theorem 1 *There exists a linear-space data structure that can be used to answer the (static) reverse nearest neighbor problem in two dimensions. Such a data structure for a set of n points can be constructed in $\mathcal{O}(n \log n)$ time, and a query can be answered in $\mathcal{O}(\log n)$ time.*

3 The dynamic case

In this section, we make an attempt to design a data structure for a dynamically changing point set to answer reverse nearest neighbor queries efficiently. We wish to support the following operations:

Insert(p,T): Insert the point p (and the associated disc) in the data structure T.

Delete(p,T): Delete the point p (and the associated disc) from the data structure T.

Search(q,T): Search for all the discs in T that contain the query point q.

To state our results, we assume that a data structure (using $\mathcal{O}(S(n))$ space) exists that requires $\mathcal{O}(Q(n))$ time to update $\mathcal{S}$ while at the same time maintaining for each point *the distance* to its nearest neighbor. One of the main problems, however, is that designing a data structure for maintaining *nearest neighbors* in a dynamic setting using $\mathcal{O}(n \log^{\mathcal{O}(1)} n)$ space and $\mathcal{O}(\log^{\mathcal{O}(1)} n)$ update time is regarded an open problem [7]. Consequently, the relevance of our result is somewhat dependent on a solution to this open problem, and we state all complexities including the terms $S(n)$ and $Q(n)$.

Our approach for designing the dynamic data structure for the reverse nearest neighbor problem is as follows. We will split a disc into four equal sized quarter-discs using coordinate axis centered at the center of the disc. The boundary of each of these quarter-discs is monotone with respect to both coordinate axes. We build four data structures, one for each kind of quarter-discs. As the situations are symmetric, we only describe the data structure for maintaining north-east quarter-discs.

We now prove that the set of quarter-discs has a very useful property:

Theorem 2 *For every quarter-disc Q, all quarter-discs Q_1, Q_2, ..., Q_q of the same type that overlap Q are pairwise disjoint.*

Proof. We show that for every quarter disc Q, all quarter-discs $Q_1, \dots, Q_q$ of the same type that overlap Q are pairwise disjoint. We prove the claim in the L_1-metric. The proofs generalize to the L_2 and L_∞-metrics using rather simple observations.

W.l.o.g., let Q be the north-east sector of an L_1-disc C with center O. Let Q' be the north-east sector of another disc C' with center O' so that $Q \cap Q' \neq \emptyset$. We denote points O and O' as the centers of quarter discs Q and Q'. We first partition the plane into seven regions w.r.t. disc C and show that point O' can lie in only two of them. These regions are defined as follows (see Figure 5):

1. The first region is disc C itself.
2. Region D is the part of the north-west quadrant of the coordinate system with origin O which lies outside the y-range of disc C.
3. Region A is the north-west quadrant of the coordinate system with origin O minus regions C and D.
4. Region E is the north-east quadrant of the coordinate system with origin O minus disc C.
5. Region F is the part of the south-east quadrant of the coordinate system with origin O which lies outside the x-range of disc C.
6. Region B is the south-east quadrant of the coordinate system with origin O minus regions C and F.
7. Region G is the south-west quadrant of the coordinate system with origin O minus disc C.

Lemma 2 *For any quarter disc Q' with center O' and so that $Q \cap Q' \neq \emptyset$, $O' \in A \cup B$.*

Proof. To prove the lemma we show that point O' cannot lie in any of the other five regions. Point O' cannot lie inside disc C, as no disc

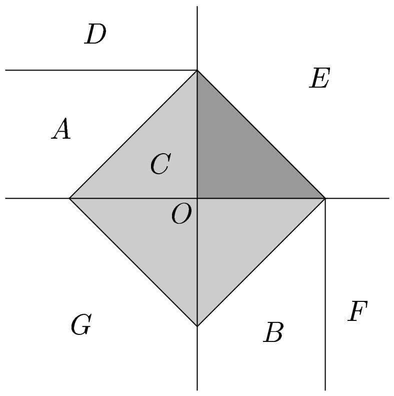


Figure 5: The different regions for point O'.

contains the center of any other disc. Point O' cannot lie in region D because otherwise $y(p') \geq y(O') > y(p)$, for any two points $p \in Q$ and $p' \in Q'$, so that $Q \cap Q' = \emptyset$. A similar argument shows that $O' \notin F$.

Now assume that $O' \in E$, and $Q \cap Q' \neq \emptyset$. Then let $p \in Q \cap Q'$. Since $p \in Q$, Q contains the axes-parallel rectangle defined by points O and p. Since every point in Q' dominates O', $x(O) \leq x(O') \leq x(p)$ and $y(O) \leq y(O') \leq y(p)$. However, this implies that O' is contained in the axes-parallel rectangle defined by O and p and hence in Q, which contradicts the condition that C cannot contain O'.

Finally, assume that $O' \in G$ and $Q \cap Q' \neq \emptyset$. Then $x(O') \leq x(O) \leq x(p)$ and $y(O') \leq y(O) \leq y(p)$, for every point $p \in Q \cap Q'$. Hence, $\mathbf{d}_1(O', O) \leq \mathbf{d}_1(O', p)$, so that $O \in Q'$, which again leads to a contradiction. $\square$

By Lemma 2, any north-east quarter-disc overlapping Q can have its center only in regions A or B. Lemma 3 shows that if $O' \in B$, then for any quarter-disc Q'' with center $O'' \in A$, $Q' \cap Q'' = \emptyset$. Lemma 4 shows that the same is true for any quarter-disc Q'' with center $O'' \in B$.

Lemma 3 *Let Q' and Q'' be two quarter-discs overlapping Q with centers $O' \in B$ and $O'' \in A$, respectively. Then $Q' \cap Q'' = \emptyset$.*

Proof. Consider the line ℓ containing the bisector of the north-east sector of C, as shown in

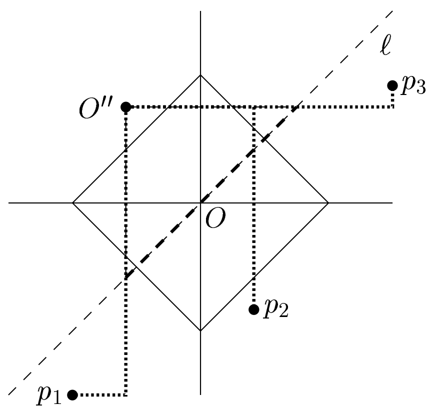


Figure 6: A disc C'' with center $O'' \in A$ and containing a point p below line ℓ also contains point O.

Figure 6. We claim that if disc C'' contains a point to the right of line ℓ, then it contains point O. Since the latter is impossible, disc C'' and hence Q'' is completely to the left of line ℓ. A similar argument shows that disc Q' is completely below line ℓ, so that $Q' \cap Q'' = \emptyset$. It remains to show the claim.

Point O is contained in the part of line ℓ between the two projections of point O'' in x and y-directions onto line ℓ. All points in this part of line ℓ have the same L_1-distance from O'' as point O. Moreover, there exists a shortest L_1-path from O'' to any point below ℓ which intersects ℓ in a point between these two projections of O'' onto line ℓ. Hence, $\mathbf{d}_1(O'',O) < \mathbf{d}_1(O'',p)$, and disc C'' contains point O. □

Lemma 4 *Let Q' and Q'' be two quarter-discs overlapping Q with centers $O' \in B$ and $O'' \in B$, respectively. Then $Q' \cap Q'' = \emptyset$.*

Proof. Consider the disc C' with center O' of which Q' is the north-east sector. We show that disc C' partitions region B into three regions as shown in Figure 7. In particular, we show that point t is always to the right of the x-range of disc C, and regions X and Y are completely below the horizontal line through point O'. Given these properties, we then show that if $O'' \in X$, either $Q' \cap Q'' = \emptyset$ or $O' \in Q''$. Since the latter

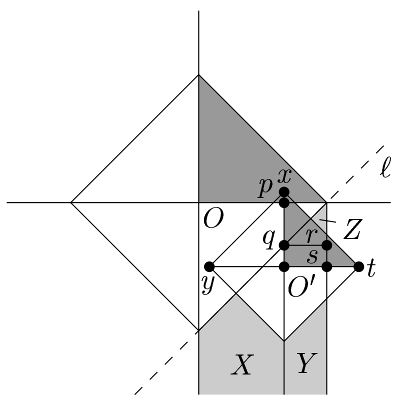


Figure 7: The three regions where the center O'' of disc C'' may lie.

is impossible, $Q' \cap Q'' = \emptyset$ in this case. We show that if $O'' \in Y$, $O' \in Q''$. And we show that if $O'' \in Z$, then $Q' \cap Q'' = \emptyset$.

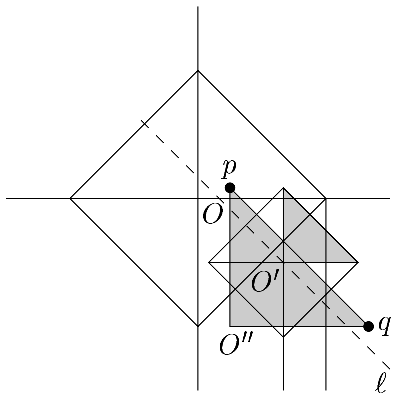
To see that t is outside of the x-range of disc C, observe that $\mathbf{d}_1(p,q) = \mathbf{d}_1(q,r) = \mathbf{d}_1(O',s)$. But $\mathbf{d}_1(O',t) = \mathbf{d}_1(O',x) > \mathbf{d}_1(p,q)$. Hence, the part of region E to the right of the vertical line through O' is divided into regions Y and Z by disc C', as shown in Figure 7. Every point in Y is south-east of O'. Every point in Z is north-east of O'.

To see that every point in X is south-west of O', it suffices to show that point y is always to the left of line ℓ. However, since Q' intersects Q, point x is above line ℓ, and the segment xy is parallel to line ℓ. Hence, y is to the left of line ℓ.

For a quarter-disc Q'' with center $O'' \in X$, there are two possibilities as shown in Figure 8. If point p is above line ℓ, then point q is to the right of line ℓ. Moreover, point p is above point O' and point q is below point O', so that Q'' contains O', which leads to a contradiction. Hence, point p is below line ℓ. But this implies that every point in Q'' is below line ℓ, so that $Q' \cap Q'' = \emptyset$.

For a quarter-disc Q'' with center $O'' \in Z$ (Figure 9(a)), every point $p \in Q''$ dominates O'', and O'' dominates O'. Thus, if there were a point $p \in Q' \cap Q''$, then $O'' \in Q'$. Since the latter is impossible $Q' \cap Q'' = \emptyset$.

Finally, for a quarter-disc Q'' with center $O'' \in Y$ (Figure 9(b)), point p has to be above



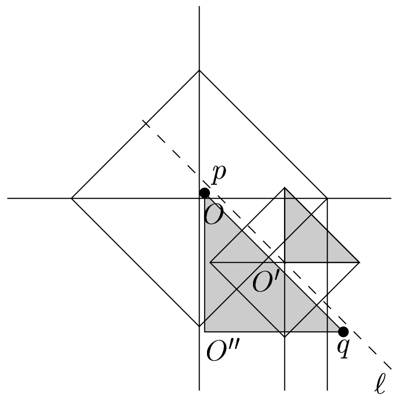
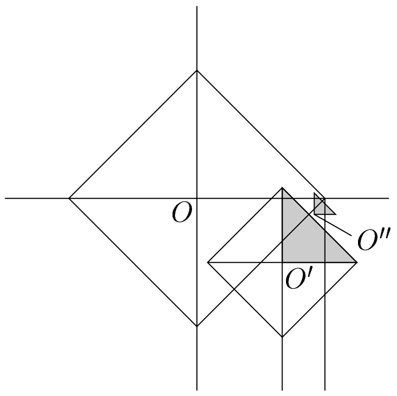
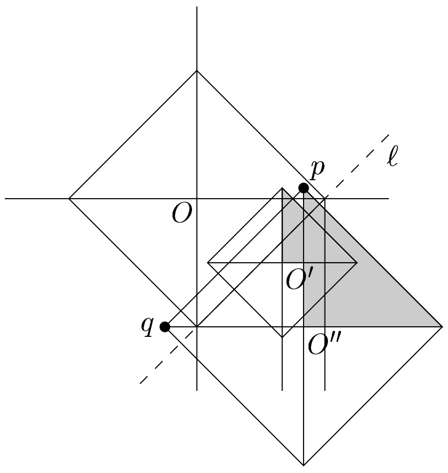


Figure 8: The two possibilities for quarter-discs with center $O'' \in X$.



(a) If $O'' \in Z$, then $Q' \cap Q'' = \emptyset$.



(b) If $O'' \in Y$, then $O' \in Q''$.

Figure 9: Quarter-discs in regions Z and Y.

line ℓ because Q'' intersects Q. Hence, q is to the left of line ℓ. Moreover, point p is above point O' and q is below point O'. Hence, $O' \in Q''$. This leads to a contradiction. $\square$

Lemma 5 *For every L_1-quarter disc Q, all L_1-quarter discs $Q_1, \ldots, Q_q$ of the same type that overlap Q are pairwise disjoint.*

Proof. By Lemma 3, there are no two quarter-discs Q' and Q'' overlapping Q so that $O' \in B$, $O'' \in A$, and $Q' \cap Q'' \neq \emptyset$. By Lemma 4, there are no two quarter-discs Q' and Q'' overlapping Q so that $O', O'' \in B$ and $Q' \cap Q'' \neq \emptyset$. A similar argument shows that there are no two quarter-discs Q' and Q'' overlapping Q so that $Q', Q'' \in A$ and $Q' \cap Q'' \in \emptyset$. By Lemma 2, regions A and B are the only two regions containing centers of quarter-discs overlapping Q. This proves the lemma. $\square$

The arguments of Lemmas 2 and 4 carry over to the L_2 and L_∞-metrics with only minor modifications. The argument of Lemma 3 requires a little closer attention. Here we argued that every L_1-disc C'' with center $O'' \in A$ has to be above line ℓ in Figure 6, and every L_1-disc C' with center $O' \in B$ has to be below that line. The argument was based on the fact that neither $O \in C'$ nor $O \in C''$. Now if C' and C'' are L_2 or L_∞-discs that do not contain point O, their largest enclosed L_1-discs do not contain point O, so that according to Lemma 3, they are completely on one side of line ℓ. Now it is easy to verify that this implies that the north-east quarter-discs of C' and C'' are also completely on one side of line ℓ, which proves Lemma 3 for the L_2 and L_∞-metrics. Summarizing these observations, we obtain the proof for Theorem 2. □

The dynamic data structure is a two-level tree structure whose base tree is a segment tree on the x-projections of the quarter-discs. Any node u of the primary tree corresponds to a vertical slab S_u, and all the quarter-discs that span the width of S_u, and not of its parent slab, are stored in the secondary structure associated with u. The secondary structure associated to the node u is an interval tree defined on the following intervals ordered according to y-coordinates. For all the quarter-discs stored at node u, intervals are the intersection of quarter-discs with the left boundary of S_u. The global (two-level) data structure is augmented such as to support dynamic fractional cascading [5].

To insert or delete a point (and its quarter-disc), we proceed as if we are doing that in the standard segment tree, except that we need to insert/delete in each of the interval trees along the search path. It is a straightforward task to modify the dynamic fractional cascading data structure of [5] to accommodate interval trees in place of "catalogue" at the expense of increasing the time complexity of these operations by a factor of $\log n$.

Reverse nearest neighbor queries are answered as follows. For a query point q use the primary segment tree structure to locate a node, say u, containing a set of quarter-discs whose x-projections encompasses the x-coordinate of q. Also recall that a node in the segment tree corresponds to an interval, say I, on the x-axis and the projections of each of the quarter-discs stored at u along x-axis covers the interval I. To search in the secondary interval tree, we locate all the intervals that are stabbed by y-coordinate of q. We will show that there are at most two such intervals, and hence we report at most two quarter-discs in each slab. Therefore the query procedure is same as that in dynamic segment trees with the exception that the search in each node of the tree is done in an interval tree at the expense of increasing the query time by a factor of $\log n$.

As a consequence of the Theorem 2, any point in the plane will be covered by at most two quarter-discs of the same type. Consider the slab S_u associated with any node u of the segment tree. Let Z be the set of quarter-discs associated with the interval tree of node u such that the y-coordinate of the query point q stabs the intervals formed by the intersections of all discs in Z with the left boundary of S_u. Let Q be the quarter-disc among the discs in Z whose horizontal axis h passing through its center is immediately below the query point q. Let α be the intersection point of h and the left boundary of S_u. As all discs in Z span the whole width of S_u, it is easy to see that the intersection point α is in all the discs in the set Z, hence, by Theorem 2, $|Z| \leq 2$. The number of discs in Z actually containing q, however, might be smaller, as can be seen from Figure 10: In this situation, the candidate set Z consists of the quarter-discs Q and Q', but only Q' actually contains the query point q.

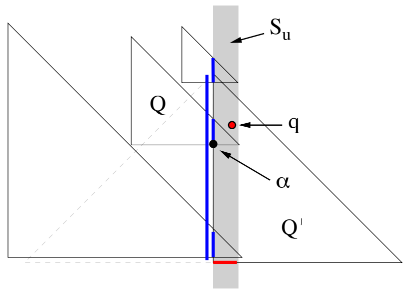


Figure 10: Querying a secondary interval tree.

It is easy to see that the complexity of insert and delete operations is $\mathcal{O}(Q(n)\log^2 n \log\log n)$ and it takes $\mathcal{O}(\log^2 n \log\log n)$ to answer a reverse nearest neighbor query. The overall space requirement for the global structure is in $\mathcal{O}(n \log n)$ [5]. We summarize the result in the following theorem.

Theorem 3 *A set $\mathcal{S}$ of n points in the plane can be maintained under insertions and deletions such that a reverse nearest neighbor query can be answered in $\mathcal{O}(\log^2 n \log\log n)$ time. The time complexity of each update operation* *is in* $\mathcal{O}(Q(n) + \log^2 n \log\log n)$*, and the space requirement for maintaining* $\mathcal{S}$ *is* $\mathcal{O}(S(n) + n\log n)$*.*

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