Discrete Mathematics I MATH/CSCI 2112 Lec 12/11 Mathematical Induction 4

- **Q.** How large do the Fibonacci numbers get? Can we obtain a "closed form" solution?
  The answer to the second question is yes. On the way to showing that, we will answer the first question.
  Imp. To try and solve a recurrence like the Fibonacci recurrence ($F_n = F_{n-1} + F_{n-2}$), try substituting $G_n = c \cdot r^n$; $(n > 2)$ into a sequence G_n which satisfies the same recurrence as F_n. Which gives:
  $cr^n = cr^{n-1} + cr^{n-2}$ simplifying gives: $r^2 = r + 1$ Solving the quadratic gives:

  $$r_1 = \frac{1+\sqrt{5}}{2} \; and \; r_2 = \frac{1-\sqrt{5}}{2}$$

  So, $G_n = c\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $G'_n = c\left(\frac{1-\sqrt{5}}{2}\right)^n$ **both** satisfy the recurrence but clearly do not match up with F_n. Notice $G_0 = c = G'_0$ but $F_0 = 0$ However, $G_0 - G'_0$ matches. In addition, $G_n - G'_n$ also, itself satisfies the recurrence.
  Matching $G_1 - G'_1 = c\sqrt{5} = F_1 = 1$ Gives $c = \frac{1}{\sqrt{5}}$.
  So, $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$
  Notice that $\left(\frac{1+\sqrt{5}}{2}\right) = \Phi > 1$ while the second term is between 0 and -1.
  This means that for large n, F_n is the integer nearest $\left(\frac{1+\sqrt{5}}{2}\right)^n$
  $\Phi = 1.618034\ldots$ is called the *Golden Ratio*, It is the limit of the ratios of consecutive Fibonacci numbers.

- We can use the same technique to bound terms in a recurrence.

  Ex. Given the recursively defined function

  $f_0 = 1;\; f_1 = 2;\; f_2 = 4 \;\&\; f_k = f_{k-1} + 2f_{k-2} + f_{k-3} \;\; k > 3$

  Find a bound on f_k.

  Sol: Try $f_n \leq r^n$.

- *Ex* If $\{x_n\}$ is a recursively defined sequence as:

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 4 \\
x_n &= 2x_{n-1} - x_{n-2} + 2
\end{aligned}
$$

  find a closed form solution for x_n and prove your answer.