Efficient Gröbner tracing in Groebner.jl

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Motivation

Earlier (or later) at ICMS 2026:

- Identifiability of Linear Compartmental Models, *Cash Bortner*
- The Euclidean Distance Degree package for Macaulay2, *Will Huang*
- Trajectory Planning and Certification for 3-DOF Robot Manipulators Using Real Quantifier Elimination Based on Comprehensive Gröbner Systems, *Akira Terui et al.*
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A common computational task..

These at some point mention or rely on **repeatedly** solving systems of nonlinear equations.

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In this talk, Groebner.jl

How to make repeated Gröbner basis computations faster and available?

One standard example from computer algebra

Example, structural identifiability of ODEs [Hong et al., 2019]

Consider the **CRN** example, a system of polynomials in 47 indeterminates:

$$
\begin{aligned}
x_{21} - x_{20}x_{10}k_{10} - x_{40}k_{20} - x_{40}k_{30} &= 0 \\
x_{20} - 544397587489487158925 &= 0 \\
&\dots \\
-6x_{52}x_{32}k_{60} - 4x_{33}x_{51}k_{60} - 4x_{53}x_{31}k_{60} - x_{34}x_{50}k_{60} - \dots &= 0.
\end{aligned}
$$

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Solving the system

We can find the solution from the Gröbner basis of the system:

$$k_{60} = 850293348699016668 38, \quad k_{50} = 5635268359634276894 29, \quad \dots$$

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Intermediate expression swell

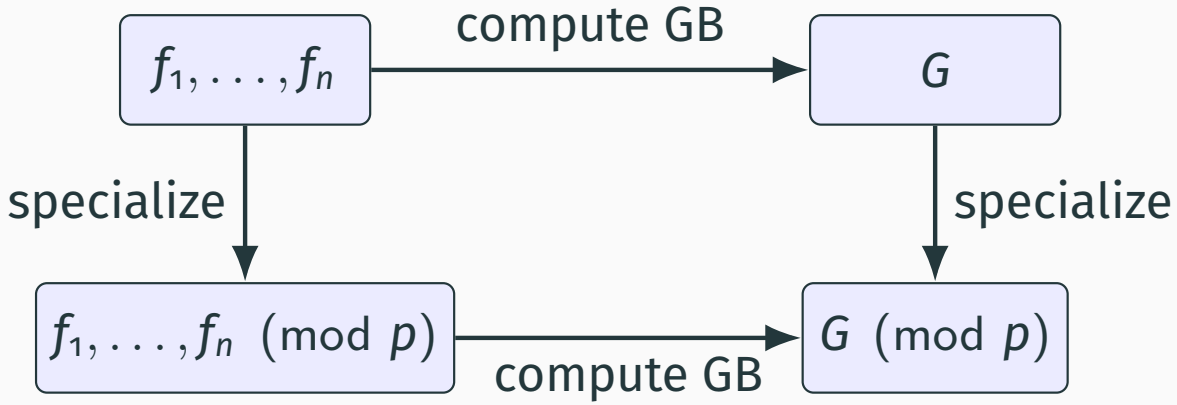
In Gröbner basis in computations there appears a term with coefficient:

389802997212380535243910008992734398822789281692823413123211432594829 24

It contains roughly **135,000 digits**.

A standard solution: multi-modular methods

Gröbner bases and specialization "almost" commute:



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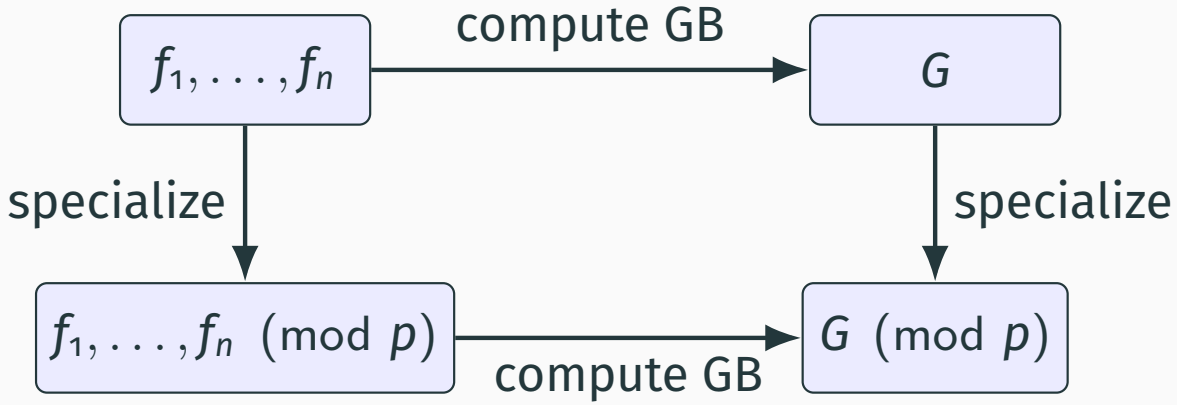

Avoiding intermediate expression swell:

1. Gröbner basis of CRN system

Compute modulo prime $\mathbf{p_1 = 65551}$:

$$k_{60} = 2196, \quad k_{50} = 36593 \pmod{p_1}$$

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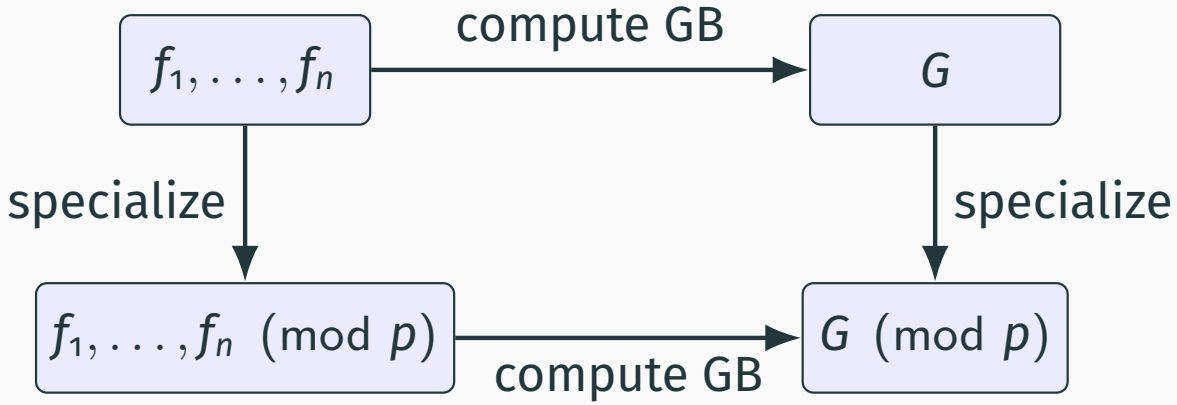

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Compute modulo prime $\mathbf{p_2 = 65557}$:

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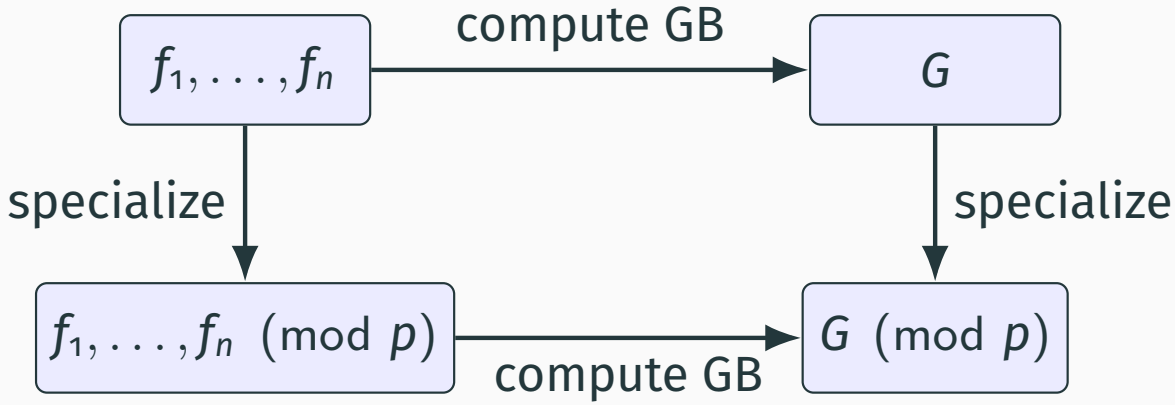

Avoiding intermediate expression swell:

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2. Gröbner basis of CRN system

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And then reconstructing the result:

Gröbner basis of CRN system mod $p_1 p_2$

Interpolate via Chinese Remainder Theorem:

$$k_{60} = 2941603321, \quad k_{50} = 3859679473 \mod \mathbf{p_1 p_2}, \quad \ldots$$

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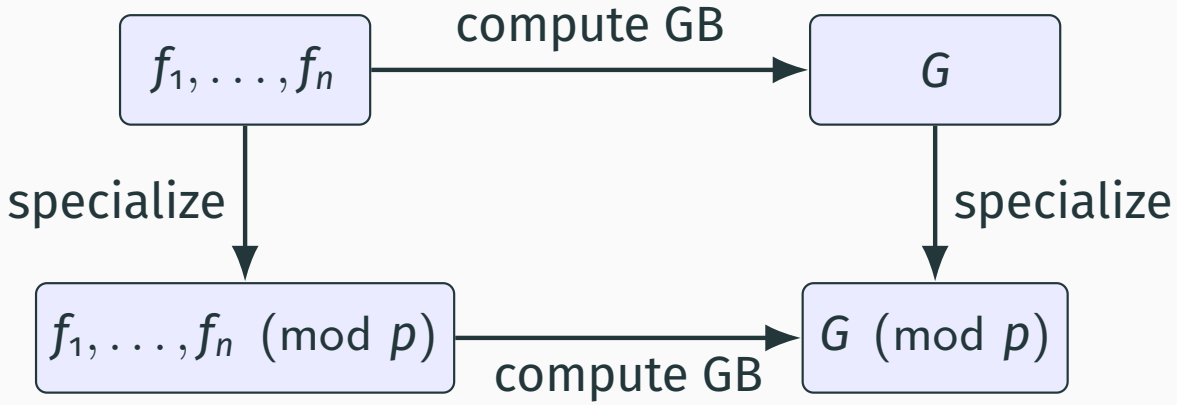

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In general, $\pmod I$ where I is an ideal

And then reconstructing the result:

Gröbner basis of CRN system mod $p_1 p_2$

Hensel lifting, rational function interpolation, etc.

Interpolate via Chinese Remainder Theorem :

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Gröbner tracing

Multi-modular method

Input: Polynomials $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$.

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Gröbner tracing

Multi-modular method with tracing [Traverso, 1989]

Input: Polynomials $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$.

1. Precompute a trace T using $f_1, \ldots, f_n$.
2. For $i = 1, 2, 3, \ldots$
   - 2.1 Let p_i be a prime number.
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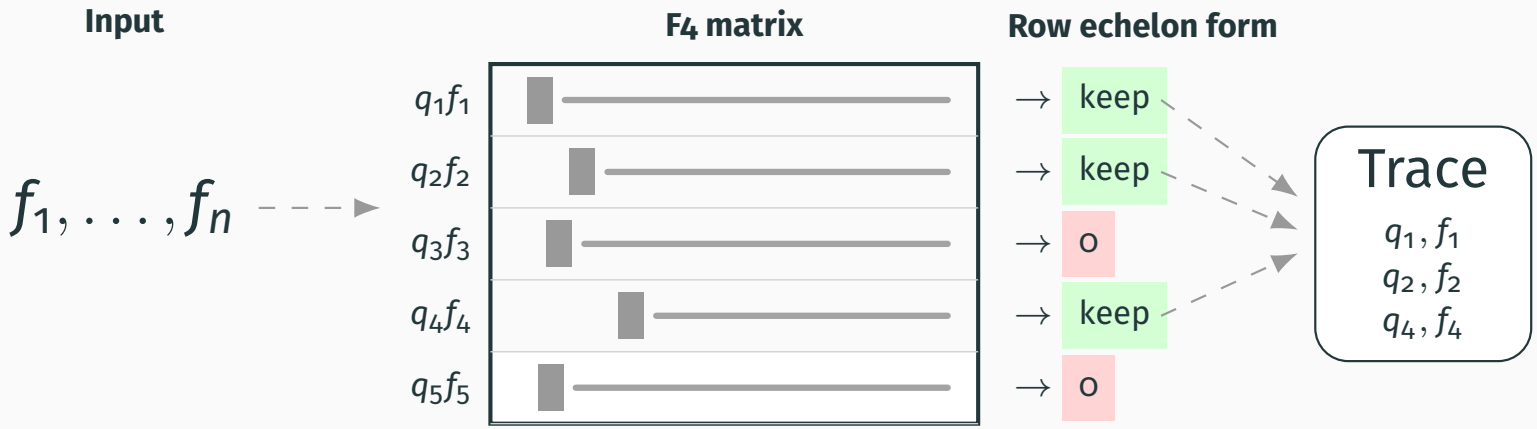
When using the F4 algorithm for computing Gröbner bases:



Adoption of Gröbner tracing

Existing software uses tracing internally to compute Gröbner bases over $\mathbb{Q}$: Giac/Xcas, msolve, Groebner.jl, etc.

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In Gröbner bases libraries, the following interface could be useful:

- `precompute` $(f_1, \ldots, f_n) \rightarrow$ a trace T.

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- `precompute` $(f_1, \ldots, f_n) \rightarrow$ a trace T.
- `apply` $(f_1, \ldots, f_n \bmod p,$ trace $T) \rightarrow$ a Gröbner basis $\bmod p$.

Our contribution

- We provide such tracing interface in Groebner.jl.

```
sys, sys_1, sys_2, sys_3 = ... # define input systems appropriately

trace, _ = groebner_learn(sys)

for i = 1 : 1000
    success, gb = groebner_apply!(trace, sys_i)
end
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- Our implementation is efficient

```
@btime groebner(sys);   # Katsura-10,  1024 solutions
#  731.000 ms (84551 allocations: 139.38 MiB)

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- We make tracing faster.

Making Gröbner tracing faster

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- Symbolic preprocessing
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Recall components of the classic F4 algorithm mod $(p_1, \ldots, p_N)$:

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- Computing row echelon form mod $(p_1, \ldots, p_N)$

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$\rightsquigarrow$ compute mod $(p_1, \ldots, p_N)$, with arithmetic done component-wise.

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In other words..

Compute over the product ring

$$\mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_N\mathbb{Z}$$

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Used by Gräbe, and de Kleine & Monagan already in **1993** and **2001**, resp.

Making Gröbner tracing faster

Arithmetic in $\mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_N\mathbb{Z}$ can be implemented with tuples:

```
julia> a1 = (1,2,3,4)::NTuple{4, Int64}
(1, 2, 3, 4)

julia> a2 = (5,6,7,8)::NTuple{4, Int64}
(5, 6, 7, 8)
```

```
julia> a1 .+ a2
(6, 8, 10, 12)
```

Typically uses SIMD automatically in Julia:

```
julia> @code_native a1 .+ a2
        push    rbp
        mov     rbp, rsp
        mov     rax, rdi
        vmovdqu ymm0, ymmword ptr [rdx]
        vpaddq  ymm0, ymm0, ymmword ptr [rsi]
        vmovdqu ymmword ptr [rdi], ymm0
        pop     rbp
        ret
```

Making Gröbner tracing faster

In the interface:

```
success, (gb1,gb2,gb3,gb4) = groebner_apply!(trace, (sys1,sys2,sys3,sys4))
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After:

1. Let $p_1, \ldots, p_N$ be prime numbers.
2. Compute a Gröbner basis of $\langle f_1, \ldots, f_n \rangle \bmod (p_1, \ldots, p_N)$ using trace T.

Results

In `StructuralIdentifiability.jl`:

Table 1: The number of Gröbner bases and the total running times for computing generators of the field of identifiable functions via rational function interpolation

Model	# Gröbner bases	F4	F4 with tracing	Speedup
EAIHRD	420	300 s	49 s	6.1
LinComp2	4388	275 s	164 s	1.7
Pharm	34	54 s	36 s	1.5

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In `RationalUnivariateRepresentation.jl`:

Table 2: Total running times of rational univariate representation computation.

System	F4	F4 with tracing	Speedup
Chandra-10	82.4 s	29.3 s	2.8
Chandra-11	851.4 s	373.5 s	2.3
Eco-10	1.0 s	0.4 s	2.5
Eco-11	11.2 s	4.2 s	2.7
Katsura-11	71.0 s	28.2 s	2.5
SEIR36	63.7 s	9.6 s	6.6

Thank you!

Table 3: Running times, and the number of computations required to amortize the cost of learning a trace compared against classic independent Gröbner basis computations.

System	ClassicF4	Precompute	Apply	Computations to break even
Chandra-11	1.0 s	3.3 s	0.1 s	4
Chandra-12	5.8 s	19.6 s	0.3 s	4
Katsura-12	2.2 s	17.2 s	0.8 s	13
Katsura-13	14.2 s	153.2 s	6.1 s	19
Cyclic-8	0.4 s	1.5 s	0.2 s	8
Cyclic-9	38.9 s	219.4 s	26.7 s	19
Cholera	13.7 s	40.9 s	7.7 s	7
Yang1	5.8 s	39.2 s	0.4 s	8