Fundamentals

Textbook reading

Chapter 1
Chapter 2
Overview

**Design principle:**
- Steady progress towards a solution

**Proof techniques:**
- Proof by contradiction

**Basics of algorithm analysis:**
- Asymptotic notation ($\mathcal{O}$, $\Omega$, and $\Theta$)
- Linear time, $n \lg n$ time, quadratic time, exponential time

**Basic data structures:**
- Arrays
- Linked lists
Stable Marriages: The Gale-Shapley Algorithm

**PROPOSAL-ALGORITHM** \((M, W)\)

1. while there is an unmarried man \(m\)

2. do \(m\) chooses his favourite woman \(w\) he has not proposed to yet

3. \(m\) proposes to \(w\)

4. if \(w\) is not married or likes \(m\) better than her current partner \(m'\)

5. then \(w\) divorces \(m'\)

6. \(w\) marries \(m\)

- Is there always a set of \(n\) stable marriages?
- Does the algorithm ever terminate?
- Does the algorithm always produce a correct answer?
- How efficient is the algorithm? Can we give an upper bound on its running time?
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**Proposal-Algorithm** \((M, W)\)

1. **while** there is an unmarried man \(m\) who has not proposed to all women
2. **do** \(m\) chooses his favourite woman \(w\) he has not proposed to yet
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Lemma: The proposal algorithm terminates after at most $n^2$ iterations.
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- There are $n$ men
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**Termination of the Proposal Algorithm**

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- In every iteration, one proposal is made.
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∴ The total number of iterations is at most $n^2$. 
Lemma: The proposal algorithm terminates after at most $n^2$ iterations.

- There are $n$ men
- Every man can propose to $n$ women
- No man proposes to the same woman twice
- Therefore, there are $n^2$ proposals to be made

- In every iteration, one proposal is made.

Therefore, the total number of iterations is at most $n^2$.

To terminate, an algorithm has to make steady progress towards a solution.
Lemma: When the proposal algorithm terminates every woman is married (and, therefore, every man is married).
**Everybody Gets Married: Proof By Contradiction**

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**Proof by contradiction:**
- Assume that what we want to prove is not correct.
- Prove that this leads to a contradiction.
  - Either to the assumption we made
  - Or to some known fact
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Contradiction.
Stable Marriages: The Gale-Shapley Algorithm

**Proposal Algorithm** ($M, W$)

1. **while** there is an unmarried man $m$ that has not proposed to all women
2. **do** $m$ chooses his favourite woman $w$ he has not proposed to yet
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5. **then** $w$ divorces $m'$
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Correctness of the Proposal Algorithm

**Lemma:** When the proposal algorithm terminates, all marriages are stable.

- Let \((w, m)\) and \((w', m')\) be two marriages so that
  - \(w\) prefers \(m'\) over \(m\) and
  - \(m'\) prefers \(w\) over \(w'\).

- Since \(m'\) prefers \(w\) over \(w'\), he must have proposed to \(w\) before getting married with \(w'\).
Let $m''$ be the man that $w$ is married to immediately after $m'$ has proposed.

- If $w$ accepts $m'$, then $m'' = m'$
- If $w$ rejects $m'$, then $m'' \succ_w m'$ (read: $w$ prefers $m''$ over $m'$)

$\therefore m'' \succeq_w m'$
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Let $m'' = m_1, m_2, \ldots, m_k = m$ be the sequence of partners $w$ has from this time on till the end of the algorithm.

$m' \preceq_w m_1 \prec_w m_2 \prec_w \cdots \prec_w m_k = m.$
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**Contradiction.**
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Contradiction.

**Corollary:** There always exists a set of stable marriages.
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- Is there always a set of \(n\) stable marriages?
- Does the algorithm ever terminate?
- Does the algorithm always produce a correct answer?
- How efficient is the algorithm? Can we give an upper bound on its running time?
Does the final set of marriages depend on the order in which the men propose?
- It does not.

Is the process fair?
- The men may fare much better than the women.

Can the algorithm be implemented efficiently?

Can we design a faster algorithm?
- Yes, using randomization.
Computational Tractability

Criterion of efficiency = resource consumption:

- Running time
- Memory usage
- Disk usage
- Amount of communication in distributed computations
- ...

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Definition of Efficiency — First Attempt

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- Efficiency of algorithm depends on quality of implementation and speed of machine.
- Does not capture the dependence of running time on input size.
- What are “real input instances”?
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- What are “real input instances”?

Therefore: Need a definition that is platform-independent, instance-independent, and captures the dependence on the input size.
Model of Computation

**RAM Model:**

Elementary operations take constant time:
- Arithmetic operations: addition, subtraction, multiplication, division
- Boolean operations: and, or, not
- If-statements
- Checking of loop conditions
- Assignment

*By counting elementary operations, we can compare the running times of two algorithms when run on the same computer.*
Running Time Depends on Structure of Input

An algorithm’s running time may differ greatly for different input instances of the same size.

*Example*: Insertion Sort

```
INSERTIONSORT(A, n)
1  for i ← 2 to n
2    do key ← A[i]
3         j ← i - 1
4     while j > 0 and A[j] > key
6     A[j + 1] ← key
```
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**Example:** Insertion Sort

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```

How do we unify this into one figure?
Worst-Case and Average-Case Analysis

The **worst-case running time** of an algorithm $\mathcal{A}$ is a function $T_A(n)$ such that $T_A(n)$ is the maximal running time of $\mathcal{A}$ over all inputs of length $n$.

The **average-case running time** of an algorithm $\mathcal{A}$ is a function $T_A(n)$ such that $T_A(n)$ is the average of the running times of $\mathcal{A}$ on all inputs of length $n$. 
An algorithm is efficient if, at an analytical level, it achieves a better worst-case performance than the brute-force method.
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Pros:

- Platform-independent
- Instance-independent
- Captures dependence on input size
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**Pros:**

- Platform-independent
- Instance-independent
- Captures dependence on input size
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**Cons:**

- Vague
An algorithm is efficient if its worst-case running time is polynomial in the input size.

Motivation:

- If input size doubles, running time should increase by only a constant factor.
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- Is $n^{1+0.02 \log n}$ inefficient?
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Justification: Overwhelmingly, polynomial-time algorithms are fast in practice and exponential-time algorithms are not.
**Scenario:** Given two algorithms $A_1$ and $A_2$ which we want to compare.

Do we care which algorithm is faster for small inputs?
### Asymptotic Running Time

**Scenario:** Given two algorithms $A_1$ and $A_2$ which we want to compare.

Do we care which algorithm is faster for small inputs?

What matters is which algorithm is faster for large inputs.
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What matters is which algorithm is faster for large inputs.

**Formally:**

We want that $T_1(n) < T_2(n)$, for all $n \geq n_0$, where $n_0$ is a constant.
$f(n) = \mathcal{O}(g(n))$

$\iff$

$\exists c > 0, n_0 \geq 0 \forall n \geq n_0 : f(n) \leq c \cdot g(n)$
Ω-Notation

\[ f(n) = \Omega(g(n)) \]

\[ \Leftrightarrow \]

\[ \exists c > 0, n_0 \geq 0 \forall n \geq n_0 : f(n) \geq c \cdot g(n) \]
\( f(n) = \Theta(g(n)) \)

\[ \exists c_1 > 0, c_2 > 0, n_0 \geq 0 \forall n \geq n_0 : c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \]
$o$-Notation

$f(n) = o(g(n))$

$\implies$

$\forall c > 0 \exists n_0 \geq 0 \forall n \geq n_0 : f(n) \leq c \cdot g(n)$
**ω-Notation**

\[ f(n) = \omega(g(n)) \]

\[ \forall c > 0 \exists n_0 \geq 0 \forall n \geq n_0 : f(n) \geq c \cdot g(n) \]
A Few Simple Facts

\[ f(n) = \mathcal{O}(f(n)) \quad f(n) = \Omega(f(n)) \quad f(n) = \Theta(f(n)) \]

\[ f(n) = \mathcal{O}(g(n)) \text{ and } g(n) = \mathcal{O}(h(n)) \implies f(n) = \mathcal{O}(h(n)) \]

\[ f(n) = \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \implies f(n) = \Omega(h(n)) \]

\[ f(n) = \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \implies f(n) = \Theta(h(n)) \]

\[ f(n) = \mathcal{O}(g(n)) \iff g(n) = \Omega(f(n)) \]

\[ f(n) = o(g(n)) \iff g(n) = \omega(f(n)) \]

\[ f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Omega(g(n)) \implies f(n) = \Theta(g(n)) \]

\[ f_1(n) = \mathcal{O}(g_1(n)) \text{ and } f_2(n) = \mathcal{O}(g_2(n)) \implies f_1(n) + f_2(n) = \mathcal{O}(g_1(n) + g_2(n)) \]

\[ f(n) = \mathcal{O}(g(n)) \implies f(n) + g(n) = \mathcal{O}(g(n)) \]
Asymptotic Analysis and Limits

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \iff f(n) = o(g(n))
\]

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = c \iff f(n) = \Theta(g(n))
\]

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \implies a^{f(n)} = o(a^{g(n)}), \text{ for any } a > 1
\]

\[
f(n) = o(g(n)) \implies a^{f(n)} = o(a^{g(n)}), \text{ for any } a > 1
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\[
f(n) = \Theta(g(n)) \iff a^{f(n)} = \Theta(a^{g(n)})
\]
What does it mean if $T_1(n) = \mathcal{O}(T_2(n))$?
Asymptotic Notation and Algorithm Performance

What does it mean if $T_1(n) = \mathcal{O}(T_2(n))$?
- Algorithm $A_1$ is by at most a constant factor slower than $A_2$.
- The factor may be large!!!
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- For sufficiently large $n$, algorithm $A_1$ will outperform algorithm $A_2$. 
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Can we ignore constants?
- In a first filter step of possible candidates, this is desirable.
- Subsequent choices have to be based on our experience or analyses that take constants into account.
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What do we gain?
- A simple, succinct expression of the performance of an algorithm.
Example: Merging two sorted sequences
**Linear Time (2)**

\[ \text{MERGE}(A, B) \]

1. \( C \leftarrow \emptyset \)
2. \textbf{while} \( A \) and \( B \) are non-empty \textbf{do} Let \( a \) be the first element in \( A \)
3. \quad Let \( b \) be the first element in \( B \)
4. \quad \textbf{if} \( a < b \)
5. \quad \quad \textbf{then} Remove \( a \) from \( A \)
6. \quad \quad Append \( a \) to \( C \)
7. \quad \quad \textbf{else} Remove \( b \) from \( B \)
8. \quad \quad Append \( b \) to \( C \)
9. \quad \quad \textbf{if} \( A \) is non-empty \textbf{then} Append \( A \) to \( C \)
10. \quad \quad \textbf{else} Append \( B \) to \( C \)
11. \quad \textbf{return} \( C \)
**Linear Time (3)**

```latex
\textbf{MERGE}(A, p, q, r)
\begin{align*}
1 & \quad n_1 \leftarrow q - p + 1 \\
2 & \quad n_2 \leftarrow r - q \\
3 & \quad \text{for } i \leftarrow 1 \text{ to } n_1 \\
4 & \quad \text{do } L[i] \leftarrow A[p + i - 1] \\
5 & \quad \text{for } i \leftarrow 1 \text{ to } n_2 \\
6 & \quad \text{do } R[i] \leftarrow A[q + i] \\
7 & \quad L[n_1 + 1] \leftarrow \infty \\
8 & \quad R[n_2 + 1] \leftarrow \infty \\
9 & \quad i \leftarrow 1 \\
10 & \quad j \leftarrow 1 \\
11 & \quad \text{for } k \leftarrow 1 \text{ to } r - p + 1 \\
12 & \quad \text{do if } L[i] < R[j] \\
13 & \quad \text{then } A[k] \leftarrow L[i] \\
14 & \qquad i \leftarrow i + 1 \\
15 & \quad \text{else } A[k] \leftarrow R[j] \\
16 & \qquad j \leftarrow j + 1
\end{align*}
```
**Example:** Sorting a sequence

```
MERGE-SORT(A, p, r)
1   if r > p
2      then q ← ⌊ \frac{p+r}{2} ⌋
3      Merge-Sort(A, p, q)
4      Merge-Sort(A, q + 1, r)
5      Merge(A, p, q, r)
```
**Example:** Sorting a sequence

\[
\text{MERGE-SORT}(A, p, r)
\]

1. **if** \( r > p \)
2. **then** \( q \leftarrow \lfloor \frac{p+r}{2} \rfloor \)
3. \( \text{Merge-Sort}(A, p, q) \)
4. \( \text{Merge-Sort}(A, q + 1, r) \)
5. \( \text{Merge}(A, p, q, r) \)

**Comments:**
- \( \mathcal{O}(n \lg n) \) is the best we can do for sorting.
- \( \mathcal{O}(n \lg n) \) is a typical time bound for divide-and-conquer algorithms.
Example: Closest pair

Given a set $P$ of $n$ points, find the two points in $P$ whose distance is smallest.
Quadratic Time (2)

\textbf{CLOSESTPAIR}(P)
1 \hspace{0.5em} d \leftarrow +\infty
2 \hspace{0.5em} \textbf{for every point } p \in P
3 \hspace{0.5em} \textbf{do for every point } q \in P \text{ such that } q \neq p
4 \hspace{0.5em} \textbf{do if } \text{dist}(p, q) < d
5 \hspace{0.5em} \textbf{then } d \leftarrow \text{dist}(p, q)
6 \hspace{0.5em} \text{pair} \leftarrow (p, q)
7 \hspace{0.5em} \textbf{return } \text{pair}
**Quadratic Time (2)**

**CLOSESTPAIR**($P$)
1. $d \leftarrow +\infty$
2. for every point $p \in P$
3. do for every point $q \in P$ such that $q \neq p$
4. do if \( \text{dist}(p, q) < d \)
5. then $d \leftarrow \text{dist}(p, q)$
6. pair $\leftarrow (p, q)$
7. return pair

**Comments:**
- Quadratic time is easy to recognize here because there are two nested loops iterating over the whole input.
- Clever use of divide-and-conquer and geometric insight leads to an \( O(n \lg n) \) solution.
Example: Matrix multiplication

\[
\text{MATRIX-MULTIPLY}(A, B, n)
\]

1. \textbf{for } \(i \leftarrow 1\) \textbf{ to } \(n\)
2. \hspace{1em} \textbf{do for } \(j \leftarrow 1\) \textbf{ to } \(n\)
3. \hspace{2em} \textbf{do } \(x \leftarrow 0\)
4. \hspace{3em} \textbf{for } \(k \leftarrow 1\) \textbf{ to } \(n\)
5. \hspace{4em} \textbf{do } \(x \leftarrow x + A[i][k] \times B[k][j]\)
6. \hspace{3em} \(C[i][j] \leftarrow x\)
7. \textbf{return } C
Cubic Time

**Example:** Matrix multiplication

Matrix-Multiply \((A, B, n)\)

1. for \(i \leftarrow 1\) to \(n\)
2. do for \(j \leftarrow 1\) to \(n\)
3. do \(x \leftarrow 0\)
4. for \(k \leftarrow 1\) to \(n\)
5. do \(x \leftarrow x + A[i][k] \times B[k][j]\)
6. \(C[i][j] \leftarrow x\)
7. return \(C\)

**Comments:**

- Again, cubic time is easy to recognize here because there are three nested loops iterating over a complete row or column.
- Clever use of divide-and-conquer leads to a sub-cubic algorithm (Strassen’s algorithm).
**Example:** Finding an independent set of size $k$

An independent set is a subset, $S$, of vertices such that no two vertices in $S$ are adjacent.

![Diagram of an independent set and a non-independent set](image)

**Goal:** Find an independent set of size $k$, if one exists.
INDEPENDENT-SET($G$, $k$)
1 for every set $S$ of $k$ vertices
2 do if $S$ is independent
3 then return $S$
4 return $\emptyset$
**INDEPENDENT-SET**\((G, k)\)

1. for every set \(S\) of \(k\) vertices
2. do if \(S\) is independent
3. then return \(S\)
4. return \(\emptyset\)

**Analysis:**

- There are \(\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{1\cdot2\cdot...\cdot k} = \mathcal{O}(n^k)\) sets of \(k\) vertices.
Polynomial Time (2)

**INDEPENDENT-SET**$(G, k)$

1. for every set $S$ of $k$ vertices
2. do if $S$ is independent
3. then return $S$
4. return $\emptyset$

**Analysis:**

- There are \( \binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot2\ldots k} = O(n^k) \) sets of $k$ vertices
- Testing each for independence takes $O(n^2)$ time
**Analysis:**

- There are $\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot2\cdot\ldots\cdot k} = \mathcal{O}(n^k)$ sets of $k$ vertices.
- Testing each for independence takes $\mathcal{O}(n^2)$ time.

**Total:** $\mathcal{O}(n^{k+2})$
**Example:** Finding a largest independent set

\[
\text{MAXIMUM-INDEPENDENT-SET}(G)
\]

1. for every set \( S \) of vertices, by decreasing cardinality
2. do if \( S \) is independent
3. then return \( S \)
**Example:** Finding a largest independent set

**MAXIMUM-INDEPENDENT-SET**(G)

1. for every set S of vertices, by decreasing cardinality
2. do if S is independent
3. then return S

**Analysis:**
- There are $2^n$ sets of vertices
  - Running time = $\mathcal{O}(n^2 \cdot 2^n)$
Exponential Time

**Example:** Finding a largest independent set

```plaintext
MAXIMUM-INDEPENDENT-SET(G)
1  for every set S of vertices, by decreasing cardinality
2   do if S is independent
3     then return S
```

**Analysis:**
- There are $2^n$ sets of vertices
- Running time = $\mathcal{O}(n^2 \cdot 2^n)$

**Comments:**
- Independent set is an NP-complete problem, that is, it is unlikely to have a polynomial-time solution.
- $n! = \text{brute-force solution to sequencing problems}$
- $2^n = \text{brute-force solution to subset selection problems}$
**Example:** Binary search

```plaintext
BINARY-SEARCH(A, p, r, x)
1  if p > r
2    then return false
3  else q ← \lfloor\frac{p+r}{2}\rfloor
4     if A[q] = x
5       then return q
6     else if x < A[q]
7       then Binary-Search(A, p, q - 1, x)
8       else Binary-Search(A, q + 1, r, x)
```
## Sublinear Time

### Example: Binary search

**Binary-Search** ($A$, $p$, $r$, $x$)

1. **if** $p > r$
2. **then** return false
3. **else** $q \leftarrow \lfloor \frac{p+r}{2} \rfloor$
4. **if** $A[q] = x$
5. **then** return $q$
6. **else if** $x < A[q]$
7. **then** Binary-Search($A$, $p$, $q - 1$, $x$)
8. **else** Binary-Search($A$, $q + 1$, $r$, $x$)

### Analysis:

- Every iteration halves the set of candidate positions
- Running time is $\mathcal{O}(\lg n)$
**Implementation of Gale-Shapley Algorithm**

**Proposition-AlGORITHM** \((M, W)\)

1. while there is an unmarried man \(m\) 
2. do \(m\) chooses his favourite woman \(w\) he has not proposed to yet 
3. \(m\) proposes to \(w\) 
4. if \(w\) is not married or likes \(m\) better than her current partner \(m'\) 
5. then \(w\) divorces \(m'\) 
6. \(w\) marries \(m\)

**Lemma:** The Gale-Shapley algorithm can be implemented using elementary data structures so that it runs in \(O(n^2)\) time.

**Tools:**
- Arrays
- A linked list or another array
Summary

Design principle:
■ Steady progress towards a solution

Proof techniques:
■ Proof by contradiction

Basics of algorithm analysis:
■ Asymptotic notation ($\mathcal{O}$, $\Omega$, and $\Theta$)
■ Linear time, $n \log n$ time, quadratic time, exponential time

Basic data structures:
■ Arrays
■ Linked lists