Data Structuring

Textbook reading

Lecture Notes
Overview

**Design principles:**
- Use data structures to do the job
- Plane sweeping
- Augment data structures to support new operations

**Data structures:**
- \((a, b)\)-trees
- Order statistics trees
- Priority search trees
- Range trees

**Problems:**
- Line segment intersection reporting and counting
- Range searching and counting
The Dictionary ADT

Goal: Store a set $S$ of elements in a structure $T$ that supports the following operations:

- **Insert**($T, x$) Update $T$ so that it represents the set $S \cup \{x\}$
- **Delete**($T, x$) Update $T$ so that it represents the set $S \setminus \{x\}$
- **Find**($T, x$) Decide whether $x \in S$ and, if so, report all information associated with $x$ in $T$
If the elements in $S$ come from a total order, we may also want to support:

- **Range-Query** ($T, a, b$) : Report the set $\{x \in S \mid a \leq x \leq b\}$
- **Predecessor** ($T, x$) : If $x \in S$, report the next smaller element in $S$
- **Successor** ($T, x$) : If $x \in S$, report the next greater element in $S$
- **Minimum** ($T, x$) : Report $\min S$
- **Maximum** ($T, x$) : Report $\max S$
Examples of Dictionaries
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Simple dictionaries:
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- (Sorted) arrays
- (Sorted) doubly-linked lists
- Hash tables
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Simple dictionaries:

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Efficient dictionaries:

- Hash tables
- Balanced binary trees (AVL, red-black, BB$^{[\alpha]}$, AA, ...)
- $(a, b)$-trees
(a, b)-Trees

- All leaves are at the same depth
- All data elements are stored at the leaves
- The root has degree between 2 and b
- Any other non-leaf node has degree between a and b

2 \leq a \text{ and } 2a - 1 \leq b
\((a, b)\)-Trees

For a leaf \(v\) storing element \(x\), \(\text{key}(v) = x\)

For an internal node \(v\) with children \(w_1, w_2, \ldots, w_k\),
\(\text{key}(v) = \min\{\text{key}(w_i) \mid 1 \leq i \leq k\}\).

\[2 \leq a \text{ and } 2a - 1 \leq b\]
The Height and Size of an \((a, b)\)-Tree

**Lemma:** An \((a, b)\)-tree storing \(n\) items has height \(\mathcal{O}(\log_a n) = \mathcal{O}(\log n)\).
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Lemma: An \((a, b)\)-tree storing \(n\) items has less than \(2n\) nodes.
**Problem:** Storing $b$ child pointers per node seems wasteful.
Representing an \((a, b)\)-Tree

**Problem:** Storing \(b\) child pointers per node seems wasteful.

**Better representation:**
- Key of \(v\)
- Degree of \(v\)
- Pointer to \(v\)'s leftmost child
- Pointer to \(v\)'s parent
- Pointers to \(v\)'s left and right sibling
Lemma: An \((a, b)\)-tree storing \(n\) items uses \(\mathcal{O}(n)\) space.
The Find Operation

- **While** \( v \) **is not a leaf:**
  - Locate the leftmost child \( w \) of \( v \) such that
    - \( w \) has no right sibling or
    - The key of \( w \)'s right sibling is greater than \( x \).
  - Let \( v \) be this child.
- **When** \( v \) **is a leaf**, report \( v \) if \( \text{key}(v) = x \) and nil otherwise.
Lemma: Operation Find is correct.
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**Lemma:** Operation Find takes $\mathcal{O}(\lg n)$ time.
The Minimum Operation

- While $v$ is not a leaf, proceed to $v$’s left child.
- When $v$ is a leaf, report $v$. 

```
81 1 77 77 97 92 90
1 15 34 41 43 66 76
1 34 41 43 71 78
1 11 23 42 51 81
1 7 23 42
1 7 23
1
```
The Minimum Operation

- While $v$ is not a leaf, proceed to $v$'s left child.
- When $v$ is a leaf, report $v$.

Lemma: Procedure Minimum takes $O(\log n)$ time.
The Predecessor Operation

- While $v$ has no left sibling and is not the root, proceed to $v$’s parent.
- When $v$ is the root, report $\text{nil}$.
- When $v$ has a left sibling, $u$, report $\text{Maximum}(u)$. 
The Predecessor Operation

- While $v$ has no left sibling and is not the root, proceed to $v$'s parent.
- When $v$ is the root, report $\text{nil}$.
- When $v$ has a left sibling, $u$, report $\text{Maximum}(u)$.

Lemma: Procedure Predecessor takes $O(\log n)$ time.
The Insert Operation

- Use the Find procedure to locate the rightmost leaf \( v \) storing an element no greater than \( x \).
- Create a new leaf, store \( x \) at this leaf, and make it the right sibling of \( v \).
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Is the result still an $(a, b)$-tree?
The Insert Operation

- Use the Find procedure to locate the rightmost leaf \( v \) storing an element no greater than \( x \).
- Create a new leaf, store \( x \) at this leaf, and make it the right sibling of \( v \).

Is the result still an \((a, b)\)-tree? No.

How do we rebalance?
Node Splitting
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Lemma: An insertion into an $(a, b)$-tree takes $O(\lg n)$ time.
What do we do when we split the root?
Splitting the Root

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Note: This is the situation why we have to allow the root to have degree less than $a$. 
The Delete Operation

- Remove the leaf storing $x$ from its parent’s child list.
The Delete Operation

- Remove the leaf storing $x$ from its parent’s child list.
- Update the keys of all its ancestors.
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- Rebalance using node fusions.
Node Fusing
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CSci 3110 • Data Structuring • 18/81
Node Fusing

A diagram of a tree structure with nodes labeled from 1 to 81.
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**Lemma:** A deletion from an $(a, b)$-tree takes $O(\lg n)$ time.
What do we do when the root degree becomes 1?
What do we do when the root degree becomes 1?

- We remove the root.

![Tree Diagram]
What do we do when the root degree becomes 1?

- We remove the root.
What if node $v$ and its sibling together have more than $b$ children?
Node Sharing

What if node $v$ and its sibling together have more than $b$ children?

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Node Sharing

What if node \( v \) and its sibling together have more than \( b \) children?

- We fuse and then split (essentially borrowing children from \( v \)’s sibling).

**Lemma:** After a fusion followed by a split, the tree is an \((a, b)\)-tree again.
The Range-Query Operation

- Perform a depth-first traversal of the tree:
  - At every internal node, recursively visit all children
    - whose key is no greater than $b$ and
    - whose right sibling does not exist or has a key no less than $a$.
  - At every leaf, report the stored point if it is in the query range.

Query range: $[35, 76]$
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Lemma: Procedure Range-Query reports all points in the query range (and only those).
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- Visiting every brown or green node takes constant time.
- There are at most two brown nodes per level $\Rightarrow O(lg \ n)$ time.
- Green nodes form $(a, b)$-trees with at most $t$ leaves $\Rightarrow O(t)$ time.
Disk-Based Dictionaries

Disk storage:
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- Disks are slow.
**Disk-Based Dictionaries**

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Goal: Develop an efficient ordered dictionary for storing data on disk.

- Every computation step takes $O(1)$ disk accesses.
  - Balanced search trees require $O(\log n)$ disk accesses per operation.
B-Trees are a variant of \((a, b)\)-trees:

- \(B\) = number of items/pointers that can be stored in a disk block.
- \(a = \frac{B}{4}\) and \(b = \frac{B}{2}\).
- Do not store parent pointers.
- Every node stores direct pointers to its children.
- Every node stores the keys of its children.
- Every leaf stores between \(\frac{B}{2}\) and \(B\) items.
Lemma: B-Trees support operations Insert, Delete, Find, Predecessor, Successor, Minimum, and Maximum in $O(\log_B n)$ disk accesses. Procedure Range-Search takes $O(\log_B n + t/B)$ disk accesses.
(a, b)-Trees: Summary

(a, b)-Trees are efficient ordered dictionaries:

- Operations Insert, Delete, Find, Predecessor, Successor, Minimum, and Maximum take $\mathcal{O}(\lg n)$ time.
- Operation Range-Query takes $\mathcal{O}(\lg n + t)$ time.

B-trees, a variant of (a, b)-trees, are efficient ordered dictionaries for storage on disk:

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- Operation Range-Query takes $\mathcal{O}(\log_B n + t/B)$ disk accesses.
The data structuring paradigm:

Delegate the non-trivial work to one or more data structures.

Most common application:

- Model the computation as a sequence of transformations.
- Use data structures to represent the current state.
- Update data structure to reflect the effect of applied transformations.
Graph exploration algorithms use data structures to maintain the unexplored vertices adjacent to explored ones. (BFS, DFS, Dijkstra’s algorithm, Prim’s algorithm)

Kruskal’s algorithm uses a union-find structure to maintain the set of spanning forests.

Huffman’s algorithm maintains the current set of characters in a priority queue.
Map Overlay

Core of the problem: Find the intersection points.
Orthogonal Line-Segment Intersection (OLSI)

**Special case:** Find all intersections between

- a set, $V$, of $n$ vertical segments, $v_1, v_2, \ldots, v_n$, and
- a set, $H$, of $m$ horizontal segments, $h_1, h_2, \ldots, h_m$. 
How many intersections are there in the worst case?
Output-Sensitivity

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So the trivial algorithm of testing every pair of segments is optimal.

Can we still do better?

- Yes, we can aim to spend little time unless the output is big.
- This is called output-sensitivity.
The Sweep-Line Paradigm

**Idea:**

- Sweep a horizontal *sweep line* across the plane, bottom-up.
- Use *sweep-line structure* to maintain the interaction between scene and sweep line.
**Event Points**

*Discretization of sweep-line paradigm:*

- Update sweep-line structure only at certain *event points*.
- Solve problem by asking queries on sweep-line structure at other event points.
Sweep-line status: \((a, b)\)-tree \(T\) storing intersections between sweep-line and vertical segments, sorted from left to right.
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Event points:

- Bottom endpoint of vertical segment \(v_i\):
- Top endpoint of vertical segment \(v_i\):
- Horizontal segment \(h_j\):
**OLSI and Plane Sweeping**

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**Event points:**

- **Bottom endpoint of vertical segment** \(v_i\):
  - Sweep line starts to intersect \(v_i\).
  - Insert \(v_i\) into \(T\).

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OLSI and Plane Sweeping

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- Bottom endpoint of vertical segment \(v_i\):
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  - \(\therefore\) Insert \(v_i\) into \(T\).

- Top endpoint of vertical segment \(v_i\):
  - Sweep line stops intersecting \(v_i\).
  - \(\therefore\) Delete \(v_i\) from \(T\).

- Horizontal segment \(h_j\):
  - Sweep line intersects all vertical segments whose \(y\)-range includes the \(y\)-coordinate of \(h_j\).
  - These are the segments in \(T\).
  - \(\therefore\) Find intersections with \(h_j\) by answering a range query on \(T\).
Orthogonal Line-Segment Intersection: Analysis

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Cost per insertion or deletion $= \mathcal{O}(\lg n)$.
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- Cost per range query with segment $h_j$ is $\mathcal{O}(\log n + t_j)$, where $t_j$ is the number of segments intersecting $h_j$. 
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Total cost:

$$\mathcal{O}((n + m) \lg n) + \sum_{j=1}^{m} \mathcal{O}(t_j) = \mathcal{O}((n + m) \lg n + t)$$
Orthogonal Line-Segment Intersection: Analysis

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Total cost:

$$\mathcal{O}((n + m) \lg n) + \sum_{j=1}^{m} \mathcal{O}(t_j) = \mathcal{O}((n + m) \lg n + t)$$

Theorem: The orthogonal line-segment intersection problem can be solved in $\mathcal{O}((n + m) \lg n + t)$ time.
Line-Segment Intersection
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Questions:
■ What is the sweep-line status?
■ How do we order the segments?
■ Where does the sweep-line status change?
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- How do we order the segments?
  By the $x$-coordinates of their intersections with the sweep line.
- Where does the sweep-line status change?
Line-Segment Intersection

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  All segments intersecting the sweep line.

- How do we order the segments?
  By the $x$-coordinates of their intersections with the sweep line.

- Where does the sweep-line status change?
  At segment endpoints and intersection points!
**Apparent problem:** We want to compute intersections points, but they are part of the event schedule.
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**Consequence:** We cannot generate all event points before we start the sweep.
The Event Schedule

Apparent problem: We want to compute intersections points, but they are part of the event schedule.

Consequence: We cannot generate all event points before we start the sweep.

Solution:

- Maintain set of event points in a priority queue $Q$, sorted by $y$-coordinates.
- Initially, $Q$ contains all segment endpoints.
- As we detect intersections, we insert them into $Q$. 
Observation: If two segments $s_1$ and $s_2$ intersect, the sweep-line must intersect $s_1$ and $s_2$ simultaneously at some point.
**Detecting Intersection Points: First Attempt**

**Observation:** If two segments $s_1$ and $s_2$ intersect, the sweep-line must intersect $s_1$ and $s_2$ simultaneously at some point.

**Idea:**

- As in the orthogonal case, insert and delete segments into and from $T$ when the sweep line passes their endpoints.
- When inserting a segment into $T$, test for intersections with all other segments in $T$. 
**Problem:** We may perform a quadratic number of intersection tests only to discover that there are no intersections.
Observation: Two segments $s_1$ and $s_2$ that intersect are adjacent in $T$ immediately before they intersect.
**Bottom endpoint:**

- Insert $s$ into $T$ and test for intersections with its two neighbours, $s_1$ and $s_2$.
- If there are intersections, insert them into event schedule.
- If $s_1$ and $s_2$ intersect after the current $y$-coordinate, remove the intersection point from event schedule.
**Top endpoint:**

- Delete $s$ from $T$.
- Test for intersection between the two segments that become adjacent in $T$.
- If they intersect, insert intersection point into event schedule.
**Intersection point:**

- Report the intersection.
- Swap the order of the two intersecting segments.
- Remove intersections with their old neighbours from the event schedule.
- Test for intersections with their new neighbours and insert them into the event schedule if necessary.
Line-Segment Intersection: Analysis

\(2n + t\) event points:

- \(n\) bottom endpoints
- \(n\) top endpoints
- \(t\) intersection points

Each incurs \(O(1)\) updates and queries on \(T\) and \(Q\).

\[\therefore\text{ Cost per event point is } O(\lg n).\]

**Theorem:** The line-segment intersection problem can be solved in \(O((n + t) \lg n)\) time.
**Problem:** Maintain a set $S$ of numbers under insertion and deletion and support the following two types of queries:

- **Rank($S, x$):** Report \( \text{rank}_S(x) = 1 + |\{y \in S \mid y < x\}| \).
- **Select($S, k$):** Report the $k$-th order statistic of $S$. 

```
Delete(8)  Insert(18)
rank(29) = 7  select(5) = 12  rank(29) = 7  select(5) = 18
```
Orthogonal Line-Segment Intersection Counting

*Problem:* Instead of reporting all intersections between a set of vertical and a set of horizontal segments, only count how many there are.
Orthogonal Line-Segment Intersection Counting

**Problem:** Instead of reporting all intersections between a set of vertical and a set of horizontal segments, only count how many there are.

Segment $h$ intersects $\text{rank}(b) - \text{rank}(a) = 3$ vertical segments.
Orthogonal Line-Segment Intersection Counting

**Problem:** Instead of reporting all intersections between a set of vertical and a set of horizontal segments, only count how many there are.

Instead of asking a range query for every horizontal segment, ask two rank queries.
**Orthogonal Line-Segment Intersection Counting**

**Problem:** Instead of reporting all intersections between a set of vertical and a set of horizontal segments, only count how many there are.

Instead of asking a range query for every horizontal segment, ask two rank queries.

**Lemma:** If Insert, Delete, and Rank operations can be supported in $O(\lg n)$ time, the orthogonal line-segment intersection counting problem can be solved in $O(n \lg n)$ time.
Observation: The rank of an element is one more than the number of leaves to the left of the path to the corresponding leaf.
An Augmented \((a, b)\)-Tree

- In addition to the normal information, store at every node \(v\) the number of leaves in \(v\)'s subtree.
Lemma: Rank queries can be answered in $O(\lg n)$ time using the leaf counts.

rank(77) = 17 = 1 + (5 + 5 + 3 + 2 + 1)
Lemma: Select queries can be answered in $O(\lg n)$ time using the leaf counts.

\[
\text{rank}(77) = 17 = 1 + (5 + 5 + 3 + 2 + 1)
\]
After an insertion of a new leaf $v$, which leaf counts need to be updated?
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Those of $v$’s ancestors must be increased by one.
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Deletions

After a deletion of a leaf $v$, which leaf counts need to be updated?
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Those of \( v \)’s ancestors must be decreased by one.
After a deletion of a leaf $v$, which leaf counts need to be updated?

Those of $v$’s ancestors must be decreased by one.
Node Splits
Node Splits

- The leaf counts of nodes $v_1$ and $v_2$ are the sums of the leaf counts of their children.
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Lemma: A node split takes $O(1)$ time, including the time to recompute leaf counts.
Node Splits

- The leaf counts of nodes \( v_1 \) and \( v_2 \) are the sums of the leaf counts of their children.
- All other leaf counts remain unchanged.

**Lemma:** A node split takes \( \mathcal{O}(1) \) time, including the time to recompute leaf counts.

**Corollary:** An insertion into a dynamic order statistics tree takes \( \mathcal{O}(\log n) \) time.
Node Fusions

- The leaf counts of the fused node $v$ is the sum of the leaf counts of its children.
- All other leaf counts remain unchanged.

**Lemma:** A node fusion takes $\mathcal{O}(1)$ time, including the time to recompute leaf counts.

**Corollary:** A deletion from a dynamic order statistics tree takes $\mathcal{O}(\lg n)$ time.
Dynamic Order Statistics: Summary

**Theorem:** There exists a data structure that can maintain a set, $S$, of numbers under insertions and deletions and supports Rank and Select queries. Each update or query takes $O(\log n)$ time.
**Problem:** Maintain a set, $S$, of points in the plane under insertions and deletions and support three-sided range queries, that is, queries of the type:

*Report all points whose $x$-coordinates are between $x_l$ and $x_r$, and whose $y$-coordinates are at least $y_b$.***
3-Sided Range Searching and \((a, b)\)-Trees

**First attempt:**

- Store points in an \((a, b)\)-tree based on their \(x\)-coordinates.
- To answer a 3-sided range query, answer a standard range query, but output only those points whose \(y\)-coordinates are above the bottom boundary of the query.
3-Sided Range Searching and \((a, b)\)-Trees

**First attempt:**

- Store points in an \((a, b)\)-tree based on their \(x\)-coordinates.
- To answer a 3-sided range query, answer a standard range query, but output only those points whose \(y\)-coordinates are above the bottom boundary of the query.

**Problem:**

We inspect too many nodes without reporting points.
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- Store points in an \((a, b)\)-tree based on their \(x\)-coordinates.
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Problem:

We inspect too many nodes without reporting points.
Priority search tree:

- Start by building an \((a, b)\)-tree on the \(x\)-coordinates.
Priority search tree:

- Start by building an \((a, b)\)-tree on the \(x\)-coordinates.
- Then propagate points up the tree to establish max-heap property on \(y\)-coordinates.
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**Priority Search Trees**

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  up the tree to establish 
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  \(y\)-coordinates.
Three-Sided Range Queries

- As in a standard range query, traverse the two paths to leftmost and rightmost leaves in the $x$-range of the query.
- For every node on these two paths, inspect the point stored at the node and report it if it is in the query range.
For every node node \( v \) between the extreme paths:

- Inspect the point stored at \( v \).
- If there is no point or its \( y \)-coordinate is below the bottom boundary of the query, stop.
- Otherwise, report the point and recursively inspect \( v \)'s children.
We spend constant time per node on the two extreme paths.

There are $O(\lg n)$ such nodes, at most two per level.

$\therefore$ Traversing the two extreme paths takes $O(\lg n)$ time.
Three-Sided Range Queries: Analysis

- We spend constant time per node on the two extreme paths.
- There are $O(\lg n)$ such nodes, at most two per level.
  \[\therefore\] Traversing the two extreme paths takes $O(\lg n)$ time.

- For every node we visit between the two extreme paths, we spend constant time.
- **We visit at most** $b(\lg n + t) = O(\lg n + t)$ such nodes.
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Three-Sided Range Queries: Analysis

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  \[\therefore\text{ Visiting nodes between the two extreme paths takes } O(lg\ n + t) \text{ time.}\]

\textbf{Lemma:} A priority search tree supports three-sided range queries in \(O(lg\ n + t)\) time.
Insertions

- Insert $p$ as into a standard $(a, b)$-tree, based on $x$-coordinates.
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Perform a Heapify-Up operation starting at the new leaf.
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Insert \( p \) as into a standard \((a, b)\)-tree, based on \( x\)-coordinates.

Locate the lowest ancestor \( v \) whose parent does not store a point \( q \) with \( y_q < y_p \).
Insertions

- Insert $p$ as into a standard $(a, b)$-tree, based on $x$-coordinates.
- Locate the lowest ancestor $v$ whose parent does not store a point $q$ with $y_q < y_p$.
- While $p \neq \text{nil}$:
  - Replace point $q$ at current node $v$ with $p$
  - $p \leftarrow q$
  - Proceed to $v$'s child that is an ancestor of $p$'s leaf.
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Deletions
Delete the leaf corresponding to $p$ as from a standard $(a, b)$-tree.
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- Delete the leaf corresponding to $p$ as from a standard $(a, b)$-tree.
- Delete $p$ from the node where it is stored.
- While the current node $v$ has a child that stores a point:
  - Choose the child $w$ whose point $p$ has maximum $y$-coordinate
  - Store $p$ at $v$
  - $v \leftarrow w$
Deletions

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- Choose the child \( w \) whose point \( p \) has maximum \( y \)-coordinate
- Store \( p \) at \( v \)
- \( v \leftarrow w \)
Node Splits
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Where do we store $p$?

What do we store at the two nodes resulting from the split?
Node Splits

- Where do we store \( p \)?
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- The one that is an ancestor of \( p \)'s leaf stores \( p \).
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Node Splits

- Where do we store $p$?
- What do we store at the two nodes resulting from the split?
  - The one that is an ancestor of $p$'s leaf stores $p$.
  - For the other one, we apply the same bubble-up operation as after a deletion.
Node Fusions
Which of the two points do we store at the merged node?
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- Which of the two points do we store at the merged node?
- The one with greater $y$-coordinate.
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![Diagram showing node fusions with points p and q]
Node Fusions

Which of the two points do we store at the merged node?

- The one with greater $y$-coordinate.
- The other one is pushed down as after an insertion.
Node Fusions

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- The one with greater $y$-coordinate.
- The other one is pushed down as after an insertion.
Priority Search Tree Updates: Analysis

**Insertion:**
- Insertion as into an \((a, b)\)-tree \(\Rightarrow \mathcal{O}(\lg n)\)
- Locate ancestor where to store \(p\) \(\Rightarrow \mathcal{O}(\lg n)\)
- Trickle-down operation \(\Rightarrow \mathcal{O}(\lg n)\)
- \(\mathcal{O}(\lg n)\) node splits

**Deletion:**
- Deletion as from an \((a, b)\)-tree \(\Rightarrow \mathcal{O}(\lg n)\)
- Locate ancestor where \(p\) is stored and delete \(p\) \(\Rightarrow \mathcal{O}(\lg n)\)
- Bubble-up operation \(\Rightarrow \mathcal{O}(\lg n)\)
- \(\mathcal{O}(\lg n)\) node fusions and at most one node split
Node split:

- Regular split $\Rightarrow \mathcal{O}(1)$
- Bubble-up operation $\Rightarrow \mathcal{O}(\lg n)$

Node fusion:

- Regular fusion $\Rightarrow \mathcal{O}(1)$
- Trickle-down operation $\Rightarrow \mathcal{O}(\lg n)$

Lemma: A priority search tree supports insertions and deletions in $\mathcal{O}(\lg^2 n)$ time.
Weight-Balanced \((a, b)\)-Trees

**Old balancing condition**: (Degree balancing)

- The root has between 2 and \(b\) children
- Any other internal node has between \(a\) and \(b\) children

This can still be quite unbalanced, but served our purpose so far.
Weight-Balanced \((a, b)\)-Trees

**New balancing condition:** (Weight balancing)

- The root has at least two children.
- Any non-leaf node at height \(h\) has weight at most \(\beta \gamma^h\).
- Any non-leaf, non-root node at height \(h\) has weight at least \(\alpha \gamma^h\).

\[
\alpha = \sqrt{a}, \quad \beta = \sqrt{b}, \quad \text{and} \quad \gamma = \alpha \beta.
\]
Lemma: In a weight-balanced \((a, b)\)-tree, the root has degree between 2 and \(b\). Any other internal node has degree between \(a\) and \(b\).
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- For any other node \(v\):
  - \(\text{weight}(v) \geq \alpha \gamma^h\)
  - \(\text{weight}(w) \leq \beta \gamma^{h-1}\), for every child, \(w\), of \(v\)
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\[
\therefore \deg(v) \geq \frac{\alpha \gamma^h}{\beta \gamma^{h-1}} = \frac{\alpha \gamma}{\beta} = \frac{\alpha^2 \beta}{\beta} = a.
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- For the root, obvious.
- For any other node \(v\):
  - \(\text{weight}(v) \geq \alpha \gamma^h\)
  - \(\text{weight}(w) \leq \beta \gamma^{h-1}\)
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\[\therefore \text{deg}(v) \geq \frac{\alpha \gamma^h}{\beta \gamma^{h-1}} = \frac{\alpha \gamma}{\beta} = a.\]

**Upper bound:**
- \(\text{weight}(v) \leq \beta \gamma^h\)
- \(\text{weight}(w) \geq \alpha \gamma^{h-1}\)
  for every child, \(w\), of \(v\)

\[\therefore \text{deg}(v) \leq \frac{\beta \gamma^h}{\alpha \gamma^{h-1}} = \frac{\beta \gamma}{\alpha} = \frac{\beta^2 \alpha}{a}.\]
Lemma: The nodes produced by a split or node fusion at height $h$ in a weight-balanced $(a, b)$-tree have weight at most $\frac{3}{4} \beta \gamma^h$. If $h$ is not the root level, these nodes also have weight at least $\frac{3}{2} \alpha \gamma^h$. 
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Proof: (Only splits)
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Splitting to distribute the leaves 50-50 may require splitting a child of $v$. 

50-50 split
**Lemma:** The nodes produced by a split or node fusion at height $h$ in a weight-balanced $(a, b)$-tree have weight at most $\frac{3}{4} \beta \gamma^h$. If $h$ is not the root level, these nodes also have weight at least $\frac{3}{2} \alpha \gamma^h$.

**Proof:** (Only splits)

- Splitting to distribute the leaves 50-50 may require splitting a child of $v$.
- Make this child a child of the left node.
For $u \in \{v', v''\}$, $\frac{\beta \gamma^h + 1}{2} - \beta \gamma^{h-1} \leq \text{weight}(u) \leq \frac{\beta \gamma^h + 1}{2} + \beta \gamma^{h-1}$. 
For $u \in \{v', v''\}$, 
\[
\frac{\beta \gamma^h + 1}{2} - \beta \gamma^{h-1} \leq \text{weight}(u) \leq \frac{\beta \gamma^h + 1}{2} + \beta \gamma^{h-1}.
\]

**Lower bound:**

\[
\text{weight}(u) \geq \frac{\beta \gamma^h + 1}{2} - \beta \gamma^{h-1}
\]
\[
\geq \beta \gamma^h \left( \frac{1}{2} - \frac{1}{\gamma} \right)
\]
\[
\geq 4\alpha \gamma^h \left( \frac{1}{2} - \frac{1}{8} \right)
\]
\[
= \frac{3}{2} \alpha \gamma^h.
\]
For $u \in \{v', v''\}$, $\frac{\beta \gamma^h + 1}{2} - \beta \gamma^{h-1} \leq \text{weight}(u) \leq \frac{\beta \gamma^h + 1}{2} + \beta \gamma^{h-1}$.

**Lower bound:**

\[
\text{weight}(u) \geq \frac{\beta \gamma^h + 1}{2} - \beta \gamma^{h-1} \\
\geq \beta \gamma^h \left( \frac{1}{2} - \frac{1}{\gamma} \right) \\
\geq 4 \alpha \gamma^h \left( \frac{1}{2} - \frac{1}{8} \right) \\
= \frac{3}{2} \alpha \gamma^h.
\]

**Upper bound:**

\[
\text{weight}(u) \leq \frac{\beta \gamma^h + 1}{2} + \beta \gamma^{h-1} \\
\leq \beta \gamma^h \left( \frac{1}{2} + \frac{1}{\gamma} \right) + \frac{1}{2} \\
\leq \beta \gamma^h \left( \frac{1}{2} + \frac{1}{8} \right) + \frac{1}{2} \\
= \frac{5}{8} \beta \gamma^h + \frac{1}{2} \\
\leq \frac{3}{4} \beta \gamma^h.
\]
Amortization

Given:

- A type of data structure $\mathcal{T}$ that supports operations $o_1, o_2, \ldots, o_k$ (e.g., Insert, Delete, Find, ... )
Amortization

Given:

- A type of data structure \( T \) that supports operations \( o_1, o_2, \ldots, o_k \) (e.g., Insert, Delete, Find, \ldots)

We say that operation \( o_i \) has **amortized cost** \( T_i(n) \), for \( 1 \leq i \leq n \), if any sequence of \( n \) operations on an initially empty data structure of type \( T \) takes at most

\[
\sum_{i=1}^{k} n_i T_i(n)
\]

time, where \( n_i \) is the number of operations of type \( o_i \) in the sequence.
Given:

- A type of data structure $\mathcal{T}$ that supports operations $o_1, o_2, \ldots, o_k$ (e.g., Insert, Delete, Find, ...)

We say that operation $o_i$ has **amortized cost** $T_i(n)$, for $1 \leq i \leq n$, if any sequence of $n$ operations on an initially empty data structure of type $\mathcal{T}$ takes at most

$$\sum_{i=1}^{k} n_i T_i(n)$$

time, where $n_i$ is the number of operations of type $o_i$ in the sequence.

What we get is a **worst-case bound**, but not for individual operations, but for the cost of any **sequence** of operations on the data structure.
Credit Method

Idea:

- Associate a credit with the nodes of the data structure.
- An operation can pay credit to certain nodes, in addition to paying for its own cost.
- An operation can cover all or part of its actual cost by taking from the credit of some nodes.
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- Define the amortized cost of an operation as its actual cost plus the total change in credit.
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Note: The algorithm does not actually maintain credits. They are an analysis technique.
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- Make every insertion or deletion pay one credit to every ancestor of the affected leaf.

∴ the amortized cost per insertion or deletion, excluding rebalancing costs, is $O(\lg n)$. 
Lemma: A weight-balanced priority-search tree supports insertions and deletions in $O(\lg n)$ amortized time.

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- The amortized cost per insertion or deletion, excluding rebalancing costs, is $O(\lg n)$.

- A node split or node fusion triggered by a node $v$ at height $h$ costs $O(h) = O(\beta \gamma^h) = O(\text{weight}(v))$ time.
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- After the creation of $v$, $\Omega(\text{weight}(v))$ insertions or deletions below $v$ are required to make $v$ split or fuse.
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- After the creation of $v$, $\Omega(\text{weight}(v))$ insertions or deletions below $v$ are required to make $v$ split or fuse.

- Each insertion or deletion below $v$ has paid one credit to $v$.
  \[ \therefore v \text{ can pay for the split or fusion with its credits.} \]
  \[ \therefore \text{the amortized cost per node split or fusion is } 0! \]
  (We have already paid for it by charging earlier insertions and deletions.)
Theorem: Priority search trees can be used to solve the dynamic three-sided range search problem. Queries take $O(lg n + t)$ time in the worst case. Updates take $O(lg^2 n)$ time in the worst case and $O(lg n)$ amortized time.

Note: By using more advanced techniques or by building priority search trees on top of red-black trees, the update bound can be made $O(lg n)$ in the worst case.
**Higher-Dimensional Range Searching**

**Goal:**

- Build a *static* data structure over a point set $S$ in $\mathbb{R}^d$ that allows to report all the points in $S$ that fall in a given ($d$-dimensional) query rectangle.
- Queries should be fast.
- Data structure should be small.
- Data structure should be fast to build.
One-Dimensional Range Searching
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**Straightforward solution:**

- Data structure = search tree on $x$-coordinates
- Search down paths to leftmost and rightmost nodes in $x$-range
- Report all the points stored at the leaves between these two paths.

**Query time:**
One-Dimensional Range Searching

*Straightforward solution:*

- Data structure = search tree on \( x \)-coordinates
- Search down paths to leftmost and rightmost nodes in \( x \)-range
- Report all the points stored at the leaves between these two paths.

*Query time:* \( \mathcal{O}(\lg n + k) \)

*Space:*
One-Dimensional Range Searching

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- Search down paths to leftmost and rightmost nodes in $x$-range
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**Query time:** $O(\lg n + k)$

**Space:** $O(n)$

**Build time:**
One-Dimensional Range Searching

**Straightforward solution:**
- Data structure = search tree on $x$-coordinates
- Search down paths to leftmost and rightmost nodes in $x$-range
- Report all the points stored at the leaves between these two paths.

**Query time:** $O(\lg n + k)$

**Space:** $O(n)$

**Build time:** $O(n \lg n)$
- Sort points by $x$-coordinates
- Build the tree bottom-up in *linear time.*
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Observations:
Two-Dimensional Range Searching (1)

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- Points between paths are guaranteed to be in $x$-range.
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**Observations:**

- Easy to check whether leftmost and rightmost leaves are in query range.
- Again, expensive to inspect subtrees between paths.
- Points between paths are guaranteed to be in $x$-range.
- $y$-coordinates are distinguishing factor.
Two-Dimensional Range Searching (2)

Solution:
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- Every node stores a search tree over the points corresponding to its leaf descendants, sorted by $y$-coordinate.
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Query time: $O(\lg^2 n + k)$

Space:
Solution:

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Query time: $O(\lg^2 n + k)$

Space: $O(n \lg n)$

Build time:
Two-Dimensional Range Searching (2)

**Solution:**
- Every node stores a search tree over the points corresponding to its leaf descendants, sorted by $y$-coordinate.

**Query time:** $O(\lg^2 n + k)$

**Space:** $O(n \lg n)$

**Build time:** $O(n \lg n)$

- Sort points by $x$-coordinates
- Build the tree bottom-up in $O(n \lg n)$ time.
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**Solution:**

- Every node stores a $(d - 1)$-dimensional search tree over the points corresponding to its leaf descendants.
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What about $d$-dimensional range searching?

**Solution:**

- Every node stores a $(d - 1)$-dimensional search tree over the points corresponding to its leaf descendants.

**Query time:** $O(\lg^d n + k)$

**Space:** $O(n \lg^{d-1} n)$

**Build time:**
What about $d$-dimensional range searching?

**Solution:**

- Every node stores a $(d-1)$-dimensional search tree over the points corresponding to its leaf descendants.

**Query time:** $O(\lg^d n + k)$

**Space:** $O(n \lg^{d-1} n)$

**Build time:** $O(n \lg^{d-1} n)$
Theorem: \( d \)-dimensional range trees can be used to solve the \( d \)-dimensional range searching problem in \( \mathcal{O}(\lg^d n + t) \) time per query. For \( d \geq 2 \), a \( d \)-dimensional range tree uses \( \mathcal{O}(n \lg^{d-1} n) \) space and can be built in \( \mathcal{O}(n \lg^{d-1} n) \) time.

Notes:

- Using a weight-balanced \((a, b)\)-tree as the primary tree, updates can be supported in \( \mathcal{O}(\lg^d n) \) amortized time.
- Using an advanced technique known as fractional cascading, the query time can be reduced to \( \mathcal{O}(\lg^{d-1} n) \). Fractional cascading as such is static; but a dynamic version with slightly worse performance has also been developed.
Data Structuring: Summary

Data structuring is a very useful paradigm that can be seen as the algorithmic equivalent of two important strategies in software engineering: modularization and code reuse.

- **Modularization**: By using the data structuring paradigm, we can often split the problem into manageable units. The nasty details are hidden inside each data structure.

- **Code reuse**: Once we have a powerful data structure, it can often be used to solve more than one problem.

*To build a new data structure often does not mean to start from scratch, but to augment an existing one:*  

- Store additional information at each node (e.g., order statistics).
- Change the rules where data items are stored (e.g., priority search trees).
- Replicate data and define recursive structures (e.g., range trees).