Sample Solutions

Assignment 6

CSCI 3110 — Summer 2014

Question 1

(a) **Claim:** \( T(n) \in O(n^{\log_3 3}) \), that is, \( T(n) \leq cn^{\log_3 3} \), for some \( c > 0 \) and \( n_0 \geq 0 \) and all \( n \geq n_0 \). In fact, we claim that \( T(n) \leq c n^{\log_3 3} - dn \), for some constants \( c > 0 \) and \( d > 0 \). This is necessary to make the inductive proof work.

**Proof:** For \( n = 1 \), we have \( T(n) \leq c - d \leq c n^{\log_3 3} - dn \), for some constant \( c > 0 \) and \( d = c/2 \) because \( T(n) \in \Theta(1) \).

For \( n > 1 \), we have \( n/2 \geq 1 \), so we can apply the inductive hypothesis to \( T(n/2) \). This gives

\[
T(n) = 3T\left(\frac{n}{2}\right) + n \\
\leq 3 \left[c \left(\frac{n}{2}\right)^{\log_3 3} - \frac{dn}{2}\right] + n \\
= cn^{\log_3 3} - \frac{3dn}{2} + n \\
= cn^{\log_3 3} - \left(\frac{3d}{2} - 1\right)n \\
\leq cn^{\log_3 3} - dn \text{, for all } 3d/2 - 1 \geq d, \text{ that is, } d \geq 2.
\]

Thus, the claim holds for \( n_0 = 1 \), \( c \geq 4 \), \( d = c/2 \), and \( c \) large enough to ensure that \( T(1) \leq c - d \).

**Claim:** \( T(n) \in \Omega(n^{\log_3 3}) \), that is, \( T(n) \geq cn^{\log_3 3} \), for some \( c > 0 \) and \( n_0 \geq 0 \) and all \( n \geq n_0 \).

**Proof:** For \( n = 1 \), we have \( T(n) \geq c = cn^{\log_3 3} \), for some constant \( c > 0 \) because \( T(n) \in \Theta(1) \) and \( n^{\log_3 3} \in \Theta(1) \).
For $n > 1$, we have $n/2 \geq 1$, so we can apply the inductive hypothesis to $T(n/2)$. This gives

\[
T(n) = 3T \left( \frac{n}{2} \right) + n \\
\geq 3c \left( \frac{n}{2} \right)^{\log_2 3} + n \\
= cn^{\log_2 3} + n \\
> cn^{\log_2 3}
\]

Thus, the claim holds for $c > 0$ small enough that $T(1) \geq c$ and for $n_0 = 1$.

(b) **Claim:** $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$, $n_0 \geq 0$, and all $n \geq n_0$.

**Proof:** For $1 \leq n < 5$, we have $T(n) \in \Theta(1)$, that is, $T(n) \leq c \leq cn$, for $c$ sufficiently large.

For $n \geq 5$, we have $n/4 > n/5 \geq 1$, that is, we can apply the inductive hypothesis to $T(n/4)$ and $T(n/5)$. This gives

\[
T(n) = 3T \left( \frac{n}{4} \right) + 3T \left( \frac{n}{5} \right) + n \\
\leq \frac{3cn}{4} + \frac{5}{5}n + n \\
= \left( \frac{19c}{20} + 1 \right) n \\
\leq cn, \text{ for all } c \geq 20.
\]

Thus, the claim holds for $c \geq 20$ and sufficiently large to ensure that $T(1) \leq c$ and for $n_0 = 1$.

**Claim:** $T(n) \in \Omega(n)$, that is, $T(n) \geq cn$, for some $c > 0$, $n_0 \geq 0$, and all $n \geq n_0$.

**Proof:** This is trivial for $c = 1$ and all $n$ because $T(n) \geq n$ by definition.

(c) **Claim:** $T(n) \in O(n \log n)$, that is, $T(n) \leq cn \log n$, for some $c > 0$, $n_0 \geq 0$, and all $n \geq n_0$.

For the inductive proof to work, we do in fact prove the stronger claim that $T(n) \leq cn \log n - dn$, for some $c > 0$ and $d > 0$.

**Proof:** For $2 \leq n < 4$, we have $T(n) \in \Theta(1)$, that is, $T(n) \leq c - d \leq cn \log n - d n$, for $c$ large enough and $d = c/2$.

For $n \geq 4$, we have $\sqrt{n} \geq 2$, that is, we can apply the inductive hypothesis to $T(\sqrt{n})$. 


This gives

\[
T(n) = 2\sqrt{n}T(\sqrt{n}) + n \\
\leq 2\sqrt{n} \left[ c\sqrt{n} \lg \sqrt{n} - d\sqrt{n} \right] + n \\
= 2cn \lg \sqrt{n} - 2dn + n \\
= cn \lg n - (2d - 1)n \quad \text{because } \lg \sqrt{n} = \frac{1}{2} \lg n \\
\leq cn \lg n - dn, \text{ for all } d \geq 1.
\]

Thus, the claim holds for \( n_0 = 2, c \geq 2, d = c/2, \) and \( c \) large enough that \( T(n) \leq c - d \) for all \( 2 \leq n < 4. \)

**Claim:** \( T(n) \in \Omega(n \lg n), \) that is, \( T(n) \geq cn \lg n, \) for some \( c > 0, \) \( n_0 \geq 0, \) and all \( n \geq n_0. \)

**Proof:** For \( 2 \leq n < 4, \) we have \( T(n) \in \Theta(1), \) so \( T(n) \geq 8c > cn \lg n, \) for some \( c > 0. \)

For \( n \geq 4, \) we have \( \sqrt{n} \geq 2, \) so we can apply the inductive hypothesis to \( T(\sqrt{n}). \) This gives

\[
T(n) = 2\sqrt{n}T(\sqrt{n}) + n \\
\geq 2\sqrt{n}c\sqrt{n} \lg \sqrt{n} + n \\
= cn \lg n + n \quad \text{because } \lg \sqrt{n} = \frac{1}{2} \lg n \\
> cn \lg n.
\]

Thus, the claim holds for \( n_0 = 2 \) and \( c \) small enough that \( T(n) \geq 8c \) for \( 2 \leq n < 4. \)

**Question 2**

(a) Here we have \( n \lg n \in o(n^{1.1}) \subset O(n^{1.1}) = O(n^{\log_3 4 - \epsilon}), \) where \( \epsilon = \log_3 4 - 1.1 > 0. \) This holds because \( \lg n \in o(n^\delta) \) for all \( \delta > 0. \) Thus, the second case of the Master Theorem applies and \( T(n) \in \Theta(n^{\log_3 4}). \)

(b) Here we have \( n^2 / \lg n = n^{\log_3 4 / \lg n}. \) Thus, the third case of the Master Theorem applies and \( T(n) \in \Theta((n^2 / \lg n) \cdot \lg n) = \Theta(n^2). \)

(c) Here we have \( n^2 = n^{\log_3 9}. \) Thus, once again, the third case of the Master Theorem applies and \( T(n) \in \Theta(n^2 \lg n). \)
(d) Here we have \( n = n^{\log_4 3 + \varepsilon} \), where \( \varepsilon = 1 - \log_4 3 > 0 \). Since we also have \( 3n/4 \leq cn \) for \( c = 3/4 \), the first case of the Master Theorem applies and \( T(n) \in \Theta(n) \).

(e) Here we have \( n \lg n = n^{\log_2 2 \lg n} \). Thus, the third case of the Master Theorem applies and \( T(n) \in \Theta(n \lg n \cdot \lg n) = \Theta(n \lg^2 n) \).