Assignments are due on the due date before class and have to include this cover page. Plagiarism in assignment answers will not be tolerated. By submitting their answers to this assignment, the authors named above declare that its content is their original work and that they did not use any sources for its preparation other than the class notes, the textbook, and ones explicitly acknowledged in the answers. Any suspected act of plagiarism will be reported to the Faculty's Academic Integrity Officer and possibly to the Senate Discipline Committee. The penalty for academic dishonesty may range from failing the course to expulsion from the university, in accordance with Dalhousie University's regulations regarding academic integrity.
In this assignment, we investigate the robustness of the stable marriage algorithm we discussed in class. It turns out that, assuming the definition of a valid solution is sensible enough, it can handle quite a bit more general inputs than we discussed in class.

Let us start by generalizing the notion of preference lists so that it allows for ties in the ranking and for men and women to indicate that they’d rather not marry a particular person. Once again, we have a set $M = \{m_1, m_2, \ldots, m_n\}$ of $n$ men and a set $W = \{w_1, w_2, \ldots, w_n\}$ of $n$ women. Instead of a single preference list that ranks all women in decreasing order of preference, the preferences of each man $m_i$ are now represented as a sequence of sets $(W_{i,0}, W_{i,1}, \ldots, W_{i,s_i})$ that form a partition of $W$, that is, $W_{i,j} \cap W_{i,k} = \emptyset$ for all $0 \leq j < k \leq s_i$ and $\bigcup_{j=0}^{s_i} W_{i,j} = W$. The meaning of this partition is as follows: $W_{i,0}$ is the set of women $m_i$ absolutely does not want to be married to; he’d rather stay single. For two women $w$ and $w'$ in the same set $W_{i,j}$ with $j > 0$, $m_i$ does not have any clear preference; he likes both of them equally well. Finally, for two women $w \in W_{i,j}$ and $w' \in W_{i,k}$ with $j > k$, $m_i$ would rather be married to $w$ than to $w'$. We write $w \succ_m w'$ if $w \in W_{i,j}$ and $w' \in W_{i,k}$ for some $j > k$, that is, $m_i$ likes $w$ strictly better than $w'$. If $j \geq \max(k, 1)$, that is, $m_i$ likes $w$ at least as well as $w'$, we write $w \succeq_m w'$. The preferences of each woman $w_i$ are represented by a sequence of sets $(M_{i,0}, M_{i,1}, \ldots, M_{i,t_i})$ that form a partition of $M$ and whose meaning is analogous to the sets in the preference lists of the men. To avoid excessively verbose discussions of different cases of married and unmarried people, we consider any unmarried person to be married to a dummy person Nobody and define $w \succ_m$ Nobody and $m \succ_w$ Nobody for every woman $w \notin W_{i,0}$ and every man $m \notin M_{i,0}$.

In words: If $m_i$ ($w_i$) is currently not married and $w$ ($m$) is not on his (her) “do-not-marry list” $W_{i,0}$ ($M_{i,0}$), then he (she) would much rather marry $w$ ($m$) than stay single. Now we define an instability in a set of marriages to be either (i) a pair of marriages $(m, w)$ and $(m', w')$ such that $m$ and $w$ would much rather be married to each other than to their current partners or (ii) a single marriage $(m_i, w_j)$ such that either $m_i \in M_{j,0}$ or $w_j \in W_{i,0}$. This notion of instability captures all scenarios where there is a pair $(m, w)$ who seeks to change the current set of marriages in the interest of increasing their own happiness. Indeed, (i) is the notion of instability we already considered in class and which reflects the fact that $m$ and $w'$ would leave their current partners in order to seek happiness with each other. Given the introduction of Nobody and the representation of being unmarried as a quite undesirable marriage to Nobody, it also captures the situation where, for example, a currently unmarried woman $w_j$ gets a proposal from a man $m_i$ not on her do-not-marry list $M_{j,0}$ because she prefers to be married to $m_i$ to not being married at all ($m_i \succ_w$ Nobody) and thus would marry $m_i$. (ii) captures that a marriage $(m_i, w_j)$ where at least one of the partners, say $w_j$, finds $m_i$ undesirable enough that she’d rather be single than be married to $m_i$ will inevitably end in divorce, that is, the set of marriages will once again change. With this notion of instability, a set of marriages is stable if there are no instabilities.

We can now define a number of different variants of the stable marriage problem. Given that some people prefer being single to being married to certain other people, it may be sensible to allow a solution to include people that aren’t married, as long as this doesn’t result in an instability. This gives us a strict and a non-strict version of the stable marriage problem. In the strict version, we do require everybody to be married. In the non-strict version, we allow for single people.

Given that we allow ties in people’s preference lists, there are different possibilities for what happens when one party has a clear preference while the other one doesn’t. Consider two marriages $(m, w)$ and $(m', w')$ where we allow $w = \text{Nobody}$ and/or $m' = \text{Nobody}$. If $w' \succ_m w$ and $m \succ_w m'$, then both $m$ and $w'$ have something to gain by getting married to each other, so it is safe to assume that they will. This clearly constitutes an instability. Similarly, if $w' \succ_m w$ and $m' \succ_w m$, then it is unlikely that $w'$ will be very receptive to $m'$’s advances because she prefers her current partner $m'$, so this is not an instability. We can also assume that, if neither $m$ nor $w'$ prefers the other over their current partner, then neither will take the initiative and woo the other, so again there is not instability. But what about the case when say $w' \succ_m w$ but $w'$ has no clear preference between $m$ and $m'$. Will she be open to $m'$’s courtship or will she value her current relationship and see no reason to switch partners, given that there is no clear advantage for her in it? In the eager stable marriage problem, we assume that $w'$ will eventually give in to $m'$’s advances because she wouldn’t mind a change even if there is no clear advantage for her in it. In this version of the problem, a pair of marriages $(m, w)$ and $(m', w')$ constitutes an instability if either $w' \succ_m w$ and $m \succeq_m m'$ or $w' \succeq_m w$ and $m \succ_w m'$. (One of the two inequalities must be strict in order for one partner to take the initiative.) In the inert stable marriage problem, we assume that $w'$ will be reluctant to
switch partner unless she sees a clear advantage. In this version of the problem, a pair of marriages \((m, w)\) and \((m', w')\) constitutes an instability only if \(w' \succeq_m w\) and \(m \succ_w m'\) (both inequalities must be strict). We can now ask which versions of the stable marriage problem have solutions at all and which ones can be solved using efficient algorithms:

(a) **Don’t force people to marry.** (2 marks) Provide an example where some men and/or women have non-empty do-not-marry lists \(M_{i,0}\) or \(W_{i,0}\) and where the strict stable marriage problem defined by these preference lists has no solution. Inertia or eagerness should not matter. Explain why there is no solution for this example. (*Hint:* If you need lots of men and women with complicated preference lists in your example, you’re overcomplicating things.)

(b) **Change for the sake of change leads to chaos.** (2 marks) Provide an example of preference lists where there is no solution to the non-strict eager stable marriage problem. Again, explain why there is no solution for this example. The same hint as for (a) applies.

(a) and (b) show that we can only hope to be able to guarantee the existence of a solution for the non-strict inert stable marriage problem, which seems to reflect the real world quite well. Our goal now is to show that the Gale-Shapley algorithm is powerful enough to solve this problem.

(c) (6 marks) Consider a variant of the Gale-Shapley algorithm where a man gives up and decides to stay single if he has proposed to all women not on his do-not-marry list and he is still single and where a woman accepts a proposal from a man if and only if he is not on her do-not-marry list and she prefers him over her current partner (who might be Nobody if she’s currently unmarried). Prove that this algorithm correctly solves the non-strict inert stable marriage problem. (*Hint:* Take the correctness proof of the Gale-Shapley algorithm discussed in class as a starting point and figure out what you need to change to prove that the algorithm solves the non-strict inert stable marriage problem.)

Finally, it is not entirely obvious whether the solution the algorithm produces “maximizes happiness” by ensuring that as many people as possible among those who are interested in getting married do in fact get married. If we allow ties in the preference lists, then it is not hard to construct an example where the wrong proposal order leads to a non-maximal number of marriages. If there are no ties, however, the algorithm always finds the maximum number of marriages. The next question asks you to prove this. This is a bonus question because it requires some non-obvious arguments.

(d) (5 marks, bonus) Prove that the solution of the non-strict inert stable marriage problem produced by the Gale-Shapley algorithm computes a set of stable marriages with the maximum number of married men and women, provided there are no ties in the preference lists, that is, provided that \(|M_{i,j}| = 1\) and \(|W_{i,j}| = 1\) for all \(1 \leq i \leq n\) and \(j > 0\). (*Hint:* You don’t need to consider the algorithm at all. Instead, you should prove that two different stable sets of marriages for the same input consist of exactly the same number of marriages.)