The Life of Pi

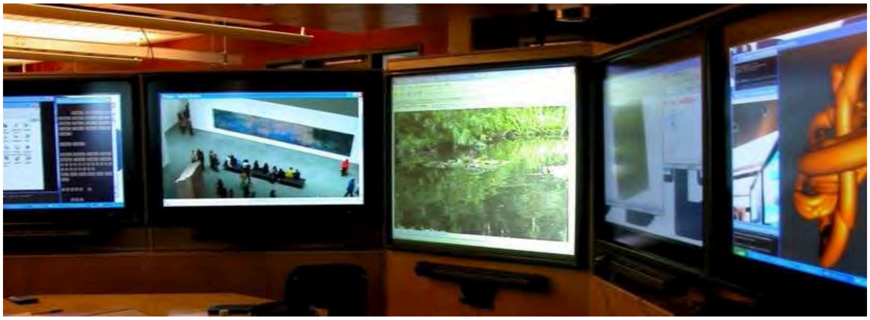
History and Computation

Jonathan M. Borwein, FRSC

Prepared for

2007/08 Colloquia

Canada Research Chair in IT & Director



Dalhousie University, Halifax NS Canada

`www.ddrive.cs.dal.ca`

`www.cecm.sfu.ca/~jborwein/pi_cover.html`

Revised: February 20, 2008

The Life of Pi (2001)

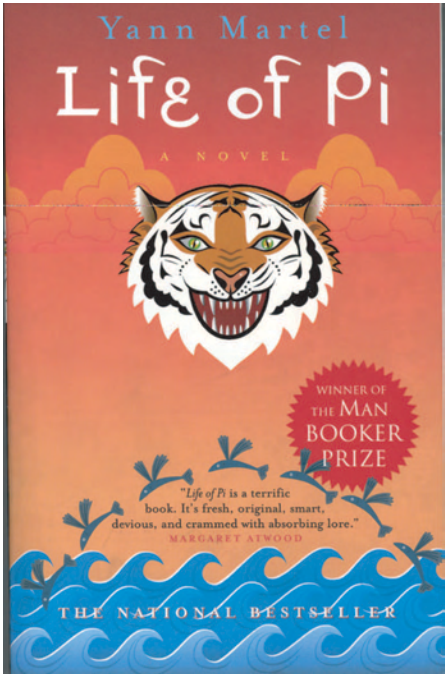
“My name is
Piscine Molitor Patel
known to all as Pi Patel

For good measure I added

$$\pi = 3.14^*$$

and I then drew a large circle which I sliced in two with a diameter, to evoke that basic lesson of geometry.”

*The Notation of π was introduced by Euler in 1737.



Abstract. The desire, and originally need, to calculate ever more accurate values of π, the ratio of the circumference of a circle to its diameter, has challenged mathematicians for centuries and, especially recently, π has provided fascinating examples of computational math. Pi, is also uniquely part of

The popular imagination:*

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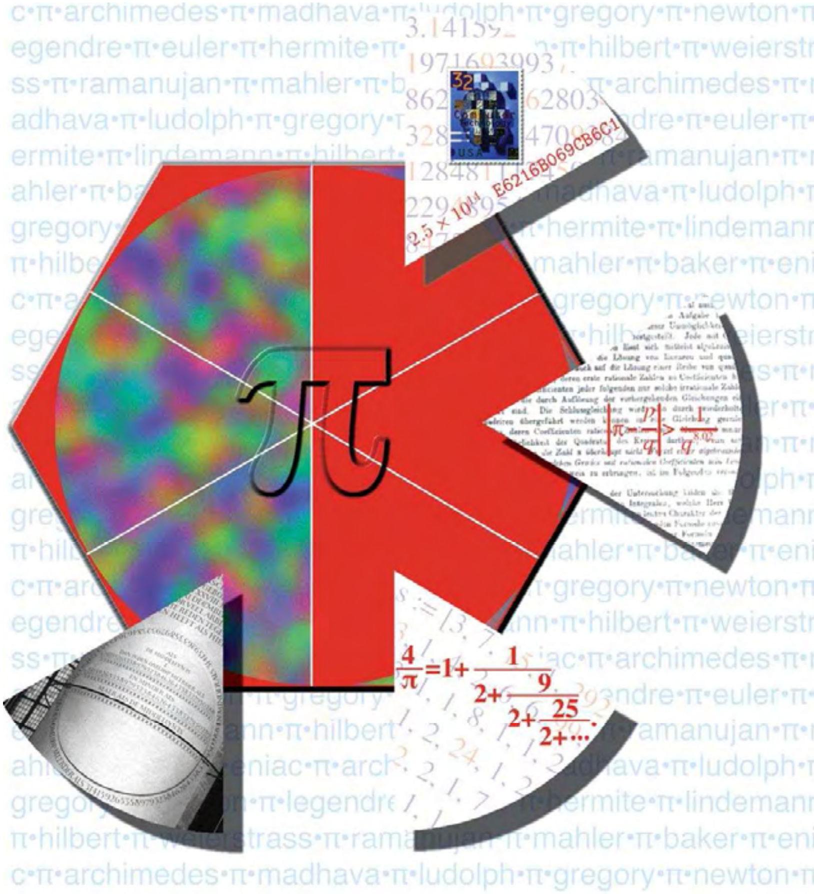
[*] The "MacTutor" website, at the University of St. Andrews — my home town in Scotland — `http://www-gap.dcs.st-and.ac.uk/~history` is rather a good history source.

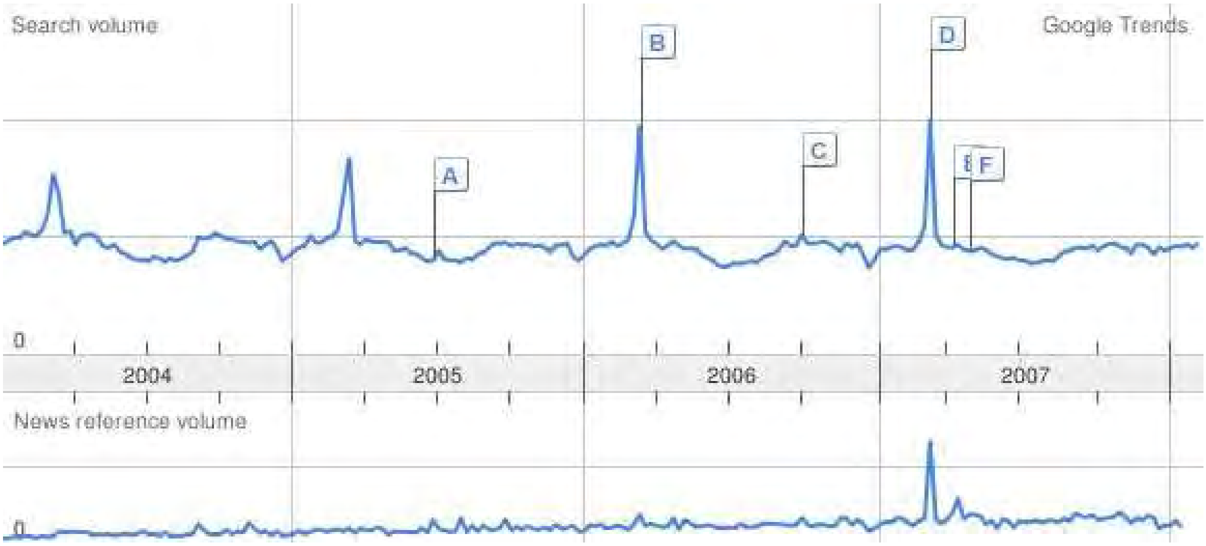
Pi: A Source Book (1997)



- Berggren, Borwein and Borwein, 3rd Ed, pp. 797, Springer, 2004. (80 selections and 3,650 years of copyright releases.)

- See also `www.cecm.sfu.ca/~jborwein/pi_cover.html`

Pi Day Turns Twenty One (also π+AGM)



- From `www.google.com/trends?q=Pi+`

- **B**: "Student recites 8,784 digits of pi," March 15, 2006.

- **1987**: *Pi Day* was a gag in SF. **2003**: it nearly crashed SFU when many schools ran `http://oldweb.cecm.sfu.ca/pi/yapPing.html` which recites Pi *fast* in many languages.

Pi the URL

Pi to 1,000,000 places

Pi to one MILLION decimal places

```
3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679
  8214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196
  4428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273
  7245870066063155881748815209209628292540917153643678925903600113305305488204665213841469519415116094
  3305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912
  9833673362440656643086021394946395224737190702179860943702770539217176293176752384674818467669405132
  0005681271452635608277857713427577896091736371787214684409012249534301465495853710507922796892589235
  4201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859
  5024459455346908302642522308253344685035261931188171010003137838752886587533208381420617177669147303
  5982534904287554687311595628638823537875937519577818577805321712268066130019278766111959092164201989
  3809525720106548586327886593615338182796823030195203530185296899577362259941389124972177528347913151
  5574857242454150695950829533116861727855889075098381754637464939319255060400927701671139009848824012
  8583616035637076601047101819429555961989467678374494482553797747268471040475346462080466842590694912
  9331367702898915210475216205696602405803815019351125338243003558764024749647326391419927260426992279
  6782354781636009341721641219924586315030286182974555706749838505494588586926995690927210797509302955
  3211653449872027559602364806654991198818347977535663698074265425278625518184175746728909777727938000
  8164706001614524919217321721477235014144197356854816136115735255213347574184946843852332390739414333
  4547762416862518983569485562099219222184272550254256887671790494601653466804988627232791786085784383
  8279679766814541009538837863609506800642251252051173929848960841284886269456042419652850222106611863
  0674427862203919494504712371378696095636437191728746776465757396241389086583264599581339047802759009
  9465764078951269468398352595709825822620522489407726719478268482601476990902640136394437455305068203
```

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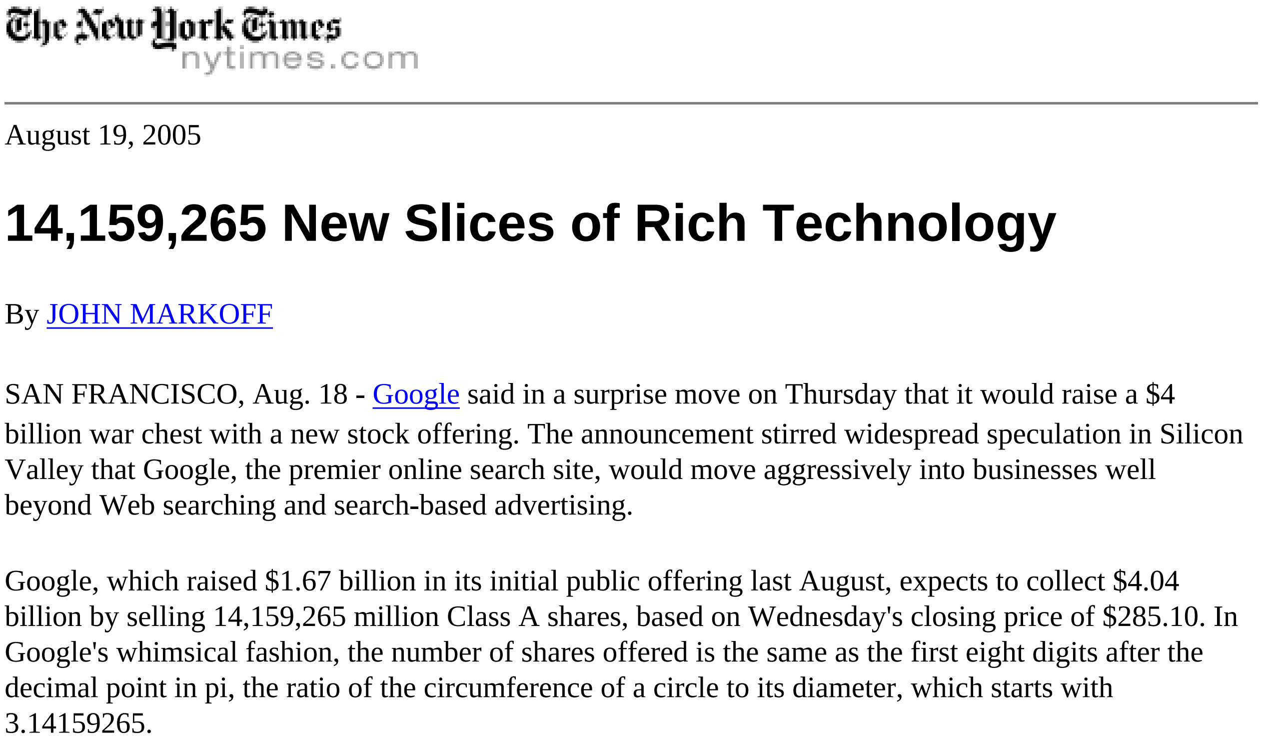
- From

3.14159265358979323846264338327950288419716939937510582097494459**2**.com/

Google Buys Pi



The New York Times
nytimes.com

August 19, 2005

14,159,265 New Slices of Rich Technology

By JOHN MARKOFF

SAN FRANCISCO, Aug. 18 - Google said in a surprise move on Thursday that it would raise a $4 billion war chest with a new stock offering. The announcement stirred widespread speculation in Silicon Valley that Google, the premier online search site, would move aggressively into businesses well beyond Web searching and search-based advertising.

Google, which raised $1.67 billion in its initial public offering last August, expects to collect $4.04 billion by selling 14,159,265 million Class A shares, based on Wednesday's closing price of $285.10. In Google's whimsical fashion, the number of shares offered is the same as the first eight digits after the decimal point in pi, the ratio of the circumference of a circle to its diameter, which starts with 3.14159265.

- Why does *Google* really want precisely this many pieces of the pie?

FINAL JEOPARDY, 20 SEPTEMBER 2005

CATEGORY: By the numbers

CLUE: The phrase "*How I want a drink, alcoholic of course*" is often used to help memorize this

ANSWER: **What is Pi?**

FINAL SCORES:
Ray: $7,200 + $7,000 = $14,200 (What is *Pi*) (New champion: $14,200)

Stacey: $11,400 - $3,001 = $8,399 (What is no *clue!?*) (2nd place: $2,000)

Victoria: $12,900 - $9,901 = $2,999 (What is *quadratic for*) (3rd place: $1,000)

`http://boards.sonypictures.com/boards/showthread.php?t=17319&page=1`

Pi the Song

Singer-songwriter Kate Bush released a **2005** album with the first single entitled “Pi”.

The lyrics can be found at:

`www.oldielyrics.com/lyrics/kate_bush/pi.html`

- The theme to the song is her singing out Pi to at least 117 decimal places, although the exact number is difficult to tell because the song fades out.

- Also of note is that she starts singing the incorrect digits partway through—so her R&D people need to be replaced.

Crossword Pi—NYT March 14, 2007

- A crossword puzzle with a π theme.

  - To solve the puzzle, first note that the clue for 28 `DOWN` is

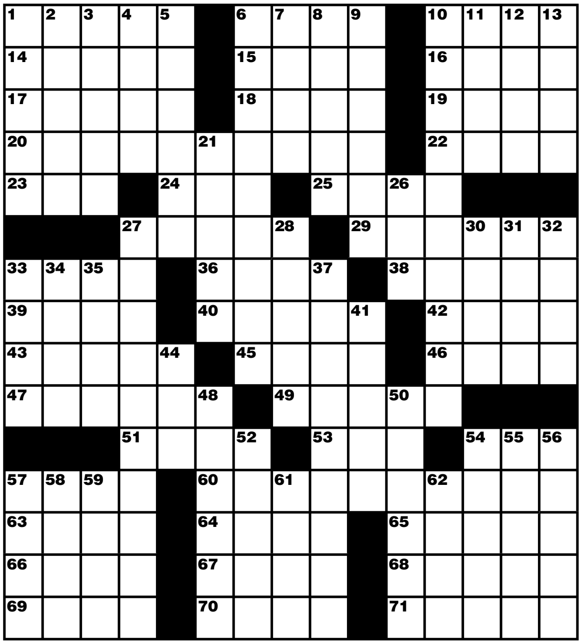
    `March 14, to Mathematicians`,

    to which the answer is `PIDAY`. Moreover, roughly a dozen other characters in the puzzle are π=**PI**.

  - For example, the clue for 5 down was `More pleased` with the six character answer `HAP`π`ER`.

The Puzzle (By Permission)

The New York Times Crossword Edited by Will Shortz No. 0314



Puzzle by Peter A. Collins

Across

1 Enlighten
6 A couple CBS spinoffs
10 1972 Broadway musical
14 Metal giant
15 Evict
16 Area
17 Surface again, as a road
18 Pirate or Padre, briefly
19 Camera feature
20 Barracks artwork, perhaps
22 River to the Ligurian Sea
23 Keg necessity
24 "… ___ he drove out of sight"
25 ___ St. Louis, Ill.
27 Preen
29 Greek peak
33 Vice president after Hubert
36 Patient wife of Sir Geraint
38 Action to an ante
39 Gain ___
40 French artist Odilon ___
42 Grape for winemaking
43 Single-dish meal
45 Broad valley
46 See 21-Down
47 Artery inserts
49 Offspring
51 Mexican mouse catcher
53 Medical procedure, in brief
54 "Wheel of Fortune" option
57 Animal with striped legs
60 Editorial
63 It gets bigger at night
64 "Hold your horses!"
65 Idiots
66 Europe/Asia border river
67 Suffix with launder
68 Leaning
69 Brownback and Obama, e.g.: Abbr.
70 Rick with the 1976 #1 hit "Disco Duck"
71 Yegg's targets

Down

1 Mastodon trap
2 "Mefistofele" soprano
3 Misbehave
4 Pen
5 More pleased
6 Treated with disdain
7 Enterprise crewman
8 Rhone feeder
9 Many a webcast
10 Mushroom, for one
11 Unfortunate
12 Nevada's state tree
13 Disney fish
21 Colonial figure with 46-Across
26 Poker champion Ungar
27 Self-medicating excessively
28 March 14, to mathematicians
30 Book part
31 Powder, e.g.
32 007 and others: Abbr.
33 Drains
34 Stove feature
35 Feet per second, e.g.
37 Italian range
41 Prefix with surgery
44 Captain's announcement, for short
48 Tucked away
50 Stealthy fighters
52 Sedative
54 Letter feature
55 Jam
56 Settles in
57 Symphony or sonata
58 Japanese city bombed in W.W. II
59 Beelike
61 Evening, in ads
62 Religious artwork

ANSWER TO PREVIOUS PUZZLE

A	R	F	S	■	A	C	H	E	■	O	R	G	A	N
C	O	R	K	■	T	R	E	X	■	K	E	R	R	Y
O	D	A	Y	■	L	A	I	T	■	S	T	A	T	S
L	E	N	■	R	A	N	D	O	M	■	I	P	S	E
D	O	C	T	O	R	K	I	L	D	A	R	E	■	■
■	■	■	R	A	G	■	■	■	S	I	E	S	T	A
T	Y	R	O	N	E	P	O	W	E	R	■	O	H	M
R	U	E	D	■	■	A	L	I	■	■	I	D	E	S
I	M	P	■	H	O	L	D	T	H	E	M	A	Y	O
B	A	A	B	A	A	■	■	■	O	R	E	■	■	■
■	■	I	R	I	S	H	C	O	U	N	T	I	E	S
P	E	R	I	■	T	I	R	A	D	E	■	D	X	C
A	R	M	E	D	■	T	A	T	I	■	Y	I	P	E
C	L	A	R	E	■	I	T	E	N	■	D	O	W	N

For answers, call 1-900-285-5656, $1.20 a minute; or, with a credit card, 1-800-814-5554.

Annual subscriptions are available for the best of Sunday crosswords from the last 50 years: 1-888-7-ACROSS.

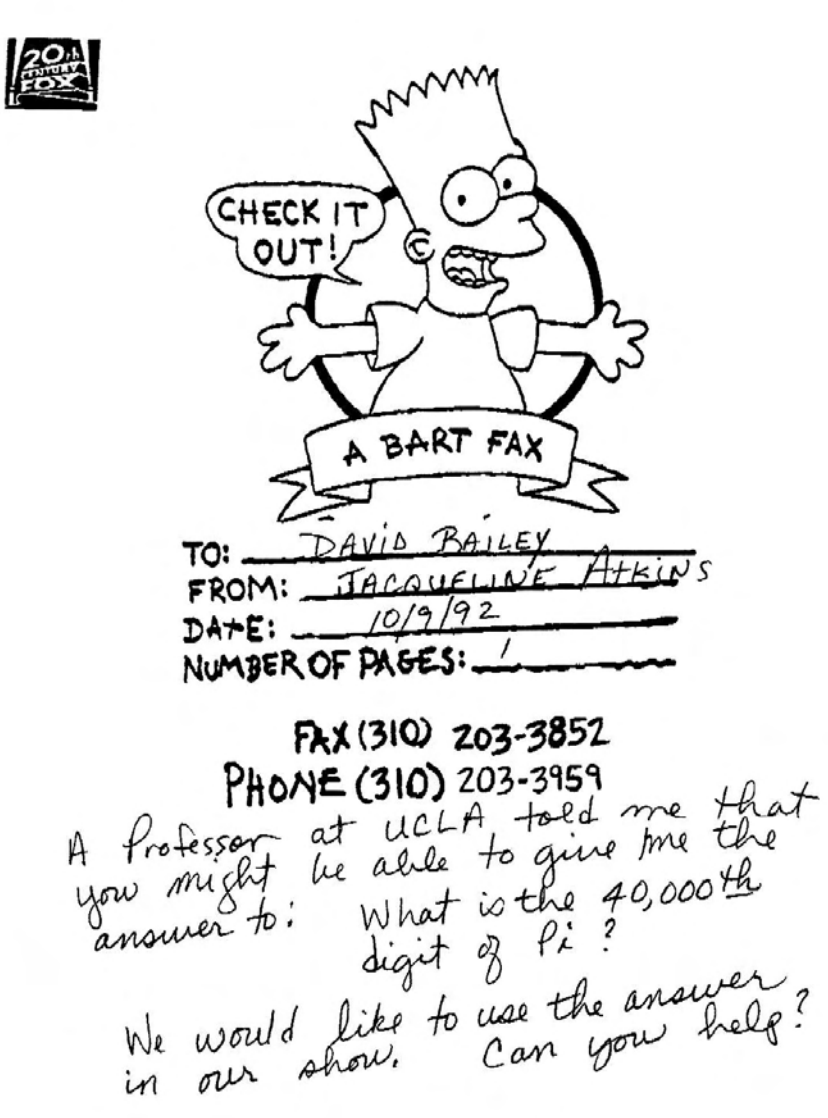
Online subscriptions: Today's puzzle and more than 2,000 past puzzles, nytimes.com/crosswords ($34.95 a year).

Share tips: nytimes.com/puzzleforum. Crosswords for young solvers: nytimes.com/learning/xwords.

ANSWER TO PREVIOUS PUZZLE

T	E	A	C	H	■	C	S	I	S	■	π	P	π	N
A	L	C	O	A	■	O	U	S	T	■	Z	O	N	E
R	E	T	O	P	■	N	L	E	R	■	Z	O	O	M
π	N	U	P	π	C	T	U	R	E	■	A	R	N	O
T	A	P	■	E	R	E	■	E	A	S	T	■	■	■
■	■	■	P	R	I	M	P	■	M	T	O	S	S	A
S	π	R	O	■	E	N	I	D	■	U	P	π	N	G
A	L	A	P	■	R	E	D	O	N	■	π	N	O	T
P	O	T	π	E	■	D	A	L	E	■	N	E	W	S
S	T	E	N	T	S	■	Y	O	U	N	G	■	■	■
■	■	■	G	A	T	O	■	M	R	I	■	S	π	N
O	K	A	π	■	O	π	N	I	O	N	π	E	C	E
P	U	π	L	■	W	A	I	T	■	J	E	R	K	S
U	R	A	L	■	E	T	T	E	■	A	T	I	L	T
S	E	N	S	■	D	E	E	S	■	S	A	F	E	S

The Simpsons (Permission refused by Fox)



20th CENTURY FOX

CHECK IT OUT!

A BART FAX

TO: DAVID BAILEY
FROM: JACQUELINE ATKINS
DATE: 10/9/92
NUMBER OF PAGES: 1

FAX (310) 203-3852
PHONE (310) 203-3959

A Professor at UCLA told me that you might be able to give me the answer to: What is the 40,000th digit of Pi?

We would like to use the answer in our show. Can you help?

- See also *Springfield Theory* (Science News, June 10, 2006) at `www.aarms.math.ca/ACMN/links`

Why π is not $\frac{22}{7}$

Even *Maple or Mathematica* ‘knows’ this since

$$(1) \qquad 0 \;<\; \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi,$$

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

Assume we trust it. Then the integrand is strictly positive on $(0,1)$, and the answer in (1) is an area and so strictly positive, despite millennia of claims that π is $22/7$.

Of course $22/7$ is one of the early continued fraction approximations to π. The first 4 are

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}.$$

In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4\,(1-x)^4}{1+x^2}\,dx \;=\; \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4\,t \;-\; 4\,\mathrm{arctan}\,(t)\,, \tag{2}$$

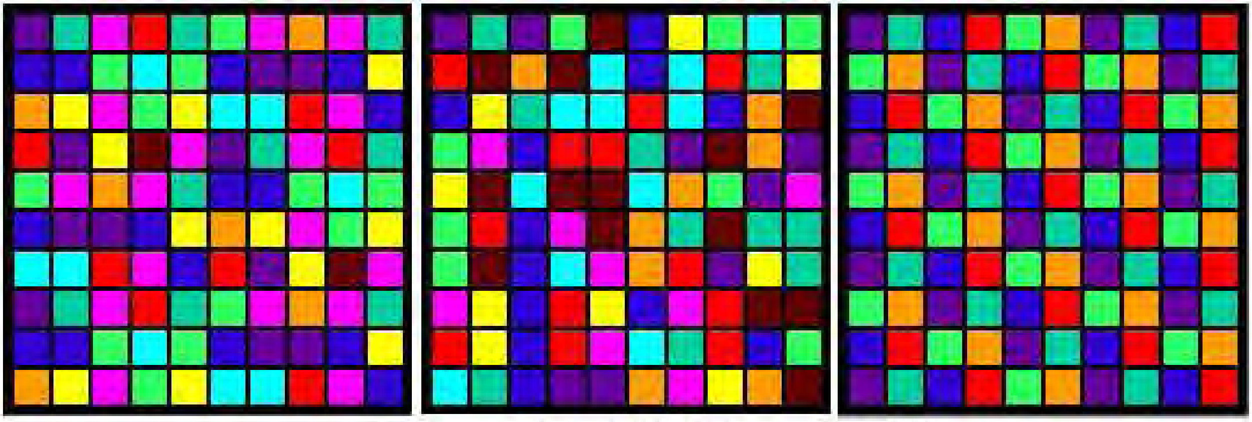
as differentiation easily confirms, and the fundamental theorem of calculus proves (1).

One can take this idea a bit further. Note that

$$\int_0^1 x^4\,(1-x)^4\,dx \;=\; \frac{1}{630}, \tag{3}$$

and we observe that

$$\begin{aligned} \frac{1}{2}\int_0^1 x^4\,(1-x)^4\,dx \;&<\; \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx \\ &<\; \int_0^1 x^4\,(1-x)^4\,dx. \end{aligned} \tag{4}$$



Archimedes: 223/71 < π < 22/7

Combine this with (1) and (3) to derive: $223/71 < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7$ and so re-obtain Archimedes famous computation

$$3\frac{10}{71} < \pi < 3\frac{10}{70}. \tag{5}$$

The Figure shows the estimate graphically.

- See Dalziel in *Eureka* (1971), a Cambridge student journal. Integral (1) was on the 68 Putnam, an early 60's Sydney exam, and traces back to 1944 (Dalziel).

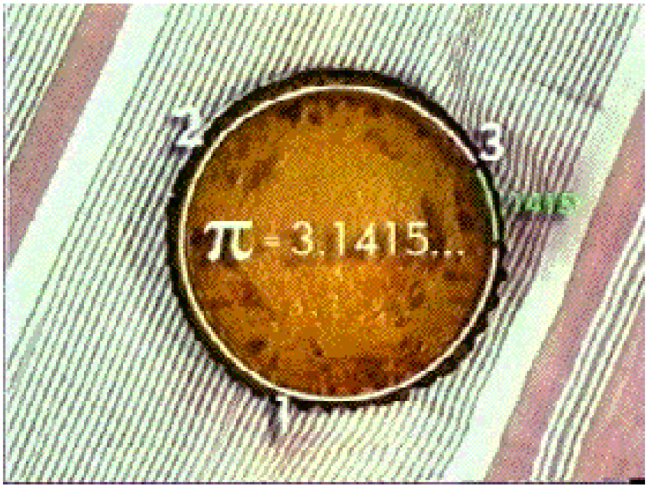
The Childhood of Pi

About 2000 BCE, the Babylonians used the approximation $3\frac{1}{8} = 3.125$. At this same time or earlier, according to an ancient papyrus, Egyptians assumed a circle with diameter nine has the same area as a square of side eight, which implies $\pi = \frac{256}{81} = 3.1604\ldots$.

Some have argued that the ancient Hebrews used $\pi = 3$:

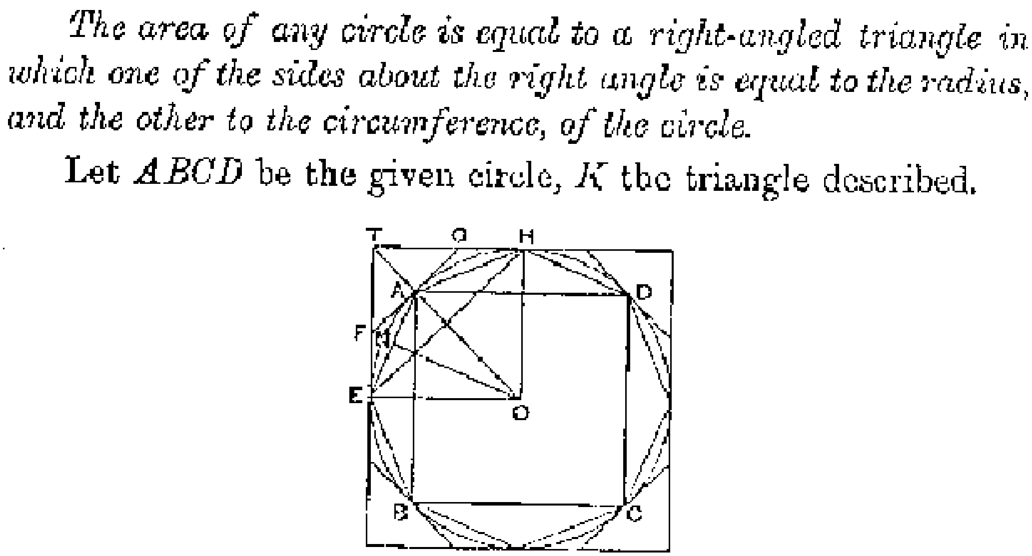
> "Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about." (I Kings 7:23; see also 2 Chron. 4:2)

Pi(es)



Archimedes (ca. 250 BCE) was the first to show that the ‘two Pi’s‘ are the same:

$$\textbf{Area}=\pi_1\,r^2 \text{ and } \textbf{Perimeter} = 2\,\pi_2\,r.$$



The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, K the triangle described.

The first rigorous mathematical calculation of π was also due to Archimedes, who used a brilliant scheme based on **doubling inscribed and circumscribed polygons** ($6 \mapsto 12 \mapsto 24 \mapsto 48 \mapsto 96$) to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

Archimedes' scheme constitutes the first true algorithm for π, in that it is capable of producing an arbitrarily accurate value for π.

As discovered in the 19th century, this scheme can be stated as a simple recursion, as follows. Set $a_0 := 2\sqrt{3}$ and $b_0 := 3$. Then define

$$\begin{aligned} a_{n+1} &= \frac{2\mathbf{a_n b_n}}{\mathbf{a_n + b_n}} && (H) \\ b_{n+1} &= \sqrt{a_{n+1} b_n} && (G) \end{aligned} \tag{6}$$

This converges to π, with the error decreasing by a factor of four with each iteration.

Variations of Archimedes' geometrical scheme were the basis for all high-accuracy calculations of π for the next 1800 years — well beyond its 'best before' date.

For example, in fifth century CE China, Tsu Chung-Chih used a variation of this method to get π correct to seven digits.

A millennium later, Al-Kashi in Samarkand "**who could calculate as eagles can fly**" computed 2π in **sexagecimal**:

$$
\begin{aligned}
2\pi = 6 \ & + \ \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} \\
& + \ \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},
\end{aligned}
$$

good to 16 decimal places (using $3 \cdot 2^{28}$-gons).

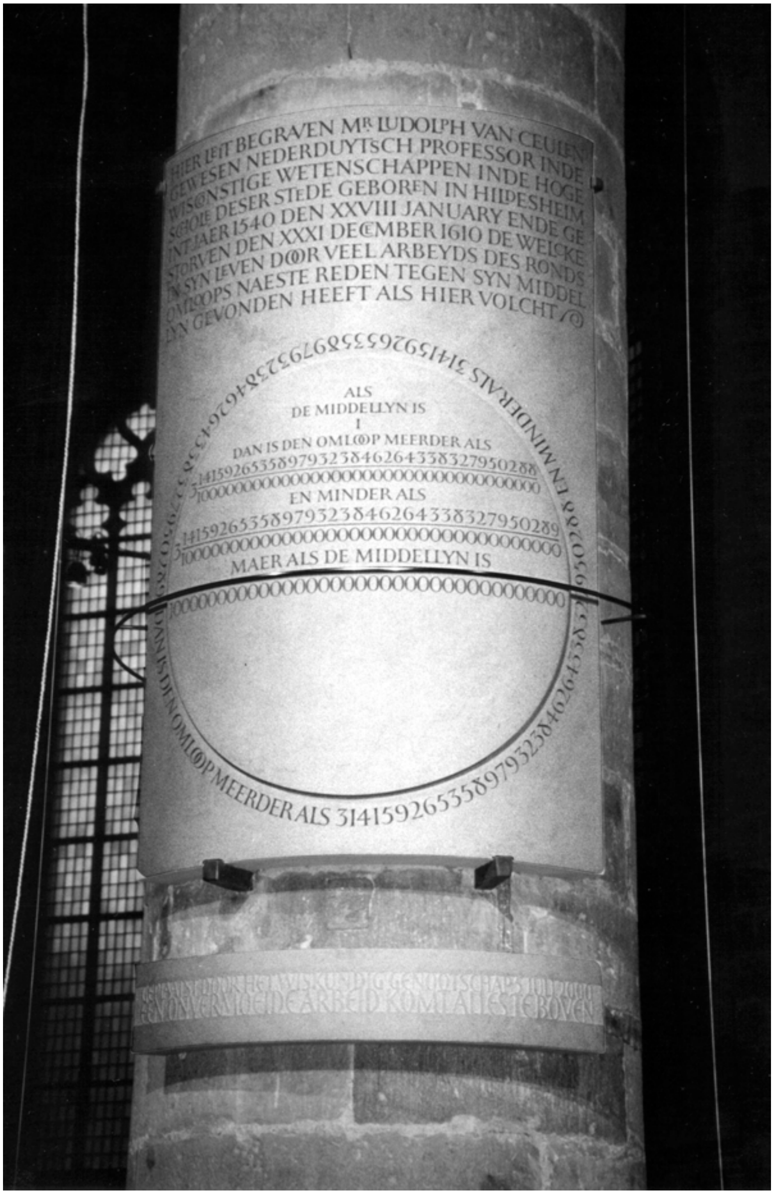
Precalculus π Calculations

Name	Year	Digits
Babylonians	2000? BCE	1
Egyptians	2000? BCE	1
Hebrews (1 Kings 7:23)	550? BCE	1
Archimedes	250? BCE	3
Ptolemy	150	3
Liu Hui	263	5
Tsu Ch'ung Chi	480?	7
Al-Kashi	1429	14
Romanus	1593	15
Van Ceulen (**Ludolph's number***)	1615	35

- * Using 2^{62}-gons—to 39 places with 35 correct—published posthumously.

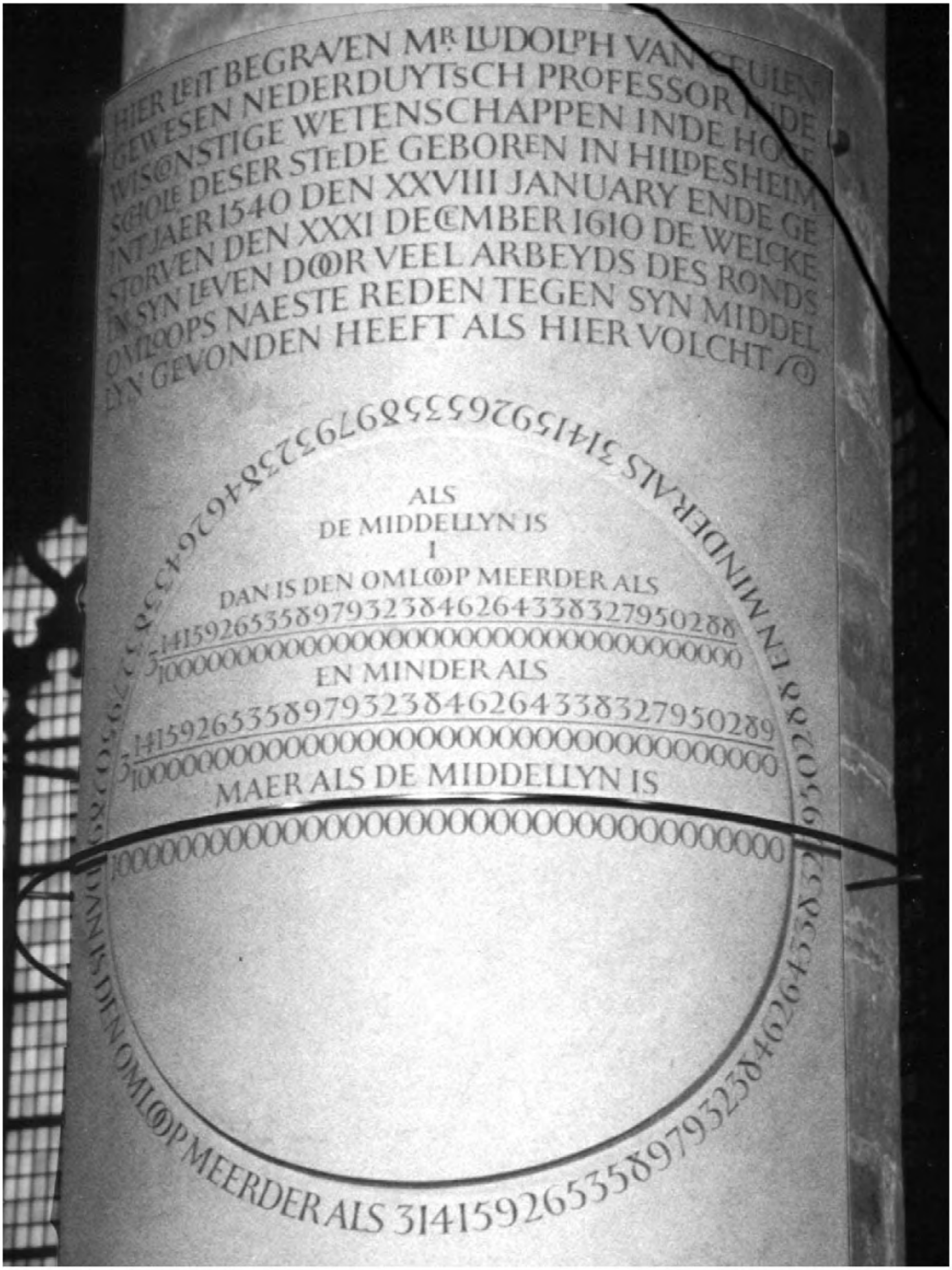
- Little progress was made in Europe during the 'dark ages', but a significant advance arose in India (450 CE): *modern positional, zero-based decimal arithmetic* — the "Indo-Arabic" system. This greatly enhanced arithmetic in general, and computing π in particular.

Ludolph's Rebuilt Tombstone in Leiden



Ludolph van Ceulen (1540-1610)

- Tombstone reconsecrated July 5, 2000.



HIER LEIT BEGRAVEN MR. LUDOLPH VAN CEULEN
GEWESEN NEDERDUYTSCH PROFESSOR INDE
WISCONSTIGE WETENSCHAPPEN INDE HOOGE
SCHOLE DESER STEDE GEBOREN IN HILDESHEIM
INT JAER 1540 DEN XXVIII JANUARY ENDE GE
STORVEN DEN XXXI DECEMBER 1610 DE WELCKE
IN SYN LEVEN DOOR VEEL ARBEYDS DES RONDS
OMLOOPS NAESTE REDEN TEGEN SYN MIDDEL
LYN GEVONDEN HEEFT ALS HIER VOLCHT

ALS
DE MIDDELLYN IS
1
DAN IS DEN OMLOOP MEERDER ALS

$$3\frac{14159265358979323846264338327950288}{100000000000000000000000000000000000}$$

EN MINDER ALS

$$3\frac{14159265358979323846264338327950289}{100000000000000000000000000000000000}$$

MAER ALS DE MIDDELLYN IS
100000000000000000000000000000000000

DAN IS DEN OMLOOP MEERDER ALS 314159265358979323846264338327950288 EN MINDER ALS 314159265358979323846264338327950289

The Indo-Arabic system came to Europe around 1000 CE. Resistance ranged from accountants who didn't want their livelihood upset to clerics who saw the system as 'diabolical,' since they incorrectly assumed its origin was Islamic. European commerce resisted until the 18th century, and even in scientific circles usage was limited into the 17th century.

The prior difficulty of doing arithmetic* is indicated by college placement advice given a wealthy German merchant in the 16th century:

> "If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division — assuming that he has sufficient gifts — then you will have to send him to Italy." (George Ifrah, p. 577)

*Claude Shannon had 'Throback 1' built to compute in Roman, at Bell Labs in 1953.

Pi's Adolescence

The dawn of modern mathematics appears in *Viéte's product* (1579)

$$\frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots=\frac{2}{\pi}$$

considered to be the first truly infinite formula; and in the *first continued fraction* for $2/\pi$ given by Lord Brouncker (1620-1684):

$$\frac{2}{\pi}=\cfrac{1}{1+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cdots}}}}$$

based on *John Wallis's 'interpolated' product*

$$\prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2}=\frac{2}{\pi}, \tag{7}$$

which lead to the discovery of the Gamma function and much more.

(7) may be derived from Euler's product formula for π, (8) with $x = 1/2$, or by repeatedly integrating $\int_0^{\pi/2} \sin^{2n}(t)\,dt$ by parts.

One may divine (8) as Euler did by *considering $\sin(\pi x)$ as an 'infinite' polynomial* and obtaining a product in terms of the roots $0, \{1/n^2\}$. It is thus plausible that

$$(8) \quad \zeta(2) = \frac{\sin(\pi\, x)}{x} \;=\; c \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Euler argued that, like a polynomial, c was the value at zero, and the coefficient of x^2 in the Taylor series the sum of the roots:

$$\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This also leads to the evaluation of $\zeta(2n)$ as a rational multiple of π^{2n}: $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945, \zeta(8) = \pi^8/9450, \ldots$ (in terms of Bernoulli numbers).

- In 1976 Apéry showed $\zeta(3)$ irrational; and we now know *one of* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is.

François Viéte (1540-1603)

Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by rational numbers, and irrational [magnitudes] by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.

Rather, says Proclus, ARITHMETIC IS MORE EXACT THAN GEOMETRY. To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial [i.e. square root of the difference] $72 - \sqrt{3888}$. Whoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.

- The inventor of 'x' and 'y'

Pi’s Adult Life with Calculus

> “I am ashamed to tell you to how many figures I carried these computations, having no other business at the time.”
> (Issac Newton, 1666)

In the 17th century, Newton and Leibniz discovered calculus, and this powerful tool was quickly exploited to find new formulas for π. One early calculus-based formula comes from the integral: $\tan^{-1} x$

$$= \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Substituting $x = 1$ *formally* gives the well-known **Gregory–Leibniz formula** (1671–74)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Calculus π Calculations

Name	Year	Correct Digits
Sharp (and Halley)	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
Shanks	1874	(707) 527
Ferguson (**Calculator**)	1947	808
Reitwiesner et al. (**ENIAC**)	1949	2,037
Genuys	1958	10,000
Shanks and Wrench	1961	100,265
Guilloud and Bouyer	1973	1,001,250

- Done naively, this is useless — so slow that hundreds of terms are needed to compute two digits. [Sharp used $\tan^{-1}(1/\sqrt{3})$.]

However, Euler's (1738) trigonometric identity

$$(9)\quad \tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

produces the geometrically convergent

$$\begin{aligned} \frac{\pi}{4} &= \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots \\ (10) \qquad & + \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \cdots \end{aligned}$$

An even faster formula, found earlier by John Machin, lies similarly in the identity

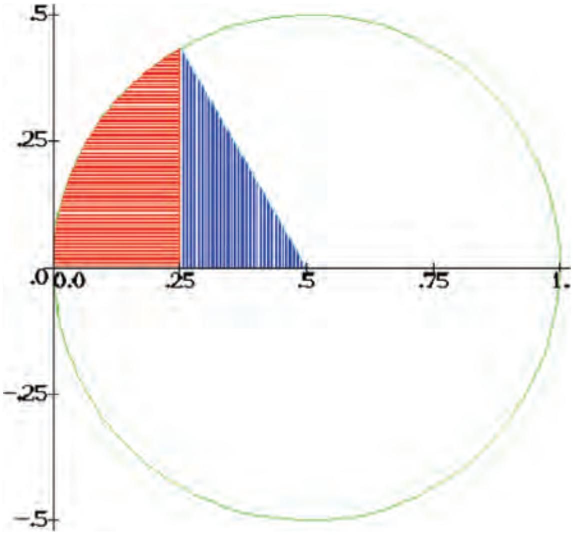
$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right). \tag{11}$$

- This was used in numerous computations of π (starting in 1706) and culminating with Shanks' computation of π to 707 decimal digits accuracy in 1873 (although it was *found in 1945 to be wrong* after the 527-th decimal place, by Ferguson).

Newton discovered a different (disguised arcsin) formula. He considering the area A of the left-most red region shown in the next Figure. Now, A is the integral

$$A = \int_0^{1/4} \sqrt{x - x^2}\, dx. \tag{12}$$

Newton's arcsin



Also, A is the area of the circular sector, $\pi/24$, less the area of the triangle, $\sqrt{3}/32$. Newton used his **binomial theorem** in (12):

$$\begin{aligned} A &= \int_0^{\frac{1}{4}} x^{1/2}(1-x)^{1/2}\,dx \\ &= \int_0^{\frac{1}{4}} x^{1/2}\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\cdots\right)dx \\ &= \int_0^{\frac{1}{4}}\left(x^{1/2}-\frac{x^{3/2}}{2}-\frac{x^{5/2}}{8}-\frac{x^{7/2}}{16}-\frac{5x^{9/2}}{128}\cdots\right)dx \end{aligned}$$

Integrate term-by-term and combining the above:

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3\cdot 8}-\frac{1}{5\cdot 32}-\frac{1}{7\cdot 128}-\frac{1}{9\cdot 512}\cdots\right).$$

Newton used this formula to compute 15 digits of π. As noted, he later ‘apologized’ for “having no other business at the time.”*

- The Viennese *computer* Johan Zacharias Dase demonstrated his computational skill by multiplying

  $$79532853 \times 93758479 = 7456879327810587$$

  in 54 seconds; two 20-digit numbers in six minutes; two 40-digit numbers in 40 minutes; two 100-digit numbers in $8\frac{3}{4}$ hours.

- In 1844, after being shown

  $$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

  he calculated π to 200 places *in his head* in two months.

---

*The great fire of London that ended the plague year took place in September 1666. A standard chronology says ”Newton never tried to compute π.”

- Dase later calculated a seven-digit logarithm table, and extended a table of integer factorizations to 10,000,000. Gauss requested that Dase be permitted to assist, but Dase died shortly afterwards.

One motivation for computations of π was very much in the spirit of modern experimental mathematics: to see if the decimal expansion of π repeats, which would mean that π is the ratio of two integers (i.e., rational), or to recognize π as an algebraic constant.

The question of the *rationality of* π was settled in the late 1700s, when Lambert and Legendre proved (using continued fractions) that the constant is irrational.

The question of whether π is algebraic was settled in 1882, when Lindemann proved that π *is transcendental*.

- Lindemann's proof also settled, once and for all, the ancient Greek question of whether the circle could be squared with ruler and compass.

  It cannot, because numbers that are the lengths of lines that can be constructed using ruler and compasses (often called *constructible numbers*) are necessarily algebraic, and squaring the circle is equivalent to constructing the value π.

- Aristophanes knew this and derided 'circle-squarers' ($\tau\varepsilon\tau\rho\alpha\gamma\omega\sigma\iota\varepsilon\iota\nu$) in his play "The Birds" of 414 BCE.

The Irrationality of π

Niven's 1947 proof that π is irrational. Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n!f(x)$ has integral coefficients and terms in x of degree not less than n, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$; also for $x = \pi = a/b$, since $f(x) = f(a/b - x)$. By elementary calculus we have

$$\frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\}$$

$$= \quad F''(x)\sin x + F(x)\sin x = f(x)\sin x$$

and

$$\begin{aligned} \int_0^\pi f(x)\sin x\,dx &= [F'(x)\sin x - F(x)\cos x]_0^\pi \\ (13) \qquad &= F(\pi) + F(0). \end{aligned}$$

Now $F(\pi) + F(0)$ is an *integer*, since $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers. But for $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (13) is *positive but arbitrarily small* for n sufficiently large. Thus (13) is false, and so is our assumption that π is rational. **QED**

- This is an excellent intimation of more sophisticated irrationality and transcendence proofs.

- With the development of computer technology in the 1950s, π was computed to thousands and then millions of digits. These computations were facilitated by the discovery of advanced algorithms for the underlying high-precision arithmetic operations.

- For example, in 1965 it was found that the newly-discovered *fast Fourier transform* (FFT) could be used to perform high-precision multiplications much more rapidly than conventional schemes.

  Such methods (e.g., for $\div, \sqrt{x}$) dramatically lowered the time required for computing π and other constants to high precision.

- In spite of these advances, until the 1970s all computer evaluations of π still employed classical formulas, usually of Machin-type.

Ballantine's (1939) Series for π

Another formula of Euler for arccot is

$$x\sum_{n=0}^{\infty}\frac{(n!)^2\,4^n}{(2\,n+1)!\,\left(x^2+1\right)^{n+1}} \;=\; \arctan\left(\frac{1}{x}\right)$$

This allows one to rewrite the formula, used by Guilloud and Boyer in 1973 to compute a million digits of Pi, viz, $\pi/4 =$

$$12\arctan\left(\frac{1}{18}\right)+8\arctan\left(\frac{1}{57}\right)-5\arctan\left(\frac{1}{239}\right)$$

in the efficient form

$$\begin{aligned}\pi \;&=\; 864\sum_{n=0}^{\infty}\frac{(n!)^2\,4^n}{(2\,n+1)!\,\mathbf{325}^{n+1}}\\ &+\; 1824\sum_{n=0}^{\infty}\frac{(n!)^2\,4^n}{(2\,n+1)!\,\mathbf{3250}^{n+1}}\\ &-\; 20\arctan\left(\frac{1}{239}\right),\end{aligned}$$

where the terms of the second series are just decimal shifts of the first.

ENIAC: Integrator and Calculator

SIZE/WEIGHT: ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.

The ENIAC in the Smithsonian

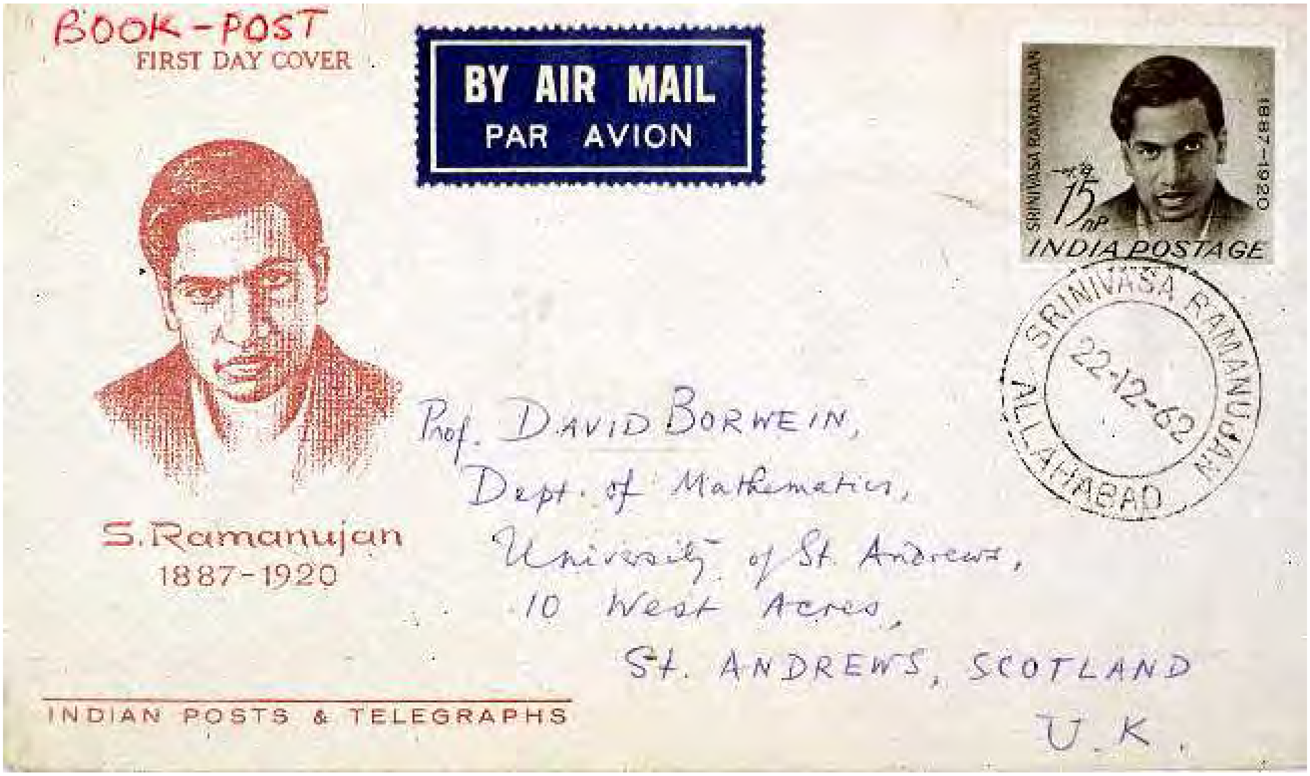
SPEED/MEMORY: A 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000 — 1,000 times faster than any earlier machine. The first stored-memory computer, ENIAC could store 200 digits.

ARCHITECTURE: Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line. The accumulators were connected to each other manually.

- The 1949 computation of π to 2,037 places took 70 hours.

- Origin of the term 'bug'?

- The Smithsonian picture would require $100,000$ ENIACs to store. [**Moore's Law!**]



Ramanujan's Seventy-Fifth Birthday Stamp.

Truly new infinite series formulas were discovered by Ramanujan around 1910, but were not well known (nor fully proven) until quite recently when his writings were widely published.

One of these series is the remarkable:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4 396^{4k}}$$

(14)

Each term of this series produces an additional *eight* correct digits in the result. When Gosper used this formula to compute 17 million digits of (the continued fraction for) π in 1985, **this concluded the first proof of (14)**!

At about the same time, David and Gregory Chudnovsky found the following variation of Ramanujan's formula:

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k\,(6k)!(13591409+545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}$$

Each term of this series produces an additional 14 correct digits.

The Chudnovskys implemented this formula using a clever scheme that enabled them to utilize the results of an initial level of precision to extend the calculation to even higher precision.

They used this in several large calculations of π, culminating with a **then record computation** to over four billion decimal digits in 1994.

- Relatedly, the Ramanujan-type series

$$(15) \qquad \frac{1}{\pi} = \sum_{n=0}^{\infty} \left(\frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42\,n+5}{16}.$$

  allows one to compute the billionth binary digit of $1/\pi$, or the like, *without computing the first half* of the series.

- While the Ramanujan and Chudnovsky series are considerably more efficient than classical formulas, they share the property that the number of terms needed increases linearly with the number of digits desired.

  That is, **if you want to compute π to twice as many digits, you have to evaluate twice as many terms** of the series.

- In 1976, Eugene Salamin and Richard Brent independently discovered a *reduced complexity* algorithm for π.

  It is based on the **arithmetic-geometric mean iteration** (AGM) and some other ideas due to Gauss and Legendre around 1800 (although Gauss never directly saw the connection to computing π).

The **Salamin–Brent algorithm** is:

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Calculate

$$\mathbf{a_k} = \frac{\mathbf{a_{k-1}} + \mathbf{b_{k-1}}}{\mathbf{2}}, \qquad \mathbf{b_k} = \sqrt{a_{k-1} b_{k-1}}$$

$$c_k = a_k^2 - b_k^2, \qquad s_k = s_{k-1} - 2^k c_k$$

$$\text{and compute} \quad p_k = \frac{2a_k^2}{s_k}. \tag{16}$$

Then p_k converges *quadratically* to π.

- Each iteration *doubles* the correct digits —**successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 digits** of π, and takes $\log N$ operations for N digits.

- Twenty-five iterations computes π to over 45 million decimal digit accuracy. However, each of these iterations must be performed to the precision of the final result.

In 1985, my brother Peter and I discovered other algorithms of this type.

A1: set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1+2(1-s_k^3)^{1/3}} \\ s_{k+1} &= \frac{r_{k+1}-1}{2} \\ \text{and} & \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k(r_{k+1}^2 - 1). \end{aligned}$$

Then $1/a_k$ **converges cubically** to π — each iteration triples the number of correct digits.

A2: set $a_0 = 6-4\sqrt{2}$ and $y_0 = \sqrt{2}-1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \qquad \text{and} \\ a_{k+1} &= a_k(1+y_{k+1})^4 \\ &\quad - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned}$$

Then $1/a_k$ **converges quartically** to π.

With $a_0 = 6 - 4\sqrt{2}, \qquad y_0 = \sqrt{2} - 1$ **and**

$$y_1 = \frac{1 - \sqrt[4]{1 - {y_0}^4}}{1 + \sqrt[4]{1 - {y_0}^4}}, a_1 = a_0 \left(1 + y_1\right)^4 - 2^3 y_1 \left(1 + y_1 + {y_1}^2\right)$$

$$y_2 = \frac{1 - \sqrt[4]{1 - {y_1}^4}}{1 + \sqrt[4]{1 - {y_1}^4}}, a_2 = a_1 \left(1 + y_2\right)^4 - 2^5 y_2 \left(1 + y_2 + {y_2}^2\right)$$

$$y_3 = \frac{1 - \sqrt[4]{1 - {y_2}^4}}{1 + \sqrt[4]{1 - {y_2}^4}}, a_3 = a_2 \left(1 + y_3\right)^4 - 2^7 y_3 \left(1 + y_3 + {y_3}^2\right)$$

$$y_4 = \frac{1 - \sqrt[4]{1 - {y_3}^4}}{1 + \sqrt[4]{1 - {y_3}^4}}, a_4 = a_3 \left(1 + y_4\right)^4 - 2^9 y_4 \left(1 + y_4 + {y_4}^2\right)$$

$$y_5 = \frac{1 - \sqrt[4]{1 - {y_4}^4}}{1 + \sqrt[4]{1 - {y_4}^4}}, a_5 = a_4 \left(1 + y_5\right)^4 - 2^{11} y_5 \left(1 + y_5 + {y_5}^2\right)$$

$$y_6 = \frac{1 - \sqrt[4]{1 - {y_5}^4}}{1 + \sqrt[4]{1 - {y_5}^4}}, a_6 = a_5 \left(1 + y_6\right)^4 - 2^{13} y_6 \left(1 + y_6 + {y_6}^2\right)$$

$$y_7 = \frac{1 - \sqrt[4]{1 - {y_6}^4}}{1 + \sqrt[4]{1 - {y_6}^4}}, a_7 = a_6 \left(1 + y_7\right)^4 - 2^{15} y_7 \left(1 + y_7 + {y_7}^2\right)$$

$$y_8 = \frac{1 - \sqrt[4]{1 - {y_7}^4}}{1 + \sqrt[4]{1 - {y_7}^4}}, a_8 = a_7 \left(1 + y_8\right)^4 - 2^{17} y_8 \left(1 + y_8 + {y_8}^2\right)$$

$$y_9 = \frac{1 - \sqrt[4]{1 - {y_8}^4}}{1 + \sqrt[4]{1 - {y_8}^4}}, a_9 = a_8 \left(1 + y_9\right)^4 - 2^{19} y_9 \left(1 + y_9 + {y_9}^2\right)$$

$$y_{10} = \frac{1 - \sqrt[4]{1 - {y_9}^4}}{1 + \sqrt[4]{1 - {y_9}^4}}, a_{10} = a_9 \left(1 + y_{10}\right)^4 - 2^{21} y_{10} \left(1 + y_{10} + {y_{10}}^2\right)$$

$$y_{11}=\frac{1-\sqrt[4]{1-{y_{10}}^4}}{1+\sqrt[4]{1-{y_{10}}^4}}, a_{11}=a_{10}\left(1+y_{11}\right)^4-2^{23}y_{11}\left(1+y_{11}+{y_{11}}^2\right)$$

$$y_{12}=\frac{1-\sqrt[4]{1-{y_{11}}^4}}{1+\sqrt[4]{1-{y_{11}}^4}}, a_{12}=a_{11}\left(1+y_{12}\right)^4-2^{25}y_{12}\left(1+y_{12}+{y_{12}}^2\right)$$

$$y_{13}=\frac{1-\sqrt[4]{1-{y_{12}}^4}}{1+\sqrt[4]{1-{y_{12}}^4}}, a_{13}=a_{12}\left(1+y_{13}\right)^4-2^{27}y_{13}\left(1+y_{13}+{y_{13}}^2\right)$$

$$y_{14}=\frac{1-\sqrt[4]{1-{y_{13}}^4}}{1+\sqrt[4]{1-{y_{13}}^4}}, a_{14}=a_{13}\left(1+y_{14}\right)^4-2^{29}y_{14}\left(1+y_{14}+{y_{14}}^2\right)$$

$$y_{15}=\frac{1-\sqrt[4]{1-{y_{14}}^4}}{1+\sqrt[4]{1-{y_{14}}^4}}, a_{15}=a_{14}\left(1+y_{15}\right)^4-2^{31}y_{15}\left(1+y_{15}+{y_{15}}^2\right)$$

$$y_{16}=\frac{1-\sqrt[4]{1-{y_{15}}^4}}{1+\sqrt[4]{1-{y_{15}}^4}}, a_{16}=a_{15}\left(1+y_{16}\right)^4-2^{33}y_{16}\left(1+y_{16}+{y_{16}}^2\right)$$

$$y_{17}=\frac{1-\sqrt[4]{1-{y_{16}}^4}}{1+\sqrt[4]{1-{y_{16}}^4}}, a_{17}=a_{16}\left(1+y_{17}\right)^4-2^{35}y_{17}\left(1+y_{17}+{y_{17}}^2\right)$$

$$y_{18}=\frac{1-\sqrt[4]{1-{y_{17}}^4}}{1+\sqrt[4]{1-{y_{17}}^4}}, a_{18}=a_{17}\left(1+y_{18}\right)^4-2^{37}y_{18}\left(1+y_{18}+{y_{18}}^2\right)$$

$$y_{19}=\frac{1-\sqrt[4]{1-{y_{18}}^4}}{1+\sqrt[4]{1-{y_{18}}^4}}, a_{19}=a_{18}\left(1+y_{19}\right)^4-2^{39}y_{19}\left(1+y_{19}+{y_{19}}^2\right)$$

$$y_{20}=\frac{1-\sqrt[4]{1-{y_{19}}^4}}{1+\sqrt[4]{1-{y_{19}}^4}}, \mathbf{a_{20}}=a_{19}\left(1+y_{20}\right)^4-2^{41}y_{20}\left(1+y_{20}+{y_{20}}^2\right).$$

Then the **transcendental number**

$\boxed{\mathrm{Pi}}$

and the **algebraic number**

$$1/a_{20}$$

actually agree for more than

1.5 trillion decimal places !!

Star Trek

Kirk asks:

> " **Aren't there some mathematical problems that simply can't be solved**?"

And Spock 'fries the brains' of a rogue computer by telling it:

> " **Compute to the last digit the value of Pi**."

- This algorithm, together with the Salamin–Brent scheme, has been employed by Yasumasa Kanada in Tokyo in various computations of π over the past 15 years or so, including 200 billion decimal digits in 1999.

- Shanks in 1963 was confident that a billion digit computation was *forever impossible*.

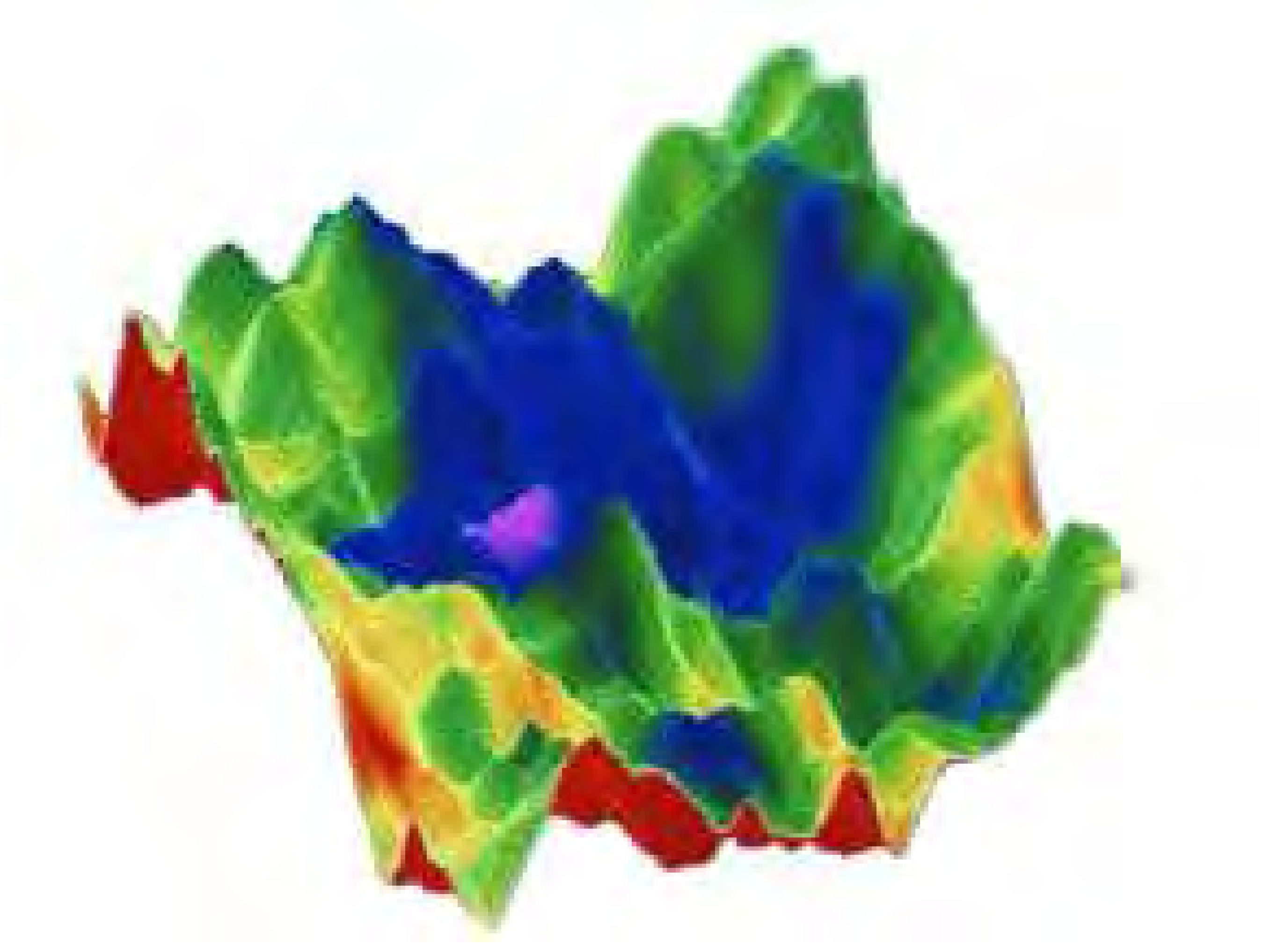
- In 1997 the *first occurrence of the sequence* **0123456789** was found (late) in the decimal expansion of π starting at the

  $\mathbf{17,387,594,880}$-th digit

  after the decimal point.

  In consequence the status of several famous intuitionistic examples *due to Brouwer and Heyting has changed.*

The First Million Digits of π



A random walk on π

(courtesy David and Gregory Chudnovsky)

- See Richard Preston's the *Mountains of Pi*.

Modern π Calculations

Name	Year	Correct Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Kanada and Takahashi	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jul. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada-Ushiro-Kuroda	Dec. 2002	1,241,100,000,000

Back to the Future

In December 2002, Kanada computed π to over **1.24 trillion decimal digits**. His team first computed π in hexadecimal (base 16) to 1,030,700,000,000 places, using arctangent relations [Takano (1982) and Störmer (1896)]:

$$\pi = 48\tan^{-1}\frac{1}{49} + 128\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239} + 48\tan^{-1}\frac{1}{110443}$$

$$\pi = 176\tan^{-1}\frac{1}{57} + 28\tan^{-1}\frac{1}{239} - 48\tan^{-1}\frac{1}{682} + 96\tan^{-1}\frac{1}{12943}$$

Kanada verified these 2 computations agreed, and converted the hex sequence to decimal.

- The resulting decimal expansion was checked by converting it back to hex. Such conversions require massive computation.

- This scheme is quite different from earlier ones. One reason is that the Salamin-Brent and Borwein quartic algorithms, used in the past, require full-scale multiply, divide and square root operations, which in turn require large-scale FFT operations.

  These require huge amounts of memory, and massive all-to-all communication between nodes of a large parallel system.

- For this computation, even the very large system available did not have sufficient memory and network bandwidth to perform these operations at reasonable efficiency levels—at least not for trillion-digit computations.

- This used a 1 Tbyte main memory system, as the previous computation, yet got six times as many digits. Hex and decimal evaluations included, it ran 600 hours on a 64-node Hitachi, the main segment at nearly 1 Tflop/sec.

Yasumasa Kanada

手にしているのはπの値が入ったカートリッジテープ

Why Pi?

- What is the motivation behind modern computations of π, given that questions such as the irrationality and transcendence of π were settled more than 100 years ago?

- One motivation is the raw challenge of harnessing the stupendous power of modern computer systems. Programming such calculations are definitely not trivial, especially on large, distributed memory computer systems.

- There have been substantial practical spin-offs. For example, some new techniques for performing the fast Fourier transform (FFT), heavily used in modern science and engineering computing, had their roots in attempts to accelerate computations of π.

- Beyond practical considerations is the abiding interest in the fundamental question of the *normality (digit randomness)* of π.

  Kanada, for example, has performed detailed statistical analysis of his results to see if there are any statistical abnormalities that suggest π is not normal.

- Indeed the first computer computation of π and e on ENIAC was so motivated by John von Neumann.

- The digits of π have been studied more than any other single constant, in part because of the widespread fascination with π.

  Both Kanada's counts are entirely reasonable.

Decimal Digit	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	**1000000000000**

- According to Kanada, the 10 decimal digits ending in position one trillion are `6680122702`, while the 10 hexadecimal digits ending in position one trillion are `3F89341CD5`.

Hex Digit	Occurrences
0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	62500216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	**1000000000000**

- In retrospect, I wonder why in antiquity π was not *measured* to an accuracy in excess of $22/7$?

  Perhaps it reflects not an inability to do so but a very different mind set to a modern (Baconian) experimental one.

- In the same vein, one reason that Gauss and Ramanujan did not further develop the ideas in their identities for π is that an iterative algorithm, as opposed to explicit results, was not as satisfactory for them (especially Ramanujan).

  Ramanujan much preferred formulae like

  $$\frac{3}{\sqrt{163}} \log(640320) \approx \pi$$

  correct to *15 decimal places* and

  $$\frac{3}{\sqrt{67}} \log(5280) \approx \pi$$

  correct to *9 decimal places*.

Discovering the Cubic & Quartic Iterations

The genesis of the π algorithms and related material is an illustrative example of experimental mathematics. For positive integer N, the function

$$\boldsymbol{\alpha}(\mathbf{N}) = \frac{\mathbf{E}'(\mathbf{k_N})}{\mathbf{K}(\mathbf{k_N})} - \frac{\boldsymbol{\pi}}{\mathbf{4\,K^2(K_n)}}$$

arose, where k_N is the N-th *singular value* and K and K and E' are complete *elliptic integrals*.

For present purposes it suffices that $\alpha(N)$ is very easy to compute.

For example, the first few non–composite values are (to 20 digit accuracy):

$$\begin{aligned}
\alpha(1) &\approx 0.49999999999999999999 \\
\alpha(2) &\approx 0.41421356237309504880 \\
\alpha(3) &\approx 0.36602540378443864678 \\
\alpha(5) &\approx \underline{0.33188261099247156221} \\
\alpha(7) &\approx 0.32287565553229529536
\end{aligned}$$

- It is obvious that $\boldsymbol{\alpha(1) = 1/2}$ and easy to spot that $\boldsymbol{\alpha(2) = \sqrt{2} - 1}$, from which it was quickly observed that $\boldsymbol{\alpha(3) = (\sqrt{3} - 1)/2}$ and that $\boldsymbol{\alpha(7) = (\sqrt{7} - 2)/2}$, but $\alpha(5)$ *did not appear* to be a quadratic.

- Twenty years ago such identification was not easy and it was only when it occurred to us that quadratic fields congruent to ±1 mod 4 behave differently that we stumbled upon (experimentally) the identity

$$\alpha(5) = \frac{\sqrt{5} - \sqrt{2\sqrt{5} - 2}}{2}. \tag{17}$$

- Nowadays this is almost trivial: a "Minpoly calculation" immediately returns

$$29 - 80x - 24x^2 + 16x^4 = 0$$

  and this has the surd above as its smallest positive root.

- At this point we could have used known results only to prove the value of $\alpha(1), \alpha(2)$ and $\alpha(3)$. Those for $\alpha(5)$ and $\alpha(7)$ remained conjectural.

There was however an empirical family of algorithms for π: let $\alpha_0 = \alpha(N)$ and $k_0 := k'_N$ (where $k' = \sqrt{1-k^2}$) and iterate

$$\begin{aligned} k_{n+1} &= \frac{1-k'_n}{1+k'_n} \\ &\text{and} \\ \alpha_{n+1} &= (1+k_{n+1})^2 \alpha_n - \sqrt{N}\, 2^{n+1} k_{n+1}. \end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n^{-1} = \pi. \tag{18}$$

Again, (18) was provable for $N = 1, 2, 3$ and only conjectured for $N = 5, 7$.

- In each case the algorithm *appeared* to converge quadratically to π. On closer inspection while the provable cases were correct to $5,000$ digits, the empirical versions of (18) agreed with π to roughly 100 places only.

- Now in many ways to have discovered a “natural” number that agreed with π to that level — and no more — would have been more interesting than the alternative. That seemed unlikely but recoding and rerunning the iterations kept producing identical results.

- Twenty years ago very high precision calculation was less accessible, and the code was being run remotely over a phone-line in a Berkeley Unix integer package.

After about six weeks, it transpired that the package's **square root algorithm was badly flawed**, but **only if run with an odd precision of more than sixty digits**!

- And for idiosyncratic reasons that had only been the case in the two unproven cases.

- Needless to say, tracing the bug was a salutory and somewhat chastening experience.

Borweins and Plouffe (MSNBC, 1997)

Computing Individual Digits of π

An outsider might be forgiven for thinking that essentially everything of interest with regards to π has been discovered.

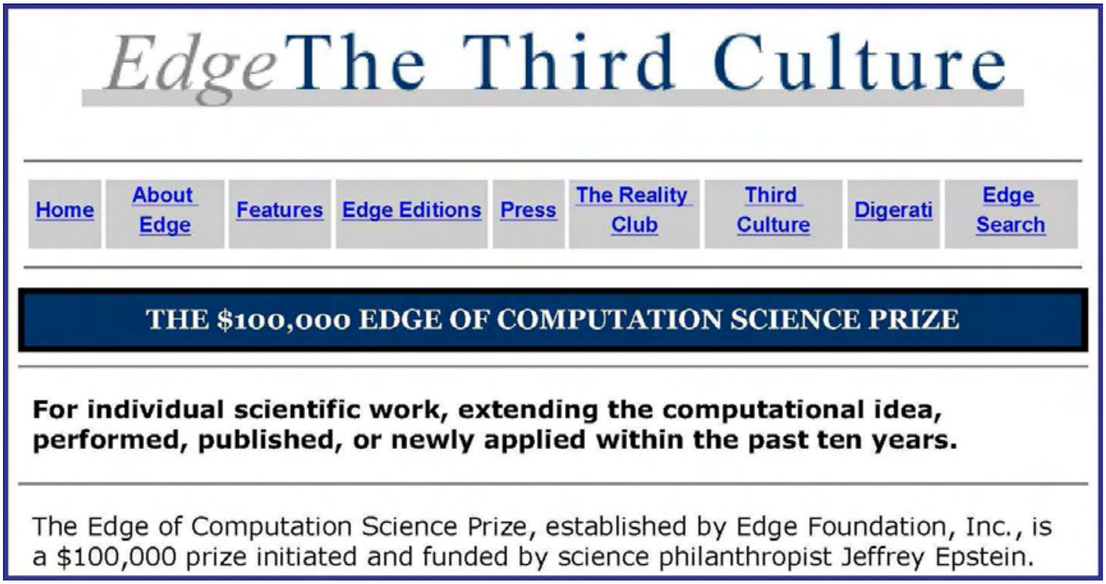
This sentiment is suggested in the closing chapters of Beckmann's 1971 book *A History of* π.

- Ironically, the Salamin–Brent quadratically convergent iteration was discovered only five years later, and the higher-order convergent algorithms followed in the 1980s.

- In 1990, Rabinowitz and Wagon discovered a 'spigot'' algorithm for π. This permits successive digits of π (in any desired base) to be computed by a relatively simple recursive algorithm based on the *all previously* generated digits.

But even insiders are sometimes surprised by a new discovery.

Edge of Computation Prize Finalist



THE $100,000 EDGE OF COMPUTATION SCIENCE PRIZE

For individual scientific work, extending the computational idea, performed, published, or newly applied within the past ten years.

The Edge of Computation Science Prize, established by Edge Foundation, Inc., is a $100,000 prize initiated and funded by science philanthropist Jeffrey Epstein.

- BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**

- Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...

- Won by David Deutsch — discoverer of Quantum Computing

Prior to 1996, most folks thought if you want to determine the d-th digit of π, you had to generate the (order of) the entire first d digits.

This is not true, at least for hex (base 16) or binary (base 2) digits of π. In 1996, P. Borwein, Plouffe, and Bailey found an algorithm for computing individual hex digits of π. It:

(1) produces a modest-length string hex or binary digits of π, beginning at an arbitrary position, using no prior bits;

(2) is implementable on any modern computer;

(3) requires no multiple precision software;

(4) requires very little memory; and

(5) has a computational cost growing only slightly faster than the digit position.

For example, the millionth hexadecimal digit (four millionth binary digit) of π can be found in under a minute on a present computer.

The new algorithm is not fundamentally faster than best known schemes for computing all digits of π up to some position, but its elegance and simplicity are of considerable interest.

It is based on the following new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{19}$$

which was *discovered numerically* using *integer relation methods* for several months in CECM in the form:

$$\pi = 4F(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}) + 2\tan^{-1}(\frac{1}{2}) - \log 5$$

where $F(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a *hypergeometric function*.

Maple, Mathematica and Human

Proof. For $0 < k < 8$,

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx = \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}$$

Thus one can write

$$\sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\,dx,$$

which on substituting $y := \sqrt{2}x$ becomes

$$\int_0^1 \frac{16\,y - 16}{y^4 - 2\,y^3 + 4\,y - 4}\,dy = \int_0^1 \frac{4y}{y^2-2}\,dy - \int_0^1 \frac{4y-8}{y^2-2y+2}\,dy = \pi.$$

QED

In 1997, Fabrice Bellard of INRIA computed 152 binary digits of π starting at the trillionth position.

The computation took 12 days on 20 workstations working in parallel over the Internet.

Bellard's scheme is actually based on the following variant of (19):

$$\begin{aligned}\pi \; = \; & 4\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k+1)} \\ & - \frac{1}{64}\sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k}\left(\frac{32}{4k+1} + \frac{8}{4k+2} + \frac{1}{4k+3}\right)\end{aligned}$$

This formula permits individual hex or binary digits of π to be calculated roughly 43% faster than (19).

In 1998 Colin Percival, a 17-year-old student at Simon Fraser University, utilized 25 machines to calculate first the five trillionth hexadecimal digit, and then the ten trillionth hex digit.

In September, 2000, he found the quadrillionth binary digit is **0**, a computation that required **250 CPU-years, using 1734 machines in 56 countries**.

The table below gives computational results.

Position	Hex Digits Beginning At This Position
10^6	`26C65E52CB4593`
10^7	`17AF5863EFED8D`
10^8	`ECB840E21926EC`
10^9	`85895585A0428B`
10^{10}	`921C73C6838FB2`
10^{11}	`9C381872D27596`
1.25×10^{12}	`07E45733CC790B`
2.5×10^{14}	`E6216B069CB6C1`

BBP Formulas

Constants α of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)2^k}, \tag{20}$$

where $p(k)$ and $q(k)$ are integer polynomials, are said to be in the class of *binary BBP numbers*.

I illustrate for $\log 2$ why this permits one to calculate isolated digits in the binary expansion:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}. \tag{21}$$

We wish to compute a few binary digits beginning at position $d+1$.

This is equivalent to calculating $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part.

We can write

$$\begin{aligned} \{2^d \log 2\} &= \left\{\left\{\sum_{k=0}^{d} \frac{2^{d-k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\}\right\} \\ &= \left\{\left\{\sum_{k=0}^{d} \frac{\mathbf{2^{d-k}} \bmod \mathbf{k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{\mathbf{2^{d-k}}}{\mathbf{k}}\right\}\right\}. \qquad (22) \end{aligned}$$

The key observation is: the numerator of the first sum in (22), $2^{d-k} \bmod k$, can be calculated rapidly by the *binary algorithm for exponentiation*, performed modulo k.

That is, exponentiation is economically performed by a factorization based on the binary expansion of the exponent. For example,

$$3^{17} = ((((3^2)^2)^2)^2) \cdot 3$$

uses only five multiplications, not the usual 16.

- It is important to reduce each product module k — $3^{17} \bmod 10$ is done $3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3$.

One question that arose in the wake of this discovery is whether there is a formula of this type and an associated computational scheme to compute individual *decimal* digits of π.

Searches conducted by numerous researchers have been unfruitful.

- Recently D. Borwein (my father) W. Gallway and I have shown that there are no BBP formulas of the *Machin-type* of (19) unless the base is a power of two.

- Bailey and Crandall have shown exciting connections between the existence of a b-ary BBP formula for α and its *normality* base b.

David Bailey

- A *ternary BBP formula*

$$
\begin{aligned}
\pi^2 \;=\; & \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{9k}}\left\{\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}\right.\\
- & \frac{81}{(12k+4)^2}-\frac{27}{(12k+5)^2}-\frac{72}{(12k+6)^2}\\
- & \frac{9}{(12k+7)^2}-\frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}\\
+ & \left.\frac{1}{(12k+11)^2}\right\}
\end{aligned}
$$

. . . **Life of Pi.**

- At the end, Piscine (Pi) Molitor writes

> I am a person who believes in form, in harmony of order. Where we can, we must give things a meaningful shape. For example—I wonder—could you tell my jumbled story in exactly one hundred chapters, not one more, not one less? I'll tell you, that's one thing I hate about my nickname, the way that number runs on forever. It's important in life to conclude things properly. Only then can you let go.

We may not share the sentiment, but we should celebrate that Pi knows Pi to be irrational.

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