

# Future Challenges for Variational Analysis

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## Nonlinear Optimization

**ABSTRACT** Modern non-smooth analysis is now roughly thirty-five years old. I shall briefly assess where the subject stands today both as theory and regarding applications.

I will also discuss open problems and current challenges for the subject.



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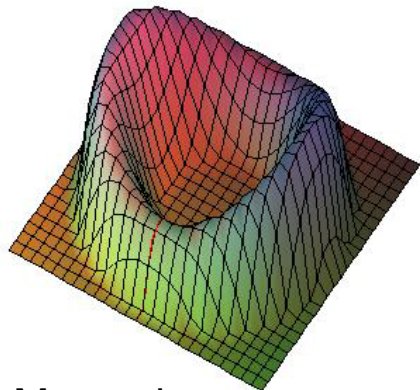
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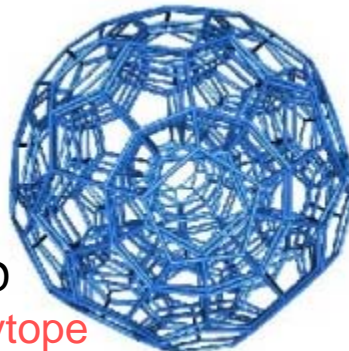
# OUTLINE

- **First Order Theory**
- **High Order Theory**
  - second (and higher)
- **Applications**
  - inside Mathematics
  - outside Mathematics
- I'll mention **successes** and **failures**
- Each item has open questions even in the convex case (**CA**)
  - some technical and specialized
  - some broader and general
- To work fruitfully in **VA** it helps to understand **CA** and smooth analysis (**SA**)
  - they are the motivating foundations
  - and often provide the key technical tools



Mountain pass

A facet of  
Coxeter's  
favourite 4D  
convex polytope



# LIPSCHITZ PRECURSORS

## Pshenichnyi (1968)

- **descriptive**
- **Large class of good “quasi-differentiable” functions**

$$f'(x; h) := \limsup_{t \rightarrow 0^+} \frac{f(x + th) - f(x)}{t}$$

- required to be convex in h

## Clarke (1972)

- **prescriptive**
- **New directional derivative**

$$f^\circ(x; h) := \limsup_{t \rightarrow 0^+, y \rightarrow x} \frac{f(y + th) - f(y)}{t}$$

- built to be convex in h

Both capture smooth and convex functions and are closed under + and  $\vee$

# FIRST ORDER THEORY

- **Subgradient**

- one-sided Fréchet  $\partial_F f(x)$   
(Gâteaux or Hadamard)

- **Viscosity subgradient**  $\partial_F^v f(x)$

- derivative of smooth (local) minorants
- in nice space  $\partial_F f(x) = \partial_F^v f(x)$

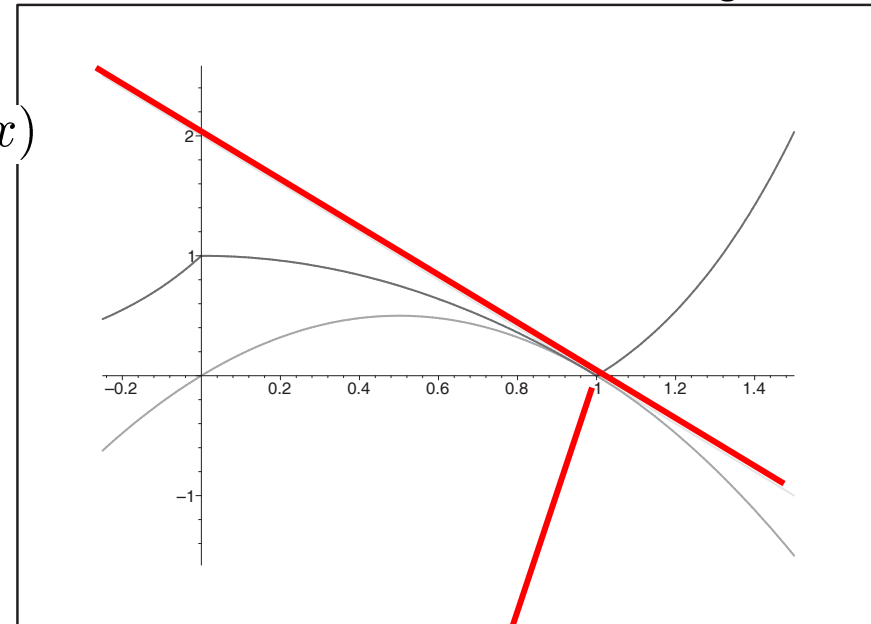
- **(Fuzzy) Sum Rule**

$$\partial_F(f + g)(x) \subseteq \partial_F f(x_1) + \partial_F g(x_2) + \varepsilon B_{X^*}$$

- **Normal cones**  $N_{\text{epi}f} := \partial \nu_{\text{epi}f}$

- **Variational principles (VP)**  $f(y) - g(y) \geq f(x) - g(x)$  for  $y$  near  $x$

A lsc function  $f$ ,  
a viscosity subgradient in red,  
and a smooth minorant  $g$



$$f(y) - g(y) \geq f(x) - g(x) \text{ for } y \text{ near } x$$

# STATE OF THE THEORY

## ACHIEVEMENTS

- Limiting subgradients

$$\partial f(x) := \lim_{y \rightarrow_f x} \partial_F f(x)$$

- Coderivatives of multis

$$D^*\Omega(x, y)(y^*) = \{x^* : (x^*, -y^*) \in N_{\text{gph}(\Omega)}(x, y)\}$$

- VP+Viscosity+Sum Rule

- yield **fine** 1<sup>st</sup> order theory for real-valued functions
- esp. in Lipschitz case or in finite dimensions
- **seq normal compactness needed more generally**; also for

- Metric regularity: locally

$$Kd(\Omega(x), y) \geq d(x, \Omega^{-1}(y))$$

- and its extensions
- implicit functions (Dontchev-Rock)
- alternating projections (Bauschke-Combettes)

I now ignore epsilons and the like

## LIMITATIONS

- Inapplicable outside of Asplund space (reflexive, ...)

–  $\partial_H f$  extensions fiddly, limited

- Theoretically very beautiful

– hard to compute even for ‘nice’ functions

Generically non-expansive functions have  $\partial f(x) \equiv B_{X^*}$

– **SNC** restriction is fundamental not technical

- better results rely on restricting classes of functions (and spaces)

– E.g., prox-normal, lower  $C_2$ ,  
– essentially smooth (B-Moors)

B-Sciffer (08) separable explicit construction for  $\partial f(x) \equiv B_{X^*}$

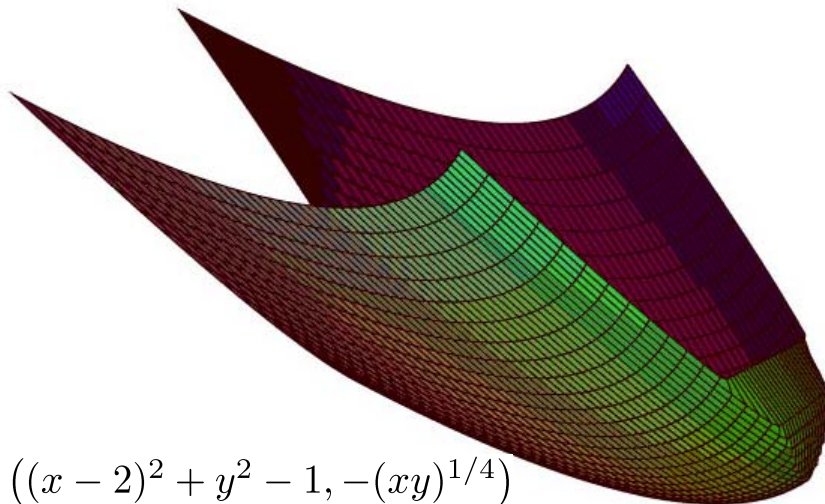
# OPEN QUESTIONS

## CONVEX

- Find a reflexive **Legendre function** generalization for convex functions without points of continuity such (neg) **Shannon entropy**

$$\mathcal{B}(x) := \int_0^1 x(t) \log x(t) dt$$

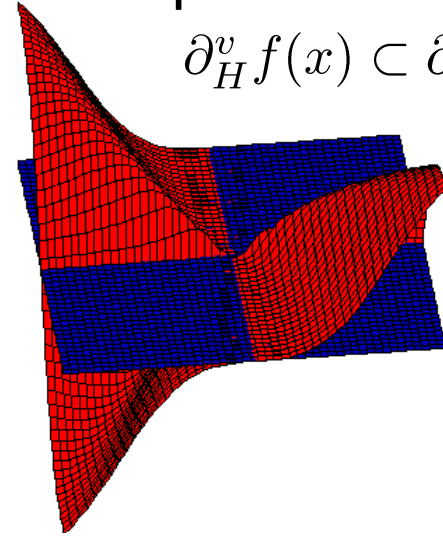
AN ESSENTIALLY STRICTLY CONVEX FUNCTION WITH  
NONCONVEX SUBGRADIENT DOMAIN  
AND WHICH IS NOT STRICTLY CONVEX



$$\max((x-2)^2 + y^2 - 1, -(xy)^{1/4})$$

## NONCONVEX

Is there a Lipschitz  $f$  on  $\ell^2$   
with  $\partial_H^v f(x) \subset \partial_H f(x)$ ?



$\frac{xy^3}{x^2+y^4}$  has  $0 \in \partial_H f(0)$  but  $0 \notin \partial_H^v f(0)$

- Is the product of a separable and a weak Asplund space still weak Asplund?
  - true for GDS, but WASP is the largest class on which refined NA calculus could be expected to work

# SECOND ORDER THEORY

Recall for **convex functions** the *difference quotient* of  $f$  by

$$\Delta_t f(x) : h \mapsto \frac{f(x + th) - f(x)}{t};$$

and the *second-order difference quotient* of  $f$  by

$$\Delta_t^2 f(x) : h \mapsto \frac{f(x + th) - f(x) - t\langle \nabla f(x), h \rangle}{\frac{1}{2}t^2}.$$

Analogously let

$$\Delta_t[\partial f](x) : h \mapsto \frac{\partial f(x + th) - \nabla f(x)}{t}.$$

For any  $t > 0$ ,

$$\Delta_t f(x)$$

is closed, proper, convex and nonnegative.

Quite beautifully

$$\partial \left[ \frac{1}{2} \Delta_t^2 f(x) \right] = \Delta_t[\partial f](x).$$

**Theorem** Alexandrov (1939) *In Euclidean space a real-valued continuous convex function admits a second order Taylor expansion at almost all points (w.r.t. Lebesgue measure).*

My favourite proof is a specialization of Mignot's 1976 extension to monotone operators [RW98, BV09]

The result relies on many happy coincidences in Euclidean space

The convex case is quite subtle and so the paucity of definitive non-convex results is no surprise

# STATE OF THE THEORY

## ACHIEVEMENTS

- Some lovely patterns and fine theorems in Euclidean space
  - but no definitive corpus of results
  - nor even canonical definitions outside of the convex case
  - interesting work by Jeyakumar 2006, Duta and others.
  - Many have noted

$$\partial^2 f(x) := \partial \nabla_G f(x)$$

is a fine object when function is Lipschitz smooth in separable Banach space (Rademacher's Thm. applies)

- Fundamental results by Ioffe-Penot (TAMS 1997) on limiting 2-subjets (and coderivatives)
- Fine refined calculus of 'efficient subset' of sub-hessians (Eberhard-Wenczel, SVA 2007)

## LIMITATIONS

- Little 'deep' work in infinite dim.
  - i.e., if obvious extensions fail
  - even in Hilbert space
- Outside separable Hilbert space general results are not to be expected [BV09]
  - research should focus on structured classes: integral functionals (Moussaoui-Seeger TAMS 1999)
  - composite convex functions

2-subjet: all order two expansions of  $C^2$  minorants agreeing at  $x$ . Now take limits

$$\bar{\partial}^2 f(x) := \limsup_{y \rightarrow_f x} \partial_-^2 f(y)$$

yields sum rule in para-convex case, etc.

- extensions by Eberhard-Ralph, others

# OPEN QUESTIONS

## CONVEX

- Does every continuous convex function on separable Hilbert space admit a **second order Gateaux expansion** at at least one point (or on a dense set of points)?
  - fails in non-separable Hilbert space or in  $\ell_p(\mathbf{N}), p \neq 2$
  - fails in the Fréchet sense even in  $\ell_2(\mathbf{N})$

## NONCONVEX

- Are there sizeable classes of functions for which **subjets** or other useful second order expansions can be built in separable Hilbert space?
  - I have no precise idea what “useful” means
  - even in convex case this is a tough request; then use
  - Lasry-Lions regularization (Penot, Eberhard, ...)

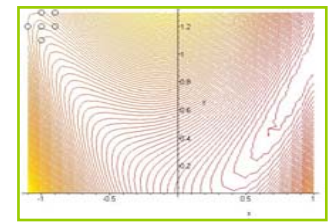
# APPLICATIONS

## SUCCESSSES

- Tools now part of pure non-linear and functional analysis
- Convergence theory for “pattern search” derivative-free optimization algorithms [Ma08]
- Eigenvalue and singular value optimization theory [BL05]
  - 2<sup>nd</sup> order (Lewis and Sendov)
- Differential Inclusions and Optimal Control
  - approx Maximum Principle
  - Hamilton-Jacobi equations
- Non-convex mathematical economics and MPECS
- Exact penalty and universal barrier methods [BV09]
  - oodles more counting convex analysis [BV04]

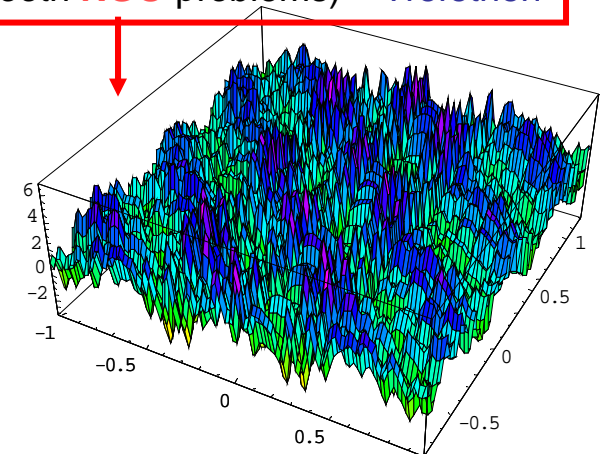
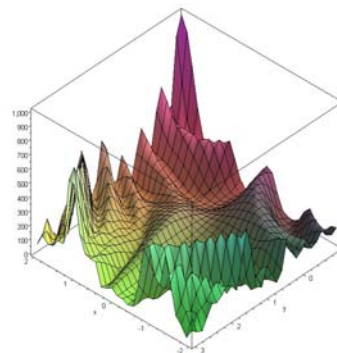
## FAILURES

- Limited numeric successes
  - even in convex case excluding spectral & SDP code (somewhat)
  - bundle methods (Lemaréchal et al)
  - Vanderbei’s LOCO package
  - composite convex & smoothing
  - need for “structured nonsmooth” optimization (a la Boyd)



Pattern search

Pinter (Smooth **NGO** problems) Trefethen



$$(x^*, y^*) \approx (-0.024627, 0.211789)$$

with minimum  $\approx -3.30687$

$$\min \exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2 + y^2)/4?$$

# THREE MORE OPEN QUESTIONS

## 1. The Chebyshev problem (Klee 1961)

If every point in  $H$  has a unique nearest point in  $C$  is  $C$  convex? (Yes: Motzkin-Bunt in Euclidean space [BL05])

## 2. Existence of nearest points (proximal boundary?)

Do some (many) points in  $H$  have a nearest point in  $C$  in every renorm of  $H$ ? (Yes: if the set is bounded or the norm is Kadec-Klee [BZ05])

## 3. Universal barrier functions in infinite dimensions

Is there an analogue for  $H$  of the universal barrier function so important in Euclidean space? (I doubt it [BV09])





# Convex Functions

**The Chebyshev problem** (Klee 1961) **A set is Chebyshev if every point in H has a unique nearest point in C**

**Theorem** If C is weakly closed and Chebyshev then C is convex. So in Euclidean space **Chebyshev iff convex.**

**Four Euclidean variational proofs** ([BL05], Opt Letters 07, [BV09])

1. **Brouwer's theorem** (Cheb. implies **sun** implies convex)
2. **Ekeland's theorem** (Cheb. implies **approx. convex** implies convex)
3. **Fenchel duality** (Cheb. iff  $d_C^2$  is Frechet) use  $f^*$  smooth implies  $f$  convex for

$$\left( \frac{\iota_C + \|\cdot\|^2}{2} \right)^* = \frac{\|\cdot\|^2 + d_C^2}{2}$$

**4. Inverse geometry** also shows if there is a counter-example it can be a **Klee cavern** (Asplund) the closure of the complement of a convex body. **WEIRD**

- **Counterexamples** exist in incomplete inner product spaces. #2 seems most likely to work in Hilbert space.
- **Euclidean case** is due to Motzkin-Bunt

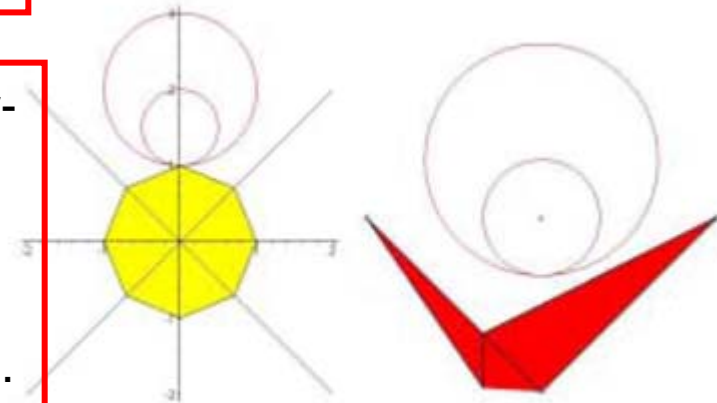


FIGURE 1. Suns and approximate convexity.



# Convex Functions

## Existence of nearest points

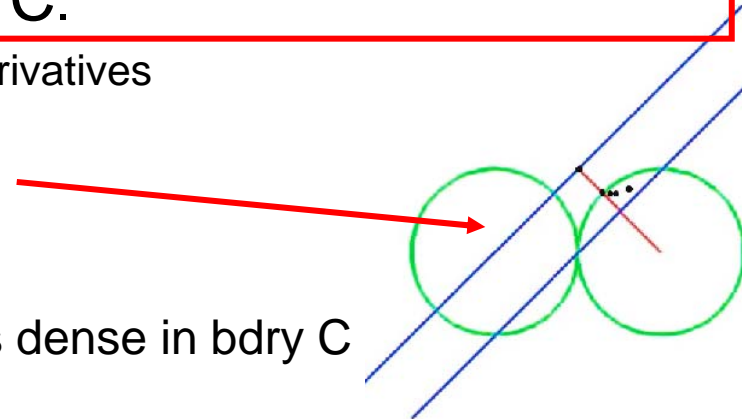
Do some (many) points in  $H$  have a nearest point in  $C$  in every renorm of  $H$  ?

**Theorem** (Lau-Konjagin, 76-86) A norm on a reflexive space is Kadec-Klee iff for every norm-closed  $C$  in  $X$  best approximations exist generically (densely) in  $X \setminus C$ .

Nicest proof is via dense existence of Frechet subderivatives

$$\phi \in \partial_F d_C(x)$$

The KK property forces approximate minimizers to line up.



- There are non KK norms with proximal points dense in bdry  $C$
- If  $C$  is closed and bounded then there are some points with nearest points (RNP)
- So a counterexample has to be a weird unbounded set in a rotten renorm (BFitzpatrick 89, [BZ09])

A norm is **Kadec-Klee** norm if weak and norm topologies agree on the unit sphere.

Hence all LUR norms are Kadec-Klee.



# Convex Functions

## Universal barrier functions in infinite dimensions

- Is there an analogue for  $H$  of the universal barrier function that is so important in Euclidean space?

**Theorem** (Nesterov-Nemirovskii) For any open convex set  $A$  in  $n$ -space, the function

$$F(x) := \lambda_N ((A - x)^o)$$

is an essentially smooth, log-convex barrier function for  $A$ .

- This relies heavily on the existence of **Haar measure** (Lebesgue).
- Amazingly for  $A$  the semidefinite matrix cone we **recover** – log det, etc

In Hilbert space the only really nice examples I know are similar to:

$$\phi(T) := \text{trace}(T) - \log \det(I + T)$$

is a strictly convex Frechet differentiable barrier function for the **Hilbert-Schmidt operators** with  $I+T > 0$ .

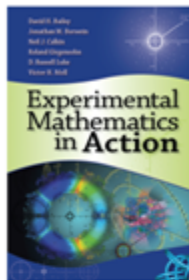
We are able to build barriers in great generality but not “universally” [BV09]

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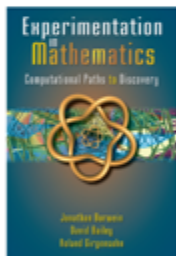


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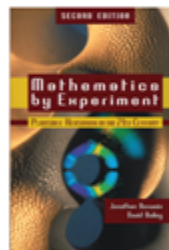
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