Hilbert's Inequality and Witten's Zeta-Function

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1 Introduction.

In this article we explore a variety of pleasing connections between analysis, number theory, and operator theory, while revisiting a number of beautiful inequalities originating with Hilbert, Hardy and others. We shall first the aforementioned Hilbert inequality [14], [18] and then apply it to various multiple zeta values. In consequence we obtain the norm of the classical Hilbert matrix, in the process illustrating the interplay of numerical and symbolic computation with classical mathematics.

2 Hilbert's (easier) inequality.

The inequality in question is:

Theorem 1 **(Hilbert Inequality)** *For nonnegative sequences* (a_n) *and* (b_n)*, not both zero, and for* p *and* q *satisfying* $1 \le p, q \le \infty$ *and* $1/p + 1/q = 1$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n\, b_m}{n+m} < \pi \csc\left(\frac{\pi}{p}\right) \|a_n\|_p \, \|b_n\|_q. \tag{1}$$

Here and throughout, we write $\|a_n\|_p := \left\{\sum_{n=1}^{\infty} \|a_n\|^p\right\}^{1/p}$ for the *p-norm* of the sequence (a_n). A preparatory lemma is needed.

Lemma 1 *If* $0 < a < 1$ *and* n *is a positive integer, then* (a)

$$\sum_{m=1}^{\infty} \frac{1}{(n+m)(m/n)^a} < \int_0^{\infty} \frac{1}{(1+x)\, x^a}\, dx < \frac{(1/n)^{1-a}}{1-a} + \sum_{m=1}^{\infty} \frac{1}{(n+m)(m/n)^a},$$

and (b)

$$\int_0^{\infty} \frac{1}{(1+x)\, x^a}\, dx = \pi \csc\left(a\, \pi\right).$$

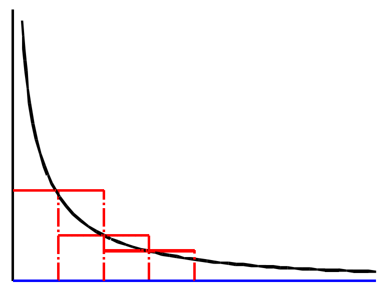

Figure 1: Riemann sums for $x^{-a}/(1+x)$.

Proof. (a) The inequalities comes from using standard rectangular approximations to a monotonic-decreasing integrand, as in Figure 2, and overestimating the integral from 0 to $1/n$ by $\int_0^{1/n} x^{-a}\,dx$ to produce

$$0 < \int_0^t \frac{1}{(1+x)\,x^a}\,dx \le \frac{t^{1-a}}{1-a}.$$

(b) The integral is found in various tables such as Abromovitz and Stegun [1] or Gradshteyn and Ryzhik [12] and is known to *Maple* or *Mathematica*. We offer two other proofs.

(i) For the first we exploit the geometric series and the monotone convergence theorem to compute

$$\begin{aligned}\int_0^\infty \frac{1}{(1+x)\,x^a}\,dx &= \int_0^1 \frac{x^{-a}+x^{a-1}}{1+x}\,dx \\ &= \sum_{n=0}^{\infty}(-1)^n\left\{\frac{1}{n+1-a}+\frac{1}{n+a}\right\} \\ &= \sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n+a}-\frac{1}{n-a}\right\}+\frac{1}{a} \\ &= \frac{1}{a}+\sum_{n=1}^{\infty}\frac{(-1)^n\,2a}{a^2-n^2}=\pi\csc(a\pi),\end{aligned}$$

since the last equality is the classical partial fraction identity for $\pi\csc(a\pi)$, (see [19, p. 255]).

(ii) Alternatively, we begin by letting $1+x=1/y$,

$$\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \int_0^1 y^{a-1}\,(1-y)^{-a}\,dy = B(a,1-a),$$

where B is the beta function, $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, which is expressible in terms of the gamma function Γ,

$$B(a, 1-a) = \Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin(a\pi)},$$

by using the product representation for Γ. □

Remark 1 Combining the arguments in (i) and (ii) above actually derives the identity

$$\Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin(a\,\pi)},$$

from the partial fraction expansion for cosecant

$$\pi \csc(a\pi) = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 - n^2}$$

or vice versa—especially if we appeal to the Bohr-Mollerup theorem [2], [19] to establish $B(a, 1-a) = \Gamma(a)\,\Gamma(1-a)$.

Proof of Theorem 1. Fix $\lambda > 0$. We apply Hölder's inequality with what Hardy calls "compensating difficulties" (inserting a term and its reciprocal) to obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n\,b_m}{n+m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n}{(n+m)^{1/p}(m/n)^{\lambda/p}}\,\frac{b_m}{(n+m)^{1/q}(n/m)^{\lambda/p}} \tag{2}$$

$$\le \left(\sum_{n=1}^{\infty}|a_n|^p \sum_{m=1}^{\infty}\frac{1}{(n+m)(m/n)^{\lambda}}\right)^{1/p} \left(\sum_{m=1}^{\infty}|b_m|^q \sum_{n=1}^{\infty}\frac{1}{(n+m)(n/m)^{\lambda q/p}}\right)^{1/q}$$

$$< \pi\ |\csc(\pi\,\lambda)|^{1/p}\ |\csc((q-1)\pi\,\lambda)|^{1/q}\ \|a_n\|_p\,\|b_m\|_q,$$

where the strict inequality follows from Lemma 1(a). That the left-hand side of (1) is no greater than $\pi\csc(\pi/p)\,\|a_n\|_p\,\|b_n\|_q$ is seen by setting $\lambda = 1/q$ and appealing to symmetry in p and q. □

The integral analogue of (1) may likewise be established. There are numerous extensions. One of interest for us later is

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n\,b_m}{(n+m)^{\sigma}} < \left\{\pi\csc\left(\frac{\pi(q-1)}{\sigma q}\right)\right\}^{\tau} \|a_n\|_p\,\|b_n\|_q, \tag{3}$$

true when $p, q > 1, \sigma > 0, 1/p + 1/q \ge 1$, and $\sigma + 1/p + 1/q = 2$. The best constant $C(p,q,\tau) \le \{\pi\csc(\pi(q-1)/(\sigma q))\}^{\tau}$ in (3) is called a *Hilbert constant* [11, sec.3.4].

For $p = 2$, (1) yields Hilbert's original inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n\,b_m}{n+m} \le \pi \sqrt{\sum_{n=1}^{\infty}|a_n|^2}\,\sqrt{\sum_{n=1}^{\infty}|b_n|^2}, \tag{4}$$

though Hilbert only obtained the constant 2π [13].

A fine direct Fourier analytic proof of (4) due to Toeplitz in 1912 starts from the observation that

$$\frac{1}{2\pi\, i}\int_0^{2\,\pi}(\pi-t)\, e^{int}\, dt=\frac{1}{n}$$

for $n=1,2\ldots$, and deduces that

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_n\, b_m}{n+m}=\frac{1}{2\pi\, i}\int_0^{2\,\pi}(\pi-t)\sum_{k=1}^{N}a_k\, e^{ikt}\sum_{k=1}^{N}b_k\, e^{ikt}\, dt. \tag{5}$$

We recover (4) by applying the integral form of the Cauchy-Schwarz inequality to the integral side of the representation in (5).

Example 1 Identity (5) has a quadratic counterpart:

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_n\, b_m}{(n+m)^2}=\frac{1}{2\pi}\int_0^{2\pi}\left(\zeta(2)-\frac{\pi t}{2}+\frac{1}{4}\right)\sum_{k=1}^{N}a_k\, e^{ikt}\sum_{k=1}^{N}b_k\, e^{ikt}\, dt,$$

where ζ signifies the Riemann zeta-function. Moreover, for larger integral σ, on setting

$$\psi_{2n}(x):=\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^{2n}},\qquad \psi_{2n+1}(x):=\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k^{2n+1}},$$

we have

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_n\, b_m}{(n+m)^\sigma}=\frac{1}{2\pi\, i^\sigma}\int_0^{2\pi}\psi_\sigma\left(\frac{t}{2\pi}\right)\sum_{k=1}^{N}a_k\, e^{ikt}\sum_{k=1}^{N}b_k\, e^{ikt}\, dt$$

where $\psi_\sigma(x)$ are related to the *Bernoulli polynomials* [1], [19] by

$$\psi_\sigma(x)=(-1)^{\lfloor(1+\sigma)/2\rfloor}\, B_\sigma(x)\,\frac{(2\pi)^\sigma}{2\,\sigma!},\qquad (0<x<1).$$

It follows that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n\, b_m}{(n+m)^\sigma}\le\|\psi_\sigma\|_{[0,1]}\,\|a\|_2\,\|b\|_2,$$

where $\|\psi_\sigma\|_{[0,1]}$ denotes the supremum norm. Finally, when $n>0$ we can compute

$$\|\psi_{2n}\|_{[0,1]}=\psi_{2n}(0)=\zeta(2n),\qquad \|\psi_{2n+1}\|_{[0,1]}=\psi_{2n+1}(1/4)=\beta(2n+1),$$

in terms of the classical *zeta-functions* $\zeta(2n):=\sum_{k>0}1/k^{2n}$ and $\beta(2n+1):=\sum_{k>0}(-1)^k/k^{2n+1}$. We should note that most of these upper-bounds are not optimal. $\square$

3 A BRIGHT AND AMUSING SUBJECT.

Hilbert's inequality and much more of the early twentieth-century history—and philosophy—of the " 'bright' and amusing" subject of inequalities charmingly discussed in Hardy's retirement lecture as London Mathematical Society Secretary,[13]. There is much in this article to reward close reading, especially on the nature of appropriateness of proof methods and the like. Hardy comments [13, p. 474] that

> Harald Bohr is reported to have remarked "Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove."

This remains true, though more recent inequalities often involve less-symmetric and less-linear objects such as entropies, divergences, and log-barrier functions [2], [6] such as as in the following *divergence estimate* [5, p. 63] for two discrete distributions:

Theorem 2 (**Kullback-Leibler**) *For two strictly positive sequences* $(p_i)_{i=1}^N$ *and* $(q_i)_{i=1}^N$ *with* $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 1$ *one has*

$$\sum_{n-1}^{\mathbf{N}} p_i \log\left(\frac{p_i}{q_i}\right) \geq \frac{1}{2}\left(\sum_{n=1}^{\mathbf{N}} |p_i - q_i|\right)^2, \tag{6}$$

Proof. Inequality (6) follows from establishing that the function $\phi : (0, \infty) \to \mathbb{R}$,

$$\phi(t) := 2\,(2+t)\,\{1 + t\log t - t\} - 3(t-1)^2,$$

is convex and is minimized at $t = 1$. One now lets $t_i = p_i/q_i$, homogenizes and sums. An application of the Cauchy-Schwarz inequality yields

$$\left(\sum_{i=1}^{n} |p_i - q_i|\right)^2 \leq 3\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + 2q_i} \leq 2\ \sum_{i=2}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

□

Two other high-spots in Hardy's essay are Carleman's inequality which states that for $a_i \geq 0$ and not all zero

$$\sum_{n=1}^{\infty} (a_1\, a_2 \cdot \ldots \cdot a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n$$

(see the recent survey [9] or [19, p. 63] for a proof, and also [3, p. 284] for an indication of why the constant e is best possible), and one of Hardy's own discoveries:

Theorem 3 (**Hardy**) *For a positive sequence* (a_k) *and* $p > 1$

$$\sum_{n=1}^{\infty}\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{7}$$

Hardy remarks [13, p. 485]:

> [My] own theorem was discovered as a by-product of my own attempt to find a really simple and elementary proof of Hilbert's.

Remark 2 For $p = 2$, Hardy reproduces Elliott's proof of (7), writing "it can hardly be possible to find a proof more concise or elegant."

Proof. This proof runs as follows. Set $A_n = a_1 + a_2 + \cdots + a_n$ (with $A_0 := 0$) and write

$$\begin{aligned} \frac{2a_n\,A_n}{n} - \left(\frac{A_n}{n}\right)^2 &= \frac{A_n^2}{n} - \frac{A_{n-1}^2}{n-1} + (n-1)\left(\frac{A_n}{n} - \frac{A_{n-1}}{n-1}\right)^2 \\ &\geq \frac{A_n^2}{n} - \frac{A_{n-1}^2}{n-1}. \end{aligned} \tag{8}$$

Today, this is something easy to check symbolically. Now sum to obtain

$$\sum_n \left(\frac{A_n}{n}\right)^2 \leq 2\sum_n \frac{a_n\,A_n}{n} \leq 2\sqrt{\sum_n a_n^2}\,\sqrt{\sum_n \left(\frac{A_n}{n}\right)^2}, \tag{9}$$

which proves (7) for $p = 2$. Indeed, this argument easily adapts to the general case. □

A pre-history of Hardy's inequality may be found in a very recent issue of this MONTHLY [16].

Finally we record the (harder) bilateral Hilbert inequality is

$$\left|\sum_{n\neq m\in\mathbf{Z}} \frac{a_n\,b_m}{n-m}\right| < \pi\sqrt{\sum_{n=1}^{\infty}|a_n|^2}\,\sqrt{\sum_{n=1}^{\infty}|b_n|^2}, \tag{10}$$

the best constant π being due to Schur in (1911) (see [17]). There are many extensions—with applications to prime number theory [17].

4 WITTEN ζ-FUNCTIONS.

We turn to a seemingly unrelated topic that, in the next section, will allow us to take a new perspective regarding the Hilbert constants. The sum

$$\mathcal{W}(r,s,t) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^r\,m^s\,(n+m)^t} \qquad (r,s,t>0)$$

is called a *Witten ζ-function* [21], [10], [8]. The double sum clearly converges for $r > 1$ and $s > 1$. We refer to [21] for a description of the uses of more general

Witten ζ-functions. Ours are also called *Tornheim double sums* [10], in honour of Tornheim who first carefully studied this specific case [20]. Correspondingly-

$$\zeta(t,s) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^s\,(n+m)^t} = \sum_{n>m>0} \frac{1}{n^t\,m^s}$$

is an *Euler double sum*. A sizable online set of references on multiple zeta values and Euler sums is found at `www.usna.edu/Users/math/meh/biblio.html`. For many illustrative proofs of the basic identity $\zeta(2,1) = \zeta(3)$ due to Euler, and of its generalizations, we refer to [4].

There is a simple algebraic relation

$$\mathcal{W}(r,s,t) = \mathcal{W}(r-1,s,t+1) + \mathcal{W}(r,s-1,t+1). \tag{11}$$

This is based on writing

$$\frac{m+n}{(m+n)^{t+1}} = \frac{m}{(m+n)^{t+1}} + \frac{n}{(m+n)^{t+1}}.$$

Clearly

$$\mathcal{W}(r,s,t) = \mathcal{W}(s,r,t), \tag{12}$$

and it is straight-forward to check that

$$\mathcal{W}(r,s,0) = \zeta(r)\,\zeta(s), \qquad \mathcal{W}(r,0,t) = \zeta(t,r). \tag{13}$$

Hence, $\mathcal{W}(s,s,t) = 2\,\mathcal{W}(s,s-1,t+1)$, so

$$\mathcal{W}(1,1,1) = 2\,\mathcal{W}(1,0,2) = 2\,\zeta(2,1) = 2\,\zeta(3).$$

We note that the analogue to (11), $\zeta(s,t) + \zeta(t,s) = \zeta(s)\,\zeta(t) - \zeta(s+t)$, shows that

$$\mathcal{W}(s,0,s) = 2\,\zeta(s,s) = \zeta^2(s) - \zeta(2s).$$

In particular, $\mathcal{W}(2,0,2) = 2\,\zeta(2,2) = \pi^4/36 - \pi^4/90 = \pi^4/72$.

Example 2 Let $a_n := 1/n^r$ and $b_n := 1/n^s$. Then inequality (4) becomes

$$\mathcal{W}(r,s,1) \le \pi\,\sqrt{\zeta(2r)}\,\sqrt{\zeta(2s)}. \tag{14}$$

Similarly, inequality (1) translates into

$$\mathcal{W}(r,s,1) \le \pi \csc\left(\frac{\pi}{p}\right)\,\sqrt[p]{\zeta(pr)}\,\sqrt[q]{\zeta(qs)}. \tag{15}$$

Indeed, (3) can be used to estimate $\mathcal{W}(r,s,\tau)$ for somewhat broader $\tau (\neq 1)$. Thence, (14) implies that $\zeta(3) \le \pi^3/3$, on appealing to equation (18). □

More generally, recursive use of (11) and (12), along with the initial conditions (13), shows that *all integer $\mathcal{W}(s,r,t)$ values are expressible in terms of double (and single) Euler sums.* As we shall see in (20) the representations are necessarily homogeneous polynomials of weight $r+s+t$. All double sums of weights less than eight and all those of odd weights reduce to sums of products of single variable zeta-values, [3]. The first impediments arise because $\zeta(6,2)$, and $\zeta(5,3)$ are not so *reducible*.

We next observe that in terms of the *polylogarithm* defined by $Li_s(t) = \sum_{n>0} t^n/n^s$ for real s, the representation (5) yields

$$\mathcal{W}(r,s,1) = \frac{1}{2\pi\, i} \int_{-\pi}^{\pi} \sigma\,\mathrm{Li}_r(-e^{i\sigma})\,\mathrm{Li}_s(-e^{i\sigma})\,d\sigma. \tag{16}$$

This representation is not numerically effective. It is better to start with

$$\frac{\Gamma(s)}{(m+n)^t} = \int_0^1 (-\log\sigma)^{t-1}\,\sigma^{m+n-1}\,d\sigma$$

and so to obtain

$$\mathcal{W}(r,s,t) = \frac{1}{\Gamma(t)} \int_0^1 \mathrm{Li}_r(\sigma)\,\mathrm{Li}_s(\sigma)\,\frac{(-\log\sigma)^{t-1}}{\sigma}\,d\sigma. \tag{17}$$

This real-variable analogue of (16) is somewhat more satisfactory computationally. For example, we recover from it an analytic proof of

$$2\,\zeta(2,1) = \mathcal{W}(1,1,1) = \int_0^1 \frac{\ln^2(1-\sigma)}{\sigma}\,d\sigma = 2\,\zeta(3). \tag{18}$$

Moreover, we can now discover analytic as opposed to algebraic relations. Integration by parts yields

$$\mathcal{W}(r,s+1,1) + \mathcal{W}(r+1,s,1) = \mathrm{Li}_{r+1}(1)\,\mathrm{Li}_{s+1}(1) = \zeta(r+1)\,\zeta(s+1). \tag{19}$$

In particular, $\mathcal{W}(s+1,s,1) = \zeta^2(s+1)/2$.

Example 3 Symbolically, *Maple* immediately evaluates $\mathcal{W}(2,1,1) = \pi^4/72$, and while it fails directly with $\mathcal{W}(1,1,2)$, we know that $\mathcal{W}(1,1,2)$ must be a multiple of π^4 or equivalently $\zeta(4)$. Numerically (working to twenty places) we obtain

$$\mathcal{W}(1,1,2)/\zeta(4) = .49999999999999999998\ldots.$$

Continuing, for $r+s+t=5$ the only terms to consider are $\zeta(5)$ and $\zeta(2)\zeta(3)$. The *integer relation method PSLQ* as implemented in *Maple* yields the weight five relations:

$$\mathcal{W}(2,2,1) = \int_0^1 \frac{\mathrm{Li}_2\,(x)^2}{x}\,dx = 2\,\zeta\,(3)\,\zeta(2) - 3\,\zeta\,(5)\,,$$

$$\mathcal{W}(2,1,2) = \int_0^1 \frac{\mathrm{Li}_2(x)\log(1-x)\log(x)}{x}\,dx = \zeta(3)\,\zeta(2) - \frac{3}{2}\,\zeta(5)\,,$$

$$\mathcal{W}(1,1,3) = \int_0^1 \frac{\log^2(x)\,\log^2(1-x)}{2\,x}\,dx = -2\,\zeta(3)\,\zeta(2) + 4\,\zeta(5)\,,$$

$$\mathcal{W}(3,1,1) = \int_0^1 \frac{\mathrm{Li}_3(x)\log(1-x)}{x}\,dx = -\zeta(3)\,\zeta(2) + 3\,\zeta(5)\,,$$

as predicted.

Likewise, for $r+s+t=6$ the only terms we need to consider are $\zeta(6), \zeta^2(3)$ since $\zeta(6), \zeta(4)\,\zeta(2)$ and $\zeta^3(2)$ are all rational multiples of π^6. We recover identities like

$$\mathcal{W}(3,2,1) = \int_0^1 \frac{\mathrm{Li}_3(x)\ \mathrm{Li}_2(x)}{x}\,dx = \frac{1}{2}\,\zeta^2(3)\,,$$

consistent with equation (19). □

The general form of the reduction for integers r, s, and t is due to Tornheim, and expresses $\mathcal{W}(r,s,t)$ in terms of $\zeta(a,b)$ with *weight* $a+b=N:=r+s+t$ [20], [10]:

Theorem 4 *For positive integers* $r, s,$ *and* t

$$\mathcal{W}(r,s,t) = \sum_{i=1}^{r\vee s} \left\{ \binom{r+s-i-1}{s-1} + \binom{r+s-i-1}{r-1} \right\} \zeta(i, N-i)\,. \tag{20}$$

Various other general formulas are given in [10] for classes of sums such as $\mathcal{W}(2n+1, 2n+1, 2n+1)$ and $\mathcal{W}(2n, 2n, 2n)$.

5 The best Hilbert constant.

It transpires that the constant π used in Theorem 1 is best possible [14].

Example 4 Let us numerically explore the ratio

$$\mathcal{R}(s) := \frac{\mathcal{W}(s,s,1)}{\pi\,\zeta(2s)}$$

as $s \to 1/2^+$. Note that $\mathcal{R}(1) = 12\,\zeta(3)/\pi^3 \sim 0.4652181552\ldots$.

Further numerical explorations seem to be in order. Unfortunately, when $1/2 < s < 1$, (17) is very hard to exploit numerically. This fact led us to look for a more sophisticated attack along the line of the Hurwitz zeta and Bernoulli polynomial integrals used in [10], or the expansions in [8]. Namely, we appeal to the identity

$$\mathcal{W}(r,s,t) = \int_0^1 E(r,x)\,E(s,x)\,\overline{E(t,x)}\,dx \tag{21}$$

where $E(s,x) := \sum_{n=1}^{\infty} e^{2\pi i n x}\, n^{-s} = \mathrm{Li}_s\left(e^{2\pi i x}\right)$, using the formulae

$$E(s,x) = \sum_{m=0}^{\infty} \zeta(s-m)\, \frac{(2\pi i\, x)^m}{m!} + \Gamma(1-s)\,(-2\pi i\, x)^{s-1} \qquad (|x|<1)$$

and

$$E(s,x) = -\sum_{m=0}^{\infty} \eta(s-m)\, \frac{(2x-1)^m\,(\pi i)^m}{m!} \qquad (0<x<1)$$

with $\eta(s) := (1-2^{1-s})\,\zeta(s)$, as given in [8, (2.6)(2.9)]. □

Ultimately, carefully expanding (21) with a free parameter θ in $(0,1)$, led Crandall to the following efficient formula, in terms of the *incomplete Gamma-function*, which is given by $\Gamma(a,z) := \int_z^{\infty} \exp(-t)\, t^{a-1}\, dt$ when $\mathrm{Re}\, a > 0$ [1]. Of course $\Gamma(a,0) = \Gamma(a)$.

Proposition 1 *If neither r nor s is an integer, then*

$$\begin{aligned} \Gamma(t)\mathcal{W}(r,s,t) &= \sum_{m,n\ge 1} \frac{\Gamma(t,(m+n)\theta)}{m^r n^s (m+n)^t} \\ &+ \sum_{u,v\ge 0} (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)\theta^{u+v+t}}{u!v!(u+v+t)} \\ &+ \Gamma(1-r)\sum_{v\ge 0}(-1)^v \frac{\zeta(s-v)\theta^{r+v+t-1}}{v!(r+v+t-1)} \\ &+ \Gamma(1-s)\sum_{u\ge 0}(-1)^u \frac{\zeta(r-u)\theta^{s+u+t-1}}{u!(s+u+t-1)} \\ &+ \Gamma(1-r)\Gamma(1-s)\frac{\theta^{r+s+t-2}}{r+s+t-2}. \end{aligned} \tag{22}$$

When at least one of r,s is an integer, a limit formula with a few more terms results. As is often the case, the analytically attractive and the computationally effective representations are quite different.

We can now use (22) to give an accurate plot of $\mathcal{R}$ on $[1/3, 2/3]$, as shown in Figure 2. Note that Figure 2 shows that, while the functions $\mathcal{R}$ and $\mathcal{I}$ do agree at $1/2$, the one is increasing but the other is decreasing. In various ways we are thus led to the following conjecture; and in turn to a proof thereof.

Conjecture 1 $\lim_{s\to 1/2} \mathcal{R}(s) = 1$.

Proof of Conjecture 1. (a) To establish this, we introduce $\sigma_n(s) := \sum_{m=1}^{\infty} n^s m^{-s}/(n+m)$ and invoke Lemma 1 to write

$$\mathcal{L} : \;=\; \lim_{s\to 1/2}(2s-1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n^{-s}\,m^{-s}}{n+m} = \lim_{s\to 1/2}(2s-1)\sum_{n=1}^{\infty}\frac{1}{n^{2s}}\,\sigma_n(s)$$

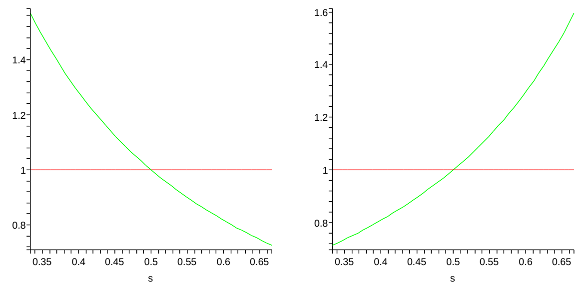

Figure 2: $\mathcal{R}$ (left) and $\mathcal{I}$ (right) on $[1/3, 2/3]$.

$$\begin{aligned}
&= \lim_{s\to 1/2} (2s-1) \sum_{n=1}^{\infty} \frac{\{\sigma_n(s) - \pi \csc(\pi s)\}}{n^{2s}} \\
&+ \lim_{s\to 1/2} \pi\,(2s-1)\zeta(2s)\csc(\pi\, s) \\
&= 0 + \pi = \pi.
\end{aligned}$$

Here, by another appeal to Lemma 1, the bracketed term in the series is $O(n^{s-1})$ while $\zeta(2s) \sim 1/(2s-1)$ as $s \to 1/2^{+}$, using the standard asymptotic for ζ [2]. In consequence, we see that $\mathcal{L} = \lim_{s\to 1/2} \mathcal{R}(s) = 1$, and—at least to first-order—inequality (4) is best possible (see also [15]).

(b) Alternatively, we can sum directly as follows:

$$\begin{aligned}
\mathcal{W}(s,s,1) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m^{-s}n^{-s}}{m+n} &= 2\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m=1}^{n-1} \frac{1/n}{(m/n)^s(m/n+1)} + \frac{\zeta(2s+1)}{2} \\
&\le 2\,\zeta(2s) \int_0^1 \frac{x^{-s}}{1+x}\,dx + \frac{\zeta(2s+1)}{2} \\
&\le 2\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m=1}^{n} \frac{1/n}{(m/n)^s(m/n+1)} + \frac{\zeta(2s+1)}{2} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m=1}^{n-1} \frac{1/n}{(m/n)^s(m/n+1)} + \frac{3\zeta(2s+1)}{2} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m^{-s}n^{-s}}{m+n} + \zeta(2s+1).
\end{aligned}$$

We deduce that

$$\mathcal{R}(s) \sim \mathcal{I}(s) := 2/\pi \int_0^1 x^{-s}/(1+x)\,dx$$

as $s \to 1/2$. Also $\mathcal{I}(1/2) = 1$. $\square$

Likewise, the constant in (1) is best possible.

Proof. Motivated by the foregoing argument we consider

$$\mathcal{R}_p(s) := \frac{\mathcal{W}((p-1)s, s, 1)}{\pi\,\zeta(ps)},$$

and observe that with $\sigma_n^p(s) := \sum_{m=1}^{\infty}(n/m)^{-(p-1)s}/(n+m)$—which satisfies $\sigma_n^p(s) \to \pi\csc(\pi/q)\,, (1/q + 1/p = 1)$ as $n \to \infty$ and $s \to 1/p$—we have

$$\begin{aligned}
\mathcal{L}_p : &= \lim_{s\to 1/p}(ps-1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n^{-s}\,m^{-(p-1)s}}{n+m} = \lim_{s\to 1/p}(ps-1)\sum_{n=1}^{\infty}\frac{1}{n^{ps}}\,\sigma_n^p(s)\\
&= \lim_{s\to 1/p}(ps-1)\sum_{n=1}^{\infty}\frac{\{\sigma_n^p(s) - \pi\csc(\pi/q))\}}{n^{ps}}\\
&+ \lim_{s\to 1/p}(2s-1)\zeta(ps)\,\pi\csc\left(\frac{\pi}{q}\right) = 0 + \pi\csc\left(\frac{\pi}{q}\right).
\end{aligned}$$

Setting $r = (p-1)s$, for s near $1/p$ we check that $\zeta(ps)^{1/p}\,\zeta(qr)^{1/q} = \zeta(ps)$, whence the best constant possible in (15) is the one given. □

To recapitulate our narrative, in terms of the celebrated infinite *Hilbert matrices* [3, pp. 250–252],

$$\mathcal{H}_0 := \left\{\frac{1}{m+n}\right\}_{m,n=1}^{\infty}$$

and

$$\mathcal{H}_1 := \left\{\frac{1}{m+n-1}\right\}_{m,n=1}^{\infty},$$

we have actually proven:

Theorem 5 *If $1 < p, q < \infty$ and with $1/p + 1/q = 1$, then the Hilbert matrices $\mathcal{H}_0$ and $\mathcal{H}_1$ determine bounded linear mappings from the sequence space ℓ^p to itself such that*

$$\|\mathcal{H}_1\|_{p,p} = \|\mathcal{H}_0\|_{p,p} = \lim_{s\to 1/p}\frac{\mathcal{W}(s,(p-1)s,1)}{\zeta(ps)} = \pi\csc\left(\frac{\pi}{p}\right).$$

Proof. Appealing to the isometry between $(\ell^p)^*$ and ℓ^q, and given our earlier evaluation of $\mathcal{L}_p$, we directly compute the operator norm of $\mathcal{H}_0$ as follows:

$$\begin{aligned}
\|\mathcal{H}_0\|_{p,p} : &= \sup_{\|x\|_p=1}\|\mathcal{H}_0 x\|_p\\
&= \sup_{\|y\|_q=1}\sup_{\|x\|_p=1}\langle\mathcal{H}_0 x, y\rangle = \pi\csc\left(\frac{\pi}{p}\right).
\end{aligned}$$

Now clearly $\|\mathcal{H}_0\|_{p,p} \le \|\mathcal{H}_1\|_{p,p}$. For $n \ge 2$ we have

$$\sum_{m=1}^{\infty}\frac{1}{(n+m-1)(m/n)^a} \le \sum_{m=1}^{\infty}\frac{1}{(n-1+m)(m/(n-1))^a} \le \pi\csc(\pi a),$$

and so Lemma 1 and Theorem 1 in tandem show that $\|\mathcal{H}_0\|_{p,p} \geq \|\mathcal{H}_1\|_{p,p}$. □A

delightful operator-theoretic introduction to the Hilbert matrix $\mathcal{H}_0$ is given by Choi in his Chauvenet-prize winning article [7] while a recent set of notes by G. J. O. Jameson (see [15]) is also well worth accessing.

In the case of (3), Finch [11, §4.3] comments that the issue of best constants is unclear in the literature. He remarks that even the case $p = q = 4/3$ and $\sigma = 1/2$ appears to be open. It seems improbable that the techniques of this article can be used to resolve the question. Indeed, consider

$$\mathcal{R}_{1/2}(s, \alpha) := \frac{\mathcal{W}(s, s, 1/2)}{\zeta(4s/3)^{\alpha}},$$

with the critical point in this case being $s = 3/4$. Numerically, using (22), we discover that $\log(\mathcal{W}(s, s, 1/2))/\log(\zeta(4s/3)) \to 0$. Hence, for any positive α, the requisite limit is given by $\lim_{s\to 3/4} \mathcal{R}_{1/2}(s, \alpha) = 0$, which is certainly not the desired norm. What we are exhibiting is that the subset of sequences $(a_n) = (n^{-s})$ for $s > 0$ is *norming* in ℓ^p for $\sigma = 1$ but not apparently for general $\sigma > 0$.

Example 5 One may also study the corresponding behaviour of Hardy's inequality (7). For example, setting $a_n = 1/n$ and writing $H_n = \sum_{k=1}^{n} 1/k$ in (7) yields

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \zeta(p).$$

Application of the integral test shows that

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^p \sim \int_1^{\infty} \left(\frac{\log x}{x}\right)^p dx = \frac{\Gamma(1+p)}{(p-1)^{p+1}},$$

when $p > 1$. Also

$$\lim_{p\to 1+} \left(\frac{p}{p-1}\right)^p \zeta(p) \frac{(p-1)^{1+p}}{\Gamma(1+p)} = 1.$$

(This is a limit that both *Maple* and *Mathematica* will compute.) This shows the constant is again best possible. □

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